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PURE MAPPING CLASS GROUP ACTING ON TEICHMÜLLER SPACE

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Abstract. For a Riemann surface of analytically infinite type, the action of the quasiconformal mapping class group on the Teichmüller space is not discontinuous in general. In this paper, we consider pure mapping classes that fix all topological ends of a Riemann surface and prove that the pure mapping class group acts on the Teichmüller space discontinuously under a certain geometric condition of a Riemann surface. We also consider the action of the quasiconformal mapping class group on the asymptotic Teichmüller space. Non-trivial mapping classes can act on the asymptotic Teichmüller space trivially. We prove that all such mapping classes are contained in the pure mapping class group.

1. Introduction and results

The *Teichmüller space* $T(R)$ of a Riemann surface R is the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms f of R. Here we say that two quasiconformal homeomorphisms f_1 and f_2 of R are *equivalent* if there exists a conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. The homotopy is considered to be relative to the ideal boundary at infinity. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d([f_1], [f_2]) = (1/2) \log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d is a complete distance on $T(R)$ which is called the Teichmüller distance. The Teichmüller space $T(R)$ can be embedded in the complex Banach space of all bounded holomorphic quadratic differentials on R', where R' is the complex conjugate of R. In this way, $T(R)$ is endowed with the complex structure; for details, see [21] and [26].

A *quasiconformal mapping class* is the homotopy equivalence class $[g]$ of quasiconformal automorphisms g of a Riemann surface, and the *quasiconformal mapping class group* $\mathrm{MCG}(R)$ of R is the set of all quasiconformal mapping classes of R. Here the homotopy is again considered to be relative to the ideal boundary at infinity. Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_*$ of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$, which is also isometric with respect to the Teichmüller distance. Let $\mathrm{Aut}(T(R))$ be the group of all biholomorphic automorphisms of $T(R)$. Then

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we have a homomorphism

$$\iota : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$$

by $[g] \mapsto [g]_*$, and define the *Teichmüller modular group* of R by

$$\mathrm{Mod}(R) = \iota(\mathrm{MCG}(R)).$$

It was proved in [4], [8] and [23] that the homomorphism ι is injective (faithful) for all Riemann surfaces R of non-exceptional type. Here we say that a Riemann surface R is of *exceptional type* if R has finite hyperbolic area and satisfies $2g+n \leq 4$, where g is the genus of R and n is the number of punctures of R. The homomorphism ι is also surjective for every Riemann surface R of non-exceptional type, namely $\mathrm{Mod}(R) = \mathrm{Aut}(T(R))$. The proof is a combination of the results in [3] and [22]; see [13] for a survey of the proof.

We say that a subgroup $G \subset \mathrm{MCG}(R)$ acts at a point $p \in T(R)$ *discontinuously* if there exists a neighborhood U of p such that the number of elements $[g] \in G$ satisfying $[g]_*(U) \cap U \neq \emptyset$ is finite. This is equivalent to that there exist no distinct elements $[g_n] \in G$ such that $d([g_n]_*(p), p) \to 0$ as $n \to \infty$, namely the orbit $G(p)$ is discrete and the stabilizer subgroup $\mathrm{Stab}_G(p) := \{[g] \in G \mid [g]_*(p) = p\}$ is finite. We say that G acts on $T(R)$ discontinuously if G acts at all points in $T(R)$ discontinuously. We also say that a subgroup $G \subset \mathrm{MCG}(R)$ acts at a point $p \in T(R)$ *freely* if $\mathrm{Stab}_G(p)$ consists only of $[id]$ and G acts on $T(R)$ freely if G acts at all points in $T(R)$ freely.

For a Riemann surface R of analytically finite type, the quasiconformal mapping class group $\mathrm{MCG}(R)$ acts on $T(R)$ discontinuously. However, for a Riemann surface of analytically infinite type, the action of $\mathrm{MCG}(R)$ is, in general, not discontinuous. On the basis of this fact, a special planar Riemann surface, the complement $\hat{\mathbb{C}} - \mathrm{C}$ of the standard middle-thirds Cantor set C in the unit interval as a subset of the complex sphere $\hat{\mathbb{C}}$, was considered in [18]. Then the pure mapping class group $P(\hat{\mathbb{C}} - \mathrm{C})$ was defined as the group of all homotopy classes of quasiconformal automorphisms of $\mathbb{C}$ that fix all points of C, and the following statement was proved.

Proposition 1.1. *The pure mapping class group $P(\hat{\mathbb{C}} - \mathrm{C})$ acts on the Teichmüller space $T(\hat{\mathbb{C}} - \mathrm{C})$ discontinuously.*

In this paper, we extend Proposition 1.1 to general Riemann surfaces. First, we define the pure mapping class group for all Riemann surfaces; see Section 2 for a definition of an end of a Riemann surface.

Definition 1.2. The *pure mapping class group* $P(R)$ of a Riemann surface R is the group of all quasiconformal mapping classes $[g] \in \mathrm{MCG}(R)$ such that g fixes all non-cuspidal ends of R.

Our main result is the following; see Section 3 for a definition of the bounded geometry condition.

