CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 12, Pages 133–152 (August 27, 2008)
S 1088-4173(08)00181-1

AREA, CAPACITY AND DIAMETER VERSIONS OF SCHWARZ'S LEMMA

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Abstract. The now canonical proof of Schwarz's Lemma appeared in a 1907 paper of Carathéodory, who attributed it to Erhard Schmidt. Since then, Schwarz's Lemma has acquired considerable fame, with multiple extensions and generalizations. Much less known is that, in the same year 1907, Landau and Toeplitz obtained a similar result where the diameter of the image set takes over the role of the maximum modulus of the function. We give a new proof of this result and extend it to include bounds on the growth of the maximum modulus. We also develop a more general approach in which the size of the image is estimated in several geometric ways via notions of radius, diameter, perimeter, area, capacity, etc.

1. Introduction

1.1. **Schwarz's Lemma.** First, let us set the following standard notations: $\mathbb{C}$ denotes the complex numbers, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk, and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. Moreover, for $r > 0$, we let $r\mathbb{D} := \{z \in \mathbb{C} : |z| < r\}$ and $r\mathbb{T} := \{z \in \mathbb{C} : |z| = r\}$. Also, we will say that a function is *linear* if it is of the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ (in particular, it may be constant).

Given a function f that is analytic in $\mathbb{D}$, and given the exhaustion $\{r\mathbb{D}\}_{0\leq r\leq 1}$, consider the corresponding image-sets

$$f(r\mathbb{D}) = \{w \in \mathbb{C} : \text{ there is at least one } z \in r\mathbb{D} \text{ such that } f(z) = w\}.$$

Let us emphasize that we will not consider "multiplicity" in this paper. So $f(r\mathbb{D})$ denotes a family of open connected sets in $\mathbb{C}$ that are increasing with r. The goal is to fix a geometric quantity so as to measure the size of $f(r\mathbb{D})$ and study how it varies with r. In particular, it turns out that linear functions seem always to exhibit a uniquely exceptional behavior.

To illustrate this point of view, we first consider the famous Schwarz's Lemma. We introduce the following notion of "radius":

$$\operatorname{Rad} f(r\mathbb{D}) := \sup_{|z|<r} |f(z) - f(0)|. \tag{1.1}$$

Received by the editors July 17, 2007.
2000 *Mathematics Subject Classification.* Primary 30C80.
The second author was supported by NSF grant DMS 0602509.
The fifth author was supported by grants from NSERC, FQRTN, and the Canada research chairs program.

Geometrically, $\operatorname{Rad} f(r\mathbb{D})$ is the radius of the smallest disk centered at $f(0)$ which contains $f(r\mathbb{D})$. As a point of warning, the way Schwarz's Lemma will be presented below might look unusual, but the proof is exactly the same.

Theorem 1.1 (Schwarz's Lemma). *Suppose f is analytic on the unit disk $\mathbb{D}$. Then the function $\phi_{\mathrm{Rad}}(r) := r^{-1} \operatorname{Rad} f(r\mathbb{D})$ is strictly increasing for $0 < r < 1$, except when f is linear, in which case ϕ_{Rad} is constant. Moreover, $\lim_{r\downarrow 0} \phi_{\mathrm{Rad}}(r) = |f'(0)|$.*

Corollary 1.2. *Suppose f is analytic on the unit disk $\mathbb{D}$ with $\operatorname{Rad} f(\mathbb{D}) = 1$. Then*

$$\operatorname{Rad} f(r\mathbb{D}) \leq r \qquad \text{for every } 0 < r < 1, \tag{1.2}$$

and

$$|f'(0)| \leq 1. \tag{1.3}$$

Moreover, equality holds in (1.2) for some $0 < r < 1$, or in (1.3), if and only if $f(z)$ is an Euclidean isometry $a + cz$ for some constants $a \in \mathbb{C}$, $c \in \mathbb{T}$.

The standard way to prove Schwarz's Lemma is to factor $f(z) - f(0) = zg(z)$, for some analytic function g and then apply the maximum modulus theorem to g to deduce that

$$r^{-1} \operatorname{Rad} f(r\mathbb{D}) = \sup_{|z|<r} |g(z)|. \tag{1.4}$$

This argument first appeared in a paper of Carathéodory [Cara1907] where the idea is attributed to E. Schmidt; see Remmert [Re1991, p. 272–273] and Lichtenstein [Li1919, footnote 427] for historical accounts.

1.2. **The theorem of Landau and Toeplitz.** In a 1907 paper, Landau and Toeplitz replaced the radius (1.1) by the diameter of the image set

$$\operatorname{Diam} f(r\mathbb{D}) := \sup_{z,w\in r\mathbb{D}} |f(z) - f(w)|. \tag{1.5}$$

Theorem 1.3 (Landau-Toeplitz [LaT1907]). *Suppose f is analytic on the unit disk $\mathbb{D}$ and $\operatorname{Diam} f(\mathbb{D}) = 2$. Then*

$$\operatorname{Diam} f(r\mathbb{D}) \leq 2r \qquad \text{for every } 0 < r < 1, \tag{1.6}$$

and

$$|f'(0)| \leq 1. \tag{1.7}$$

Moreover, equality holds in (1.6) for some $0 < r < 1$, or in (1.7), if and only if $f(z)$ is an Euclidean isometry $a + cz$ for some constants $a \in \mathbb{C}$, $c \in \mathbb{T}$.

Remark 1.4. The main contribution of the Landau-Toeplitz paper is perhaps its elucidation of the extremal case. Pólya and Szegő mention the inequality (1.7) on p. 151 and p. 356 of the classic book [PolS1972], and they cite the paper of Landau and Toeplitz. However, they say nothing about when equality holds. It is also worth mentioning the proof of Lemma 2.9 in [GeH1999], where F.W. Gehring and K. Hag essentially treat the case of equality in Theorem 1.3 in the special case of one-to-one maps, using the Cauchy-Schwarz inequality and the so-called "isodiametric" inequality.

Remark 1.5. The growth estimate on the diameter (1.6) should be viewed in analogy with the classical growth bound (1.2). Notice, however, that Theorem 1.3 covers the case when $f(\mathbb{D})$ is an equilateral triangle of side-length 2, which is of course not contained in a disk of radius 1; likewise, when $f(\mathbb{D})$ is contained in the so-called Reuleaux triangle that is obtained from the equilateral triangle by joining adjacent vertices by a circular arc having center at the third vertex.

We start by giving a new proof of the Landau-Toeplitz Theorem that can be used to prove more general cases as well. Later we will show how the original proof of Landau and Toeplitz can be adapted to some of these more general cases. The Landau-Toeplitz approach is more direct but seems to accomplish less.

1.3. **Higher-diameters and log-convexity.** As we already mentioned in Remark 1.5, the Landau-Toeplitz result generalizes the bounds on $|f'(0)|$ that can be deduced from Schwarz's Lemma. It is therefore natural to ask if there are other conditions on analytic functions $f : \mathbb{D} \to \mathbb{C}$, weaker than $\operatorname{Diam} f(\mathbb{D}) \leq 2$, which imply $|f'(0)| \leq 1$ with equality if and only if f is an Euclidean isometry. Also, it follows from (1.4) and Hadamard's three-circles theorem that $\phi_{\mathrm{Rad}}(r)$ is not only strictly increasing (except when f is linear), but it is also log-convex, i.e., it is a convex function of $\log r$. In fact, even more is true: its logarithm is log-convex. Thus, a more general question arises: assuming that f is not linear, is the function $\phi_{\mathrm{Diam}}(r) := (2r)^{-1} \operatorname{Diam} f(r\mathbb{D})$ strictly increasing and log-convex?

Other geometric quantities may be used to measure the size of the image of an analytic function. In this paper we will focus on n-diameter, capacity, area and perimeter. In [PolS1951] Pólya and Szegő also consider other quantities such as the moment of inertia, the torsional rigidity, and the principal frequency. Such topics deserve to be explored but we reserve to do this in another paper.

