z-CLASSES OF ISOMETRIES OF THE HYPERBOLIC SPACE

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ABSTRACT. Let G be a group. Two elements x, y are said to be *z-equivalent* if their centralizers are conjugate in G. The class equation of G is the partition of G into conjugacy classes. Further decomposition of conjugacy classes into z-classes provides important information about the internal structure of the group; cf. J. Ramanujan Math. Soc. 22 (2007), 35-56, for the elaboration of this theme.

Let $I(\mathbb{H}^n)$ denote the group of isometries of the hyperbolic n-space, and let $I_o(\mathbb{H}^n)$ be the identity component of $I(\mathbb{H}^n)$. We show that the number of z-classes in $I(\mathbb{H}^n)$ is finite. We actually compute their number; cf. theorem 1.3. We interpret the finiteness of z-classes as accounting for the finiteness of "dynamical types" in $I(\mathbb{H}^n)$. Along the way we also parametrize conjugacy classes. We mainly use the linear model of the hyperbolic space for this purpose. This description of parametrizing conjugacy classes appears to be new; cf. Academic Press, New York, 1974, 49–87 and Conformal geometry (Bonn, 1985/1986), 41–64, Aspects Math., E12, Vieweg, Braunschweig, 1988, for previous attempts. Ahlfors (Differential Geometry and Complex Analysis (Springer, 1985), 65–73) suggested the use of Clifford algebras to deal with higher dimensional hyperbolic geometry; cf. Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985), 15–27, Quasiconformal Mappings and Analysis (Springer, 1998), 109–139, Complex Variables Theory Appl. 15 (1990), 125–133, and Adv. Math. 101 (1993), 87–113. These works may be compared to the approach suggested in this paper.

In dimensions 2 and 3, by remarkable Lie-theoretic isomorphisms, $I_o(\mathbb{H}^2)$ and $I_o(\mathbb{H}^3)$ can be lifted to $GL_o(2, \mathbb{R})$, and $GL(2, \mathbb{C})$ respectively. For orientation-reversing isometries there are some modifications of these liftings. Using these liftings, in the appendix A, we have introduced a single numerical invariant $c(A)$, to classify the elements of $I(\mathbb{H}^2)$ and $I(\mathbb{H}^3)$, and explained the classical terminology.

Using the "Iwasawa decomposition" of $I_o(\mathbb{H}^n)$, it is possible to equip $\mathbb{H}^n$ with a group structure. In the appendix B, we visualize the stratification of the group $\mathbb{H}^n$ into its conjugacy and z-classes.

1. Introduction

Let $\mathbb{H}^n$ denote the n-dimensional hyperbolic space, and $I(\mathbb{H}^n)$ its full isometry group. In the disk-model of the hyperbolic space, an isometry is said to be *elliptic* if it has a fixed point in the space. A non-elliptic isometry is called *hyperbolic,* resp. *parabolic* if it has 2, resp. 1 fixed points on the ideal conformal boundary of the hyperbolic space. We shall build on this classification, and obtain a finer classification up to conjugacy, and beyond. A remarkable feature of $\mathbb{H}^n$ is that the

group $I(\mathbb{H}^n)$ is infinite. But the "dynamical types" that our minds can perceive are just finite in number. Can we account for this fact in terms of the internal structure of the group alone?

Let $I(\mathbb{H}^n)$ act on itself by conjugacy. For x in $I(\mathbb{H}^n)$, let $Z(x)$ denote the centralizer of x in $I(\mathbb{H}^n)$. For x, y in $I(\mathbb{H}^n)$ we shall say $x \sim y$ if $Z(x)$ and $Z(y)$ are conjugate in $I(\mathbb{H}^n)$. We call the equivalence class defined by this relation, the *z-class* of x. It turns out that the number of z-classes is finite, and this fact is interpreted as accounting for the finiteness of dynamical types. The z-classes are pairwise disjoint, and are manifolds; cf. theorem 2.1 [8], and so give a stratification of $I(\mathbb{H}^n)$ into finitely many strata. Each stratum has two canonical fibrations, and they explain the "spatial" and "numerical" invariants which characterize the transformation. The $I(\mathbb{H}^n)$ provides a significant example of the philosophy that was suggested in [8].

In this paper we work mostly in the linear model. Let $\mathbb{V}$ be a real vector space of dimension $n + 1$ equipped with a quadratic form of signature $(1, n)$, and $O(Q)$ its full group of isometries. This group has four components. The hyperboloid $\{v \in \mathbb{V} \mid Q(v) = 1\}$ has two components. One of the components is the model for $\mathbb{H}^n$. Suggestively, let $I(Q)$ denote the group which preserves the component, and we identify it with $I(\mathbb{H}^n)$. Now $O(Q)$ is a linear algebraic group. A significant property of a linear algebraic group over characteristic zero is that each T in a linear algebraic group has the Jordan decomposition $T = T_s T_u$ where T_s is semisimple, (that is, every T_s-invariant subspace has a T_s-invariant complement), and T_u is unipotent, (that is, all eigenvalues of T_u are 1); cf. [7]. Then T_s and T_u are also in $O(Q)$. T_s and T_u commute. They are polynomials in T. The decomposition is unique. A systematic use of Jordan decomposition leads to a neat and more refined classification of elements of $I(Q)$ up to conjugacy, and as we shall see, also z-equivalence.

Let T be in $O(Q)$. Let $\mathbb{V}_c = \mathbb{V} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. We identify T with $T \otimes_{\mathbb{R}} id$ and also consider it as an operator on $\mathbb{V}_c$. We say that an eigenvalue λ of T is *pure* if the corresponding eigenspace $\{v \in \mathbb{V} | (T - \lambda)v = 0\}$, coincides with the generalised eigenspace $\{v \in \mathbb{V} | (T - \lambda)^{n+1}v = 0\}$. Otherwise λ is *mixed*.

It is customary to call $v \in V$ *time-like* (resp. *space-like*, resp. *light-like*) if $Q(v) > 0$, (resp. $Q(v) < 0$, resp. $Q(v) = 0$). A subspace $\mathbb{W}$ is *time-like* (resp. *space-like*, resp. *light-like*) if $Q|_{\mathbb{W}} > 0$ (resp. $Q|_{\mathbb{W}} < 0$, resp. $Q|_{\mathbb{W}} = 0$).

The roots of a polynomial of the form $x^2 - 2ax + 1$, with $|a| < 1$ are of the form $e^{\pm i\theta}$. With the restriction $0 < \theta < \pi$, the θ is uniquely determined. When $x^2 - 2ax + 1$ is a factor of the characteristic polynomial of a transformation, the θ will be called *the rotation angle* of the transformation.

Theorem 1.1. *Let T be in $I(Q)$.*

(i) *Only the eigenvalue 1 can be mixed.*

(ii) *The eigenvectors corresponding to eigenvalue -1 are necessarily space-like. T is orientation-preserving iff the multiplicity of the eigenvalue -1 is even. Equivalently, T is orientation-preserving iff $\det T = 1$.*

(iii) *Suppose T has a time-like eigenvector. Then T is semisimple, and its characteristic polynomial $\chi_T(x)$ has the form*

$$\chi_T(x) = (x-1)^l (x+1)^m \chi_{oT}(x),$$

where

$$\chi_{oT}(x) = \Pi_{j=1}^s (x^2 - 2a_j x + 1)^{r_j}, \ |a_j| < 1.$$

Here a_j's are distinct, and $l \geq 1$. Let $k = \sum_{j=1}^s r_j + [\frac{m}{2}]$. We agree to call T a k-rotatory elliptic if m is even, and a k-rotatory elliptic inversion if m is odd.

(iv) T can have only one pair of real eigenvalues (r, r^{-1}), $r \neq \pm 1$. Such r is necessarily positive. The eigenspaces corresponding to r, r^{-1} are 1-dimensional, and are light-like. Suppose this is the case. Then T is semisimple, and its characteristic polynomial $\chi_T(x)$ has the form

$$\chi_T(x) = (x-1)^l (x+1)^m \chi_{oT}(x),$$

where

$$\chi_{oT}(x) = (x-r)(x-r^{-1})\Pi_{j=1}^s (x^2 - 2a_j x + 1)^{r_j}, \ |a_j| < 1.$$

Here a_j's are distinct. Let $k = \sum_{j=1}^s r_j + [\frac{m}{2}]$. We agree to call T a k-rotatory hyperbolic if m is even, and a k-rotatory hyperbolic inversion if m is odd.

(v) Suppose T is not semisimple. Then T has a 1-dimensional light-like eigenspace with eigenvalue 1. Let $T = T_s T_u$ be its Jordan decomposition, $T_u \neq Id$. The characteristic polynomial $\chi_T(x)$ has the form

$$\chi_T(x) = (x-1)^l (x+1)^m \chi_{oT}(x),$$

where

$$\chi_{oT}(x) = \Pi_{j=1}^s (x^2 - 2a_j x + 1)^{r_j}, \ |a_j| < 1.$$

Here a_j's are distinct, and $l \geq 3$. Let $k = \sum_{j=1}^s r_j + [\frac{m}{2}]$. We agree to call T a k-rotatory parabolic if m is even, and a k-rotatory parabolic inversion if m is odd. T_s is elliptic, and the minimal polynomial of T_u is $(x-1)^3$.