Theorem 1.3. *Let R be a Riemann surface that satisfies the bounded geometry condition and has more than two non-cuspidal ends. Then the pure mapping class group $P(R)$ acts on the Teichmüller space $T(R)$ discontinuously.*

Our next concern is the action of the quasiconformal mapping class group on the asymptotic Teichmüller space. The asymptotic Teichmüller space is a quotient space of the Teichmüller space and it was introduced in [19] when R is the upper half-plane and in [5] and [17] when R is an arbitrary hyperbolic Riemann surface. We say that a quasiconformal homeomorphism f of R is *asymptotically conformal* if for every $\epsilon > 0$, there exists a compact subset V of R such that the maximal dilatation $K(f|_{R-V})$ of the restriction of f to $R-V$ is less than $1+\epsilon$. We say that two quasiconformal homeomorphisms f_1 and f_2 of R are *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity by a homotopy that keeps every point of the ideal boundary at infinity fixed throughout. The *asymptotic Teichmüller space* $AT(R)$ of a Riemann surface R is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal homeomorphisms f of R. The asymptotic Teichmüller space $AT(R)$ is of interest only when R is analytically infinite. Otherwise $AT(R)$ is trivial; that is, it consists of just one point. Conversely, if R is analytically infinite, then $AT(R)$ is not trivial. In fact, it is infinite dimensional. Since a conformal homeomorphism is asymptotically conformal, there is a natural projection $\pi : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $[f] \in T(R)$ to the asymptotic Teichmüller equivalence class $[[f]] \in AT(R)$. The asymptotic Teichmüller space $AT(R)$ has a complex manifold structure such that π is holomorphic; see also [7].

For a quasiconformal homeomorphism f of R, the *boundary dilatation* of f is defined by $H^*(f) = \inf K(f|_{R-V})$, where the infimum is taken over all compact subsets V of R. Furthermore, for a Teichmüller equivalence class $[f] \in T(R)$, the *boundary dilatation* of $[f]$ is defined by $H([f]) = \inf H^*(g)$, where the infimum is taken over all elements $g \in [f]$. A distance between two points $[[f_1]]$ and $[[f_2]]$ in $AT(R)$ is defined by $d_A([[f_1]], [[f_2]]) = (1/2) \log H([f_2 \circ f_1^{-1}])$, where $[f_2 \circ f_1^{-1}]$ is a Teichmüller equivalence class of $f_2 \circ f_1^{-1}$ in $T(f_1(R))$. Then d_A is a complete distance on $AT(R)$, which is called the asymptotic Teichmüller distance. For every point $[[f]] \in AT(R)$, there exists an asymptotically extremal element $f_0 \in [[f]]$ in the sense that $H([f]) = H^*(f_0)$.

Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_{**}$ of $AT(R)$ by $[[f]] \mapsto [[f \circ g^{-1}]]$, which is also isometric with respect to d_A; see [6]. Let $\mathrm{Aut}(AT(R))$ be the group of all biholomorphic automorphisms of $AT(R)$. Then we have a homomorphism

$$\iota_A : \mathrm{MCG}(R) \to \mathrm{Aut}(AT(R))$$

by $[g] \mapsto [g]_{**}$, and define the *asymptotic Teichmüller modular group* (the geometric automorphism group) of R by

$$\mathrm{Mod}_A(R) = \iota_A(\mathrm{MCG}(R)).$$

It is different from the case of $\iota : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$ that the homomorphism ι_A is not injective (faithful), namely $\mathrm{Ker}\, \iota_A \neq \{[id]\}$, unless R is either the unit disc or a once-punctured disc. We call an element of $\mathrm{Ker}\, \iota_A$ asymptotically trivial and call $\mathrm{Ker}\, \iota_A$ the *asymptotically trivial mapping class group*. A Riemann surface R satisfying $\mathrm{MCG}(R) = \mathrm{Ker}\, \iota_A$ was constructed in [24]. Moreover, in [12], we gave a sufficient condition of a Riemann surface R such that the quasiconformal mapping class group $\mathrm{MCG}(R)$ acts on $AT(R)$ non-trivially, namely $\mathrm{Ker}\, \iota_A \subsetneqq \mathrm{MCG}(R)$. We also gave a condition of a quasiconformal homeomorphism g satisfying $[g] \notin \mathrm{Ker}\, \iota_A$.

We characterize elements of $\operatorname{Ker}\iota_A$ topologically. To state our result, we define a subgroup of the pure mapping class group which is called the stable quasiconformal mapping class group or the essentially trivial mapping class group.

Definition 1.4. A quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ is said to be *essentially trivial* if there exists a compact subsurface V_g of R such that, for each connected component W of $R - V_g$ that is not a cusp neighborhood, the restriction $g|_W : W \to R$ is homotopic to the inclusion map $id|_W : W \hookrightarrow R$. The group of all essentially trivial mapping classes of R is called the *stable quasiconformal mapping class group* or the *essentially trivial mapping class group* and is denoted by $G_\infty(R)$.

Our second result in this paper is the following.