We focus at first on the so-called higher-order diameters, which are defined for sets $E \subset \mathbb{C}$ as follows: fix $n = 2, 3, 4, \ldots$, then

$$d_n(E) := \sup \left(\prod_{j<k} |\zeta_j - \zeta_k| \right)^{\frac{2}{n(n-1)}},$$

where the supremum is taken over all n-tuples of points from E. We say $d_n(E)$ is the n-diameter of E. Note that $d_2(E) = \operatorname{Diam} E$, and that $d_n(E)$ is weakly decreasing in n. Hence $d_\infty(E) := \lim_{n\to\infty} d_n(E)$ is well defined and is called the transfinite diameter of E. It turns out that the transfinite diameter $d_\infty(E)$ coincides with the logarithmic capacity $\operatorname{Cap}(E)$; see the Fekete-Szegő Theorem of [Ra1995, p. 153]. It can also be shown, see Fact 3.2 below, that $d_n(\mathbb{D}) = n^{1/(n-1)}$.

The following inequality is due to Pólya; see [Pol1928] or [Ra1995, p. 145]. For a compact set E in $\mathbb{C}$,

$$\operatorname{Area} E \leq \pi \operatorname{Cap}^2 E. \tag{1.8}$$

Equality holds for a closed disk. Moreover, Corollary 6.2.4 of [Ra1995] asserts that

$$\operatorname{Cap}(E) \leq \frac{d_n(E)}{n^{1/(n-1)}}.$$

Hence, combining with (1.8), we obtain, for $n = 2, 3, ...$,

$$\text{Area}\, E \leq \pi \frac{d_n(E)^2}{n^{2/(n-1)}}. \tag{1.9}$$

The $n = 2$ case is sometimes called the "isodiametric" inequality.

Therefore, we see that the condition $\text{Area}\, f(\mathbb{D}) = \pi$ ($= \text{Area}(\mathbb{D})$) is more general than $\text{Cap}\, f(\mathbb{D}) = 1$ ($= \text{Cap}(\mathbb{D})$), which in turn is more general than $d_n(f(\mathbb{D})) = n^{1/(n-1)}$ ($= d_n(\mathbb{D})$).

The first main result of this paper is the following generalization of the Landau-Toeplitz Theorem. We consider the following ratios:

$$\phi_{\text{n-Diam}}(r) := \frac{d_n(f(r\mathbb{D}))}{d_n(\mathbb{D})r} \qquad \text{and} \qquad \phi_{\text{Cap}}(r) := \frac{\text{Cap}(f(r\mathbb{D}))}{\text{Cap}(r\mathbb{D})} = \frac{d_\infty(f(r\mathbb{D}))}{d_\infty(r\mathbb{D})}.$$

Theorem 1.6. *Suppose f is analytic on $\mathbb{D}$. The functions $\phi_{\text{Cap}}(r)$ and $\phi_{\text{n-Diam}}(r)$ are increasing and log-convex. Moreover, they are strictly increasing for $0 < r < 1$ except in the special case that f is linear.*

It can be checked from the power series expansion of f that the following limits hold:

$$\lim_{r\downarrow 0} \phi_{\text{Rad}}(r) = \lim_{r\downarrow 0} \phi_{\text{n-Diam}}(r) = \lim_{r\downarrow 0} \phi_{\text{Cap}}(r) = |f'(0)|. \tag{1.10}$$

Hence, we leave as an exercise to show that Theorem 1.6 implies the following corollary.

Corollary 1.7. *Suppose f is analytic on $\mathbb{D}$ and $d_n(f(\mathbb{D})) = d_n(\mathbb{D})$ (or $\text{Cap}\, f(\mathbb{D}) = \text{Cap}\, \mathbb{D}$). Then*

$$d_n(f(r\mathbb{D})) \leq d_n(\mathbb{D})r, \qquad \textit{for every } r \in (0,1) \tag{1.11}$$

$$(\textit{resp. } \text{Cap}\, f(r\mathbb{D}) \leq (\text{Cap }\, \mathbb{D})r) \tag{1.12}$$

and

$$|f'(0)| \leq 1. \tag{1.13}$$

Moreover, equality holds in (1.11) (resp. in (1.12)) for some $0 < r < 1$, or in (1.13), if and only if $f(z)$ is an Euclidean isometry $a + cz$ for some constants $a \in \mathbb{C}$, $c \in \mathbb{T}$.

Remark 1.8. It follows from the proof of Theorem 1.6 (see the proof of Lemma 2.1), that $\phi_{\text{n-Diam}}(r)$ and $\phi_{\text{Cap}}(r)$ have the stronger property that their logarithm is a convex function of $\log r$. This is also how Hadamard's Theorem is usually phrased.

1.4. **An area Schwarz Lemma.** As mentioned above, the condition $\text{Area}\, f(\mathbb{D}) = \pi$ is weaker than $\text{Diam}\, f(\mathbb{D}) = 2$. We can prove the following analog of Schwarz's Lemma.

Theorem 1.9 (Area Schwarz's Lemma)**.** *Suppose f is analytic on the unit disk $\mathbb{D}$. Then the function $\phi_{\text{Area}}(r) := (\pi r^2)^{-1} \text{Area}\, f(r\mathbb{D})$ is strictly increasing for $0 < r < 1$, except when f is linear, in which case ϕ_{Area} is constant.*

Moreover, by the power series expansion, $\lim_{r\downarrow 0} \phi_{\text{Area}}(r) = |f'(0)|$. So the following corollary ensues.

Corollary 1.10. *Suppose f is analytic on the unit disk $\mathbb{D}$ with $\text{Area}\, f(\mathbb{D}) = \pi$. Then*

$$\text{Area}\, f(r\mathbb{D}) \leq \pi r^2 \qquad \textit{for every } 0 < r < 1, \tag{1.14}$$

and

$$|f'(0)| \le 1. \tag{1.15}$$

Moreover, equality holds in (1.14) *for some* $0 < r < 1$*, or in* (1.15)*, if and only if* $f(z)$ *is an Euclidean isometry* $a + cz$ *for some constants* $a \in \mathbb{C}$*,* $c \in \mathbb{T}$*.*

One might ask whether $\phi_{\text{Area}}(r)$ is also log-convex as with the growth functions ϕ_{Rad}, $\phi_{\text{n-Diam}}$, and ϕ_{Cap}. This is true for univalent functions, but fails in general. In Section 5 we give an explicit example for which $\phi_{\text{Area}}(r)$ is not log-convex.

1.5. **Structure of the paper and other results.** The structure of the paper is as follows. In Section 2 we prove Theorem 1.6 about n-diameter and capacity generalizations of Schwarz's Lemma. In Section 3 we dust off the original approach of Landau and Toeplitz and show that it can be made to work for n-diameters, hence giving an alternative, more direct proof of Theorem 1.6 for n-diameter. In Section 4 we explore an even further generalization of Schwarz's Lemma using area instead and prove Theorem 1.9. In Section 5, however, we give an example where log-convexity fails. In Section 6, we formulate a generalization using perimeter. This is our weakest result because log-convexity is missing and extra conditions must be imposed on the range. In Section 7, we give some applications of these Schwarz lemmas to hyperbolic geometry. In particular, we obtain the global lower bound (7.2) for the Poincaré density on arbitrary domains. In Section 8, we study bounds on the growth of $|f(z)|$ under conditions on the image $f(\mathbb{D})$ that involve diameter instead of radius. In Section 9 we describe a related result of Poukka, obtained around the same time as the Landau-Toeplitz paper, which involves higher derivatives. Finally, in Section 10 we state some open problems.

2. Higher and transfinite diameter generalizations of Schwarz's Lemma

In this section we prove Theorem 1.6. The stronger notion of log-convexity turns out to be essential to prove the sharp result.

Lemma 2.1. *For* f *analytic on* $\mathbb{D}$ *and* $n = 2, 3, \ldots$*, both* $\phi_{\text{n-Diam}}(r)$ *and* $\phi_{\text{Cap}}(r)$ *are increasing convex functions of* $\log r$*,* $0 < r < 1$*.*

Proof. Let f be analytic on $\mathbb{D}$; we may assume that $f(0) = 0$ and that f is not linear. It suffices to prove that $\phi_{\text{n-Diam}}(r)$ is an increasing convex function of $\log r$, since the corresponding result for $\phi_{\text{Cap}}(r)$ then follows by a limit argument.

