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WILD KNOTS IN HIGHER DIMENSIONS AS LIMIT SETS OF KLEINIAN GROUPS

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Abstract. In this paper we construct infinitely many wild knots, $\mathbb{S}^n \hookrightarrow \mathbb{S}^{n+2}$, for $n = 1, 2, 3, 4$ and 5, each of which is a limit set of a geometrically finite Kleinian group. We also describe some of their properties.

1. Introduction

Kleinian groups were introduced by Henri Poincaré in the 1880's [27], as the monodromy groups of certain second order differential equations on the Riemann sphere $\widehat{\mathbb{C}}$. They have played a major role in many parts of mathematics throughout the twentieth and the present centuries, as for example in Riemann surfaces and Teichmüller theory, automorphic forms, holomorphic dynamics, conformal and hyperbolic geometry, number theory, and topology (for instance, the study of 3-manifolds).

The higher dimensional analogue of Kleinian groups are certain discrete subgroups of the group of diffeomorphisms of the $(n + 2)$-sphere $(n \geq 1)$, with its standard metric, consisting of those diffeomorphisms which preserve angles and which we denote by $M\ddot{o}b\,(\mathbb{S}^{n+2})$. The subgroup of index two of $M\ddot{o}b\,(\mathbb{S}^{n+2})$ which consists of those elements which are orientation-preserving is called the *Conformal or Möbius Group* and is denoted by $M\ddot{o}b_+\,(\mathbb{S}^{n+2})$; see the book by Ahlfors [1]. If $\Gamma \subset M\ddot{o}b\,(\mathbb{S}^{n+2})$ is a discrete subgroup acting on the $(n+2)$-sphere, then this action extends naturally to a conformal action on the disk $\mathbb{D}^{n+3}$. Its limit set, $\Lambda(\Gamma)$, is the set of points of $\mathbb{S}^{n+2}$ which are accumulation points of some orbit of Γ in $\mathbb{D}^{n+3}$. If $\Omega(\Gamma) := \mathbb{S}^{n+2} - \Lambda(\Gamma) \neq \emptyset$ one says that Γ is a *Kleinian group*. The set $\Omega(\Gamma)$ is called the *discontinuity set* of Γ.

One interesting question is whether a topological n-sphere $(n \geq 1)$ which is not a round sphere can be the limit set of a higher dimensional *geometrically finite* Kleinian group. In this case one can show that the sphere is necessarily fractal (possibly unknotted). Examples of wild knots in $\mathbb{S}^3$, which are limit sets of geometrically finite Kleinian groups, were obtained by Maskit [22], Kapovich [17], Hinojosa [13], and Gromov, Lawson and Thurston [9]. An example of a wild 2-sphere in $\mathbb{S}^4$ which is the limit set of a geometrically finite Kleinian group was obtained by the

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second-named author [14] and, independently, by Belegradek [5] (see also [4] for a wild limit set $\mathbb{S}^2 \to \mathbb{S}^3$). Such wild knots are examples of self-similar fractal sets and they are extremely beautiful to contemplate. For instance, one can admire the pictures in the classic book by R. Fricke and F. Klein [10] or the pictures of limit sets of classical Kleinian groups in the book *Indra's Pearls: The vision of Felix Klein* by D. Mumford, C. Series, D. Wright [25].

In this paper we construct an infinite number of wild knots $\mathbb{S}^n \hookrightarrow \mathbb{S}^{n+2}$ for $n = 1, \cdots, 5$ which are limit sets of geometrically finite Kleinian groups. If there existed a way to picture and travel through the spheres of high dimensions, we could contemplate our examples as the analogue of Indra's Pearls in all dimensions!

This paper is organized as follows. In section 2 we give the preliminaries: the definition of oriented tangles and knots in high dimensions and some basic facts of Kleinian groups. In section 3 we introduce the notion of *orthogonal ball covering* (OBC) of $\mathbb{R}^{n+2}$ as a covering by round balls satisfying certain conditions of orthogonality. Using results by L. Potyagailo, E.B. Vinberg in [28], we describe explicit OBCs for $n = 1, \dots, 5$. In section 4 we construct n-knots as limit sets of geometrically finite Kleinian groups for $n = 1, \dots, 5$. In section 5 we show that if we start with a nontrivial tame fibered n-knot K, then the complement of the corresponding limit n-knot $\Lambda(K)$ also fibers over $\mathbb{S}^1$ and gives a description of the fibers. In section 6 we describe the monodromy of $\Lambda(K)$ in terms of the monodromy of K. We prove that $\Lambda(K)$ is wildly embedded in $\mathbb{R}^{n+2}$. Therefore there exist infinitely many wild knots which are limit sets of discrete, geometrically finite Kleinian groups in dimensions 3, 4, 5, 6 and 7.