Theorem 1.5. *The inclusion relation* $G_\infty(R) \subset \operatorname{Ker}\iota_A \subset P(R)$ *holds.*

In Section 2, we give another equivalent definition of the pure mapping class group. We prove Theorem 1.3 in Section 3 and Theorem 1.5 in Section 4. In Section 5, we consider the discontinuous action of the stable quasiconformal mapping class group on the Teichmüller space.

2. Equivalent definition of the pure mapping class group

Throughout this paper, we assume that a Riemann surface R is hyperbolic. Namely, it is represented by a quotient space Δ/H of the unit disk $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ by a torsion free Fuchsian group H. We also assume that R is non-elementary, namely H is not cyclic.

For a Riemann surface R of infinite type, a canonical exhaustion $\{R_n\}_{n=1}^\infty$ of R is an increasing sequence of compact subsurfaces R_n satisfying that $R = \bigcup_{n=1} R_n$ and each boundary component of R_n is a dividing simple closed curve and each connected component of the complement of R_n is not relatively compact. Let $\{X_i^{(n)}\}_{i=1}^{N(n)}$ be the connected components of the complement of R_n. A determining sequence for a canonical exhaustion $\{R_n\}_{n=1}^\infty$ is a sequence $\{X_{i_n}^{(n)}\}_{n=1}^\infty$ such that $X_{i_n}^{(n)} \supset X_{i_{n+1}}^{(n+1)}$ for all n. Another canonical exhaustion $\{R'_n\}_{n=1}^\infty$ gives another determining sequence $\{{X'}_{i'_n}^{(n)}\}_{n=1}^\infty$. We say that two determining sequences $\{X_{i_n}^{(n)}\}_{n=1}^\infty$ and $\{{X'}_{i'_n}^{(n)}\}_{n=1}^\infty$ are equivalent if for every n, there exists an m such that $X_{i_n}^{(n)} \supset {X'}_{i'_m}^{(m)}$ and vice versa.

A *Stoïlow end* is an equivalence class of determining sequences, and the *Stoïlow compactification* R^* of R is the union of R and the set of all Stoïlow ends endowed with canonical topology. A *Stoïlow ideal boundary point* is a point in $R^* - R$; for details, see [27]. We call a Stoïlow ideal boundary point simply an *end* of R. Furthermore we say that an end is *non-cuspidal* if it does not correspond to a puncture. We say that a compactification R^* is of type S when, for every domain $G^* \subset R^*$ whose boundary is in R, the set $G^* - (R^* - R)$ is connected. The Stoïlow compactification is the smallest compactification of type S; see [2]. A homeomorphism of R extends to a homeomorphism of R^* by the correspondence of determining sequences.

For a Riemann surface R, let $\dot{R}$ be the Riemann surface obtained by filling all the punctures of R. Let $\Gamma(R)$ be the group of all elements $[g] \in \mathrm{MCG}(R)$ such that, for all dividing simple closed oriented curves c on $\dot{R}$, the image $g(c)$ is homologous to c in $\dot{R}$. Then we have the following.

Theorem 2.1. *The pure mapping class group $P(R)$ is coincident with $\Gamma(R)$.*

Before we prove this theorem, we note the following fact. Fix a canonical exhaustion $\{R_n\}_{n=1}^{\infty}$ of a Riemann surface R. Then for each n, the subsurface R_n is compact and the boundary ∂R_n consists of finitely many dividing curves $\gamma_1, \ldots, \gamma_{N(n)}$ on R. The boundary ∂R_n has a counter-clockwise orientation when we view from the inside of R_n. Then ∂R_n is represented by the sum $\gamma_1 + \cdots + \gamma_{N(n)}$ as an element of the homology group. For two dividing simple closed oriented curves c and c' on R, there exists an n such that R_n contains both c and c'. The curve c (resp. c') is homologous to the partial sum $\gamma = \gamma_{i_1} + \cdots + \gamma_{i_k}$ (resp. $\gamma' = \gamma_{i'_1} + \cdots + \gamma_{i'_\ell}$) of components of ∂R_n, and the partial sums are uniquely determined. Then the curves c and c' are homologous if and only if $\gamma = \gamma'$. By using this fact, we prove Theorem 2.1.

Proof of Theorem 2.1. First we assume that R has no non-cuspidal ends or only one non-cuspidal end. Then $P(R) = \mathrm{MCG}(R)$. Let $\dot{R}$ be the Riemann surface obtained by filling all the punctures of such a surface R, and we take an arbitrary dividing simple closed curve c on $\dot{R}$. Then at least one component of $\dot{R} - c$ has no ends. This means that the component is compact and hence c is null-homologous in $\dot{R}$. For a quasiconformal automorphism g, the image $g(c)$ is also null-homologous in $\dot{R}$. Thus we also have $\Gamma(R) = \mathrm{MCG}(R)$.

Hereafter, we assume that R has at least two non-cuspidal ends. We take a canonical exhaustion $\{R_n\}_{n=1}^{\infty}$ of R, and let $\{X_i^{(n)}\}_{i=1}^{N(n)}$ be the connected components of the complement of R_n for each n. The boundary ∂R_n consists of dividing simple closed curves on R and the boundary components have counter-clockwise orientations when we view from the inside of R_n. For a quasiconformal automorphism g of R, the image $\{g(R_n)\}_{n=1}^{\infty}$ is also a canonical exhaustion of R.