So fix $n = 2, 3, \ldots$, and consider the auxiliary function

$$F_{w_1,\ldots,w_n}(z) := d_n(\mathbb{D})^{-\frac{n(n-1)}{2}} \prod_{j<k} (f(w_k z) - f(w_j z)), \tag{2.1}$$

for fixed distinct $w_1, \ldots, w_n \in \overline{\mathbb{D}}$. Then $F_{w_1,\ldots,w_n}(z) = z^{\frac{n(n-1)}{2}} g(z)$, where g is analytic in $\mathbb{D}$. So

$$\log\left(r^{-\frac{n(n-1)}{2}} \operatorname{Rad} F_{w_1,\ldots,w_n}(r\mathbb{D})\right) = \max_{|z|<r} \log|g(z)|$$

is strictly increasing for $0 < r < 1$, except in the special case when $g(z) \equiv g(0)$; in fact, by Hadamard's three-circles Theorem it is also log-convex. Moreover, for fixed $r \in (0,1)$ we have

$$\max_{w_1,\ldots,w_n \in \overline{\mathbb{D}}} \operatorname{Rad} F_{w_1,\ldots,w_n}(r\mathbb{D}) = \left(\frac{d_n(f(r\mathbb{D}))}{d_n(\mathbb{D})}\right)^{\frac{n(n-1)}{2}}. \tag{2.2}$$

So the function

$$\log \phi_{\text{n-Diam}}(r) = \max_{w_1,\ldots,w_n \in \overline{\mathbb{D}}} \log \left(r^{-\frac{n(n-1)}{2}} \operatorname{Rad} F_{w_1,\ldots,w_n}(r\mathbb{D})\right)^{\frac{2}{n(n-1)}} \tag{2.3}$$

is the pointwise maximum of a family of increasing log-convex functions, hence it is increasing and log-convex for $0 < r < 1$. This implies that $\phi_{\text{n-Diam}}(r)$ itself must be increasing and log-convex. So Lemma 2.1 is proved. □

Finally, we will need the following elementary lemma.

Lemma 2.2. *Let f be analytic in $\mathbb{D}$ and not linear. Then there is $0 < r_0 < 1$ such that for $0 < r < r_0$,*

$$\phi_{\text{Area}}(r) > |f'(0)|^2.$$

Proof. The statement is clear if $f'(0) = 0$. So assume $f'(0) \neq 0$. Then f is one-to-one near the origin and for $r > 0$ small

$$\operatorname{Area} f(r\mathbb{D}) = \int_{r\mathbb{D}} |f'(z)|^2 d\mathcal{H}^2(z) = \pi \sum_{n=0}^{\infty} n|a_n|^2 r^{2n}$$

(where $\mathcal{H}^2$ is two-dimensional Lebesgue measure). So $\phi_{\text{Area}}(r) = \sum_{n=1}^{\infty} n|a_n|^2 r^{2(n-1)}$ is strictly increasing unless f is linear. □

Proof of Theorem 1.6. We do the proof for $\phi_{\text{Cap}}(r)$, since the one for $\phi_{\text{n-Diam}}(r)$ is the same except for the obvious changes, e.g., use (1.9) in place of (1.8) below.

By Lemma 2.1 the function $\phi_{\text{Cap}}(r)$ is an increasing convex function of $\log r$. Suppose it fails to be strictly increasing. Then by monotonicity it must be constant on an interval $[s,t]$ for some $0 < s < t < 1$. By log-convexity, it then would have to be constant and equal to $|f'(0)|$ on all of the interval $(0,t)$. But, for $0 < r < \min\{r_0, t\}$, with r_0 as in Lemma 2.2,

$$|f'(0)|^2 \leq \phi_{\text{Area}}(r) = \frac{\operatorname{Area} f(r\mathbb{D})}{\pi r^2} \leq \phi^2_{\text{Cap}}(r) = |f'(0)|^2,$$

where Pólya's inequality (1.8) has been used. Therefore, $\phi_{\text{Area}}(r)$ is constant on $(0,t)$, so, by Lemma 2.2, f must be linear. □

3. The original Landau-Toeplitz approach

In this section we revive the original method of Landau and Toeplitz. We show that it can be used to give a direct proof of Theorem 1.6 for n-diameters. However, it seems that for capacity one really needs to use log-convexity and Pólya's inequality.

The proof hinges on the following lemma.

Lemma 3.1. *Suppose g is analytic in $\mathbb{D}$, $0 < r < 1$, $|w| = r$,*

$$w = g(w) \text{ and } r = \max_{|z|=r} |g(z)|.$$

Then, $\operatorname{Im} g'(w) = 0$.

Proof. Actually, the stronger conclusion $g'(w) \geq 0$ is geometrically obvious because when $g'(w) \neq 0$, the map g is very close to the rotation-dilation centered at w given by $\zeta \mapsto w + g'(w)(\zeta - w)$. Since g can't rotate points inside $D(0,|w|)$ to points outside, the derivative must be positive.

For the sake of rigor, we instead give a "calculus" proof of the weaker statement, along the lines of the original paper of Landau and Toeplitz, which they credit to F. Hartogs.

For $\theta \in \mathbb{R}$, we introduce

$$\phi(\theta) := |g(we^{i\theta})|^2 = g(we^{i\theta})\overline{g(we^{i\theta})}.$$

The function $g^\star(z) := \overline{g(\bar{z})}$ is also analytic in $\mathbb{D}$, and ϕ may be written

$$\phi(\theta) = g(we^{i\theta})g^\star(\bar{w}e^{-i\theta}),$$

enabling us to compute $\phi'(\theta)$ via the product and chain rules. We get routinely,

$$\phi'(\theta) = -2\,\mathrm{Im}\left[we^{i\theta}g'(we^{i\theta})\overline{g(we^{i\theta})}\right]$$

and setting $\theta = 0$,

$$\phi'(0) = -2\,\mathrm{Im}\left[wg'(w)\overline{g(w)}\right] = -2\,\mathrm{Im}\left[wg'(w)\overline{w}\right] = -2|w|^2\,\mathrm{Im}\,g'(w).$$

Since ϕ realizes its maximum over $\mathbb{R}$ at $\theta = 0$, we have $\phi'(0) = 0$, so the preceding equality proves Lemma 3.1. □

The following fact will also be important in the sequel.

Fact 3.2. *Given n points $\{w_j\}_{j=1}^n \subset \overline{\mathbb{D}}$,*

$$\prod_{j<k} |w_j - w_k| \leq n^{\frac{n}{2}}$$

with equality if and only if, after relabeling, $w_j = u\alpha^j$ for some $u \in \mathbb{T}$, where α^j are the n-th roots of unity: i.e., $\alpha^j := \exp(i(2\pi j)/n)$.

We briefly sketch here why this is so. Recall that given n complex numbers $\{w_j\}_{j=1}^n$, one may form the Vandermonde matrix $V_n := [w_j^{k-1}]_{j,k=1}^n$, and that

$$\det V_n = \prod_{1\leq j<k\leq n} (w_k - w_j). \tag{3.1}$$

Indeed, $\det V_n$ is a polynomial of degree at most $n-1$ in w_n, vanishing at $w_1, \ldots,$ w_{n-1} with coefficient of w_n^{n-1} equal to $\det V_{n-1}$, so that (3.1) follows by induction. Hadamard's inequality states that for every $n \times n$ matrix $A = [a_{jk}]$ with complex entries:

$$|\det(A)| \leq \prod_{j=1}^n \left(\sum_{k=1}^n |a_{jk}|^2\right)^{\frac{1}{2}},$$

with equality if and only if the rows of A are orthogonal.

Since all the entries of the matrix V_n are bounded by 1 in modulus, we find that

$$|\det(V_n)| \leq n^{\frac{n}{2}} \tag{3.2}$$

with equality if and only if the rows of V_n are orthogonal, i.e., if and only if

$$0 = \sum_{k=1}^{n} w_j^{k-1} \overline{w_l}^{k-1} = \frac{(w_j \overline{w_l})^n - 1}{w_j \overline{w_l} - 1}$$

whenever $j \neq l$. Fact 3.2 then follows.

It follows that the n-diameter of $\mathbb{D}$ is $d_n(\mathbb{D}) = n^{1/(n-1)}$ (which strictly decreases to $d_\infty(\mathbb{D}) = 1$), and it is attained exactly at the n-th roots of unity (modulo rotations).

