CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 12, Pages 58–66 (March 13, 2008)
S 1088-4173(08)00176-8

HOLES AND MAPS OF EUCLIDEAN DOMAINS

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Abstract. We study the behavior of the quasiconvexity and bounded turning of holes of domains under quasisymmetric and bilipschitz maps.

1. Introduction

1.1. *Holes of a domain.* Let $G \subset \dot{\mathsf{R}}^n = \mathsf{R}^n \cup \{\infty\}$ be a domain (open connected nonempty set). By a *hole* of G we mean a component of the complement $\complement G = \dot{\mathsf{R}}^n \setminus G$ of G. For each hole E of G, the boundary ∂E is a component of ∂G; see [Ne, V.14.5].

Let f be a homeomorphism of G onto another domain $G' \subset \dot{\mathsf{R}}^n$. It is well known that to each hole E of G there corresponds a unique hole E' of G' such that if (x_i) is a sequence in G converging to a point in E, then the sequence $(f(x_i))$ clusters to E'. In fact, we shall only consider the case where f is quasisymmetric, and then f extends to a homeomorphism $\bar{f}\colon \bar{G} \to \bar{G}'$, and E' is determined by $\partial E' = \bar{f}\partial E$.

In this article we are interested in the following question: If $f\colon G \to G'$ is quasisymmetric or bilipschitz, what can we say of the metric properties (bounded turning or quasiconvexity) of the hole E' if E has the corresponding property? For quasiconvexity, we only consider the case $n = 2$. In Section 4 we change the roles of open and closed sets and consider continua instead of domains.

1.2. **Definitions**. We let $\dot{\mathsf{R}}^n = \mathsf{R}^n \cup \{\infty\}$ denote the one-point extension of the Euclidean n-space R^n. For $c \geq 1$, a set $A \subset \dot{\mathsf{R}}^n$ is of *c-bounded turning* if each pair of points $a, b \in A$ can be joined by a continuum $F \subset A$ with diameter $d(F) \leq c|a-b|$. If one of the points a, b is ∞, the inequality is always understood to hold in the form $\infty \leq \infty$. The continuum F can be chosen to be an arc if A is a G_δ-set and if c is replaced by any number $c_1 > c$; see [NV, 4.3].

A set $A \subset \dot{\mathsf{R}}^n$ is *c-quasiconvex* if each pair $a, b \in A$ can be joined by a path (equivalently an arc) γ in A with length $l(\gamma) \leq c|a-b|$.

For $A \subset \mathsf{R}^n$ and $L \geq 1$, a map $f\colon A \to \mathsf{R}^n$ is *L-bilipschitz* if

$$|x-y|/L \leq |f(x)-f(y)| \leq L|x-y|$$

for all $x, y \in A$. If $A \subset \dot{\mathsf{R}}^n$, a map $f\colon A \to \dot{\mathsf{R}}^n$ is L-bilipschitz if $f(\infty) = \infty$ and if $f|A \setminus \{\infty\}$ is L-bilipschitz.

Let $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism. If $A \subset \mathsf{R}^n$, an injective map $f\colon A \to \mathsf{R}^n$ is *η-quasisymmetric* if

$$\frac{|f(a)-f(x)|}{|f(b)-f(x)|} \leq \eta\left(\frac{|a-x|}{|b-x|}\right)$$

Received by the editors November 13, 2007.
2000 *Mathematics Subject Classification.* Primary 30C65.
Key words and phrases. Quasisymmetric, bilipschitz, bounded turning, quasiconvex.

for each triple x, a, b of distinct points in A. If $A \subset \dot{\mathsf{R}}^n$, an injective map $f\colon A \to \dot{\mathsf{R}}^n$ is η-quasisymmetric if $f(\infty) = \infty$ and if $f|A \setminus \{\infty\}$ is η-quasisymmetric.