(vi) Every $T \in I(Q)$ satisfies either (iii), (iv), *or* (v).

A significant consequence of theorem 1.1 is the classification of conjugacy classes.

Theorem 1.2. *The characteristic polynomial and the minimal polynomial of T in $I(Q)$ determine the conjugacy class of T.*

We call T *elliptic* if it is k-rotatory elliptic or k-rotatory elliptic inversion for some k. Similarly for *hyperbolic* or *parabolic*. It will be convenient to call $\chi_{oT}(x)$, the *reduced characteristic polynomial* of T. Its degree is always even. Let $k' = \frac{1}{2} \deg \chi_{oT}(x)$.

The number of conjugacy classes is infinite. This infinity arises roughly from the eigenvalues r and the rotation-angles θ_j with multiplicity r_j in the notation of theorem 1.1. These are the "numerical invariants" of a transformation. The "spatial invariants" of a transformation are the corresponding orthogonal decomposition of the space, and the signatures of the quadratic form of the summands. Roughly speaking the "spatial invariants" define the z-class.

We determine the centralizers of elements and z-classes in §4, and also count their number.

Theorem 1.3. *For u a natural number, let $p(u)$ denote the Eulerian partition function, $p(0) = 1$.*

(i) There are $(n+1)p(0) + (n-1)p(1) + (n-3)p(2) + \ldots + \epsilon p([\frac{n}{2}])$ where $\epsilon = 1$ or 2 according as n is even or odd z-classes of elliptic elements. For this count identity is considered as an elliptic element.

(ii) *There are* $([\frac{n-1}{2}] + 1)p(0) + ([\frac{n-1}{2}])p(1) + ([\frac{n-1}{2}] - 1)p(2) + \ldots + p([\frac{n-1}{2}])$ *z-classes of hyperbolic elements.*

(iii) *There are* $(n-1)p(0) + (n-3)p(1) + \ldots + \epsilon p([\frac{n-2}{2}])$ *where* $\epsilon = 1$ *or* 2 *according as* n *is even or odd z-classes of parabolic elements.*

Each z-class is a manifold, and so each component has a well-defined dimension. It is said to be *generic* if it has the maximum dimension $(= dim\, I(Q))$.

Theorem 1.4. (i) *Let* n *be even,* $n = 2n'$. *The generic z-classes are the z-classes of* n'-*rotatory elliptics,* $(n'-1)$-*rotatory hyperbolics, and* $(n'-1)$-*rotatory hyperbolic inversions, all with pairwise distinct rotation angles.*

(ii) *Let* n *be odd,* $n = 2n' + 1$. *The generic z-classes are the z-classes of* n'-*rotatory hyperbolics,* n'-*rotatory elliptic inversions, and* $(n'-1)$-*rotatory hyperbolic inversions, all with pairwise distinct rotation angles.*

In §5 we give some simple criteria to detect the type of an isometry of $I(\mathbb{H}^n)$ to be hyperbolic, resp. elliptic, resp. parabolic. For example, T is hyperbolic iff $\chi_{oT}(1) < 0$. T is parbolic, resp. elliptic, iff $\chi_{oT}(1) > 0$ and $(x-1)^2$ divides (resp. does not divide) the minimal polynomial of T. There are also criteria in terms of the trace of T.

Previously Chen and Greenberg [4], Kulkarni [9] analysed the conjugacy classes of hyperbolic isometries. The criterion for conjugacy provided in Theorem 1.2 appears to be new. Ahlfors, [1], suggested the use of Clifford algebras to study Möbius transformations in higher dimensions; cf. [2], [5], [13], and [14] for elaboration of this theme. The present paper presents an alternate approach. It appears that Clifford-algebraic approach has not yet been worked out for orientation-reversing transformations.

2. PROOF OF THEOREM 1.1

(1) Let $\mathbb{V}_c$ denote the complexification of $\mathbb{V}$, and Q_c the bilinear extension of Q to $\mathbb{V}_c$. Let T be a linear transformation of $\mathbb{V}_c$, and its Jordan decomposition $T = T_s T_u$. Recall some basic facts about this setup. Let $\mathbb{V}_{c,\lambda}$ denote the generalized eigenspace of T with eigenvalue λ. Then it is the (usual) eigenspace of T_s. Let T be in $O(Q_c)$, and let $\langle\,,\,\rangle$ denote the corresponding bilinear form. We have for $v, w \in \mathbb{V}_{c,\lambda}$,

$$\langle v, w \rangle = \langle Tv, Tw \rangle = \langle T_s v, T_s w \rangle = \lambda^2 \langle v, w \rangle.$$

So if $\langle v, w \rangle \neq 0$, then $\lambda = \pm 1$. Or to put it another way, if $\lambda \neq \pm 1$, then $Q|_{\mathbb{V}_{c,\lambda}} = 0$. Also for $v \in \mathbb{V}_{c,\lambda}$ and $w \in \mathbb{V}_{c,\mu}$ we have

$$\langle v, w \rangle = \langle Tv, Tw \rangle = \langle T_s v, T_s w \rangle = \lambda\mu \langle v, w \rangle.$$

So unless $\lambda\mu = 1$ we have that $\mathbb{V}_{c,\lambda}$ and $\mathbb{V}_{c,\mu}$ are othogonal with respect to Q_c. Let $\oplus$ denote the orthogonal direct sum, and $+$ the usual direct sum of subspaces. We have

$$\mathbb{V}_c = \mathbb{V}_{c,1} \oplus \mathbb{V}_{c,-1} \bigoplus \oplus_{\lambda \neq \pm 1}(\mathbb{V}_{c,\lambda} + \mathbb{V}_{c,\lambda^{-1}}).$$

Note in particular, $dim\, \mathbb{V}_{c,\lambda} = dim\, \mathbb{V}_{c,\lambda^{-1}}$.

(2) Now let T be in $I(Q)$. Note that a unipotent transformation must preserve $\{v \in \mathbb{V}|Q(v) \geq 0\}$, so T_u and hence T_s, are also in $I(Q)$.

Suppose that T has a time-like eigenvector v. Since $\langle v, v \rangle \neq 0$, we see that the corresponding eigenvalue is equal to 1. T leaves the orthogonal complement of v

invariant, and the metric is negative definite on this complement. So T must be semisimple. The characteristic polynomial $\chi_T(x)$ is of the form

$$\chi_T(x) = (x-1)^l(x+1)^m\chi_{oT}(x),$$

where

$$\chi_{oT}(x) = \Pi_{j=1}^s(x^2 - 2a_jx + 1)^{r_j}, \quad |a_j| < 1.$$

Here a_j's are distinct. We have $l \geq 1$. The minimal polynomial $m_T(x)$ divides the characteristic polynomial, it has the same factors as the characteristic polynomial, and for a semisimple element it is a product of distinct factors. So

$$m_T(x) = (x-1)(x+1)^\epsilon\Pi_{j=1}^s(x^2 - 2a_jx + 1).$$

where $\epsilon = 0$ if $m = 0$, and $\epsilon = 1$ if $m > 0$. Let $k = \sum_{j=1}^s r_j + [\frac{m}{2}]$. We have agreed to call such T a k-rotatory elliptic if m is even, and a k-rotatory elliptic inversion if m is odd.

(3) Let T be in $I(Q)$ and suppose that r is a real eigenvalue $\neq 1$. By (1) the corresponding eigenvector is light-like. Now T preserves $\{v \in \mathbb{V}|\langle v,v \rangle \geq 0\}$. So r must be positive. By (1) r^{-1} must also be an eigenvalue, since Q is non-degenerate. The dimension of a subspace on which $Q = 0$ is at most 1. So $dim\,\mathbb{V}_r = dim\mathbb{V}_{r^{-1}} = 1$, and $Q|_{\mathbb{V}_r+\mathbb{V}_{r^{-1}}}$ is non-degenerate. Now the orthogonal complement of $\mathbb{V}_r + \mathbb{V}_{r^{-1}}$ is T-invariant, and $Q|_{\mathbb{V}_r+\mathbb{V}_{r^{-1}}}$ is negative semidefinite. So T is semisimple. The characteristic polynomial $\chi_T(x)$ is of the form

$$\chi_T(x) = (x-1)^l(x+1)^m\chi_{oT}(x),$$

where

$$\chi_{oT}(x) = (x-r)(x-r^{-1})\Pi_{j=1}^s(x^2 - 2a_jx + 1)^{r_j}.$$

Here $|a_j| < 1$, and are distinct. Moreover,

$$m_T(x) = (x-1)^{\epsilon_1}(x+1)^{\epsilon_2}(x-r)(x-r^{-1})\Pi_{j=1}^s(x^2 - 2a_jx + 1),$$

where ϵ_1 (resp. ϵ_2) is 0 if l (resp. m) is 0, and equal to 1 if l (resp. m) > 0. Let $k = \sum_{j=1}^s r_j + [\frac{m}{2}]$. We have agreed to call such T a k-rotatory hyperbolic if m is even, and a k-rotatory hyperbolic inversion if m is odd.