Landau-Toeplitz-type proof of Theorem 1.6 *for n-diameter.* Consider as before the auxiliary function $F_{w_1,\ldots,w_n}(z)$ defined in (2.1), for fixed distinct $w_1, \ldots, w_n \in \overline{\mathbb{D}}$. Then $F_{w_1,\ldots,w_n}(z) = z^{\frac{n(n-1)}{2}} g(z)$, where g is analytic in $\mathbb{D}$ and

$$|g(0)| = \left| \frac{f'(0)}{d_n(\mathbb{D})} \right|^{\frac{n(n-1)}{2}} \prod_{j<k} |w_k - w_j|. \tag{3.3}$$

As shown above, in the proof of Lemma 2.1, the function $r \mapsto (d_n(\mathbb{D})r)^{-1} d_n(f(r\mathbb{D}))$ is increasing for $0 < r < 1$. Assume that it is not strictly increasing. Then we can find $0 < s < t < 1$ so that it is constant on $[s, t]$. By (2.2) we can find distinct $w_1, \ldots, w_n \in \overline{\mathbb{D}}$ so that

$$\left(\frac{d_n(f(s\mathbb{D}))}{d_n(\mathbb{D})s} \right)^{\frac{n(n-1)}{2}} = s^{-\frac{n(n-1)}{2}} \operatorname{Rad} F_{w_1,\ldots,w_n}(s\mathbb{D}). \tag{3.4}$$

But (2.2) also implies that

$$\left(\frac{d_n(f(r\mathbb{D}))}{d_n(\mathbb{D})r} \right)^{\frac{n(n-1)}{2}} \geq r^{-\frac{n(n-1)}{2}} \operatorname{Rad} F_{w_1,\ldots,w_n}(r\mathbb{D})$$

for every $0 < r \leq t$. In particular, letting $r = t$ and by Schwarz's Lemma (Theorem 1.1) applied to g (for this choice of w_j's), we find that $r^{-\frac{n(n-1)}{2}} \operatorname{Rad} F_{w_1,\ldots,w_n}(r\mathbb{D})$ is constant for $0 < r \leq t$; hence by (3.4) and the monotonicity of $\phi_{\text{n-Diam}}(r)$,

$$\begin{aligned} \left(\frac{d_n(f(r\mathbb{D}))}{d_n(\mathbb{D})r} \right)^{\frac{n(n-1)}{2}} &\equiv r^{-\frac{n(n-1)}{2}} \operatorname{Rad} F_{w_1,\ldots,w_n}(r\mathbb{D}) \\ &= \left(\frac{|f'(0)|}{d_n(\mathbb{D})} \right)^{\frac{n(n-1)}{2}} \prod_{j<k} |w_j - w_k|, \end{aligned} \tag{3.5}$$

for $0 < r < t$. In particular, either f is constant or $f'(0) \neq 0$. In what follows, assume f is not constant.

We have

$$|f'(0)|^{\frac{n(n-1)}{2}} = \lim_{z \to 0} \frac{1}{(d_n(\mathbb{D}))^{\frac{n(n-1)}{2}}} \prod_{j<k} \left| \frac{f(\alpha^k z) - f(\alpha^j z)}{z} \right| \leq \lim_{r \to 0} \left(\frac{d_n(f(r\mathbb{D}))}{r d_n(\mathbb{D})} \right)^{\frac{n(n-1)}{2}},$$

and so from (3.5),

$$\prod_{j<k} |w_j - w_k| \geq (d_n(\mathbb{D}))^{\frac{n(n-1)}{2}},$$

which implies that $w_j = u\alpha^j$ for some $u \in \mathbb{T}$ by Fact 3.2. By a rotation, we may take $u = 1$.

Therefore, we find that, for all $z \in \mathbb{D}$,

$$F_{\alpha^1,\dots,\alpha^n}(z) = d_n(\mathbb{D})^{-\frac{n(n-1)}{2}} \prod_{j<k} (f(\alpha^k z) - f(\alpha^j z)) = c(zf'(0))^{\frac{n(n-1)}{2}}, \tag{3.6}$$

where c is a constant with $|c| = 1$. In particular, notice that $f(z\alpha^k) - f(z\alpha^j) = 0$ if and only if $z = 0$.

Now, fix $0 < |z| = r < t$ and consider the function

$$h_z(\zeta) := \prod_{k=1}^{n-1} \frac{f(\zeta z) - f(z\alpha^k)}{(1-\alpha^k)zf'(0)} \prod_{1<j<l\le n-1} \frac{f(z\alpha^l) - f(z\alpha^j)}{(\alpha^l - \alpha^j)zf'(0)},$$

which is analytic for $\zeta \in \overline{\mathbb{D}}$. Then by (3.6) and Fact 3.2,

$$|h_z(1)| = 1 \ge \sup_{|\zeta|<1} |h_z(\zeta)|.$$

Note that,

$$h_z'(\zeta) = h_z(\zeta) z f'(\zeta z) \sum_{k=1}^{n-1} \frac{1}{f(z\zeta) - f(z\alpha^k)}.$$

By Lemma 3.1 applied to $h_z(\cdot)/h_z(1)$, and the Open-Mapping Theorem there is a real constant A so that

$$zf'(z) \sum_{k=1}^{n-1} \frac{1}{f(z) - f(z\alpha^k)} = A, \tag{3.7}$$

for $z \in t\mathbb{D} \setminus \{0\}$.

To show that (3.7) implies f is linear, we may suppose $f(0) = 0$, $f'(0) = 1$. In the $n = 2$ case (the one considered in the Landau-Toeplitz paper), the end-game is much simpler. Here, in the general case, we proceed as follows. If f is not linear, we may write $f(z) = z + a_p z^p + \dots$ where $a_p \neq 0$, and $p \ge 2$. Then

$$\frac{1}{f'(z)} = 1 - pa_p z^{p-1} + \dots,$$

and

$$\sum_{k=1}^{n-1} \frac{z}{f(z) - f(z\alpha^k)} = \sum_{k=1}^{n-1} \frac{1}{1-\alpha^k} - \sum_{k=1}^{n-1} \frac{1-\alpha^{kp}}{(1-\alpha^k)^2} a_p z^{p-1} + \dots.$$

This and (3.7) imply that

$$A = \sum_{k=1}^{n-1} \frac{1}{1-\alpha^k} \quad \text{and} \quad pA = \sum_{k=1}^{n-1} \frac{1-\alpha^{kp}}{(1-\alpha^k)^2}.$$

Recall that A is real and $\mathrm{Re}(1/(1-\alpha^k)) = \frac{1}{2}$, so that $A = (n-1)/2$.

For $1 \le j \le n$, by Fubini,

$$\sum_{k=1}^{n-1} \frac{1-\alpha^{jk}}{1-\alpha^k} = \sum_{k=1}^{n-1}\sum_{q=0}^{j-1} \alpha^{qk} = n - 1 + \sum_{q=1}^{j-1} \left[\frac{1-\alpha^{qn}}{1-\alpha^q} - 1\right] = n - j \tag{3.8}$$

since $\alpha^n = 1$. So, if $1 \le p \le n$, using Fubini, (3.8) and the definition of A, we get

$$\sum_{k=1}^{n-1} \frac{1-\alpha^{kp}}{(1-\alpha^k)^2} = \sum_{k=1}^{n-1}\sum_{j=0}^{p-1} \frac{\alpha^{jk}}{1-\alpha^k} = A + \sum_{j=1}^{p-1} [A - (n-j)] = pA - (n - \frac{p}{2})(p-1). \tag{3.9}$$

The earlier identification of pA then leads to the conclusion that

$$pA = pA - (n - \frac{p}{2})(p-1), \tag{3.10}$$

which is a contradiction for $2 \le p \le n$. If $p > n$ and $p \equiv p' \mod n$, with $p' \le n$, then $\alpha^{kp} = \alpha^{kp'}$ so (3.9) again shows that

$$pA = p'A - (n - \frac{p'}{2})(p'-1)$$

which is impossible. Thus, the assumption that f is not linear is untenable. □

4. Area generalization of Schwarz's Lemma

In this section we prove Theorem 1.9. This requires some preliminaries.

Let f be analytic and non-constant in a neighborhood of $\overline{\mathbb{D}}$. For every $w \in f(\mathbb{D})$ let

$$Z(w) := \{z_j(w)\}_{j=1}^{N(w)}$$

be the set of points in $f|_{\mathbb{D}}^{-1}(w)$ of minimum modulus. Note that $0 < N(w) < \infty$.