The case where all holes of G are of bounded turning is implicitly contained in [Vä2]:

1.3. **Theorem**. *Let $G \subset \dot{\mathsf{R}}^n$ be a domain such that each hole of G is of c-bounded turning and let $f\colon G \to \dot{\mathsf{R}}^n$ be η-quasisymmetric. Then each hole of the domain $G' = fG$ is of c'-bounded turning with $c'(c, \eta)$.*

Proof. From [Vä2, 3.10] it follows that the c-bounded turning of all holes of a domain G is quantitatively equivalent to the outer $(0, c)$-joinability of $\complement G$. For terminology, see [Vä2, 2.2]. By the duality theorem [Vä2, 2.7], this property is quantitatively equivalent to the inner $(n-2, c)$-joinability of G. As all joinability properties are quantitatively preserved by η-quasisymmetric maps [Vä2, 4.3], the theorem follows. □

1.4. *Remarks.* (1) For simply connected domains $G \subset \dot{\mathsf{R}}^2$, Theorem 1.3 also follows from the fact that $\complement G$ is of bounded turning iff G is a John domain and from the quasisymmetric invariance of John domains; see [NV, 3.6, 3.7, and 4.5].

(2) It is not true that if $f\colon G \to G'$ is an η-quasisymmetric homeomorphism and if a hole E of G is of c-bounded turning, then the corresponding hole E' of G' is of c'-bounded turning with $c'(c, \eta)$. A counterexample is given in 3.4.

It is natural to ask whether Theorem 1.3 remains true if bounded turning is replaced by quasiconvexity and η-quasisymmetry by L-bilipschitz. In 3.2 we prove this for $n = 2$. This will follow from a more general result 3.1 where f and G are as in 1.3 with additional conditions given on a particular hole of G. The case $n \geq 3$ remains open.

The proof makes use of some extension results of Ghamsari, Näkki and the author in [GNV].

I thank Pekka Koskela for calling my attention to this problem.

1.5. *Notation.* In the rest of the paper we shall work in the extended plane $\dot{\mathsf{R}}^2 = \mathsf{R}^2 \cup \{\infty\}$. Given a set $A \subset \dot{\mathsf{R}}^2$, its complement $\complement A$, closure $\bar{A}$ and boundary ∂A are always considered in the topology of $\dot{\mathsf{R}}^2$. For open and closed disks and for circles we use the customary notation $B(x, r), \bar{B}(x, r), S(x, r)$, where the center x may be omitted if $x = 0$. In particular, $B(1)$ is the open unit disk.

2. Preliminaries

2.1. *John disks.* Several equivalent characterizations for John disks are given in [NV]. In the present article it is convenient to say that a domain $D \subset \dot{\mathsf{R}}^2$ is a *c-John disk* if its complement $\complement D$ is of c-bounded turning and contains more than one point; see [NV, 4.5]. The concept of a John domain is not needed in this paper except in the following lemma, which is readily obtained by choosing a John center of D.

2.2. **Lemma**. *If $D \subset \mathsf{R}^2$ is a bounded c-John disk, then there is a point $x_0 \in D$ and numbers $r > 0$ and $c' = c'(c) \geq 1$ such that $\partial D \subset \bar{B}(x_0, c'r) \setminus B(x_0, r)$.*

From [GNV, 4.1 and 4.2] we get the following.

2.3. **Theorem**. *Suppose that D and D' are unbounded c-John disks in $\dot{\mathsf{R}}^2$ and that $f\colon \bar{D} \to \bar{D}'$ is a homeomorphism such that $f|\partial D$ is L-bilipschitz. Then $f|\partial D$ extends to an L'-bilipschitz homeomorphism $f_1\colon \dot{\mathsf{R}}^2 \to \dot{\mathsf{R}}^2$ with $L'(c, L)$ such that $f_1 D = D'$.* □

2.4. *Inversion.* If one of the domains D, D' in 2.3 is bounded, the theorem is not true; see [NV, 3.11]. We shall make use of auxiliary inversions to reduce the bounded case to the unbounded case.

We let $u\colon \dot{\mathsf{R}}^2 \to \dot{\mathsf{R}}^2$ denote the inversion $u(x) = x/|x|^2$. This map does not, in general, preserve the quasiconvexity or the bounded turning of a set. For example, consider a line segment with midpoint at the origin. In the next lemma we show that these properties are preserved in a special case.