(4) Now let T be in $I(Q)$, and λ a non-real complex eigenvalue. Let $v \in \mathbb{V}_c$ be a corresponding eigenvector. Write $v = v_1 + iv_2$ where v_1 resp. v_2 are real, and $\lambda = a + ib$, where a, b are real and $b \neq 0$. We have $Tv = \lambda v$, so

$$Tv_1 = av_1 - bv_2, \quad Tv_2 = bv_1 + av_2.$$

Since $b \neq 0$, we see that both v_1, v_2 are non-zero, and they are linearly independent. Let $\mathbb{W} = span\{v_1, v_2\}$. It is a 2-dimensional T-invariant subspace. It is indeed a *minimal* invariant subspace. So $Q|_\mathbb{W}$ must be either 0, or it must be non-degenerate, for otherwise the nullspace of $\mathbb{W}$ would be 1-dimensional and would be T-invariant. Also $Q|_\mathbb{W}$ cannot have signature $(1, 1)$, for an elementary fact from 2-dimensional Lorentzian geometry is that the eigenvalues of $T|_\mathbb{W}$ would be real. So $Q|_\mathbb{W}$ must be negative definite. But then the eigenvalues of T on $\mathbb{W} \otimes_\mathbb{R} \mathbb{C}$ are $(\lambda, \bar{\lambda})$, and must be of the form $(e^{i\theta}, e^{-i\theta})$. By interchanging λ with $\bar{\lambda}$ if necessary, we may take $0 < \theta < \pi$. By making this argument on $\mathbb{V}_{c,\lambda}$ we see that $\mathbb{V}_{c,\lambda} + \mathbb{V}_{c,\bar{\lambda}}$ is a complexification of a real subspace on which Q is negative definite. Let us denote this real subspace $\mathbb{V}_{\lambda,\bar{\lambda}}$, and call it the *real trace* of the eigenspace of $\mathbb{V}_{c,\lambda} + \mathbb{V}_{c,\bar{\lambda}}$.

(5) Suppose -1 is an eigenvalue of T and v a corresponding eigenvector. Since T preserves the set $\{u \in \mathbb{V}|Q(u) \geq 0\}$, v must be outside this set. In other words, v is

space-like. Now combining (3) and (4), we see that 1 is the only possible eigenvalue which is mixed.

(6) $\mathbb{V}$ is said to be *orthogonally indecomposable* with respect to T, or *T-orthogonally indecomposable* for short, if $\mathbb{V}$ is not an orthogonal direct sum of proper T-invariant subspaces. Given T, we can decompose $\mathbb{V}$ into T-orthogonally indecomposable subspaces $\mathbb{W}$, $Q|_{\mathbb{W}}$ is non-degenerate. So far we know of three types of T-orthogonally indecomposable subspaces:

(i) Dim $\mathbb{W} = 1$: Here $\mathbb{W}$ is spanned by an eigenvector of non-zero length.

(ii) Dim $\mathbb{W} = 2$: Here $T|_{\mathbb{W}}$ has no real eigenvalue and $Q|_{\mathbb{W}}$ is negative definite. The eigenvalues of T are $e^{\pm i\theta}$, where $0 < \theta < \pi$.

(iii) Dim $\mathbb{W} = 2$: Here $\mathbb{W} = \mathbb{V}_r + \mathbb{V}_{r^{-1}}$, where both $\mathbb{V}_r$ and $\mathbb{V}_{r^{-1}}$ are spanned by eigenvectors of length 0.

We now describe the fourth, and last, type of T-orthogonally indecomposable subspaces.

Lemma 2.1. *Let $\mathbb{W}$ be a T-orthogonally indecomposable subspace of dimension ≥ 3. Then T is unipotent and dim $\mathbb{W} = 3$.*

Proof. Let $dim\ \mathbb{W} = m$. From previous discussion we see that $T|_{\mathbb{W}}$ must be unipotent. Let v be an eigenvector. From (3) and (4) we must have v of length 0, and eigenvalue 1. Let $C = \{u \in \mathbb{W}|Q(u) = 0\}$. This is a right circular cone. Since $Tv = v$, T must preserve the tangent space τ at v to C. Now $dim\ \tau = m - 1$, and $Q|_{\tau}$ is degenerate. $Q|_{\tau}$ has nullity 1, spanned by v, and otherwise its signature is $(0, m-2)$. Let $\tau \to \tau/\langle v \rangle$ be the canonical projection. Q induces the canonical symmetric quadratic form $\bar{Q}$ on $\tau/\langle v \rangle$. T induces a linear transformation $\bar{T}$ on $\tau/\langle v \rangle$, and it preserves $\bar{Q}$. Now $\bar{Q}$ has signature $(0, m-2)$, so it is negative definite. T is unipotent, so is $\bar{T}$. So $\bar{T}$ is both semisimple and unipotent, so it must be I, the identity. It is now easy to see that since we assumed $\mathbb{W}$ is orthogonally indecomposable with respect to T, we must have $dim\ \tau/\langle v \rangle = m - 2 = 1$. So $dim\ \mathbb{W} = 3$. $\square$

(7) Now let T be a non-semisimple element of $I(Q)$. Let $T = T_s T_u$, $T_u \neq I$ be the Jordan decomposition of T.

Let λ be a non-real eigenvalue of T_s. In (4) we have seen that $\mathbb{V}_{c,\lambda} + \mathbb{V}_{c,\bar{\lambda}}$ is a complexification of its trace $\mathbb{V}_{\lambda,\bar{\lambda}}$ on which Q is negative definite. Now T_u commutes with T_s. So T_u keeps the eigenspace of T_s in $\mathbb{V}_c$ invariant. It follows that T_u leaves $\mathbb{V}_{c,\lambda} + \mathbb{V}_{c,\bar{\lambda}}$, and hence $\mathbb{V}_{\lambda,\bar{\lambda}}$ invariant. Now Q restricted to $\mathbb{V}_{\lambda,\bar{\lambda}}$ is negative definite. So $O(\mathbb{V}_{\lambda,\bar{\lambda}})$ is compact. A compact group does not contain a non-semisimple element. It follows that T_u restricted to $\mathbb{V}_{\lambda,\bar{\lambda}}$ is identity.

Let -1 be an eigenvalue of T_s. By (5) the corresponding eigenvector is space-like. It follows that Q restricted to $\mathbb{V}_{-1}$ is negative definite. By the above argument, T_u restricted to $\mathbb{V}_{-1}$ is identity also.

Let r be a (necessarily positive) real eigenvalue $\neq 1$ of T_s. By (4), T is hyperbolic. So $T = T_s$ and $T_u = identity$. By our hypothesis, this is not the case. So this case cannot occur.

So T_u can be different from identity only on the eigenspace $\mathbb{V}_1$ with eigenvalue 1 of T_s. But T_s being semisimple, it must be itself identity on $\mathbb{V}_1$.

We agree to call such T *parabolic*. The characteristic polynomial of such T has the form
$$\chi_T(x) = (x-1)^l (x+1)^m \chi_{oT}(x),$$

where
$$\chi_{oT}(x) = \Pi_{j=1}^s (x^2 - 2a_j x + 1)^{r_j}, \ |a_j| < 1.$$
Here a_js are distinct. We have $l \geq 3$. The minimal polynomial has the form
$$m_T(x) = (x - 1)^{l'} (x + 1)^{\epsilon} \Pi_{j=1}^s (x^2 - 2a_j x + 1).$$
Here $l' = 3$. Moreover $\epsilon = 0$ if $m = 0$, and $\epsilon = 1$ if $m > 0$. Let $k = \sum_{j=1}^s r_j + [\frac{m}{2}]$. In a more refined way, we call T a k-rotatory parabolic if m is even, and a k-rotatory parabolic inversion if m is odd. Notice also that $m_{T_u}(x) = (x - 1)^3$.

This completes the proof.

3. Conjugacy classes

If the transformations are conjugate, their characteristic and minimal polynomials are the same, as indeed this is true for elements of $GL(\mathbb{V})$. The interesting point is the converse.