Claim 4.1. The function $w \mapsto N(w)$ is Borel measurable on $f(\mathbb{D})$.

Proof. Let $C = \{z \in \mathbb{D} : f'(z) = 0\}$ be the set of critical points, which is finite, and let $P = f(C)$ be the finite post-critical set. Pick $w_0 \in f(\mathbb{D})\backslash P$. It is enough to check Borel measurability of N near w_0. By the argument principle, there is a small disk D centered at w_0 and there are M branches of the inverse of f such that $f|_{\mathbb{D}}^{-1}(w) = \{\zeta_1(w), \dots, \zeta_M(w)\}$ for every $w \in D$; see Theorem on p. 238 of [Gam2001]. Now, upon relabeling, $Z(w_0) = \{\zeta_j(w_0)\}_{j=1}^{M'}$ for some $M' \le M$. Moreover, restricting to a smaller disk $D' \subset D$ centered at w_0, we can assume using continuity of the branches that $Z(w) \subset \{\zeta_j(w)\}_{j=1}^{M'}$ for every $w \in D'$. Since each ζ_j is analytic, standard results show that N is Borel measurable. □

Now consider the set

$$E := \bigcup_{w \in f(\mathbb{D})} Z(w).$$

Claim 4.2. *The set* $\mathbb{D} \setminus E$ *is open.*

Proof. Pick $z_0 \in \mathbb{D} \setminus E$. Let $w_0 = f(z_0)$. Then we can find $z_0^\star \in f^{-1}(w_0)$ such that

$$|z_0^\star| < |z_0|.$$

Assume that z_0 is of order $m-1$ and $z_0^\star$ of order $m^\star - 1$. Then, by the argument principle, there are disks $D = D(z_0, \epsilon)$ and $D^\star = D(z_0^\star, \epsilon)$ with radius $0 < \epsilon < (|z_0| - |z_0^\star|)/3$ small enough so that for every point $z \in D$, the value $w = f(z)$ is close enough to w_0 to have at least one preimage $z^\star$ in $D^\star$. Thus $D \subset \mathbb{D} \setminus E$. □

Below we will need the following "Non-Univalent Change of Variables Formula".

Theorem 4.3 (Theorem 2, p. 99 of [EG1992])**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}^m$ *be Lipschitz,* $n \le m$. *Then for each integrable* $g : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} g(x) Jf(x) dx = \int_{\mathbb{R}^m} \left[\sum_{x \in f^{-1}(y)} g(x) \right] d\mathcal{H}^n(y),$$

whenever either side converges, and where Jf *is the Jacobian (determinant) of* f.

Proof of Theorem 1.9. Fix $0 < r < 1$ and consider the integral

$$\int_{\mathbb{D}} \frac{\chi_{r\mathbb{D}\cap E}(z)}{N(f(z))}|f'(z)|^2 d\mathcal{H}^2(z). \tag{4.1}$$

By the non-univalent change of variables (Theorem 4.3) applied to the function

$$g(z) = \frac{\chi_{r\mathbb{D}\cap E}(z)}{N(f(z))} : \mathbb{D} \to \mathbb{R},$$

it equals

$$\begin{aligned}
&\int_{f(\mathbb{D})} \frac{1}{N(w)} \sum_{z\in f^{-1}(w)} \chi_{r\mathbb{D}\cap E}(z) d\mathcal{H}^2(w) \\
&= \int_{f(r\mathbb{D}\cap E)} \frac{1}{N(w)} \sum_{z\in f^{-1}(w)} \chi_{r\mathbb{D}\cap Z(w)}(z) d\mathcal{H}^2(w) \\
&= \int_{f(r\mathbb{D}\cap E)} \frac{1}{N(w)} \sum_{z\in Z(w)} 1 d\mathcal{H}^2(w) \\
&= \operatorname{Area} f(r\mathbb{D}\cap E) \\
&= \operatorname{Area} f(r\mathbb{D}).
\end{aligned}$$

The last equality holds because $w \in f(r\mathbb{D})$ if and only if $Z(w) \subset r\mathbb{D}$ if and only if $w \in f(r\mathbb{D}\cap E)$.

Thus,

$$\operatorname{Area} f(r\mathbb{D}) = \int_{\mathbb{D}} \frac{\chi_{r\mathbb{D}\cap E}(z)}{N(f(z))}|f'(z)|^2 d\mathcal{H}^2(z),$$

for $0 < r < 1$. In particular, the function $A(r) := \operatorname{Area} f(r\mathbb{D})$ is absolutely continuous.

By Fubini,

$$A(r) = \int_{\mathbb{D}} \frac{\chi_{r\mathbb{D}\cap E}(z)}{N(f(z))}|f'(z)|^2 d\mathcal{H}^2(z) = \int_0^r \int_{s\mathbb{T}} \frac{\chi_E(z)}{N(f(z))}|f'(z)|^2 |dz| ds.$$

So we have

$$\frac{dA(r)}{dr} = \int_{r\mathbb{T}} \frac{\chi_E(z)}{N(f(z))}|f'(z)|^2 |dz|.$$

By the Cauchy-Schwarz inequality,

$$\frac{dA(r)}{dr} \geq \frac{1}{2\pi r}\left(\int_{r\mathbb{T}} \frac{\chi_E(z)}{N(f(z))}|f'(z)||dz|\right)^2.$$

Again, by the non-univalent change of variables (Theorem 4.3) applied to the function

$$g(z) = \frac{\chi_E(z)}{N(f(z))} : r\mathbb{T} \to \mathbb{R},$$

we obtain

$$\begin{aligned}
\int_{r\mathbb{T}} \frac{\chi_E(z)}{N(f(z))}|f'(z)||dz| &= \int_{f(r\mathbb{T}\cap E)} \frac{1}{N(w)} \#(r\mathbb{T}\cap Z(w)) d\mathcal{H}^1(w) \\
&= \operatorname{Length} f(r\mathbb{T}\cap E) = \operatorname{Length} \partial f(r\mathbb{D}).
\end{aligned}$$

The last equality holds because $w \in \partial f(r\mathbb{D})$ if and only if $w \in f(r\mathbb{T} \cap E)$. Thus, writing $L(r) := \text{Length}\, \partial f(r\mathbb{D})$, we have shown that

$$\frac{dA(r)}{dr} \geq \frac{L(r)^2}{2\pi r}.$$

The isoperimetric inequality [Lax1995] says that, for planar domains,

$$4\pi \,\text{Area}\, \Omega \leq (\text{Length}\, \partial\Omega)^2.$$

So, we have

$$\frac{dA(r)}{dr} \geq \frac{2A(r)}{r}. \tag{4.2}$$

Now consider the function $\phi_{\text{Area}}(r)$ defined in the statement of Theorem 1.9. We have shown that it is absolutely continuous and its derivative is

$$\frac{d\phi_{\text{Area}}(r)}{dr} = -2\pi^{-1}r^{-3}A(r) + (\pi r^2)^{-1}\frac{dA(r)}{dr} = (\pi r^2)^{-1}\left(\frac{dA(r)}{dr} - \frac{2A(r)}{r}\right) \geq 0$$

by (4.2). Therefore, $\phi_{\text{Area}}(r)$ is an increasing function of r.

If $\phi_{\text{Area}}(r)$ is not strictly increasing, then there is $0 < s < t < 1$ such that $\phi_{\text{Area}}(r) = c$ for every $s \leq r \leq t$. This implies that $\phi'_{\text{Area}}(r) \equiv 0$ on $[s, t]$. Hence,

$$\frac{dA(r)}{dr} \equiv \frac{L(r)^2}{2\pi r} \equiv \frac{2A(r)}{r}$$

on $[s, t]$. So the extremal case in the isoperimetric inequality shows that $f(r\mathbb{D})$ is a disk for $s \leq r \leq t$, with area πr^2. Hence $\text{Rad}\, f(r\mathbb{D}) \equiv r$ on $[s, t]$, and by Theorem 1.1, we conclude that f must be linear. □

We leave the proof of Corollary 1.10 as an exercise for the reader.