2.5. **Lemma**. *Let $R \ge 1$ and let $E \subset \dot{\mathsf{R}}^2$ be a closed set with $\partial E \subset A(R) = \bar{B}(R) \setminus B(1/R)$.*

(1) *The inversion u defines an R^2-bilipschitz homeomorphism $u_R\colon A(R) \to A(R)$.*

(2) *If $E \cap A(R)$ is c-quasiconvex, then E is c-quasiconvex.*

(3) *If E is c-quasiconvex, then $E \cap A(R)$ is πc-quasiconvex .*

(4) *If E is c-quasiconvex, then uE is c'-quasiconvex with $c' = \pi R^4 c$.*

(5) *If $E \cap A(R)$ is of c-bounded turning, then E is of c-bounded turning.*

(6) *If E is of c-bounded turning, then $E \cap A(R)$ is of $2c_1$-bounded turning for every $c_1 > c$.*

(7) *If E is of c-bounded turning, then uE is of c'-bounded turning with $c' = 2R^4 c_1$ for every $c_1 > c$.*

Proof. Part (1) follows from the formula $|u(x) - u(y)| = |x - y|/|x||y|$.

In (2)–(7) there are four possibilities: The set E is the union of $E \cap A(R)$ and one of the sets $B(1/R)$, $\complement\bar{B}(R)$, $B(1/R) \cup \complement\bar{B}(R)$, $\varnothing$. We prove the first case, the proof in the other cases is almost similar. The last two cases are not needed in this paper.

(2) Let $a \in B(1/R)$, $b \in E \cap A(R)$. Let $x \in [a, b]$ be the point with $|x| = 1/R$. There is an arc $\alpha \subset E \cap A(R)$ from x to b with $l(\alpha) \le c|x - b|$. Now $\gamma = [a, x] \cup \alpha$ is an arc from a to b in E with $l(\gamma) \le c|a - b|$.

(3) Let $a, b \in E \cap A(R)$. There is an arc $\alpha \subset E$ from a to b with $l(\alpha) \le c|a - b|$. We may assume that α meets $B(1/R)$. Let a_1 and b_1 be the first and the last point of α in $S(1/R)$ and let β be the shorter arc of $S(1/R)$ between a_1 and b_1. Now $\gamma = \alpha[a, a_1] \cup \beta \cup \alpha[b_1, b]$ is an arc from a to b in $E \cap A(R)$. As $l(\beta) \le \pi|a_1 - b_1| \le \pi l(\alpha[a_1, b_1])$, we obtain $l(\gamma) \le \pi l(\alpha) \le \pi c|a - b|$.

(4) By (1) and (3) we see that $(uE) \cap A(R) = u[E \cap A(R)]$ is $\pi R^4 c$-quasiconvex. Hence uE is $\pi R^4 c$-quasiconvex by (2).

(5) Similar to (2).

(6) Let $a, b \in E \cap A(R)$. By 1.2 there is an arc $\alpha \subset E$ from a to b with $d(\alpha) \le c_1|a - b|$. Let $\gamma = \alpha[a, a_1] \cup \beta \cup \alpha[b_1, b]$ be as in (3). If $x \in \alpha[a, a_1]$ and $y \in \beta$, then

$$|x - y| \le |x - a_1| + |a_1 - y| \le d(\alpha) + |a_1 - b_1| \le 2d(\alpha),$$

whence $d(\gamma) \le 2c_1|a - b|$.

Remark. The constant c_1 can be replaced by c in (6) and (7) either by a limiting process or by using continua instead of arcs together with [HY, 2-16].

(7) Similar to (4). □

2.6. *Note to the reader.* In 2.8 we prove a variant of 2.3. However, if you are only interested in domains with a finite number of holes, Theorem 2.3 is sufficient in the sequel, and the rest of the present section can be skipped; see 2.9.

2.7. *Basic assumptions.* In the rest of this section we assume that $G \subset \dot{\mathsf{R}}^2$ is a domain whose holes are of c-bounded turning and that $f\colon G \to G' \subset \dot{\mathsf{R}}^2$ is an η-quasisymmetric homeomorphism. Let $\bar{f}\colon \bar{G} \to \bar{G}'$ be the η-quasisymmetric homeomorphic extension of G; see [TV, 2.25]. Furthermore, we assume that E is a c-quasiconvex hole of G, that $f_0 = \bar{f}|\partial E$ is L-bilipschitz and that $\operatorname{int} E \cup \operatorname{int} E' \subset \mathsf{R}^2$ where E' is the hole of G' with $\partial E' = f_0 \partial E$.

2.8. **Theorem**. *In the situation of* 2.7, *the map* $f_0\colon \partial E \to \partial E'$ *extends to an* L_1*-bilipschitz homeomorphism* $f_1\colon \dot{\mathsf{R}}^2 \to \dot{\mathsf{R}}^2$ *with* $f_1 E = E'$ *and* $L_1 = L_1(c, \eta, L)$.