2. Preliminaries

In classical knot theory, a subset K of a space X is a *knot* if K is homeomorphic to a sphere $\mathbb{S}^p$. Two knots K, K' are *equivalent* if there is a homeomorphism $h : X \to X$ such that $h(K) = K'$; in other words $(X, K) \cong (X, K')$. However, a knot K is sometimes defined to be an embedding $K : \mathbb{S}^p \to \mathbb{S}^n$ (see [23] and [29]). We shall also find this convenient at times and will use the same symbol to denote either the map K or its image $K(\mathbb{S}^p)$ in $\mathbb{S}^n$.

Definition 2.1. An *oriented n-dimensional tame single-strand tangle* is a couple $D = (B^{n+2}, T)$ satisfying the following conditions:

(1) B^{n+2} is homeomorphic to the $(n+2)$-disk $\mathbb{D}^{n+2}$, and T is homeomorphic to the n-disk $\mathbb{D}^n$.
(2) The pair (B^{n+2}, T) is a proper manifold pair, i.e. $\partial T \subset \partial B^{n+2}$ and $\mathrm{Int}(T) \subset \mathrm{Int}(B^{n+2})$.
(3) (B^{n+2}, T) is locally flat ([30], p. 33).
(4) B^{n+2} has an orientation which induces the canonical orientation on its boundary ∂B^{n+2}.
(5) $(\partial B^{n+2}, \partial T)$ is homeomorphic to $(\partial \mathbb{D}^{n+2}, \partial \mathbb{D}^n)$.

Compare to Zeeman's definition of ball-pair in [36].

Two oriented tangles $D_1 = (B_1^{n+2}, T_1)$, $D_2 = (B_2^{n+2}, T_2)$ are equivalent if there exists an orientation-preserving homeomorphism of B_1^{n+2} onto B_2^{n+2} that sends T_1 to T_2. A tangle is unknotted if it is equivalent to the trivial tangle $(\mathbb{D}^{n+2}, \mathbb{D}^n)$.

Given an oriented tangle $D = (B^{n+2}, T)$, the pair $(\partial B^{n+2}, \partial T)$ is homeomorphic to the pair $(\mathbb{S}^{n+1}, \mathbb{S}^{n-1})$, via a homeomorphism f. Then D determines canonically a knot $K \subset \mathbb{S}^{n+2}$, in the following way: $(\mathbb{S}^{n+2}, K) = (B^{n+2}, T) \cup_f (\mathbb{D}^{n+2}, \mathbb{D}^n)$.

Conversely, given a smooth knot $K \subset \mathbb{S}^{n+2}$, there exists a smooth ball B^{n+2} such that $(B^{n+2}, B^{n+2} \cap K)$ is equivalent to the trivial tangle. The tangle $K_T = (\mathbb{S}^{n+2} - \mathrm{Int}(B^{n+2}), K - \mathrm{Int}(B^{n+2} \cap K))$ is called the *canonical tangle* associated to K. Notice that if K is not the trivial knot, then K_T is not equivalent to the trivial tangle. In this case, we say that K_T is knotted.

Remark 2.2. The above constructions are well defined up to isotopy.

The *connected sum* of the oriented tangles $D_1 = (B_1^{n+2}, T_1)$ and $D_2 = (B_2^{n+2}, T_2)$ for $n > 2$, denoted by $D_1 \# D_2$, can be defined as follows: Since D_i is locally flat $i = 1, 2$, there exist sets $U_i \subset \partial B_i$ closed in ∂B_i such that $\mathrm{Int}(U_i) \cap \partial T_i \neq \varnothing$ and the pair $(U_i, U_i \cap T_i)$ is homeomorphic to $(\mathbb{D}^{n+1}, \mathbb{D}^{n-1})$. Choose an orientation-reversing homeomorphism h of $(U_1, U_1 \cap T_1)$ onto $(U_2, U_2 \cap T_2)$. Then

$$D_1 \# D_2 = (B_1^{n+2}, T_1) \cup_h (B_2^{n+2}, T_2).$$

Remark 2.3. The connected sum does not depend on the choice of the homeomorphism h and the open sets U_i.