First we prove that $\Gamma(R) \subset P(R)$. Fix an arbitrary non-cuspidal end e of R, and let $\{X_{i_n}^{(n)}\}_{n=1}^{\infty}$ be a determining sequence corresponding to e. For an arbitrary element $[g] \in \Gamma(R)$, the image $\{g(X_{i_n}^{(n)})\}_{n=1}^{\infty}$ is a determining sequence corresponding to $g(e)$. To prove $g(e) = e$, we will show that, for every n, there is an m such that $g(X_{i_n}^{(n)}) \supset X_{i_m}^{(m)}$, and that for every n', there is an m' such that $X_{i'_n}^{(n')} \supset g(X_{i'_m}^{(m')})$. Then the two determining sequences $\{X_{i_n}^{(n)}\}_{n=1}^{\infty}$ and $\{g(X_{i_n}^{(n)})\}_{n=1}^{\infty}$ are equivalent.

For each n, we take an m $(\geq n)$ so that R_m contains the dividing simple closed curves $\partial X_{i_n}^{(n)}$ and $g(\partial X_{i_n}^{(n)})$. Since $[g] \in \Gamma(R)$, the curves $\partial X_{i_n}^{(n)}$ and $g(\partial X_{i_n}^{(n)})$ are homologous. Then by the above mentioned fact, there exists a partial sum α of components of the boundary ∂R_m such that it is homologous to both $\partial X_{i_n}^{(n)}$ and $g(\partial X_{i_n}^{(n)})$. Since $\partial X_{i_m}^{(m)}$ is a component of α, we have $g(X_{i_n}^{(n)}) \supset X_{i_m}^{(m)}$. By replacing g with g^{-1} in this argument, we see that for every n', there is an m' such that $g^{-1}(X_{i'_n}^{(n')}) \supset X_{i'_m}^{(m')}$. Thus $X_{i'_n}^{(n')} \supset g(X_{i'_m}^{(m')})$.

Next we prove that $P(R) \subset \Gamma(R)$. Let $[g] \in P(R)$. We will prove that for all dividing simple closed curves c on $\dot{R}$, the curve $g(c)$ is homologous to c in $\dot{R}$. We take such a curve c arbitrarily, and may assume that it is null-homologous. There is an m such that R_m contains both c and $g(c)$. Then the curve c (resp. $g(c)$) is homologous to the partial sum α (resp. β) of components of the boundary ∂R_m. Suppose to the contrary that c and $g(c)$ are not homologous. Then $\alpha \neq \beta$. Without loss of generality, we may assume that there is a component γ of ∂R_m such that γ is a component of α but it is not a component of β. We take an end e of R so that the determining sequence $\{X_{i_n}^{(n)}\}_{n=1}^{\infty}$ corresponding to e satisfies $\partial X_{i_m}^{(m)} = \gamma$. Then we have $g(X_{i_n}^{(n)}) \cap X_{i_n}^{(n)} = \emptyset$ for all $n \geq m$, and this contradicts that $g(e) = e$. □

The mapping class group $\mathrm{MCG}(R)$ acts on the homology group $\mathrm{H}_1(R)$. Then there is a homomorphism from $\mathrm{MCG}(R)$ to the group of automorphisms of $\mathrm{H}_1(R)$. For a compact Riemann surface R, the *Torelli group* is defined to be the kernel of this homomorphism. Namely, the Torelli group is the set of all elements $[g] \in \mathrm{MCG}(R)$ such that g fixes all homology classes of simple closed curves on R; see [1] and [20]. For a Riemann surface R of general type, we define $I(R)$ as the subgroup consisting of all elements $[g] \in \mathrm{MCG}(R)$ such that g fixes all homology classes of simple closed curves on $\dot{R}$. Then, by Theorem 2.1, we have the following.

Corollary 2.2. *The $I(R)$ is a subgroup of the pure mapping class group $P(R)$.*

3. Stationary property of the mapping class group

In this section, we prove Theorem 1.3. We say that a Riemann surface R satisfies the *lower bound condition* if there exists a constant $\epsilon > 0$ such that the injectivity radius at any point of R° is greater than ϵ. Here R° is the non-cuspidal part of R obtained by removing all horocyclic cusp neighborhoods whose areas are 1. Furthermore we say that R satisfies the *upper bound condition* if there exists a constant $M > 0$ and a subdomain $\check{R}$ of R such that the injectivity radius at any point of $\check{R}$ is less than M and that the simple closed curves in $\check{R}$ carry the fundamental group of R. We say that R satisfies the *bounded geometry condition* if R satisfies the lower and upper bound conditions and has no ideal boundary at infinity. The bounded geometry condition is quasiconformally invariant, and all normal covers, except the universal cover, of compact Riemann surfaces satisfy the bounded geometry condition; see [9, Proposition 3].

We have given a sufficient condition for the discontinuity of the action of subgroups of $\mathrm{MCG}(R)$ on $T(R)$, and will apply the result to our proof of Theorem 1.3. To state the sufficient condition, we give a definition of the stationary property of the quasiconformal mapping class group.