Suppose T is elliptic. Then $\mathbb{V}$ is an orthogonal direct sum of 1- or 2-dimensional indecomposable T-invariant subspaces. The metric on all 2-dimensional subspaces is negative definite. On one 1-dimensional subspace it is positive definite, and on the others it is negative definite. Given T' conjugate to T, $\mathbb{V}$ is a similar orthogonal direct sum of 1- or 2-dimensional indecomposable T'-invariant subspaces. There is an isometry in $O(Q)$ carrying one decomposition into another. The isometry may be chosen to lie in $I(Q)$ for both T, T' preserve $\{v \in \mathbb{V} | Q(v) \geq 0\}$. This isometry conjugates T into T'.

The proof in the hyperbolic case is similar. In this case $\mathbb{V}$ is an orthogonal direct sum of one 2-dimensional subspace on which Q has signature $(1, 1)$. The rest is similar to the elliptic case.

Suppose T is parabolic. Then $\mathbb{V}$ is an orthogonal direct sum of one 3-dimensional T-indecomposable subspace on which the metric has the signature $(1, 2)$, and the other 1 or 2-dimensional T-invariant subspaces on which the metric is negative definite. Now the proof is similar to the elliptic case.

Notice that if we know a transformation is not parabolic then the characteristic polynomial itself determines the conjugacy class.

4. Centralizers, and z-classes

(1) Let T be an elliptic transformation. First let us note some invariants of T from a dynamic viewpoint. Let Λ be the set of fixed points of T. Then Λ is the eigenspace of T with eigenvalue 1. Now $Q|_\Lambda$ is non-degenerate; cf. part (1) of §2. Let Π be the subspace orthogonal to Λ. Then $\mathbb{V} = \Lambda \oplus \Pi$. The centralizer $Z(T)$ in $I(Q)$ leaves each eigenspace, or a trace of a complex eigenspace, of T invariant. So it is clear that $Z(T|_\Lambda) = I(Q|_\Lambda)$.

Note that $Q|_\Pi$ is negative definite. In Π let Π_{-1} be the eigenspace with eigenvalue -1, and Π_o be its orthogonal complement. Let
$$\chi_{T|_{\Pi_o}}(x) = \Pi_{j=1}^s (x^2 - 2a_j x + 1)^{r_j}, \quad |a_j| < 1.$$
Here a_j's are distinct. Let $\Pi_j = ker \, (T^2 - a_j T + 1)^{r_j}$. Then $\Pi = \bigoplus_{j=1}^s \Pi_j$, and $dim \, \Pi_j = 2r_j$.

Let v be a vector in Π_j, $\langle v, v \rangle = -1$, and $Tv = v_1$. Then $span \, \{v, v_1\}$ is T-invariant. Choose orientation on $span \, \{v, v_1\}$ so that the angle between v and v_1 is θ, where $0 < \theta < \pi$. Let w be a vector in $span \, \{v, v_1\}$, $\langle w, w \rangle = -1$, such that (v, w)

is a positively oriented orthogonal pair in $span\ \{v, v_1\}$. Let $\mathbb{J}_j(v) = w$, $\mathbb{J}_j(w) = -v$. Then $\mathbb{J}_j$ is a T-invariant complex structure on $span\ \{v, v_1\}$. We easily see that $\mathbb{J}_j$, in $span\ \{v, v_1\}$, is independent of the choice of v. Thus this process in fact defines $\mathbb{J}_j$ as a T-invariant complex structure on Π_j. It follows that the $Z(T|_{\Pi_j})$ is isomorphic to the unitary group $U(r_j)$.

Now $Z(T|_{\Pi_{-1}})$ is clearly $O(Q|_{\Pi_{-1}})$, and it is compact.

So $Z(T)$ is isomorphic to $I(Q|_\Lambda) \times O(Q|_{\Pi_{-1}}) \times \Pi_{j=1}^s U(r_j)$.

(2) Let T be a hyperbolic transformation. Here we have a unique 2-dimensional nondegenerate T-invariant subspace $\mathbb{W}$, which is the sum of two eigenspaces with eigenvalues $\neq \pm 1$, and on which the metric is of the form $(1, 1)$. $Z(T)$ leaves $\mathbb{W}$ invariant. It is easy to see that $Z(T|_{\mathbb{W}}) = I_o(Q|_{\mathbb{W}})$, where $I_o(Q)$ is the identity component of $I(Q)$. On the orthogonal complement of $\mathbb{W}$, the metric is negative definite. Now the analysis is similar to the elliptic case. We have $Z(T)$ isomorphic to

$$I_o(Q|_{\mathbb{W}}) \times O(Q|_\Lambda) \times O(Q|_{\Pi_{-1}}) \times \Pi_{j=1}^s U(r_j).$$

Notice that in this case $O(Q|_\Lambda)$ is necessarily compact.

(3) Now let T be a parabolic transformation, $T = T_s T_u$. We know that T_s and T_u are polynomials in T. So if A commutes with T, it commutes with T_s and T_u as well. Conversely if A commutes with T_s and T_u it commutes with T. In other words, we have $Z(T) = Z(T_s) \cap Z(T_u)$.

We have a decomposition $\mathbb{V} = \mathbb{W} \oplus \mathbb{U}$, where $T_u|_{\mathbb{U}} = identity$, $dim\ \mathbb{W} = 3$, and $Q|_{\mathbb{W}}$ has signature $(1, 2)$. Also $T_s|_{\mathbb{W}} = identity$. On $\mathbb{U}$ the metric is negative definite. Decompose $\mathbb{U}$ into $\mathbb{U}_1 \oplus \mathbb{U}_{-1} \oplus \mathbb{U}_2$, where $\mathbb{U}_1$ (resp. $\mathbb{U}_{-1}$) are the eigenspaces with eigenvalue 1 (resp. -1), and $\mathbb{U}_2$ is the orthogonal sum of the real traces of the complex eigenspaces. Since $T_u|_{\mathbb{U}} = identity$, we have $Z(T|_{\mathbb{U}_{-1} \oplus \mathbb{U}_2}) = Z(T_s|_{\mathbb{U}_{-1} \oplus \mathbb{U}_2})$ is a direct product of $O(\mathbb{U}_{-1})$ and various unitary groups. This leaves $Z(T|_{\mathbb{W} \oplus \mathbb{U}_1}) = Z(T_u|_{\mathbb{W} \oplus \mathbb{U}_1})$. To compute $Z(T_u|_{\mathbb{W} \oplus \mathbb{U}_1})$, it is convenient to use the upper half space model of the hyperbolic space, where ∞ is the fixed point of T_u. Let $dim\ \mathbb{W} \oplus \mathbb{U}_1 = n' + 1$. Take the upper half space, which we conveniently again denote by $\mathbb{H}^{n'}$, with orthogonal coordinates $(x_1, x_2, \ldots, x_{n'})$, with $x_{n'} \geq 0$. The $(n' - 1)$-planes $x_{n'} = c > 0$, are parts of the horo-spheres. We may further assume that the coordinates are so chosen that T_u is given by $(x_1, x_2, \ldots, x_{n'}) \mapsto (x_1 + 1, x_2, x_3, \ldots, x_{n'})$. Now the orthogonal group in coordinates $(x_2, x_3, \ldots, x_{n'-1})$ commutes with T_u, as also the translations in the planes $x_{n'} = c$. The structure of the group may be described as follows. It is a direct product of two groups:

(i) a copy of $\mathbb{R}$; here $\mathbb{R}$ is the group of translations $(x_1, x_2, \ldots, x_{n'}) \mapsto (x_1 + c, x_2, x_3, \ldots, x_{n'})$, and

(ii) a semidirect product of $\mathbb{R}^{n'-2}$ by $O(n' - 2)$; here $\mathbb{R}^{n'-2}$ is the group of translations $(x_1, x_2, \ldots, x_{n'}) \mapsto (x_1, x_2 + c_2, x_3 + c_3, \ldots, x_{n'-1} + c_{n'-1}, x_{n'})$, and $O(n' - 2)$ is the orthogonal group in coordinates $(x_2, x_3, \ldots, x_{n'-1})$.

From this description of centralizers we see that to each centralizer there is attached a certain orthogonal decomposition of the space, and its isomorphism type is determined by it. Given any two such decompositions of the same type (meaning that the dimensions and the signatures of the quadratic form for the corresponding summands are the same) there exists an element of $I(Q)$ carrying one decomposition into the other. This element will conjugate one centralizer into the other. Thus there are only finitely many z-classes. Now the centralizers are

described in terms of group-structure itself. This fact is interpreted as accounting for the finiteness of "dynamical types".

The actual count of z-classes requires more work, and involves counting certain types of partitions of $n + 1$. Let T be in $I(Q)$, $\chi_T(x) = (x - 1)^l(x + 1)^m \chi_{oT}(x)$ be its characteristic polynomial, where

$$\chi_{oT}(x) = \Pi_{j=1}^s(x^2 - 2a_jx + 1)^{r_j},$$

or,

$$\chi_{oT}(x) = (x - r)(x - r^{-1})\Pi_{j=1}^s(x^2 - 2a_jx + 1)^{r_j},$$

$|a_j| < 1$, and distinct. We associate to T the partition $n + 1 = l + m + \sum_{j=1}^s r_j$, or $n + 1 = 2 + l + m + \sum_{j=1}^s r_j$. Notice that knowing that T is elliptic, resp. hyperbolic, resp. parabolic, and the partition of $n + 1$ determine its z-class uniquely. It is easy to see that distinct partitions (and knowing that T is elliptic, resp. hyperbolic, resp. parabolic) are associated to distinct z-classes. In considering z-classes we consider identity as elliptic. There are some restrictions on which partitions can actually occur.