Our goal is to obtain wild n-spheres, $n = 1, 2, \ldots, 5$ as limit sets of conformal Kleinian groups. We will briefly review some basic definitions about Kleinian groups.

Let $M\ddot{o}b\,(\mathbb{S}^n)$ denote the group of Möbius transformations of the n-sphere $\mathbb{S}^n = \mathbb{R}^n \cup \{\infty\}$, *i.e.*, conformal diffeomorphisms of $\mathbb{S}^n$ with respect to the standard metric. For a discrete group $G \subset M\ddot{o}b(\mathbb{S}^n)$ the *discontinuity set* $\Omega(G)$ is defined as follows:

$$\Omega(G) = \{x \in \mathbb{S}^n : \text{the point } x \text{ possesses a neighborhood } U(x) \text{ such that } U(x) \cap g(U(x)) \text{ is empty for all but a finite number of elements } g \in G\}.$$

The complement $\mathbb{S}^n - \Omega(G) = \Lambda(G)$ is called the *limit set* (see [17]). Both $\Omega(G)$ and $\Lambda(G)$ are G-invariant. The set $\Omega(G)$ is open, hence $\Lambda(G)$ is compact.

A subgroup $G \subset M\ddot{o}b\,(\mathbb{S}^n)$ is called *Kleinian* if $\Omega(G)$ is not empty.

We recall that a conformal map ψ on $\mathbb{S}^n$ has a Poincaré extension to the hyperbolic space $\mathbb{H}^{n+1}$ as an isometry with respect to the Poincaré metric. Hence we can identify the group $M\ddot{o}b(\mathbb{S}^n)$ with the group of isometries of hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$. This allows us to define the limit set of a Kleinian group through sequences. We say that a point x is a *limit point* for the Kleinian group G, if there exists a point $z \in \mathbb{S}^n$ and a sequence $\{g_m\}$ of *distinct elements* of G, with $g_m(z) \to x$. The set of limit points is $\Lambda(G)$ (see [22], section II.D).

One way to illustrate the action of a Kleinian group G is to draw a picture of $\Omega(G)/G$. For this purpose a fundamental domain is very helpful. Roughly speaking, it contains one point from each equivalence class in $\Omega(G)$ (see [18], pages 78–79 and [22], pages 29–30).

Definition 2.4. A fundamental domain D for a Kleinian group G is a codimension zero piecewise-smooth submanifold (subpolyhedron) of $\Omega(G)$ satisfying the following:

(1) $\bigcup_{g \in G} g(\mathrm{Cl}_{\Omega(G)} D) = \Omega$ where $Cl_{\Omega(G)}$ is the closure in $\Omega(G)$.
(2) $g(\mathrm{Int}(D)) \cap \mathrm{Int}(D) = \emptyset$ for all $g \in G - \{e\}$, where e is the identity in G and Int denotes the interior.

(3) The boundary of D in $\Omega(G)$ is a piecewise-smooth (polyhedron) submanifold in $\Omega(G)$, divided into a union of smooth submanifolds (convex polyhedra) which are called faces. For each face S, there is a corresponding face F and an element $g_{SF} \in G - \{e\}$ such that $g_{SF}(S) = F$ (g_{SF} is called a face-pairing transformation); $g_{SF} = g_{FS}^{-1}$.

(4) Only finitely many translates of D meet any compact subset of $\Omega(G)$.

Theorem 2.5 ([18], [22])**.** *Let $D^* = \overline{D} \cap \Omega/\sim_G$ denote the orbit space with the quotient topology. Then D^* is homeomorphic to Ω/G.*

3. Orthogonal ball coverings of $\mathbb{R}^{n+2}$

A countable collection of closed round $(n+2)$-balls $B_1, B_2, B_3, \ldots$ is an *orthogonal ball covering* (OBC) of $\mathbb{R}^{n+2}$ if:

(1) $\bigcup_i B_i = \mathbb{R}^{n+2}$.
(2) There exist $\epsilon_1, \epsilon_2 > 0$ such that $\epsilon_1 <$ diameter$(B_i) \leq \epsilon_2$, $\forall i$.
(3) For each pair (B_i, B_j) with $i \neq j$, we have that they are either disjoint, meet at only one point or their boundaries meet orthogonally.

In particular, an OBC is locally finite.