Definition 3.1. We say that a subgroup $G \subset \mathrm{MCG}(R)$ is *stationary* if there exists a compact subsurface W on R such that $g(W) \cap W \neq \emptyset$ for all representatives g of all elements of G.

The stationary property is a generalization of the property that the quasiconformal mapping class group $\mathrm{MCG}(R)$ of a compact Riemann surface R has. It is known that a sequence of normalized quasiconformal homeomorphisms whose maximal dilatations are uniformly bounded is sequentially compact in compact open topology. The stationary property of mapping classes corresponds to the normalization in this context and hence such a sequence of mapping classes also has the compactness property if they are uniformly bounded. By using this observation, we have proved the following.

Proposition 3.2 ([10, Theorem 4.8])**.** *Let R be a Riemann surface satisfying the bounded geometry condition. Then every stationary subgroup of* $\mathrm{MCG}(R)$ *acts on the Teichmüller space $T(R)$ discontinuously.*

Remark 3.3. There exists a Riemann surface R and a subgroup G of $\mathrm{MCG}(R)$ such that G is non-stationary but G acts on $T(R)$ discontinuously; see [15, Proposition 3.1]. In this paper, we have further constructed a Riemann surface R satisfying the bounded geometry condition such that $\mathrm{MCG}(R)$ is non-stationary, but $\mathrm{MCG}(R)$ acts on $T(R)$ discontinuously.

By Proposition 3.2, the following proposition completes a proof of Theorem 1.3.

Proposition 3.4. *If R has more than two non-cuspidal ends, then the pure mapping class group $P(R)$ is stationary.*

The following lemma gives a candidate for a compact subsurface W in Definition 3.1 so that $P(R)$ is stationary.

Lemma 3.5. *Let R be a Riemann surface that has more than two non-cuspidal ends. Then there exists a pair of pants Y in R with geodesic boundary such that $R - Y$ has three connected components and that each of the connected components has a non-cuspidal end of R.*

Proof. We take a dividing simple closed geodesic c_1 so that each of the connected components E_1 and R_1 of $R - c_1$ has at least one non-cuspidal end of R. Since R has more than two non-cuspidal ends, either E_1 or R_1 has at least two non-cuspidal ends of R. We may assume that R_1 has at least two non-cuspidal ends of R. Then we can take a simple closed geodesic c_2 of R so that $c_2 \subset R_1$ and that each of the connected components of $R_1 - c_2$ has at least one non-cuspidal end of R. Let R_{12} be the connected component of $R_1 - c_2$ whose boundary contains c_1. The curves c_1 and c_2 have counter-clockwise orientations as the boundary of R_{12}. We connect c_1 and c_2 by a simple curve α in R_{12} having an orientation from c_1 to c_2. We take a closed curve $\beta = \alpha \cdot c_2 \cdot \alpha^{-1} \cdot c_1^{-1}$. Let c_3 be the geodesic that is homotopic to β. The union of c_1, c_2 and α determines a pair of pants Y, and c_1, c_2 and c_3 are boundary geodesics of Y. This pair of pants Y is a desired one. □

By using Lemma 3.5, we prove Proposition 3.4.

Proof of Proposition 3.4. By Lemma 3.5, we can take a pair of pants Y on R such that $R^* - Y$ has three connected components and that each of the connected components has a non-cuspidal point of $R^* - R$. Here R^* is the Stoïlow compactification of R. We will prove that $g(Y) \cap Y \neq \emptyset$ for all representatives g of all elements in $P(R)$.

Suppose to the contrary that $g(Y) \cap Y = \emptyset$ for a representative g of an element in $P(R)$. Then $g(Y)$ belongs to a connected component E_1^* of $R^* - Y$. Let E_2^* and E_3^* be the other connected components of $R^* - Y$, and $e_2 \in E_2^*$ and $e_3 \in E_3^*$ non-cuspidal points of $R^* - R$. Then e_2 and e_3 belong to the same connected component E^* of $R^* - g(Y)$. Since e_2 and e_3 belong to different connected components of $R^* - Y$ and since g is a homeomorphism, the points $g(e_2)$ and $g(e_3)$ should belong to different connected components of $R^* - g(Y)$. However, since $g(e_i) = e_i$ $(i = 2, 3)$, both the points $g(e_2)$ and $g(e_3)$ belong to E^*. This is a contradiction. □

As the following example shows, Theorem 1.3 is not true for Riemann surfaces that do not satisfy the bounded geometry condition.

Example 3.6. There exists a Riemann surface R that does not satisfy the bounded geometry condition and the action of $P(R)$ on $T(R)$ is not discontinuous. Let R be a Riemann surface that does not satisfy the lower bound condition. Then there exists a sequence of simple closed geodesics c_n on R such that they are mutually disjoint and their hyperbolic lengths tend to 0. For each n, let $[g_n]$ be a mapping class caused by the Dehn twist with respect to c_n. Then $[g_n] \in P(R)$ and the maximal dilatation K_n of the extremal quasiconformal homeomorphism that is homotopic to g_n satisfies $K_n \to 0$ as $n \to 0$; see [9, Theorem 2]. This implies that the action of $P(R)$ on $T(R)$ is not discontinuous.