2.9. *Preparatory observations.* Replacing c by a larger constant we may assume by 1.3 that the holes of G' are of c-bounded turning. Moreover, we may assume that the holes E and E' contain more than one point. The domains $D = \complement E$ and $D' = \complement E'$ are then unbounded c-John disks, and $\partial D = \partial E$, $\partial D' = \partial E'$.

If f_0 has an extension to a homeomorphism $f_2\colon \bar{D} \to \bar{D}'$, the theorem follows directly from 2.3. By the Schoenflies theorem, this is the case if ∂E is a Jordan curve or if E is isolated in $\complement G$. In particular, this holds if G has only a finite number of holes.

In the present case, we only have the extension $\bar{f}\colon \bar{G} \to \bar{G}'$ of f_0, and 2.8 is proved by constructing a homeomorphic extension $f_2\colon \bar{D} \to \bar{D}'$ of f_0. Example 3.10 of [GNV] illustrates the difficulties that may arise if only a map $f_0\colon \partial D \to \partial D'$ is given without any extension.

2.10. *Prime ends.* As in [GNV, 2.17] and [Vä1, 3.1], we may consider the prime ends of a John disk D as equivalence classes of endcuts. We recall the basic facts. An *endcut* of D at a point $z \in \partial D$ is a continuous map $\alpha\colon [a, b) \to D$ such that $\alpha(t) \to z$ as $t \to b$. We write $|\alpha| = \operatorname{im} \alpha \subset D$ and $e(\alpha) = z$. Two endcuts α and β are *equivalent*, written $\alpha \sim \beta$, if $e(\alpha) = e(\beta)$ and if for each neighborhood U of $e(\alpha)$ there is a connected set in $U \cap D$ meeting $|\alpha|$ and $|\beta|$. An equivalence class $[\alpha]$ is a *prime end* of D. We let $\partial^* D$ denote the set of all prime ends of D and we write $D^* = D \cup \partial^* D$. The *impression* $i\colon D^* \to \bar{D}$ is defined by $i([\alpha]) = e(\alpha)$ for $[\alpha] \in \partial^* D$ and by $i(x) = x$ for $x \in D$.

Every conformal map $\varphi\colon B(1) \to D$ extends to a continuous map $\bar{\varphi}\colon \bar{B}(1) \to \bar{D}$ and to a bijection $\varphi^*\colon \bar{B}(1) \to D^*$ such that $\varphi^*(z) = [\varphi \circ \alpha]$ where α is an arbitrary endcut of $B(1)$ at $z \in S(1)$. The topology of D^* is defined so that φ^* is a homeomorphism. Then D^* is compact and the impression $i = \bar{\varphi} \circ {\varphi^*}^{-1}$ is continuous, hence an identification.

2.11. *Proof of Theorem* 2.8. Setting $G^* = G \cup \partial^* D \subset D^*$, $G'^* = G' \cup \partial^* D' \subset D'^*$ we show that the map $f\colon G \to G'$ extends to a homeomorphism $f^*\colon G^* \to G'^*$ such that $i' \circ f^* = \bar{f} \circ i|G^*$ where $i\colon D^* \to \bar{D}$ and $i'\colon D'^* \to \bar{D}^*$ are impressions. This will imply Theorem 2.8 as follows:

As D^* is homeomorphic to $\bar{B}(1)$, the map $f_0^* = f^*|\partial^* D$ extends to a homeomorphism $f_2^*\colon D^* \to D'^*$. Since $i' \circ f_0^* = f_0 \circ i|\partial^* D$, the map f_2^* defines a homeomorphism $f_2\colon \bar{D} \to \bar{D}'$ with $i' \circ f_2^* = f_2 \circ i$. Then $f_2|\partial D = f_0$, and Theorem 2.8 follows from 2.3.