One has the following theorem of Potyagailo and Vinberg (see [28]):

Theorem 3.1. *There exist right-angled hyperbolic polyhedra of finite volume, with at least one point at infinity, in $\mathbb{H}^n$, for $n = 3, \ldots, 8$.*

This theorem has, for our purposes, the following relevant corollary:

Corollary 3.2. *There exist OBCs for $\mathbb{R}^n$, $n = 2, \ldots, 7$.*

Proof. The OBC is obtained as follows: Consider the half-space model of $\mathbb{H}^{n+1} = \{(x_1, \cdots, x_{n+1}) |\ x_{n+1} > 0\} \subset \mathbb{R}^{n+1}$ with boundary the hyperplane at infinity $\mathbb{R}^n$ with equation $x_{n+1} = 0$. Let P be a right-angled hyperbolic polyhedra of finite volume in $\mathbb{H}^{n+1}$ for $n = 2, \ldots, 7$ as in the previous theorem. We can assume that one of the ideal vertices of P is ∞. Consider the facets F_i of P which are not asymptotic to ∞ and let the spheres S_i ($i = 1, \ldots, k$) be the ideal boundaries of the hyperbolic hyperplanes through F_i's. Now, apply to the spheres S_i the group G generated by the reflections in the facets of P asymptotic to ∞ (semi-hyperplanes orthogonal to the hyperplane at infinity). The action of G on the hyperplane at infinity has as fundamental domain a compact parallelepiped (in fact, as we will see afterwards, it is a regular cube). The set of images of the balls whose boundaries are the spheres S_i is the desired OBC.

3.1. **Description of some OBCs for $\mathbb{R}^n$, $n = 2, \ldots, 7$ and their nerves.** We will describe geometrically the OBCs $\mathcal{B}$ obtained in corollary 3.2 and the geometric realization of their nerves. Given a collection of round open balls $\{B_j, j \in I\}$ in $\mathbb{R}^n$, consider its nerve N. We define the canonical simplicial mapping $f : N \to \mathbb{R}^n$ by sending each vertex of N to the center of the corresponding ball and extending f linearly to the simplices of N. *The geometric realization* of N is $f(N)$.

For $n = 2$, we have that the fundamental domain for the group G is the unit square, I^2. We set two closed round balls of radius one at $(0, 0)$ and $(1, 1)$ (see Figure 1). The geometric realization of its nerve, is the closed segment joining $(0, 0)$ and $(1, 1)$.

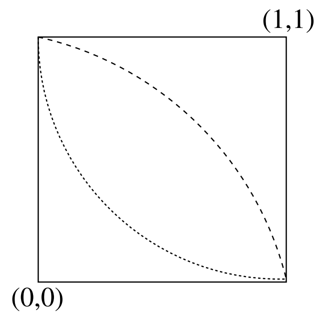

FIGURE 1. A fundamental domain for $n = 2$.

If we propagate the fundamental domain by G, we get a flower at the remaining vertices of I^2, *i.e.* we obtain a configuration consisting of four balls such that balls centered at the same straight line parallel to a coordinate axis, are tangent (see Figure 2). The geometric realization of the nerve of the above four balls is a rhombus.

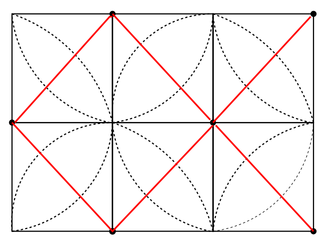

FIGURE 2. A flower for $n = 2$.

For $n = 3$, we have that the fundamental domain for the group G is the unit cube, I^3. Notice that $I^3 = I^2 \times [0, 1]$. At the face $I^2 \times \{0\}$, we set two closed round balls of radius one at $(0, 0, 0)$ and $(1, 1, 0)$. At the face $I^2 \times \{1\}$, we set two closed round balls of radius one at $(1, 0, 1)$ and $(0, 1, 1)$ (see Figure 3). The geometric realization of its nerve is a solid tetrahedron spanned by $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$.

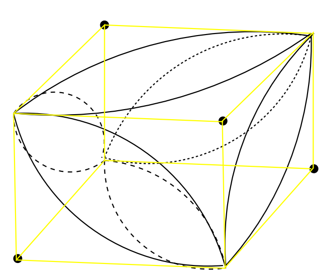

FIGURE 3. A fundamental domain for $n = 3$.