5. A counter-example to log-convexity

Notice that $\log \phi_{\text{Area}}(r)$ is a convex function of $\log r$ if and only if $\log A(r)$ is. Also, $\log A(r)$ is log-convex for all univalent functions. In fact, write $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If f is univalent, $A(r) = \sum_{n=0}^{\infty} n|a_n|^2 r^{2n}$. Then, by straight differentiation, $\log \sum_{n=1}^{\infty} n|a_n|^2 e^{2nx}$ has non-negative second derivative if and only if

$$\sum_{n,k=1}^{\infty} 4(nk^3 - n^2k^2)|a_n|^2|a_k|^2 e^{(2n+2k)x} \geq 0.$$

If we switch n and k and add the results, it doesn't affect the truth of non-negativity, so the above will be non-negative if

$$nk^3 + kn^3 \geq 2n^2k^2$$

dividing by n^2k^2 it suffices that $k/n + n/k \geq 2$, which is true. In fact equality occurs if and only if $n = k$, and hence we have a strictly positive second derivative unless $f(z) = cz^m$, and by univalence, unless f is linear.

However, as the following example shows, neither $\log A(r)$ nor even $A(r)$ is log-convex in general.

Example 5.1. We study the function

$$f(z) = \exp\left(ic \log\left(\frac{1+z}{1-z}\right)\right)$$

with $c > 0$, which is a universal cover of $\mathbb{D}$ onto the annulus $\{e^{-\pi c/2} < |z| < e^{\pi c/2}\}$. To compute $A(r)$ we first apply the conformal map $\psi(z) = i \log\left(\frac{1+z}{1-z}\right) = u(z) + iv(z)$ which sends $r\mathbb{D}$ into an oval contained in the vertical strip $\{|u| < \pi/2\}$. We then notice that $f(r\mathbb{D}) \setminus (-\infty, 0)$ is covered by the restriction of $e^{\pi z}$ to the part of the oval which is in $\{|v| < \pi/c\}$. So a computation shows that f is univalent on $r\mathbb{D}$ for $r < \tanh(\pi/(2c))$ and that for $\tanh(\pi/(2c)) \leq r < 1$,

$$A(r) = \int_0^\pi 2\sinh\left(2c \arccos\left(\frac{1-r^2}{1+r^2}\cosh(t/c)\right)\right) dt.$$

Writing $A_c(r)$ for $A(r)$ to emphasize the dependence on the parameter c, we then study the asymptotics as $c \downarrow 0$. We find that for $x \in (0,1)$,

$$\lim_{c\downarrow 0} \frac{A_c(e^{-x \log\coth(\pi/(2c))}) - 2\pi\sinh(c\pi)}{4c^2} = -\int_0^x \frac{\arcsin u}{u} du.$$

But the right-hand side is a strictly concave function of $x \in (0,1)$, since its derivative is *minus* the strictly increasing function $x^{-1}\arcsin x$. Thus, for $c > 0$ sufficiently small, $A_c(r)$ cannot be log-convex.

6. Perimeter generalization of Schwarz's Lemma

The results are not as strong when considering the notion of perimeter.

Proposition 6.1. *Suppose Ω is a simply-connected domain. If F is a one-to-one analytic map of $\mathbb{D}$ onto Ω, then $r \mapsto r^{-1}\,\mathrm{Length}\,\partial(F(r\mathbb{D}))$ is strictly increasing, unless F is linear. Hence, if $\partial\Omega$ is a Jordan curve with Euclidean length at most 2π, then*

(6.1) $\qquad \mathrm{Length}\,\partial(F(r\mathbb{D})) \leq 2\pi r \qquad$ *for every* $0 < r < 1$,

(6.2) $\qquad |F'(0)| \leq 1.$

Moreover, equality holds in (6.1) *for some* $0 < r < 1$, *or in* (6.2), *if and only if $F(z)$ is an Euclidean isometry $a + cz$ for some $a \in \mathbb{C}$, $c \in \mathbb{T}$.*

Corollary 6.2. *Suppose Ω is an simply-connected region in $\mathbb{C}$ and $\partial\Omega$ is a Jordan curve with Euclidean length at most 2π. If f is analytic on $\mathbb{D}$ with values in Ω, then $|f'(0)| \leq 1$ and equality holds if and only if $f(z)$ is an Euclidean isometry $a + cz$ for some $a \in \mathbb{C}$, $c \in \mathbb{T}$.*

Remark 6.3. The bound (6.2) also follows from the well-known result that the H^1-norm of F' is the length of the boundary of the image, together with the mean-value inequality

$$|F'(0)| \leq \int |F'(re^{it})| \frac{dt}{2\pi}.$$

The right-hand side converges to the H^1-norm as $r \uparrow 1$. Also the right side above increases with r, so that r times the right side, is bounded above by r times the right side evaluated at $r = 1$, and that gives (6.1).

Remark 6.4. The same "square root trick" used in the proof of Proposition 6.1 below can be used to prove the isoperimetric inequality; see [Carl1921] and [D1983, exercise 3, page 25].

Remark 6.5. *Lower* bounds for area and perimeter of image disks can be found in a paper of MacGregor [Mac1964] and they involve the derivative at the origin.

Proof of Proposition 6.1. Let $G(z) := \sum_{n=0}^{\infty} b_n z^n$ be an analytic square root in $\mathbb{D}$ of the zero-free function F'. Then $F'(0) = G^2(0)$ and

$$\begin{aligned} r^{-1} \operatorname{Length} \partial(F(r\mathbb{D})) &= r^{-1} \int_{|z|=r} |F'(z)||dz| \\ &= r^{-1} \int_{|z|=r} |G(z)|^2 |dz| \\ &= 2\pi \sum_{n=0}^{\infty} |b_n|^2 r^{2n}, \end{aligned} \tag{6.3}$$

which is strictly increasing for $0 < r < 1$ unless $b_n = 0$ for all $n \geq 1$, i.e., unless F is linear. The rest follows straightforwardly.

Also, the isoperimetric inequality [Lax1995] says that

$$4\pi \operatorname{Area} \Omega \leq (\operatorname{Length} \partial\Omega)^2.$$

Therefore, $\operatorname{Area} \Omega \leq \pi$ and by Corollary 1.10, $|F'(0)| \leq 1$, with equality if and only if $F(z) = a + cz$ identically for some $a \in \mathbb{C}$, $c \in \mathbb{T}$. □

Proof of Corollary 6.2. If f is analytic on $\mathbb{D}$ with values in Ω, let F be the Riemann map of $\mathbb{D}$ onto Ω with $F(0) = f(0)$ and $F'(0) > 0$. Then $g := F^{-1} \circ f$ is a self-map of the disk which fixes the origin and $f = F \circ g$. So

$$|f'(0)| = F'(0)|g'(0)| \leq F'(0) \leq 1$$

with equality if and only if g is a rotation, i.e., if and only if $f(z) = F(cz)$ for some $c \in \mathbb{T}$. Now apply Proposition 6.1. □

7. Applications to hyperbolic geometry

The hyperbolic metric on $\mathbb{D}$ is

$$\rho_{\mathbb{D}}(z)|dz| := \frac{|dz|}{1 - |z|^2}.$$

So $\rho_{\mathbb{D}}(z) \geq 1$ for every $z \in \mathbb{D}$ with equality when $z = 0$.

The associated hyperbolic distance function is

$$h_{\mathbb{D}}(z, w) := \tanh^{-1} \left| \frac{z - w}{1 - \bar{w}z} \right|.$$

The hyperbolic disk with hyperbolic center c and hyperbolic radius $R > 0$ is

$$D_{\mathbb{D}}(c, R) = \{z : h_{\mathbb{D}}(z, c) < R\}.$$

The closed hyperbolic disk $\bar{D}_{\mathbb{D}}(c, R)$ is defined similarly. For $0 < r < 1$ the Euclidean disk $r\mathbb{D}$ is the hyperbolic disk $D_{\mathbb{D}}(0, R)$, where $R = \tanh^{-1} r$, or $r = \tanh R$.