Let $p \in \partial^* D$ and choose a representative $\alpha\colon [a,b) \to D$ of p with $\alpha(a) \in G$. In the case $G = D$ of [GNV, 2.18] we simply defined $f^*(p) = [f \circ \alpha]$, but now $\bar f \circ \alpha$ is only defined in the set $J = [a,b) \cap \alpha^{-1}\bar G$. Observing that $b \in \bar J$ we let $\alpha'\colon [a,b) \to D'$ be an arbitrary continuous extension of $\bar f \circ \alpha | J$. To show that such an extension exists, consider a component $V = (u,v)$ of $[a,b) \setminus J$. Then $u, v \in J$, and the points $\bar f(\alpha(u))$ and $\bar f(\alpha(v))$ belong to a hole $E_1' \neq E'$ of G'. As E_1' is of c-bounded turning, there is a path $\alpha_V'\colon [u,v] \to E_1'$ from $\bar f(\alpha(u))$ to $\bar f(\alpha(v))$ with $d(|\alpha_V'|) \le 2c|\bar f(\alpha(u)) - \bar f(\alpha(v))|$; see 1.2. The paths α_V' give an extension $\alpha'\colon [a,b) \to D'$ of $\bar f \circ \alpha | J$, and it is easy to show that α' is continuous. Moreover, α' is an endcut of D' at $z' = \bar f(z)$ where $z = e(\alpha)$.

Let α and β be equivalent endcuts of D at z. We show that $\alpha' \sim \beta'$. If $z = \infty$, then also $z' = \infty$, and the domains D and D' are locally connected at ∞ by [NV, 2.18]. Hence we may assume that z and z' are finite. Let U' be a neighborhood of z' and choose a disk $B(z', r) \subset U'$. As $\bar f$ is continuous, there is a neighborhood U of z such that $\bar f[U \cap \bar G] \subset B(z', r/(2c+1))$. Since $\alpha \sim \beta$, there is an arc $\gamma \subset U \cap D$ with endpoints $a \in |\alpha| \cap G,\ b \in |\beta| \cap G$. Let C be a component of $\gamma \setminus \bar G$. Then C is an open subarc of γ with endpoints x, y on the boundary of a hole $E_2 \neq E$ of G. As the corresponding hole E_2' of G' is of c-bounded turning, it contains a continuum C' containing $\bar f(x)$ and $\bar f(y)$ such that $d(C') \le c|\bar f(x) - \bar f(y)| < 2cr/(2c+1)$. Hence $C' \subset U'$, and the union of $\bar f[\gamma \cap \bar G]$ and the continua C' is a connected set in $U' \cap D'$ containing the points $\bar f(a) \in |\alpha'|$ and $\bar f(b) \in |\beta'|$. Thus $\alpha' \sim \beta'$, and we get a well-defined map $f^*\colon G^* \to G'^*$ setting $f^*(p) = [\alpha']$ and $f^*|G = f$.

Similarly, the map f^{-1} extends to a map $f^{-1*}\colon G'^* \to G^*$. If α' is as above, the endcut α represents the prime end $f^{-1*}([\alpha'])$. The maps $f^* \circ f^{-1*}$ and $f^{-1*} \circ f^*$ are therefore identities, whence f^* is bijective.

To show that f^* is continuous we choose conformal maps $\varphi_1\colon B(1) \to D$ and $\varphi_2\colon B(1) \to D'$. Let $\varphi_1^*\colon \bar B(1) \to D^*$ and $\varphi_2^*\colon \bar B(1) \to D'^*$ be their homeomorphic extensions and write

$$G_0^* = \varphi_1^{*-1} G^*, \quad G_0'^* = \varphi_2^{*-1} G'^*.$$

Define $h\colon G_0^* \to G_0'^*$ by $h(x) = \varphi_2^{*-1} f^* \varphi_1^*(x)$. We must show that h is continuous at a point $z_0 \in S(1)$. Let (x_j) be a sequence in G_0^* converging to z_0. To show that $h(x_j) \to z_0' = h(z_0)$ it suffices to consider two cases:

Case 1. $|x_j| < 1$ for all j. Joining the points x_j and x_{j+1} by line segments we obtain an endcut α of $B(1)$ at z_0. Setting $\beta = \varphi_1 \circ \alpha$ we have $\varphi_1^*(z_0) = [\beta]$ and $f^*\varphi_1^*(z_0) = [\beta']$ where β' is the endcut of D' constructed above. Now $\varphi_2^{-1} \circ \beta'$ is an endcut of $B(1)$ at z_0', whence $h(x_j) \to z_0'$.

Case 2. $|x_j| = 1$ for all j. By Case 1 we can choose for each j a point $y_j \in G_0^* \cap B(1)$ such that $|y_j - x_j| < 1/j,\ |h(y_j) - h(x_j)| < 1/j$. Now $y_j \to z_0$, whence $h(y_j) \to z_0'$ by Case 1, which implies that $h(x_j) \to z_0'$.