Theorem 1.3 is also not true for Riemann surfaces that have at most two non-cuspidal ends.

Example 3.7. Let $\hat{R}$ be a compact Riemann surface, and R a normal covering surface of $\hat{R}$ whose covering transformation group is a cyclic group $\langle \phi \rangle$ generated by a conformal automorphism ϕ of R of infinite order. Then R satisfies the bounded geometry condition and has only two non-cuspidal ends. Since the element $[\phi] \in \mathrm{MCG}(R)$ belongs to $P(R)$ and since $\langle [\phi] \rangle$ does not act on $T(R)$ discontinuously, the action of $P(R)$ on $T(R)$ is not discontinuous.

Similarly, let R' be a normal covering surface of $\hat{R}$ whose covering transformation group is generated by two commuting conformal automorphisms of R' of infinite order. Then R' satisfies the bounded geometry condition and has only one non-cuspidal end. By the same reason as above, the action of $P(R')$ on $T(R')$ is not discontinuous.

We mention the action of mapping classes that fix punctures (cuspidal ends). For a Riemann surface R with a puncture x, let $\mathrm{MCG}_x(R)$ be the group of all quasiconformal mapping classes $[g] \in \mathrm{MCG}(R)$ such that $g(x) = x$.

Lemma 3.8. $\mathrm{MCG}_x(R)$ *is stationary.*

Proof. Let Z be a degenerate pair of pants in R such that the boundary of Z consists of x and two dividing simple closed geodesics. Since $g(x) = x$ and $g(Z)$ is also a pair of pants, we see that $g(Z - U) \cap (Z - U) \neq \emptyset$ for all elements $[g] \in \mathrm{MCG}_x(R)$. Here U is a cusp neighborhood of x. Thus we conclude that $\mathrm{MCG}_x(R)$ is stationary. □

Thus by Proposition 3.2, $\mathrm{MCG}_x(R)$ acts on $T(R)$ discontinuously if R satisfies the bounded geometry condition; see also [16, Theorem 2].

Since the stable quasiconformal mapping class group $G_\infty(R)$ is a subgroup of $P(R)$, the following corollary immediately follows from Theorem 1.3.

Corollary 3.9. *Let R be a Riemann surface satisfying the bounded geometry condition. If R has more than two non-cuspidal ends, then $G_\infty(R)$ acts on $T(R)$ discontinuously.*

As we have seen in Example 3.7, the assumption on the number of non-cuspidal ends is necessary for the discontinuous action of $P(R)$. However, concerning the action of $G_\infty(R)$, the assumption on the number of ends in Corollary 3.9 can be removed. We will prove this in Section 5.

4. Characterization of asymptotically trivial mapping classes

In this section, we prove Theorem 1.5. By definition, an element $[g] \in \mathrm{MCG}(R)$ belongs to $\mathrm{Ker}\,\iota_A$ if and only if $f \circ g^{-1} \circ f^{-1}$ is homotopic to an asymptotically conformal automorphism of $f(R)$ for all quasiconformal homeomorphisms f of R.

Proof of Theorem 1.5. Since a representative of an essentially trivial mapping class is asymptotically conformal, we have $G_\infty(R) \subset \mathrm{Ker}\,\iota_A$. Thus we will prove that $\mathrm{Ker}\,\iota_A \subset P(R)$. Take an element $[g] \in \mathrm{MCG}(R) - P(R)$ arbitrarily. We will show that there exists a quasiconformal homeomorphism h of R such that $h \circ g^{-1} \circ h^{-1}$ is not homotopic to any asymptotically conformal automorphism of $h(R)$. Then $[g] \notin \mathrm{Ker}\,\iota_A$ and we conclude that $\mathrm{Ker}\,\iota_A \subset P(R)$. If g is not homotopic to any asymptotically conformal automorphism of R, then we have nothing to prove since we may take $h = id$. Thus we assume that g is homotopic to an asymptotically conformal automorphism of R.

The assumption $[g] \notin P(R)$ implies that there exists a non-cuspidal end e of R such that $g(e) \neq e$. Let $\{R_n\}_{n=1}^{\infty}$ be a canonical exhaustion of R. Namely, for each n, the complement of R_n is decomposed into finitely many non-compact connected components $\{X_i^{(n)}\}_{i=1}^{N(n)}$, and the boundary of R_n are dividing curves. Let $\{X_{i_n}^{(n)}\}_{n=1}^{\infty}$ be a determining sequence corresponding to e. By the assumption that $g(e) \neq e$, there exists a sufficiently large m such that $X_{i_m}^{(m)} \cap g(X_{i_m}^{(m)}) = \emptyset$. Set $W_1 = X_{i_m}^{(m)}$, $W_2 = g(X_{i_m}^{(m)})$, $W_3 = R - (W_1 \cup W_2)$ and $c_i = \partial W_i$ $(i = 1, 2)$ which is the boundary of W_i on R.