As in the previous case, if we propagate the fundamental domain by G, we get a flower at the remaining vertices of I^3, *i.e.* we obtain a configuration consisting of six balls such that balls centered at the same straight line parallel to a coordinate axis are tangent (see Figure 4). This situation will appear in the next dimensions, therefore we will give a general definition.

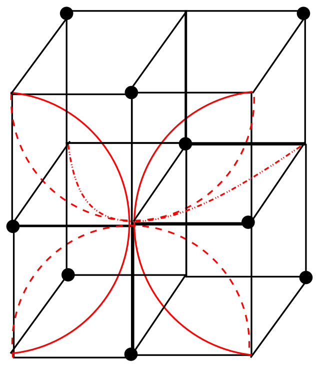

FIGURE 4. A flower for $n = 3$.

Definition 3.3. Let $B_1, B_2, \ldots, B_{2n+4}$ be round closed balls in the OBC $\mathcal{B}$ of $\mathbb{R}^{n+2}$. We say that they form a flower if:

(1) For each pair (B_i, B_j) with $i \neq j$, we have that they either meet at only one point or their boundaries meet orthogonally.
(2) The intersection $\bigcap_{i=1}^{2n+4} B_i$ consists of only one point c which we call the *center* of the flower.

The geometric realization of the nerve of the above six balls is the boundary of a solid octahedron. Observe that each face of the octahedron corresponds to a face of a solid tetrahedron previously described.

For $n = 4$, we have that the fundamental domain for the group G is the unit hypercube, I^4. Notice that $I^4 = I^3 \times [0, 1]$. At the face $I^3 \times \{0\}$, we set four closed round balls of radius one at $(0, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 0, 1, 0)$ and $(0, 1, 1, 0)$. At the face $I^3 \times \{1\}$, we set four closed round balls of radius one at $(1, 1, 1, 1)$, $(0, 1, 0, 1)$, $(1, 0, 0, 1)$ and $(0, 0, 1, 1)$.

Observe that all these balls meet at the center of I^4, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, hence they form a flower. The geometric realization of the nerve of the above eight balls is the 4-dimensional analogous of the octahedron, a hyper-octahedron, which for simplicity will be called a *diamond*. Notice that the center of this diamond is the center of the corresponding flower. This remains true in the next dimensions.

For $n = 5$ we have that the fundamental domain for the group G is the unit hypercube, I^5. Notice that $I^5 = I^4 \times [0, 1]$. In the face $I^4 \times \{0\}$, we set eight closed round balls of radius one centered at the following vertices $(0, 0, 0, 0, 0)$, $(1, 1, 0, 0, 0)$, $(1, 0, 1, 0, 0)$, $(0, 1, 1, 0, 0)$, $(1, 1, 1, 1, 0)$, $(0, 1, 0, 1, 0)$, $(1, 0, 0, 1, 0)$ and $(0, 0, 1, 1, 0)$. In the face $I^4 \times \{1\}$, we set eight closed round balls of radius one at $(1, 0, 0, 0, 1)$, $(0, 1, 0, 0, 1)$, $(0, 0, 1, 0, 1)$, $(0, 0, 0, 1, 1)$, $(0, 1, 1, 1, 1)$, $(1, 0, 1, 1, 1)$, $(1, 1, 0, 1, 1)$ and $(1, 1, 1, 0, 1)$.

Next, we set a ball at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ of radius $\frac{1}{2}$.

If we propagate the fundamental domain by G, as before we obtain flowers at vertices and also a flower at the center of each hyperface of dimension four.

For $n = 6$ we again have that the fundamental domain for the group G is the unit hypercube, I^6. At each face of I^6, we repeat the construction for $n = 5$ in such a way that, we set 32 closed round balls of radius one centered at vertices and 12 closed round balls of radius $\frac{1}{2}$, each one set at the center of each face.

Notice that the twelve closed balls at the centers of faces, form a flower at the center of I^6.

For $n = 7$ we again have that the fundamental domain for the group G is the unit hypercube, I^7. We will not describe in detail this construction, since we only need for our purpose, the OBC restricted to its faces. Now, at each face of I^7, we repeat the construction for $n = 6$.

Remark 3.4. The geometric realization of the nerve of the OBC $\mathcal{B}$ is embedded in $\mathbb{R}^{n+2}$, for $n = 1, \dots, 5$.