A region Ω in $\mathbb{C}$ is hyperbolic if $\mathbb{C} \setminus \Omega$ contains at least two points. If Ω is a hyperbolic region and $f : \mathbb{D} \to \Omega$ is an analytic covering, then the density ρ_Ω of the hyperbolic metric $\rho_\Omega(w)|dw|$ on Ω is defined so that

$$\rho_\Omega(w)|dw| = \rho_\Omega(f(z))|f'(z)||dz| = \rho_{\mathbb{D}}(z)|dz|. \tag{7.1}$$

This defines the hyperbolic density ρ_Ω independent of the covering. Let h_Ω be the associated hyperbolic distance function on Ω. Open and closed hyperbolic disks in Ω are defined in the standard way. If $f : \mathbb{D} \to \Omega$ is an analytic covering with $f(0) = c$ and $R > 0$, then $f(D_{\mathbb{D}}(0, R)) = D_\Omega(c, R)$ with the similar result for closed hyperbolic disks.

By Schwarz's Lemma (Theorem 1.1) and the Monodromy Theorem, the following monotonicity holds

$$\tilde{\Omega} \subset \Omega \qquad \Longrightarrow \qquad \rho_{\tilde{\Omega}}(z) \geq \rho_{\Omega}(z) \qquad \forall z \in \tilde{\Omega}.$$

So for $z \in \Omega$, by choosing $\tilde{\Omega}$ to be the largest Euclidean disk centered at z contained in Ω, one gets the following upper bound for hyperbolic density:

$$\rho_{\Omega}(z) \leq \frac{1}{\operatorname{dist}(z, \partial\Omega)}.$$

In [A1973, p. 16], Ahlfors states that it is a much harder problem to find lower bounds. Theorem 7.1(b) below shows, in particular, that the geometric lower bound of

$$\rho_{\Omega}(z) \geq \sqrt{\pi / \operatorname{Area}(\Omega)} \tag{7.2}$$

holds for every region Ω and every $z \in \Omega$.

Theorem 7.1. *Suppose Ω is a hyperbolic region. Then for each $c \in \Omega$ and $R > 0$, the function $R \mapsto \left(\pi \tanh^2(R)\right)^{-1} \operatorname{Area} D_{\Omega}(c, R)$ is strictly increasing except when Ω is an Euclidean disk with center c. If $\operatorname{Area} \Omega \leq \pi$, then*

(a) *for each $c \in \Omega$ and all $R > 0$,*

$$\operatorname{Area} D_{\Omega}(c, R) \leq \pi \tanh^2 R$$

with equality if and only if Ω is an Euclidean disk with center c and radius 1; and

(b) *for each $c \in \Omega$,*

$$1 \leq \rho_{\Omega}(c)$$

and equality holds if and only if Ω is an Euclidean disk with center c and radius 1.

Proof. Fix $c \in \Omega$ and let $f : \mathbb{D} \to \Omega$ be an analytic covering with $f(0) = c$. Since $f(r\mathbb{D}) = D_{\Omega}(c, R)$, where $r = \tanh R$,

$$\frac{\operatorname{Area} D_{\Omega}(c, R)}{\pi \tanh^2 R} = \frac{\operatorname{Area} f(r\mathbb{D})}{\pi r^2}.$$

Theorem 1.9 implies that this quotient is strictly increasing unless f is linear, or equivalently, Ω is a disk with center c.

If $\operatorname{Area} \Omega \leq \pi$, then parts (a) and (b) follow from Corollary 1.10; note that $\rho_{\Omega}(c) = 1/|f'(0)|$. □

Analogous theorems can be formulated for logarithmic capacity and n-diameter.

8. Modulus growth bounds

In view of the bound on the growth of the modulus in Schwarz's Lemma, it is natural to ask whether a similar statement holds in the context of 'diameter'. We offer the following result.

Theorem 8.1. *Suppose f is analytic on the unit disk $\mathbb{D}$ and $\operatorname{Diam} f(\mathbb{D}) \leq 2$. Then for all $z \in \mathbb{D}$,*

$$|f(z) - f(0)| \leq |z| \frac{2}{1 + \sqrt{1 - |z|^2}}. \tag{8.1}$$

Moreover, equality holds in (8.1) *at some point in* $\mathbb{D}\setminus\{0\}$ *if and only if* f *is a linear fractional transformation of the form*

$$f(z) = c\frac{z-b}{1-\bar{b}z} + a \tag{8.2}$$

for some constants $a \in \mathbb{C}$, $b \in \mathbb{D}\setminus\{0\}$ *and* $c \in \mathbb{T}$.

Remark 8.2. In Schwarz's Lemma, equality in (1.2) at some point in $\mathbb{D}\setminus\{0\}$ holds if and only if equality holds at every point $z \in \mathbb{D}$. This is not true any more in Theorem 8.1. Namely, when f is the linear fractional transformation in (8.2), then equality in (8.1) occurs only for $z := 2b/(1+|b|^2)$.

Remark 8.3. Since in (8.1) the origin does not play a special role, we can rewrite that inequality more symmetrically as follows:

$$|f(z) - f(w)| \le \operatorname{Diam} f(\mathbb{D})\frac{\delta}{1+\sqrt{1-\delta^2}} = \operatorname{Diam} f(\mathbb{D})\tanh(\rho/2) \qquad \forall z, w \in \mathbb{D}$$

where $\delta = \delta(z,w) := \left|\frac{z-w}{1-\bar{w}z}\right|$ is the pseudohyperbolic distance between z and w and $\rho = \rho(z,w) := (1/2)\log[(1+\delta)/(1-\delta)]$ is the hyperbolic distance between z and w.

The preceding inequality can also be rewritten using the well-known identity

$$1 - \left|\frac{z-w}{1-\bar{w}z}\right|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{w}z|^2},$$

as

$$|f(z) - f(w)| \le \operatorname{Diam} f(\mathbb{D})\frac{|z-w|}{|1-\bar{w}z| + \sqrt{(1-|z|^2)(1-|w|^2)}}.$$

Proof of Theorem 8.1. Fix $d \in \mathbb{D}$ such that $f(d) \ne f(0)$. Set

$$g = c_1 f \circ T + c_2$$

where T is a linear fractional transformation of $\mathbb{D}$ onto $\mathbb{D}$ such that $T(x) = d$, $T(-x) = 0$, for some $x > 0$ and c_1, c_2 are constants chosen so that $g(x) = x$ and $g(-x) = -x$. By elementary algebra

$$T(z) = \frac{d}{|d|}\frac{z+x}{1+xz}$$

where $x := |d|/(1+\sqrt{1-|d|^2})$,

$$c_1 := \frac{2x}{f(d)-f(0)} \qquad \text{and} \qquad c_2 := -x\frac{f(d)+f(0)}{f(d)-f(0)}.$$

Then

$$\operatorname{Diam} g(\mathbb{D}) = |c_1| \operatorname{Diam} f(\mathbb{D}) \le \frac{4}{|f(d)-f(0)|}\frac{|d|}{(1+\sqrt{1-|d|^2})}. \tag{8.3}$$

We now prove that $\operatorname{Diam} g(\mathbb{D}) \ge 2$, with equality if and only if $g(z) \equiv z$.

Set $h(z) := (g(z) - g(-z))/2$. Then $h(x) = x$ and $h(-x) = -x$. Note also that $h(0) = 0$ so that $h(z)/z$ is analytic in the disk and has value 1 at x and hence by the maximum principle $\sup_{\mathbb{D}} |h(z)| = \sup_{\mathbb{D}} |h(z)/z| \ge 1$, with equality only if $h(z) = z$ for all $z \in \mathbb{D}$. Since, by definition of h, $\operatorname{Diam} g(\mathbb{D}) \ge 2\sup_{\mathbb{D}} |h|$, we see that $\operatorname{Diam} g(\mathbb{D}) \ge 2$ and then (8.3) gives (8.1) for $z = d$.

If equality holds in (8.1) at some point in $\mathbb{D}\setminus\{0\}$, then that point is an eligible d for the preceding discussion, and (8.3) shows that $\operatorname{Diam} g(\mathbb{D}) \le 2$, while we have already shown that $\operatorname{Diam} g(\mathbb{D}) \geq 2$. Thus $\operatorname{Diam} g(\mathbb{D}) = 2$. Hence $\sup_{z\in\mathbb{D}} |h(z)| = 1$ and therefore $h(z) \equiv z$. Since h is the odd part of g, we have $g'(0) = h'(0) = 1$. Thus, by the Landau-Toeplitz Theorem 1.3 applied to g, we find that $g(z) \equiv g(0)+z$ and thus

$$f(z) = \frac{1}{c_1} T^{-1}(z) + f(T(0)).$$

Moreover, equality at $z = d$ in (8.1) says that $|f(d) - f(0)| = 2x$, hence $|c_1| = 1$. Since T is a Möbius transformation of $\mathbb{D}$, namely of the form

$$T(z) = \eta \frac{z - \xi}{1 - \bar{\xi} z}$$

for some constants $\xi \in \mathbb{D}$ and $\eta \in \mathbb{T}$, its inverse is also of this form. Therefore, we conclude that f can be written as in (8.2).