Let $\widehat{W_i}$ $(i = 1, 2)$ be the Riemann surface that is obtained by gluing a disk to W_i along c_i, and let $\widehat{W_3}$ be the Riemann surface that is obtained by gluing two disks to W_3 along c_1 and c_2. For a given quasiconformal homeomorphism f of R, let μ be the Beltrami coefficient of f. Let f_1 be a quasiconformal homeomorphism of $\widehat{W_1}$ whose Beltrami coefficient is $\mu|_{W_1}$ on W_1 and 0 on $\widehat{W_1} - W_1$. By the same method, we also have a quasiconformal homeomorphism f_i of $\widehat{W_i}$ $(i = 2, 3)$. Then there exists a biholomorphic map

$$\psi : AT(R) \to AT(\widehat{W_1}) \times AT(\widehat{W_2}) \times AT(\widehat{W_3})$$

by $[[f]] \mapsto ([[f_1]], [[f_2]], [[f_3]])$; see [25] and [28]. Since g is homotopic to an asymptotically conformal automorphism that maps W_1 onto W_2, it induces a biholomorphic map between $AT(\widehat{W_1})$ and $AT(\widehat{W_2})$ that preserves the base points. Since $\widehat{W_2}$ is analytically infinite, the asymptotic Teichmüller space $AT(\widehat{W_2})$ is non-trivial. Then we take a point $\tau = ([[id]], [[h']], [[id]]) \in AT(\widehat{W_1}) \times AT(\widehat{W_2}) \times AT(\widehat{W_3})$ for $[[h']]$ $(\neq [[id]]) \in AT(\widehat{W_2})$. Let $[[h]] = \psi^{-1}(\tau) \in AT(R)$. Then $h \circ g^{-1} \circ h^{-1}$ on $h(R)$, that maps $h(W_2)$ to $h(W_1)$, is not homotopic to any asymptotically conformal homeomorphism of $h(R)$, since it induces a map from $AT(\widehat{W_2})$ to $AT(\widehat{W_1})$ satisfying $[[h']] \mapsto [[id]]$. Hence $[g] \notin \mathrm{Ker}\,\iota_A$. $\square$

Remark 4.1. Each inclusion in Theorem 1.5 is not necessarily equal. Indeed, let a Riemann surface R and a conformal automorphism ϕ of R be the same as in Example 3.7. Then $[\phi] \in P(R)$. On the other hand, we have proved that $[\phi] \notin \mathrm{Ker}\,\iota_A$ in [12, Corollary 4.6]. Thus $P(R) - \mathrm{Ker}\,\iota_A \neq \emptyset$. Although this is an example of a Riemann surface R having only two non-cuspidal ends, a similar reasoning can be applied to make an example of R having more than two non-cuspidal ends.

Next we consider a Riemann surface R that does not satisfy the lower bound condition. Then there exists a sequence of mutually disjoint simple closed geodesics c_n whose hyperbolic lengths tend to 0. Let $[g]$ be a mapping class caused by infinitely many Dehn twists with respect to all c_n. Then $[g] \in \operatorname{Ker} \iota_A - G_\infty(R)$.

In our forthcoming paper, we will prove that $G_\infty(R) = \operatorname{Ker} \iota_A$ if R satisfies the bounded geometry condition.

In this last portion of this section, we mention the theory of dynamics of the action of the quasiconformal mapping class group on the Teichmüller space. For a subgroup G of $\mathrm{MCG}(R)$, let

$$\Omega(G) = \{p \in T(R) \mid G \text{ acts at } p \text{ discontinuously}\},$$

which is an open subset of the Teichmüller space $T(R)$ and is called the *region of discontinuity* of G. Theorem 1.3 says that $\Omega(P(R)) = T(R)$ if R satisfies the bounded geometry condition and has more than two non-cuspidal ends. However in general, $\Omega(G)$ may be a proper subset of $T(R)$. The complement of $\Omega(G)$ is denoted by $\Lambda(G)$ and is called the *limit set* of G. In our papers [9] and [10], we have investigated the dynamical behavior of the quasiconformal mapping class group by analyzing these two subsets. We also define the region of discontinuity $\Omega_A(G)$ and the limit set $\Lambda_A(G)$ on the asymptotic Teichmüller space $AT(R)$ in a similar way. In this case, $\Omega_A(P(R))$ can be a proper subset of $AT(R)$; see [12, Theorem 4.2]. In [14], we prove a certain relationship between the limit set on $T(R)$ and its projection to $AT(R)$ by using Theorem 1.5.

5. The stable quasiconformal mapping class group

In Section 3, we stated Corollary 3.9 by using the fact that the stable quasiconformal mapping class group is a subgroup of the pure mapping class group. In this section, by considering the stable quasiconformal mapping class group directly, we extend the corollary to the following theorem.

Theorem 5.1. *Let R be an analytically infinite Riemann surface satisfying the bounded geometry condition. Then the stable quasiconformal mapping class group $G_\infty(R)$ acts on the Teichmüller space $T(R)$ discontinuously and freely.*

For a proof of Theorem 5.1, we note the following proposition which gives a condition for a quasiconformal mapping class to be periodic.