4. Construction of a wild knot as limit set of a Kleinian group

As we have mentioned before, from the existence of right-angled hyperbolic polyhedra of finite volume, with at least one point at infinity in $\mathbb{H}^{n+3}$, for $n = 1, \dots, 5$, it follows that there exist OBC, $\mathcal{B}$, for $\mathbb{R}^{n+2}$, $n = 1, \dots, 5$. In this section, we will use these OBCs to construct wild knots. Our aim, in this and the next sections, is to prove the following:

Theorem 4.1. *There exist infinitely many non-equivalent knots $\psi : \mathbb{S}^n \to \mathbb{S}^{n+2}$ wildly embedded as limit sets of geometrically finite Kleinian groups, for $n = 1, 2, 3, 4, 5$.*

Let $\psi : \mathbb{S}^n \to \mathbb{R}^{n+2} \subset \mathbb{R}^{n+2} \cup \{\infty\} = \mathbb{S}^{n+2}$ be a smoothly embedded knotted n-sphere in $\mathbb{S}^{n+2}$. We denote $K = \psi(\mathbb{S}^n)$ and endow it with the Riemannian metric induced by the standard Riemannian metric of $\mathbb{R}^{n+2}$.

The general idea of our construction of wild knots is the following: Given a smooth knot $\bar{K}$, there exists an isotopic copy of it, K, in the n-skeleton of the canonical cubulation of $\mathbb{R}^{n+2}$ (see the next subsection). This cubulation is canonically associated to an OBC, $\mathcal{B}$.

Let $\mathcal{B}(K) = \{B \in \mathcal{B} : K \cap \mathrm{Int}(B) \neq \emptyset\}$. The group generated by inversion on the boundaries of balls belonging to $\mathcal{B}(K)$ whose centers are in the n-skeleton will be Kleinian and its limit set will be a wild knot.

4.1. OBCs and cubulations for $\mathbb{R}^{n+2}$.

A *cubulation* of $\mathbb{R}^{n+2}$ is a decomposition of $\mathbb{R}^{n+2}$ into a collection $\mathcal{C}$ of $(n+2)$-dimensional cubes such that any two of its hypercubes are either disjoint or meet in one common face of some dimension. This provides $\mathbb{R}^{n+2}$ with the structure of a cubic complex.

The canonical cubulation $\mathcal{C}$ of $\mathbb{R}^{n+2}$ is the decomposition into hypercubes which are the images of the unit cube $I^{n+2} = \{(x_1, \dots, x_{n+2}) \,|\, 0 \leq x_i \leq 1\}$ by translations by vectors with integer coefficients. Consider the homothetic transformation $H_m : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$, $H_m(x) = \frac{1}{m}x$, where $m > 1$ is an integer. The set $\mathcal{C}_m = H_m(\mathcal{C})$ is called a *subcubulation* or *cubical subdivision* of $\mathcal{C}$.

Observe that the n-skeleton of $\mathcal{C}$, denoted by $\mathcal{S}$, consists of the union of the n-skeletons of the cubes in $\mathcal{C}$, *i.e.,* the union of all cubes of dimension n contained in the faces of the $(n+2)$-cubes in $\mathcal{C}$. We will call $\mathcal{S}$ the *standard scaffolding* of $\mathbb{R}^{n+2}$.

By the previous section, the OBCs of $\mathbb{R}^{n+2}$ were obtained by covering the unit cube by balls which are either tangent, meet orthogonally or are disjoint and then taking all the balls obtained as images under the group generated by reflections on the faces of the cube. This is possible since the balls meet the planes which support the faces either tangentially or orthogonally. Thus we have the canonical cubulation associated in a natural way to the OBCs.

We can also associate the cubulation $\mathcal{C}_2$ to the OBCs. In this case, our fundamental cube E is the union of 2^{n+2} cubes in $\mathcal{C}$ and can be obtained as follows: take a cube I in the standard cubulation and take a vertex $v \in I$ and reflect in all the hyperplanes which support faces not containing v. The union of all the images of I under these reflections is E.

Notice that there are two types of vertices in I: Those which are centers of balls and those which are not. We will choose v such that it is not a center of ball. Since the vertices of E are the orbit of v, then all vertices of E are not centers of balls. We can assume (if necessary, after applying a translation) that the origin is a vertex of E.

We will use the following theorem [6] to embed an isotopic copy of the n-knot K in the scaffolding $\mathcal{S}_2$ of $\mathcal{C}_2$.