Finally, if f is given by (8.2), then $2b/(1 + |b|^2) \in \mathbb{D} \setminus \{0\}$, and one checks that equality is attained in (8.1) when z has this value and for no other value in $\mathbb{D} \setminus \{0\}$. □

9. Higher derivatives

We finish with a result, due to Kalle Poukka in 1907, which is to be compared with the usual Cauchy estimates that one gets from the maximum modulus. Interestingly, Poukka seems to have been the first student of Ernst Lindelöf, who is often credited with having founded the Finnish school of analysis.

Theorem 9.1 (Poukka [Pou1907]). *Suppose f is analytic on $\mathbb{D}$. Then for all positive integers n we have*

$$\frac{|f^{(n)}(0)|}{n!} \leq \frac{1}{2} \operatorname{Diam} f(\mathbb{D}). \tag{9.1}$$

Moreover, equality holds in (9.1) *for some n if and only if $f(z) = f(0) + cz^n$ for some constant c of modulus* $\operatorname{Diam} f(\mathbb{D})/2$.

Proof (Poukka): Write $c_k := f^{(k)}(0)/k!$, so that $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for every $z \in \mathbb{D}$. Fix $n \in \mathbb{N}$. For every $z \in \mathbb{D}$,

$$h(z) := f(z) - f(ze^{i\pi/n}) = \sum_{k=1}^{\infty} c_k(1 - e^{i\pi k/n}) z^k. \tag{9.2}$$

Fix $0 < r < 1$ and notice that, by absolute and uniform convergence,

$$\sum_{k=1}^{\infty} |c_k|^2 |1 - e^{i\pi k/n}|^2 r^{2k} = \int_0^{2\pi} |h(re^{i\theta})|^2 \frac{d\theta}{2\pi} \leq (\operatorname{Diam} f(\mathbb{D}))^2. \tag{9.3}$$

Therefore

$$|c_k(1 - e^{i\pi k/n})| r^k \leq \operatorname{Diam} f(\mathbb{D})$$

for every $0 < r < 1$ and every $k \in \mathbb{N}$. In particular, letting r tend to 1 and then setting $k = n$, we get $2|c_n| \leq \operatorname{Diam} f(\mathbb{D})$, which is (9.1).

If equality holds here, then letting r tend to 1 in (9.3), we get that all coefficients $c_k(1 - e^{i\pi k/n})$ in (9.2) for $k \neq n$ must be 0. Hence, $c_k = 0$ whenever k is not a multiple of n. Thus, $f(z) = g(z^n)$ for some analytic function g on $\mathbb{D}$. Moreover, $g'(0) = c_n$ and $\operatorname{Diam} g(\mathbb{D}) = \operatorname{Diam} f(\mathbb{D})$. So, by Theorem 1.3, $g(z) = cz$ for some constant c with $|c| = \operatorname{Diam} g(\mathbb{D})$, and the result follows. □

10. Further problems

Here we discuss a couple of problems that are related to these "diameter" questions.

The first problem arises when trying to estimate the distance of f from its linearization, $f(z)-(f(0)+f'(0)z)$, to give a "quantitative" version for the 'equality' case in Schwarz's Lemma (Theorem 1.1). This is done via the so-called Schur algorithm. As before, one considers the function

$$g(z) := \frac{f(z)-f(0)}{z}$$

which is analytic in $\mathbb{D}$, satisfies $g(0)=f'(0)$ and which, by the Maximum Modulus Theorem, has, say, $\sup_{\mathbb{D}}|g| \leq 1$. Now let $a := f'(0)$ and post-compose g with a Möbius transformation of $\mathbb{D}$ which sends a to 0 to find that

$$\frac{g(z)-a}{1-\bar{a}g(z)} = zh(z)$$

for some analytic function h with $\sup_{\mathbb{D}}|h| \leq 1$.

Inserting the definition of g in terms of f and solving for f shows that

$$f(z)-f(0)-az = (1-|a|^2)\frac{z^2h(z)}{1+\bar{a}zh(z)}.$$

Thus, for every $0<r<1$,

$$\max_{|z|<r}|f(z)-f(0)-f'(0)z| \leq (1-|f'(0)|^2)\frac{r^2}{1-|f'(0)|r} \tag{10.1}$$

and 'equality' holds for at least one such r if and only if $h(z) \equiv a/|a| = f'(0)/|f'(0)|$, i.e., if and only if

$$f(z) = z\frac{a}{|a|}\frac{z+|a|}{1+|a|z}+b$$

identically, for constants $a \in \overline{\mathbb{D}}$, $b \in \mathbb{C}$.

In the context of this paper, when f is analytic in $\mathbb{D}$ and $\operatorname{Diam} f(\mathbb{D}) \leq 2$, by the Landau-Toeplitz Theorem 1.3 and a normal-family argument we see that, for every $\epsilon>0$ and every $0<r<1$, there exists $\alpha>0$ such that: $|f'(0)| \geq 1-\alpha$ implies

$$|f(z)-(f(0)+f'(0)z)| \leq \epsilon \qquad \forall |z| \leq r.$$

However, one could ask for an explicit bound as in (10.1).

Problem 10.1. *If f is analytic in $\mathbb{D}$ and $\operatorname{Diam} f(\mathbb{D}) \leq 2$, find an explicit (best?) function $\phi(r)$ for $0 \leq r < 1$ so that*

$$|f(z)-(f(0)+f'(0)z)| \leq (1-|f'(0)|)\phi(r) \qquad \forall |z| \leq r.$$

Another problem can be formulated in view of Section 7. It is known (see the Corollary to Theorem 3 in [MinW1982]) that if Ω is a bounded convex domain, then the minimum

$$\Lambda(\Omega) := \min_{w\in\Omega}\rho_\Omega(w) \tag{10.2}$$

is attained at a unique point τ_Ω, which we can call the hyperbolic center of Ω. Also let us define the hyperbolic radius of Ω to be

$$R_h(\Omega) := \sup_{w\in\Omega}|w-\tau_\Omega|.$$

Now assume that $\operatorname{Diam}\Omega = 2$. Then we know, by the corresponding "diameter" version of Theorem 7.1, that $\Lambda(\Omega) \geq 1$, with equality if and only if Ω is a disk of radius 1. In particular, if $\Lambda(\Omega) = 1$, then $R_h(\Omega) = 1$.

Problem 10.2. *Given $m > 1$, find or estimate, in terms of $m - 1$,*

$$\sup_{\Omega \in \mathcal{A}_m} R_h(\Omega)$$

where $\mathcal{A}_m$ is the family of all convex domains Ω with $\operatorname{Diam}\Omega = 2$ and $\Lambda(\Omega) \leq m$.

More generally, given an analytic function f on $\mathbb{D}$ such that $\operatorname{Diam} f(\mathbb{D}) \leq 2$, define

$$M(f) := \min_{w \in \mathbb{D}} \sup_{z \in \mathbb{D}} |f(z) - f(w)|$$

and let w_f be a point where $M(f)$ is attained.

Problem 10.3. *Fix $a < 1$. Find or estimate, in terms of $1 - a$,*

$$\sup_{f \in \mathcal{B}_a} M(f)$$

where $\mathcal{B}_a$ is the family of all analytic functions f on $\mathbb{D}$ with $\operatorname{Diam} f(\mathbb{D}) \leq 2$ and

$$|f'(w_f)|(1 - |w_f|^2) \geq a.$$

Similar questions can be asked replacing diameter by area or capacity.

Also in this paper we considered analytic maps f of the unit disk $\mathbb{D}$ into a region with bounded area, diameter or capacity, and established analogs of Schwarz's Lemma. What about analogs of Schwarz's Lemma for the 'dual' situation of an analytic map $f : \Omega \to \mathbb{D}$, where Ω satisfies some geometric restriction?

Acknowledgment

The authors thank A. F. Beardon for his comments and suggestions on an earlier version.

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