Proposition 5.2 ([11]). *Let R be a Riemann surface satisfying the bounded geometry condition. More precisely, R satisfies the lower bound condition for a constant $\epsilon > 0$ as well as the upper bound condition for a constant $M > 0$ and a subdomain $\check{R} \subset R$. Then, for a given constant $\ell > 0$, there exists a constant $K_0 = K_0(\epsilon, M, \ell) > 1$ depending only on ϵ, M and ℓ that satisfies the following: Let g be a quasiconformal automorphism of R such that $g(c)$ is freely homotopic to a simple closed geodesic c on R with $c \subset \check{R}$ and $\ell(c) \le \ell$. Suppose that $K(g) < K_0$. Then $[g]$ is periodic.*

Proof of Theorem 5.1. Since R satisfies the upper bound condition, there exists a constant $M > 0$ and a subdomain $\check{R} \subset R$ such that the injectivity radius at any point of $\check{R}$ is less than M and that the simple closed curves in $\check{R}$ carry the fundamental group of R. For each essentially trivial mapping class $[g] \in G_\infty(R)$, there exists a compact subsurface $V_g \subset R$ with geodesic boundary such that, for each connected component W of $R - V_g$ that is not a cusp neighborhood, the restriction $g|_W : W \to R$ is homotopic to the inclusion map $id|_W : W \hookrightarrow R$. We take a point x_g in a connected component of $\check{R} - V_g$ that is not a cusp neighborhood so that the hyperbolic distance between x_g and V_g is greater than $3M$. Since $x_g \in \check{R}$, there exists a simple closed curve c_g passing through x_g such that the hyperbolic length of c_g is less than $2M$. By [16, Proposition 1], we may assume that c_g does not surround a puncture of R. Then $c_g \subset R - V_g$ and $g(c_g)$ is freely homotopic to c_g. We apply Proposition 5.2 for the simple closed geodesic that is freely homotopic to c_g. Then there exists a constant $K_0 > 1$ independent of $[g]$ such that if $K(g) < K_0$ for a representative $g \in [g]$, then $[g]$ is periodic. By applying Proposition 5.3 below, we have $[g] = [id]$. Thus we conclude that $G_\infty(R)$ acts at the base point $[id] \in T(R)$ discontinuously and freely.

We take an arbitrary point $[f] \in T(R)$. Since $f(R)$ also satisfies the bounded geometry condition and since the restriction $f \circ g^{-1} \circ f^{-1}|_{f(W)} : f(W) \to f(R)$ is homotopic to the inclusion map $id|_{f(W)} : f(W) \hookrightarrow f(R)$, we can apply the same argument as above to conclude that $G_\infty(R)$ acts at the point $[f]$ discontinuously and freely. Hence the action of $G_\infty(R)$ on $T(R)$ is discontinuous and free. □

The following proposition was used in the proof of Theorem 5.1.

Proposition 5.3. *Suppose that $[g] \in G_\infty(R)$ is periodic. Then $[g] = [id]$.*

Proof. For $[g] \in G_\infty(R)$, let V_g be a compact subsurface with geodesic boundary of R such that, for each connected component W of $R - V_g$ that is not a cusp neighborhood, the restriction $g|_W : W \to R$ is homotopic to the inclusion map $id|_W : W \hookrightarrow R$. We can take such a connected component W of infinite type. Let Γ_W be a subgroup of the Fuchsian model Γ of R such that it corresponds to W. Let $\Lambda(\Gamma_W)$ and $\Lambda(\Gamma)$ be the limit sets of Γ_W and Γ, respectively. The limit set $\Lambda(\Gamma_W)$ is a closed subset of $\partial\Delta$ containing more than two points.

Since g is homotopic to the identity on W and since g^n is homotopic to the identity on R for some $n \in \mathbf{N}$, we can take a lift $\tilde{g}$ to Δ of g so that its extension to $\overline{\Delta}$ which is denoted by the same letter $\tilde{g}$ fixes each point of $\Lambda(\Gamma_W)$ and that $\tilde{g}^n$ is the identity on $\Lambda(\Gamma)$. Suppose to the contrary that g is not homotopic to the identity on R. Then $\tilde{g}$ is not the identity on $\Lambda(\Gamma)$. Hence there exists a point $x \in \Lambda(\Gamma)$ such that $\tilde{g}(x) \neq x$. Let α be the connected component of $\partial\Delta - \mathrm{Fix}(\tilde{g})$ containing x. Here $\mathrm{Fix}(\tilde{g})$ $(\supset \Lambda(\Gamma_W))$ is the set of all points fixed by $\tilde{g}$. Then the end points of α are fixed by $\tilde{g}$ and α contains no other fixed points of $\tilde{g}$.

The restriction $\tilde{g}|_{\partial\Delta}$ to $\partial\Delta$ is a quasisymmetric function. In particular, it is an orientation preserving homeomorphism of $\partial\Delta$. Then we see that $\tilde{g}|_{\partial\Delta}(\alpha) = \alpha$ and that the action of $\tilde{g}|_{\partial\Delta}$ on α is monotone. Hence, the assumption $\tilde{g}(x) \neq x$ implies that $\tilde{g}^n(x) \neq x$. This is a contradiction. Thus g is homotopic to the identity on R. □

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