First observe that in any group the center forms a single z-class.

Proposition 4.1. *The center of $I(Q)$ is trivial.*

Proof. Let g be an element of the center. In the disk-model, we see that g must have a fixed point. The fixed point set of a central element is left invariant by the whole group. Suppose g has a fixed point in the open disk. Now $I(Q)$ is transitive on the open disk, so g must fix all the points. In other words, g must be identity. Now suppose g has a fixed point on the boundary. But $I(Q)$ is transitive also on the boundary. So g must fix all the boundary. But then g fixes the convex hull of the boundary. So again g has to be identity. $\square$

Case 1. Let T be an elliptic element. Let $n + 1 = l + m + 2\sum_{j=1}^s r_j$ be the associated partition. We have $l \geq 1$ and $|a_j| < 1$. Let k' be the number of rotation angles lying in $(0, \pi)$, $k' = \sum_{j=1}^s r_j$. We assume, as we may, that the rotation angles are ordered in increasing order. The rotation angle θ_j is repeated as many times as its multiplicity r_j. The contribution of this part to the centralizer is the product of unitary groups $U(r_j)$. Thus there are $p(k')$ choices of distinct rotation angles counting with multiplicities. Since $l \geq 1$, we have $[\frac{n}{2}] + 1$ choices for k' (including 0). For a fixed k', there are $n + 1 - 2k'$ choices for l. Thus setting $p(0) = 1$, there are $(n+1)p(0) + (n-1)p(1) + (n-3)p(2) + \ldots + \epsilon p([\frac{n}{2}])$ where $\epsilon = 1$ or 2 according as n is even or odd, z-classes of elliptic elements.

Case 2. Let T be a hyperbolic element. Rewrite the associated partition of $n + 1$ as $n + 1 = 2 + l + m + 2\sum_{j=1}^s r_j$, where the first "2" stands for a pair of real eigenvalues $\neq \pm 1$, and l (resp. m) is the multiplicity of the eigenvalue 1 (resp. -1), and $|a_j| < 1$. In this case we see that if l and m are interchanged, we get the same z-class. So for counting z-classes we may further assume $l \leq m$. Thus we have $[\frac{n-1}{2}] + 1$ choices for k' (including 0), and for a fixed k', there are $[\frac{n-1-2k'}{2}] + 1$ choices for l (including 0). Thus we have (setting $p(0) = 1$) $([\frac{n-1}{2}]+1)p(0) + ([\frac{n-1}{2}])p(1) + ([\frac{n-1}{2}]-1)p(2) + \ldots + p([\frac{n-1}{2}])$ z-classes of hyperbolic elements.

Case 3. Let T be a parabolic element. Rewrite the associated partition of $n + 1$ as $n + 1 = 3 + l + m + 2\sum_{j=1}^s r_j$, where $3 + l$ (resp. m) is the multiplicity of the

eigenvalue 1 (resp. -1). Now there are $[\frac{n-2}{2}] + 1$ choices for k' (including 0), and for a fixed k', we have $n - 1 - 2k'$ choices for l (including 0). Thus we have (setting $p(0) = 1$) $(n-1)p(0) + (n-3)p(1) + \ldots + \epsilon p([\frac{n-2}{2}])$ where $\epsilon = 1$ or 2, according as n is even or odd, z-classes of parabolic elements.

We have completely proved theorem 1.3.

Lastly note that each z-class is a manifold, so one can talk about its dimension for each of its components. A z-class may not be connected, but there is actually a single dimension associated to a z-class. It can be read from theorem 2.1 of [8]. From that description we see that in a semisimple Lie group, the z-class is generic, i.e. the dimension of a z-class is maximal, iff the centralizer is abelian. The stated z-classes in Theorem 1.4 are precisely the z-classes with this property. This completes the proof of Theorem 1.4.

5. SOME SIMPLE CRITERIA

In this section we note some simple criteria for detecting the type of isometries of the hyperbolic n-space, using the linear model. Let T be in $I(Q)$, and let $\chi_{oT}(x)$ be its reduced characteristic polynomial, that is the polynomial obtained after factoring all factors of the type $x \pm 1$. Note that the factors $x^2 - 2ax + 1$ for $|a| < 1$ always take positive values for real x. The factors $(x - r)(x - r^{-1})$ for $r > 1$ are present only in the case of hyperbolic transformation. The criteria involve only detecting the factors $x \pm 1$ of the characteristic or minimal polynomial of T. So they are defined over any subfield of $\mathbb{R}$ which contains the coefficients of the characteristic polynomial of T, in particular on any subfield of $\mathbb{R}$ generated by the coefficients of the matrix of T with respect to a suitable basis.

• T is hyperbolic iff $\chi_{oT}(1) < 0$.

If T is parabolic, then the minimal polynomial $m_T(x)$ must be divisible by $(x - 1)^3$. On the other hand, the elliptic transformation is semisimple.

• T is parabolic (resp. elliptic), iff $\chi_{oT}(1) > 0$, and $m_T(x)$ is divisible (resp. is not divisible), by $(x - 1)^2$.

• Note that the above two statements provide a *finite algorithm* to test whether T is hyperbolic, parabolic or elliptic.

Some other simple criteria for recognizing the type can be obtained by just looking at the trace. Note that for elliptic and parabolic transformations, all eigenvalues have absolute value 1. So,

• if $trace\, T > n + 1$, then T is hyperbolic.

The eigenvalues of T^u are u-th powers of the eigenvalues of T. Only the hyperbolic transformation has eigenvalue $r > 1$, and its multiplicity is 1. So for large u, the eigenvalue r^u of T^u will dominate the others. So,

• T is hyperbolic iff for all sufficiently large u, $trace\, T^u > n + 1$.

Proof. Let T be hyperbolic. Then it has a pair of real eigenvalues $\{\lambda, \lambda^{-1}\}$, $\lambda > 1$. All other eigenvalues have absolute value 1. Let the other eigenvalues

be $u_1, ..., u_{n-1}$. So we have,

$$trace\ T = \lambda + \lambda^{-1} + \sum_{i=1}^{n-1} u_i,$$

$$trace\ T^d = \lambda^d + \lambda^{-d} + \sum_{i=1}^{n-1} u_i{}^d.$$

As $\lambda > 1$ and $|u_i| = 1$ for $i = 1, 2, .., n-1$, we have, for any $d \in \mathbb{N}$,

$$|trace\ T^d| = |\lambda^d + \lambda^{-d} + u_1^d + ... + u_{n-1}^d| \geq \lambda^d - (n-1).$$

Since $\lambda > 1$, we can find an $u \in \mathbb{N}$ such that $\lambda^u > 2n$; i.e. $trace\ T^u > n+1$. On the other hand, if T is parabolic or elliptic, then $|u_i| = 1$ for $i = 1, 2, ..., n+1$. Hence, for any d,

$$|trace\ T^d| = \Big| \sum_{i=1}^{n+1} u_i^d \Big| \leq n + 1.$$

$\square$

For the cases $n = 2, 3$ we have very neat criteria. Compare the usual criteria given in the $PSL(2, \mathbb{R})$ model of $I_o(\mathbb{H}^2)$, or the $PSL(2, \mathbb{C})$ model of $I_o(\mathbb{H}^3)$; cf. [3], [11], or appendix A below. Note that the orientation-reversing case, either for $n = 2$ or for $n = 3$, is usually not treated in the literarture.

• Let $n = 2$.

(i) Orientation-preserving case: T is hyperbolic, resp. parabolic, resp. elliptic, iff $trace\ T > 3$, resp. $trace\ T = 3$, resp. $trace\ T < 3$.

(ii) Orientation-reversing case: T is hyperbolic, resp. elliptic, iff $trace\ T > 1$, resp. $trace\ T = 1$.

• Let $n = 3$.

Let T be in $I(Q)$, $T \neq identity$.

(i) Orientation-preserving case:

(a) T is 1-rotatory hyperbolic iff 1 is not an eigenvalue.

(b) T is 0-rotatory hyperbolic, iff 1 is an eigenvalue and $trace\ T > 4$.

(c) If 1 is an eigenvalue, then T is elliptic (resp. parabolic) iff $trace\ T < 4$, (resp. $trace\ T = 4$).

(ii) Orientation-reversing case: T is hyperbolic, resp. elliptic, iff $trace\ T > 2$, resp. $trace\ T \leq 2$.