We have proved that f^* is continuous. Similarly f^{-1*} is continuous, and f^* is therefore a homeomorphism. The proof of Theorem 2.8 is complete. □

3. Quasiconvex holes

As before, we let $\bar f\colon \bar G \to \bar G'$ denote the unique homeomorphic extension of a quasisymmetric homeomorphism $f\colon G \to G'$.

3.1. **Theorem**. *Let $G \subset \dot{\mathsf{R}}^2$ be a domain whose holes are of c-bounded turning and let $f\colon G \to G' \subset \dot{\mathsf{R}}^2$ be an η-quasisymmetric homeomorphism. Let E be a c-quasiconvex hole of G and let E′ be the hole of G′ containing $\bar{f}\partial E$. If $\bar{f}|\partial E$ is L-bilipschitz, then E′ is c′-quasiconvex with $c'(c, L, \eta)$.*

Proof. We let $c_i \geq 1$, $L_i \geq 1$ denote constants depending only on (c, L, η). We may assume that E contains more than one point. The domains $D = \complement E$ and $D' = \complement E'$ are c_1-John disks by 1.3. If D and D' are unbounded, Theorem 2.3 gives an L_1-bilipschitz extension $f_2\colon \dot{\mathsf{R}}^2 \to \dot{\mathsf{R}}^2$ of $\bar{f}|\partial D$ with $f_2 E = E'$. Hence E' is $L_1^2 c$-quasiconvex.

We next consider the case where D and D' are bounded. Let $u\colon \dot{\mathsf{R}}^2 \to \dot{\mathsf{R}}^2$ be the inversion as in 2.4 and let $A(R)$ be as in 2.5. By 2.2 we may normalize the situation so that $0 \in D \cap D'$ and $\partial D \cup \partial D' \subset A(c_2)$. Set

$$D_1 = uD,\ E_1 = uE = \complement D_1,\ D_1' = uD',\ E_1' = uE' = \complement D_1'.$$

It follows from 2.5 that E_1 is c_3-quasiconvex and that E_1' is of c_3-bounded turning. Hence D_1 and D_1' are c_3-John disks containing ∞. The map $f_2\colon \partial D_1 \to \partial D_1'$, defined by $f_2(x) = u\bar{f}u(x)$, is an L_2-bilipschitz homeomorphism, which has a homeomorphic extension $\bar{D}_1 \to \bar{D}_1'$. Hence 2.3 gives an L_3-bilipschitz extension $f_3\colon \dot{\mathsf{R}}^2 \to \dot{\mathsf{R}}^2$ of f_2, whence $E_1' = f_3 E_1$ is c_4-quasiconvex. By 2.5 this implies that E' is c_5-quasiconvex.

The cases where exactly one of the domains D, D' is bounded are treated similarly but considering the map $u \circ \bar{f}$ or $\bar{f} \circ u$ instead of $u \circ \bar{f} \circ u$. The details are omitted. □

As a corollary of Theorem 3.1 we get:

3.2. **Theorem**. *Let $G \subset \dot{\mathsf{R}}^2$ be a domain such that each hole of G is c-quasiconvex and let $f\colon G \to \dot{\mathsf{R}}^2$ be L-bilipschitz. Then each hole of the domain $G' = fG$ is c′-quasiconvex with $c'(c, L)$.*

3.3. *Remark.* A similarity maps c-quasiconvex sets onto c-quasiconvex sets. It follows that in the results of this section, the L-bilipschitz condition can be replaced by the weaker condition $M|x-y|/L \leq |f(x) - f(y)| \leq ML|x-y|$ for some $M > 0$. The constant c' does not depend on M.

3.4. **Example**. It is natural to ask whether Theorem 3.1 (or 1.3) holds without the condition that all holes of G be of c-bounded turning. The following example shows that the answer is negative.

Let $h > 0$ be a small number and let γ be the closed polygon with successive vertices $ih, 1+ih, 1+i, -1+i, -1-i, 1-i, 1-ih, -ih$. Let D be the bounded component of $\complement\gamma$. Let α be the arc of γ between $1-ih$ and $1+ih$ containing 0 and let $f\colon \gamma \to \mathsf{R}^2$ be the map such that $f|\alpha$ is the reflection in the line $\{1 + ti\colon t \in \mathsf{R}\}$ and $f|\gamma \setminus \alpha = \mathrm{id}$. Set $\gamma' = f\gamma$ and let D' be the bounded component of $\complement\gamma'$. There is a universal constant L_0 such that f has an L_0-bilipschitz extension $f_1\colon U \to \bar{D}'$ where U is a narrow connected neighborhood of γ in $\bar{D}$. The domain $G = U \setminus \gamma$ has two holes, one of which is $E = \complement D$. This hole is c_0-quasiconvex with a universal c_0. The corresponding hole of $G' = fG$ is $E' = \complement D'$, and this hole is not even of c'-bounded turning for $c' < 1 + 1/h$.

4. Variations

We give some variations of the previous results. The proof of Theorem 1.3 is valid if the outer $(0, c)$-joinability is replaced by any joinability property, and we get the following.

4.1. **Theorem**. *Let $G \subset \dot{\mathsf{R}}^n$ be a domain such that $\complement G$ is inner [or outer] (p, c)-joinable for some $p \in [0, n-2]$ in the sense of* [Vä2]*, and let $f: G \to G' \subset \dot{\mathsf{R}}^n$ be an η-quasisymmetric homeomorphism. Then $\complement G'$ is inner [or outer] (p, c')-joinable with $c'(c, \eta)$.* □

Furthermore, [Vä2, 4.6] implies that if an open set $G \subset \dot{\mathsf{R}}^n$ is (p, c)-joinable, that is, both inner and outer (p, c)-joinable, then the image of G under an η-*quasimöbius* map $f: G \to \dot{\mathsf{R}}^n$ is (p, c')-joinable with $c'(c, \eta)$. For terminology, see e.g. [Vä2, 4.4]. This implies:

4.2. **Theorem**. *Let $G \subset \dot{\mathsf{R}}^n$ be a domain such that $\complement G$ is (p, c)-joinable for some $p \in [0, n-2]$ in the sense of* [Vä2]*, and let $f: G \to G' \subset \dot{\mathsf{R}}^n$ be an η-quasimöbius homeomorphism. Then $\complement G'$ is (p, c')-joinable with $c'(c, \eta)$.*

By [Vä2, 3.5], a closed set in $\dot{\mathsf{R}}^n$ is $(0, c)$-joinable iff, quantitatively, its components are c-linearly locally connected. Hence 4.2 gives for $p = 0$:

4.3. **Theorem**. *If $f: G \to G'$ is η-quasimöbius and if all holes of G are c-linearly locally connected, then the holes of G' have the same property with a constant $c'(c, \eta)$.* □

4.4. *Holes of continua.* We change the roles of open and closed sets and consider continua instead of domains. A *continuum* is a compact connected set containing more than one point. A *hole* of a continuum $F \subset \dot{\mathsf{R}}^n$ is a component of $\complement F$. The following properties of holes are well known; see [Ne, Sec. V.14].

4.5. **Lemma**. *Let U be a hole of a continuum $F \subset \dot{\mathsf{R}}^n$. Then:*
(1) *$\complement U$ is connected,*
(2) *∂U is a component of ∂F.* □

As the duality theory is symmetric with respect to open and closed sets, we get:

4.6. **Theorem**. *Theorems* 1.3*,* 4.1*,* 4.2 *and* 4.3 *are true if the domain G is replaced by a continuum F.* □

We next turn to planar continua and their holes of bounded turning. We first show that such a hole has a simple topological structure, provided that its boundary is bounded. This makes the theory of such holes easier than in the case of domains. The case $\infty \in \partial U$ is discussed in 4.12.

4.7. **Lemma**. *Suppose that U is a hole of a continuum $F \subset \dot{\mathsf{R}}^2$ and that U is of bounded turning. If ∂U is bounded, then ∂U is a Jordan curve.*

Proof. The bounded turning of U implies that U is uniformly locally connected, and the lemma follows from [Ne, VI.2]. □

4.8. **Lemma**. *Let $F \subset \dot{\mathsf{R}}^2$ be a Jordan curve such that both holes of F are of c-bounded turning. Then both holes of F are K-quasidisks and b-quasiconvex with K and b depending only on c.*

Proof. As the closures of the holes are of c-bounded turning, both holes are c-John disks. Hence they are K-quasidisks by [NV, 9.2] and therefore b-uniform domains and thus b-quasiconvex. □

4.9. *Maps of continua.* Let $F \subset \dot{\mathsf{R}}^2$ be a continuum and let $f\colon F \to F' \subset \dot{\mathsf{R}}^2$ be a homeomorphism. If U is a hole of F, it follows from 4.5 that there is a hole U' of F' with $\partial U' = f\partial U$. However, U' is not uniquely determined, since several holes may have a common boundary; see the Lakes of Wada, [HY, p. 143]. Assume that the holes of F are of c-bounded turning. Then F' has the same property with a constant $c'(c)$ by 1.3 and 4.6. If $\infty \notin \partial U$, then U is a Jordan domain by 4.7, and U' is uniquely determined, except for the case where F is a Jordan curve, which was considered in 4.8.