In these dimensions, by remarkable isomorphisms, the group $I(\mathbb{H}^2)$, resp. $I_o(\mathbb{H}^3)$ can be identified with the group $PGL(2, \mathbb{R})$, resp. $PGL(2, \mathbb{C})$. In the appendix A, we give algebraic criteria using these isomorphisms. We have treated the orientation-reversing case also.

• In higher dimensions, although it is hard to compute individual eigenvalues of T, the $trace\ T$ and $trace\ T^u$ are easily computable by using the well-known Newton's identities.

Let $\chi_T(x) = x^{n+1} - a_1 x^n + a_2 x^{n-1} + ... + (-1)^{n+1} a_{n+1}$.

In our case, a_{n+1} is $+1$ or -1 according as T is orientation preserving or orientation reversing.

Let p_k be the sum of k-th powers of eigenvalues of T. Since the eigenvalues of T^k are the k-th powers of eigenvalues of T, we have $p_k = trace\ T^k$. Newton's identities

(cf. [10] and [12]) express p_k's in terms of the co-efficients of the characteristic polynomial of T as follows:

(1) For $k \leq n + 1$,

$$p_k = det \begin{pmatrix} a_1 & 1 & 0 & 0 & ... & 0 \\ 2a_2 & a_1 & 1 & 0 & ... & 0 \\ 3a_3 & a_2 & a_1 & 1 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ ka_k & a_{k-1} & a_{k-2} & ... & ... & a_1 \end{pmatrix}.$$

(2) For $k > n + 1$,

$$p_k = a_1 p_{k-1} - a_2 p_{k-2} + ... + (-1)^{n-1} p_{k-n}.$$

So we have,

$$trace\ T = a_1,$$
$$trace\ T^2 = p_2 = a_1^2 - 2a_2,$$
$$trace\ T^3 = p_3 = a_1^3 - 3a_1 a_2 + 3a_3,$$
$$trace\ T^4 = p_4 = a_1^4 - 4a_1^2 a_2 + 4a_1 a_3 + 2a_2^2 - 4a_4, ...$$

If either of these numbers is $> n + 1$, then T is hyperbolic.

• Suppose T is hyperbolic and has eigenvalue λ such that $\lambda \geq a > 1$. Then for any real $r > 0$, and $m > \frac{ln\ (n-1+r)}{ln\ a}$, we have $trace\ T^m > r$.

Proof. We have, for any positive integer k,

(5.1) $$trace\ T^k \geq \lambda^k - (n-1) > a^k - (n-1).$$

Since $a > 1$, choose m such that $a^m - (n-1) > r$, or, $m > \frac{ln\ (n-1+r)}{ln\ a}$.   □

Corollary 1. *An isometry T of $\mathbb{H}^n$ is hyperbolic if and only if the sequence $\{trace\ T^k\}$ is divergent.*

APPENDIX A. ALTERNATIVE CRITERIA FOR $I(\mathbb{H}^2)$ AND $I(\mathbb{H}^3)$, A LIE GROUP THEORETIC AND DYNAMIC PERSPECTIVE

In low dimensions there are two remarkable Lie theoretic isomorphisms. The group $SO_o(2, 1)$ is isomorphic to $PSL(2, \mathbb{R})$ and $SO_o(3, 1)$ is isomorphic to $PGL(2, \mathbb{C})$. These allow us to identify $I_o(\mathbb{H}^2)$ and $I_o(\mathbb{H}^3)$ with the groups $PSL(2, \mathbb{R})$ and $PGL(2, \mathbb{C})$ respectively.

A.1. **Isometries of $\mathbb{H}^3$.** Consider the disk model of the hyperbolic 3-space, and identify its conformal boundary with the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The group $I(\mathbb{H}^3)$ can be identified with the group $\mathcal{M}(2)$ of all Möbius transformations of $\hat{\mathbb{C}}$. Recall that $\mathcal{M}(2) = \mathcal{M}^+(2) \cup \mathcal{M}^-(2)$, where

$$\mathcal{M}^+(2) = \{z \mapsto \frac{az + b}{cz + d} \mid a, b, c, d \in \mathbb{C},\ ad - bc \neq 0\},$$

$$\mathcal{M}^-(2) = \{z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d} \mid a, b, c, d \in \mathbb{C},\ ad - bc \neq 0\}.$$

In particular, $I_o(\mathbb{H}^3)$ is isomorphic to $PGL(2, \mathbb{C})$. Let σ_0 be the reflection in the real line in $\mathbb{C}$, i.e. $\sigma_0(z) = \bar{z}$. The coset $PGL(2, \mathbb{C})\sigma_0$ is isomorphic to the group of orientation-reversing isometries of $\mathbb{H}^3$. Note that we consider $PGL(2, \mathbb{C})$ rather than the usual $PSL(2, \mathbb{C})$ in the literature. In general, over fields $\mathbb{F} \neq \mathbb{C}$, the

groups $PGL(2,\mathbb{F})$ and $PSL(2,\mathbb{F})$ are not always isomorphic. So we find it natural to consider the group $PGL(2,\mathbb{C})$ and to avoid the "det = 1" normalization for formulating the criteria.

Before proceeding further we introduce a new terminology of "Möbius co-ordinates". The coordinate z in $\mathbb{C}$, extended to $\hat{\mathbb{C}}$ by setting $z(\infty) = \infty$, is called a *Möbius coordinate* on $\hat{\mathbb{C}}$. It is *not* a complex coordinate in the sense the terminology is used in the theory of manifolds. Any $PGL(2,\mathbb{C})$-translate will also be termed as a Möbius coordinate.

A.2. **Isometries of** $\mathbb{H}^2$**.** In the upper-half space model, $I(\mathbb{H}^2)$ is a subgroup of $\mathcal{M}(2)$ and is given by the linear fractional transformations with real co-efficients, i.e. $I(\mathbb{H}^2) = \mathcal{M}^+(1) \cup \mathcal{M}^-(1)$, where

$$\mathcal{M}^+(1) = \{z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{R},\ ad - bc > 0\},$$

$$\mathcal{M}^-(1) = \{z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d} \mid a,b,c,d \in \mathbb{R},\ ad - bc < 0\}.$$

Thus $I(\mathbb{H}^2)$ can be identified with the group $PGL(2,\mathbb{R})$. Let $GL_+(2,\mathbb{R})$, resp. $GL_-(2,\mathbb{R})$, be the subgroup of all elements with positive, resp. negative, determinant in $GL(2,\mathbb{R})$. Then the group $I_o(\mathbb{H}^2)$ is isomorphic to $PGL_+(2,\mathbb{R})$, which can be identified with $\mathcal{M}^+(1)$. The other component of the isometry group may be identified with $GL_-(2,\mathbb{R})$.

In the disk model, the group $I_o(\mathbb{H}^2)$ consists of the usual holomorphic transformations of the unit disk $\mathbb{D}^2$. Recall that such a transformation is of the form

$$f_{a,\phi} : z \mapsto e^{i\phi}\frac{z-a}{1-\bar{a}z}, \quad 0 \le \phi \le 2\pi,\ |a| < 1.$$

The orientation-reversing isometries are given by

$$\bar{f}_{a,\phi} : z \mapsto e^{i\phi}\frac{\bar{z}-a}{1-\bar{a}\bar{z}}, \quad 0 \le \phi \le 2\pi,\ |a| < 1.$$

Let $\mathbb{R}_+$ denote the group of all positive reals, and let

$$SU(1,1) = \left\{ \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} \mid |a|^2 - |c|^2 = 1 \right\}.$$

Then it follows from the above that $I_o(\mathbb{H}^2)$ lifts to the subgroup $\mathbb{R}_+ \times SU(1,1)$ of $GL(2,\mathbb{C})$.

Observe that the co-efficient matrix M of $f_{a,\phi}$ has the following decomposition:

$$M = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} 1 & -a \\ -\bar{a} & 1 \end{pmatrix}.$$

It turns out to be the polar decomposition, or the KP decomposition, of $f_{a,\phi}$.

A.3. **The criteria.** First note that the quotient map $GL(2,\mathbb{C}) \to PGL(2,\mathbb{C})$ is surjective, and it maps the conjugacy class of an element A onto the conjugacy class of the corresponding element $\hat{A}$. Since the trace and determinant of A are the conjugacy invariants in $GL(2,\mathbb{C})$, we can take $\frac{(trace\ A)^2}{det\ A}$ as the conjugacy invariant for $\hat{A}$ in $PGL(2,\mathbb{C})$.

Now suppose A is an element in $GL(2,\mathbb{C})$ and let $\hat{A}$ be the corresponding Möbius transformation on $\hat{\mathbb{C}}$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\hat{A}(z) = \frac{az+b}{cz+d}.$$

By Jordan theory we know the conjugacy classes in $GL(2,\mathbb{C})$.