We next give a continuum version of Theorem 3.1.

4.10. **Theorem**. *Suppose that $F \subset \dot{\mathsf{R}}^2$ is a continuum whose holes are of c-bounded turning and let $f\colon F \to F' \subset \dot{\mathsf{R}}^2$ be an η-quasisymmetric homeomorphism. Let U be a c-quasiconvex hole of F with $\infty \notin \partial U$, and let U' be a hole of F' with $\partial U' = f\partial U$. If $f|\partial U$ is L-bilipschitz, then U' is c'-quasiconvex with $c'(c, L, \eta)$.*

Proof. The domains U and U' are Jordan domains of $c_1(c, \eta)$-bounded turning by 4.7 and 4.6. Hence the domains $U^* = \complement\bar{U}$ and $U'^* = \complement\bar{U}'$ are Jordan c_1-John disks. If U^* and U'^* contain ∞, we may apply Theorem 2.3 to extend $f|\partial U$ to an $L_1(c, L, \eta)$-bilipschitz homeomorphism $f_1\colon \dot{\mathsf{R}}^2 \to \dot{\mathsf{R}}^2$ with $f_1 U = U'$, and then U' is c'-quasiconvex with $c' = L_1^2 c$. If U^* or U'^* is bounded, we make use of one or two auxiliary inversions as in the proof of Theorem 3.1 to reduce the case to the previous one. □

4.11. **Corollary**. *Let $F \subset \mathsf{R}^2$ be a bounded continuum whose holes are c-quasiconvex and let $f\colon F \to F' \subset \mathsf{R}^2$ be an L-bilipschitz homeomorphism. Then the holes of F' are c'-quasiconvex with $c'(c, L)$.* □

4.12. *The case $\infty \in \partial U$.* Suppose that $F \subset \dot{\mathsf{R}}^2$ is a continuum and that U is a hole of c-bounded turning of F with $\infty \in \partial U$. Then ∂U need not be a Jordan curve. For example, a parallel strip is a convex hole of its complement, and its boundary consists of two Jordan curves meeting at ∞. By [HH], the boundary ∂U consists of a finite number of Jordan curves meeting at ∞ but not elsewhere. Moreover, the number of these holes is bounded by a number $N(c)$, and the components of $\complement\bar{U}$ are Jordan $2c$-John disks. If U_1 is another hole of F, then U_1 lies in one of these John disks. Hence $\partial U_1 \neq \partial U$, except for the case where F is a Jordan curve, which was discussed in 4.8.

Assume that F is not a Jordan curve and let $f\colon F \to F' \subset \dot{\mathsf{R}}^2$ be an η-quasisymmetric homeomorphism. Then there is a unique hole U' of F' with $\partial U' = f\partial U$. Again, U is of c'-bounded turning with $c'(c)$ by 1.3 and 4.6.

However, an *open question* remains: Is Theorem 4.10 true in the case $\infty \in \partial U$? In other words, does f preserve the quasiconvexity of U if $f|\partial U$ is bilipschitz? The following example shows that $f|\partial U$ need not extend to any homeomorphism of $\bar{U}$.

Let $A_1 = \{(x, y) \in \mathsf{R}^2 : |y| \le x - 1\}$ and set $A_2 = -A_1$, $F = A_1 \cup A_2 \cup \{\infty\}$. Then F is a continuum with one hole $U = \complement F$, and U is $\sqrt{2}$-quasiconvex. Define a homeomorphism $f\colon F \to F$ by $f(z) = \bar{z}$ for $z \in A_1$ and by $f(z) = z$ for $z \in A_2 \cup \{\infty\}$. Then f is $\sqrt{2}$-bilipschitz but $f|\partial U$ has no homeomorphic extension to $\bar{U}$.

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