Case (i). Suppose A is conjugate to the diagonal matrix $D_{\lambda,\mu} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, i.e. there exists P in $GL(2,\mathbb{C})$ such that $D_{\lambda,\mu} = PAP^{-1}$. Considering $\hat{P} : z \mapsto w$ as a Möbius change of co-ordinates on $\hat{\mathbb{C}}$, we have

$$\hat{A}(w) = \frac{\lambda}{\mu} w.$$

Further note that $D_{\lambda,\mu}$ is conjugate to $D_{\mu,\lambda}$. So the size $|\frac{\lambda}{\mu}|$ has no significance for the conjugacy problem.

Now there are the following possibilities.

(a) $\frac{\lambda}{\mu}$ is not a real number and $|\frac{\lambda}{\mu}| \neq 1$, then A acts as a 1-rotatory hyperbolic. Classically these are known as *loxodromic*.

(b) If $|\frac{\lambda}{\mu}| = 1$ and $\frac{\lambda}{\mu} \neq 1$, then A acts as a 1-rotatory elliptic.

If $\frac{\lambda}{\mu} = -1$, then A acts as a 1-rotatory elliptic with rotation angle π, and we agree to call it a half-turn.

(c) $\frac{\lambda}{\mu}$ is a real number. Suppose $|\frac{\lambda}{\mu}| \neq 1$.

If $\frac{\lambda}{\mu} > 0$, then A acts as a 0-rotatory hyperbolic. We agree to call it a stretch.

If $\frac{\lambda}{\mu} < 0$, then A acts as a 1-rotatory hyperbolic with rotation angle π, and in this case we agree to call it a stretch half-turn.

Case (ii). Suppose A is conjugate to the upper-triangular matrix $T_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. In this case $\hat{A}$ takes the form $w \mapsto w + \frac{1}{\lambda}$, and we see that A acts as a 0-rotatory parabolic. We agree to call it a translation.

In the orientation-reversing case, lifting $f : z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ to A has no dynamic significance for $I(\mathbb{H}^3)$, for the "correct" lift is $A\sigma_0$. Hence the conjugacy classes in $GL(2,\mathbb{C})$ do not determine the conjugacy classes of the orientation-reversing isometries. However one dynamic invariant of f is f^2, which corresponds to $A\sigma_0 A\sigma_0$, and is an orientation-preserving transformation. The associated matrix to f^2 is given by $B = A\bar{A}$. As we shall see in the next theorem, when $B \neq \lambda I$, $\lambda > 0$, B determines A. The elusive case is when $B = \lambda I$. It will turn out that, in this case, there are exactly two conjugacy classes. Either f is a 0-rotatory elliptic inversion, and the fixed point set of f is a circle, or f is a 1-rotatory elliptic inversion, and f is an *antipodal* map without fixed points. An *antipodal map*, by definition, is an orientation-reversing map of order 2 and without a fixed point on $\hat{\mathbb{C}}$. It turns out that these maps are conjugate to $\mathfrak{a} : z \mapsto -\frac{1}{\bar{z}}$. In the disk model of $\mathbb{H}^3$, these maps fix a unique point in the disk, and interchange the end points of a geodesic passing through the fixed-point.

A.3.1. *Criteria for $I(\mathbb{H}^3)$.*

Theorem A.1. *Let f be an isometry of $\mathbb{H}^3$ induced by $A \in GL(2, \mathbb{C})$. Define*

$$c(A) = \frac{(trace\ A)^2}{det\ A}.$$

I. *Suppose f is orientation-preserving.*

(i) *If $c(A)$ is in $\mathbb{C} - \mathbb{R}$, then f is a 1-rotatory hyperbolic and is different from a stretch half-turn.*

(ii) *If $c(A)$ is in $\mathbb{R}$, $|c(A) - 2| > 2$, then f is a stretch or a stretch half-turn according as $c(A) > 0$ or $c(A) < 0$.*

(iii) *If $c(A)$ is in $\mathbb{R} - \{0\}$, $|c(A) - 2| < 2$, then f is a 1-rotatory elliptic.*

(iv) *If $c(A) = 0$, then f is a half-turn.*

(v) *If $c(A)$ is in $\mathbb{R} - \{0\}$, $|c(A) - 2| = 2$, $A \neq \lambda I$, then f is a translation.*

(vi) *If $c(A)$ is in $\mathbb{R} - \{0\}$, $|c(A) - 2| = 2$, $A = \lambda I$, then f is the identity.*

II. *Suppose f is orientation-reversing. Let $B = A\bar{A}$. Then $c(B)$ is a real positive number.*

(i) *If $|c(B) - 2| < 2$, then f is a 1-rotatory elliptic inversion.*

(ii) *If $|c(B) - 2| > 2$, then f is a 0-rotatory hyperbolic inversion.*

(iii) *If $|c(B) - 2| = 2$ and $B \neq \lambda I$, then f is a 0-rotatory parabolic inversion.*

(iv) *If $|c(B) - 2| = 2$ and $B = \lambda I$, then f acts as an inversion in a circle, or an antipodal map. In both cases, f is an elliptic inversion. We detect these cases as follows.*

If there exists a non-zero u in $\mathbb{C}$, unique up to a non-zero real multiple, such that the matrix uA is of the form: $\begin{pmatrix} a & b \\ c & -\bar{a} \end{pmatrix}$, *where b, c are reals, then f acts as an inversion in a circle, resp. antipodal map according as $\det uA < 0$, resp. $\det uA > 0$.*

The inversion in a circle is a 0-rotatory elliptic-inversion and the antipodal map is a 1-rotatory elliptic inversion of $\mathbb{H}^3$.

Proof. I. Let the matrix A induce an orientation-preserving isometry. Let λ and μ be eigenvalues of A. So,

$$c(A) = \frac{(trace\ A)^2}{det\ A} = \frac{(\lambda + \mu)^2}{\lambda\mu} = \frac{\lambda}{\mu} + \frac{\mu}{\lambda} + 2.$$

Let $\frac{\lambda}{\mu} = re^{i\theta}$, i.e. $c(A) = re^{i\theta} + \frac{1}{r}e^{-i\theta} + 2$.

(1) If $c(A)$ is non-real, then the imaginary part of $re^{i\theta} + \frac{1}{r}e^{-i\theta}$ must be non-zero. This is possible if and only if $r \neq 1$ and $\theta \neq 0, \pi$. Thus A acts as a 1-rotatory hyperbolic and is different from a stretch half-turn.

(2) Now suppose $c(A)$ is a real number. Then either $\sin\theta = 0$, or $r = 1$. So $\theta = 0$, or π, or, $r = 1$.

If $|c(A) - 2| < 2$, then we must have $r = 1$, $\theta \neq 0, \pi$. Hence A acts as a 1-rotatory elliptic, different from a half-turn.

If $|c(A) - 2| > 2$, then we must have $\theta = 0$ or π, $r \neq 1$. Thus A acts as a stretch or a stretch half-turn according as $\theta = 0$ or $\theta = \pi$.

If $|c(A) - 2| = 2$, then we must have $r = 1$, $\theta = 0$ or π. If $\theta = \pi$, we have $\frac{\lambda}{\mu} = -1$, i.e. A acts as a half-turn. If $\theta = 0$, we have $\lambda = \mu$. If $A \neq \lambda I$, then A is upper triangulable; hence, A acts as a translation. If $A = \lambda I$, it induces the identity map.

II. Let f be an orientation-reversing isometry and the corresponding Möbius transformation, again denoted by f, is given by

$$f : z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d}.$$

Then $A\sigma_0 A\sigma_0 = A\bar{A}$ induces the orientation-preserving Möbius transformation f^2 on $\hat{\mathbb{C}}$, where

$$\sigma_0 A \sigma_0 = \bar{A} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}.$$

Let $B = A\bar{A}$. Then, by the previous case, we have:

(3) $|c(B) - 2| < 2 \Rightarrow f^2$ is a 1-rotatory elliptic $\Rightarrow f$ is a 1-rotatory elliptic inversion.

(4) $|c(B) - 2| > 2 \Rightarrow f^2$ is a stretch $\Rightarrow f$ is a 0-rotatory hyperbolic inversion.

(5) $|c(B) - 2| = 2$, $B \neq \lambda I$. Then f^2 is a translation, and hence f is a 0-rotatory parabolic inversion.

(6) Finally, consider the case when $|c(B) - 2| = 2$, $B = \lambda I$. Note that

$$trace\ B = |a|^2 + |d|^2 + (b\bar{c} + \bar{b}c) = 2\lambda,\ \det B = |ad - bc|^2.$$

Thus $c(B)$ and λ are real numbers. Since $\lambda = |a|^2 + b\bar{c} = |d|^2 + \bar{b}c$, we must have $b\bar{c} \in \mathbb{R}$.

Let $b\bar{c} \neq 0$. Then b must be a real multiple of c. Let $b = uc = ure^{i\theta}$, $u \in \mathbb{R}$. Multiplying A by $e^{-i\theta}$, without loss of generality, we may assume $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $b, c \in \mathbb{R}$. The equation $B = A\bar{A} = \lambda I$ also gives us, $c\bar{a} + d\bar{c} = 0$. Since c is a real, we have $\bar{a} + d = 0$, i.e. $\bar{a} = -d$. Hence, $A = \begin{pmatrix} a & b \\ c & -\bar{a} \end{pmatrix}$. The induced Möbius transformation by $A\sigma_0$ is

$$f : z \mapsto \frac{a\bar{z} + b}{c\bar{z} - \bar{a}}.$$

Suppose $c \neq 0$. Note that f has a fixed point on $\mathbb{C}$ if and only if $f(z) = z$ has a solution. We see that

(A.1) $$\frac{a\bar{z} + b}{c\bar{z} - \bar{a}} = z \Leftrightarrow (z - \frac{a}{c})(\bar{z} - \frac{\bar{a}}{\bar{c}}) = \frac{a\bar{a} + bc}{\bar{c}^2} = \frac{-\det A}{\bar{c}^2}.$$

Thus if $\det A < 0$, then the above equation has a solution and f has a fixed point. The fixed point set is a circle C with radius $\sqrt{\frac{-\det A}{\bar{c}^2}}$. This implies, f must be an inversion in C. If $\det A > 0$, then f has no fixed-point on $\hat{\mathbb{C}}$ and $f^2 = id$. By definition, f will be the antipodal map of $\hat{\mathbb{C}}$. An inversion in a circle is a 0-rotatory elliptic inversion of $\mathbb{H}^3$. It follows from the representation of the isometries in $O(3, 1)$ that the antipodal map acts as a 1-rotatory elliptic inversion of $\mathbb{H}^3$, with rotation angle π.

If $c = 0$, then $A\sigma_0$ acts as $f : z \mapsto a\bar{z}d^{-1} + bd^{-1}$. Since f^2 is the identity, it follows that f is an inversion in a circle of $\hat{\mathbb{C}}$, and acts as a 0-rotatory elliptic inversion of $\mathbb{H}^3$. $\qquad \square$

A.3.2. *Criteria for $I(\mathbb{H}^2)$.*

Corollary 2. *Let f be an isometry of $\mathbb{H}^2$ induced by A in $GL(2,\mathbb{R})$, resp. $\mathbb{R}_+ \times SU(1,1)$. Let $c(A) = \frac{(trace\ A)^2}{det A}$. Then $c(A)$ is a real number.*
 I. Suppose f is orientation-preserving.
 (i) If $|c(A) - 2| < 2$, then f is a 1-rotatory elliptic.
 (ii) If $|c(A) - 2| > 2$, then f is a stretch.
 (iii) If $|c(A) - 2| = 2$ and $A \neq rI$, then f is a translation.
 (iv) If $|c(A) - 2| = 2$ and $A = rI$, then f is the identity.
 II. Suppose f is orientation-reversing. Let f be induced by A in $GL(2,\mathbb{R})$, resp. $\mathbb{R}_+ \times SU(1,1)$. Let $B = A\bar{A}$.
 (i) If $|c(B) - 2| > 2$, then f is a 0-rotatory hyperbolic inversion.
 (ii) If $B \neq rI$ and $|c(B) - 2| < 2$, then f is a 1-rotatory elliptic inversion.
 (iii) If $B = rI$, then f is a 0-rotatory elliptic inversion. In this case, f is an inversion in a circle of $\hat{\mathbb{C}}$.

Proof. It is easy to see that every orientation-reversing isometry of $\mathbb{H}^2$ has exactly two fixed points on the boundary circle. Now the corollary follows from the above theorem. $\qquad\square$

APPENDIX B. z-CLASSES IN THE GROUP $\mathbb{H}^n$

We have seen the use of "KP-decomposition" above. Similarly there is a "KAN-" or the "Iwasawa decomposition" for every reductive Lie group, in particular for $I_o(\mathbb{H}^n)$. The action of AN on $\mathbb{H}^n$ is simply transitive. So modulo a choice of a base-point, which in the upper half space model we take to be $(0,0,...,1)$, $\mathbb{H}^n$ can be considered as a *group*. As a curiosity, we visualize the stratification of AN into z-classes by identifying AN with $\mathbb{H}^n$.

Let $\mathbf{x}$ denote a point in $\mathbb{R}^{n-1}$, which we take as a part of the ideal boundary (except for ∞) of $\mathbb{H}^n$. The points of $\mathbb{H}^n$ are $(\mathbf{x}, x_n)$, $x_n > 0$. Then $A = \{f_r \mid r > 0\}$ acts by

$$f_r : (\mathbf{x}, x_n) \mapsto (r\mathbf{x}, rx_n),\ r > 0,$$

and $N = \{g_{\mathbf{a}} \mid \mathbf{a} \in \mathbb{R}^{n-1}\}$ acts by

$$g_{\mathbf{a}} : (\mathbf{x}, x_n) \mapsto (\mathbf{x} + \mathbf{a}, x_n).$$

The subgroup N is normal in AN, and AN is in fact a semi-direct product of N by A, where $A \approx \mathbb{R}_{>0}$ acts on $N \approx \mathbb{R}^{n-1}$ by

$$\mathbf{x} \mapsto r\mathbf{x}.$$

A general element of AN is $g_{\mathbf{a}} \circ f_r$, which acts on $\mathbb{H}^n$ by

$$(\mathbf{x}, x_n) \mapsto (r\mathbf{x} + \mathbf{a}, rx_n).$$

By the identification of AN with $\mathbb{H}^n$ as indicated above, we see that the element $g_{\mathbf{a}} \circ f_r$ corresponds to the point $(\mathbf{a}, r)$.

Theorem B.1. (1) *The non-identity conjugacy classes in $\mathbb{H}^n$ are represented by f_r, $r \neq 1$, and $g_{\mathbf{a}}$, $\mathbf{a} \neq 0$.*
 (i) The conjugacy class of f_r, $r \neq 1$, in the above identification, corresponds to the hyperplane $x_n = r$.
 (ii) The conjugacy class of $g_{\mathbf{a}}$, $\mathbf{a} \neq 0$ corresponds to the ray $(r\mathbf{a}, 1)$, $r > 0$, in the hyperplane $x_n = 1$.

(2) *There are only two non-identity z-classes, which can be visualized as:*

(i) *the complement of the hyperplane $x_n = 1$ in $\mathbb{H}^n$, and*

(ii) *the hyperplane $x_n = 1$, with puncture at $(\mathbf{0}, 1)$.*

Proof. Let $f = g_{\mathbf{a}} \circ f_r$, $r \neq 1$. Let $\mathbf{x}_0 = -(r-1)^{-1}\mathbf{a}$. Then

$$g_{\mathbf{x}_0} f g_{\mathbf{x}_0}^{-1} = f_r.$$

It is easy to see that f_r is not conjugate to f_s, $r \neq s$.

Now suppose $r = 1$. Then $f = g_{\mathbf{a}}$. Now for any element $h = g_{\mathbf{b}} \circ f_s$, we have

$$hfh^{-1} = g_{s\mathbf{a}}.$$

Thus $g_{\mathbf{a}}$ is conjugate to $g_{s\mathbf{a}}$, $s > 0$, and if $\mathbf{b} \neq t\mathbf{a}$ for some $t > 0$, then $g_{\mathbf{b}}$ is not conjugate to $g_{\mathbf{a}}$.

Hence the conjugacy classes in AN are represented by f_r, for $r > 0$, and $g_{\mathbf{a}}$, where $\mathbf{a}$ is unique up to a multiplication by $r > 0$. In the above identification of AN with $\mathbb{H}^n$, the conjugacy class of f_r corresponds to

$$\{(\mathbf{a}, r) \mid \mathbf{a} \in \mathbb{R}^{n-1}\},$$

and the conjugacy class of $g_{\mathbf{a}}$ corresponds to

$$\{(\mathbf{x}, 1) \mid \mathbf{x} = r\mathbf{a} \text{ for } r > 0\}.$$

For $\mathbf{a} \neq 0$, the group $Z(g_{\mathbf{a}}) = N$ is normal. For $r, s \neq 1$, $Z(f_r) = Z(f_s) = A$. Since for $r \neq 1$, $g_{\mathbf{a}} \circ f_r$ is conjugate to f_r, it follows that for all $r \neq 1$, all $Z(f_r)$, and $Z(g_{\mathbf{a}} \circ f_r)$ form a single conjugacy class of subgroups. Correspondingly there are exactly two non-identity z-classes in $\mathbb{H}^n$. The z-class corresponding to f_r is given by

$$\bigcup \{x_n = s \mid s \neq 1\}.$$

The z-class of $g_{\mathbf{a}}$ is given by

$$\{(\mathbf{x}, 1) \mid \mathbf{x} \neq \mathbf{0}\}. \qquad \square$$

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