Theorem 4.2. *Let $\mathcal{C}$ be the standard cubulation of $\mathbb{R}^{n+2}$. Let $K \subset \mathbb{R}^{n+2}$ be a smooth knot of dimension n. There exists a knot $\hat{K}$ isotopic to K, which is contained in the scaffolding (n-skeleton) of the standard cubulation $\mathcal{C}$ of $\mathbb{R}^{n+2}$. The cubulation of the knot $\hat{K}$ admits a subdivision by simplexes and with this structure the knot is PL-equivalent to the n-sphere with its canonical PL-structure.*

By abuse of notation, we will denote $\hat{K}$ by K.

4.2. **Nerves and regular neighborhoods.** Let us consider the OBC $\mathcal{B}$ for $\mathbb{R}^{n+2}$ (section 3) together with the cubulation $\mathcal{C}_2$ constructed above. By theorem 4.2 we will assume that the knot K is embedded in the scaffolding $\mathcal{S}_2$ of $\mathcal{C}_2$.

Let o be a vertex of $\mathcal{C}_2$. By the above, we can assume that o is the origin. For $i, j \in \{1, 2, , \dots, n+2\}$, let $P_{i,j} = \{(x_1, \dots, x_{n+2}) : x_i = x_j = 0\}$ be the n-dimensional plane.

Proposition 4.3. *Let B_j and B_k be balls whose centers lie in $P_{i,j}$ and $P_{i,k}$, respectively. Suppose that B_j and B_k do not belong to a flower whose center lies in $P_{i,j} \cap P_{i,k}$ (see definition 3.3). Then $B_j \cap B_k = \emptyset$.*

Proof. Observe that $P_{i,j} \cap P_{i,k} = \{(x_1, \dots, x_{n+2}) : x_i = x_j = x_k = 0\}$. Let $C_j = (c_1^j, c_2^j, \dots, c_{n+2}^j)$, $C_k = (c_1^k, c_2^k, \dots, c_{n+2}^k)$ be the centers of B_j and B_k respectively, then $c_i^j = c_j^j = 0$ and $c_i^k = c_k^k = 0$. The minimum of the distance between their centers is attained as $c_r^j = c_r^k$ for $r \neq i, j, k$. It is enough to prove this proposition for this case.

Let $x = (x_1, \dots, x_{n+2}) \in P_{i,j} \cap P_{i,k}$ such that $x_r = 0$ if $r = i, j, k$ and $x_r = c_r^j = c_r^k$ if $r \neq i, j, k$. For $n < 5$ the centers C_j and C_k are vertices of the cubulation $\mathcal{C}$ therefore their coordinates are integers. Notice that c_k^j and c_j^k are bigger than one, since in the other case, B_j and B_k would be part of the flower centered at x. Hence these coordinates are either bigger or equal to two. This implies that the distance $d(C_j, C_k) \geq \sqrt{8}$. Therefore $B_j \cap B_k = \emptyset$ since their radii are one.

For $n = 5$ we have two types of balls: balls centered at vertices of the cubulation $\mathcal{C}$ and radii equal to one and, balls whose centers coincide with centers of five dimensional faces of cubes in $\mathcal{C}$ and their radii are equal to $\frac{1}{2}$. The argument to prove this case is analogous to the previous one. □

In the remaining of this and the next sections we will restrict ourselves to those balls in $\mathcal{B}$ which cover K. These balls coincide with the balls whose centers lie in the scaffolding $\mathcal{S}_2$ of $\mathcal{C}_2$. That is, let $\mathcal{B}(K) = \{B \in \mathcal{B} : K \cap \mathrm{Int}(B) \neq \emptyset\}$ and $T = \bigcup\{B \in \mathcal{B}(K)\}$. Observe that T is optimal in the sense that if we remove one, then the remaining balls do not cover K. *We will call such a T a generalized pearl necklace.*

Let E be a cube in $\mathcal{C}_2$. Suppose that $K \cap E$ consists of more than one n-dimensional face of E. Let F_1 and F_2 be n-faces of E such that $F_1, F_2 \subset K \cap E$. If F_{12} denotes the $(n-1)$-dimensional cubic simplex $F_1 \cap F_2$, then *we will say that K turns in F_{12}.*

From the construction of the OBCs we have that there are centers of flowers, C_i, $i = 1, \ldots, r$, contained in K. At each center, C_i which does not belong to an $(n-1)$-cubic complex where K turns, we will add a ball B_{C_i} of center C_i and radius less than $\frac{1}{\sqrt{n+2}}$.

Let $\mathcal{B}'(K) = \{B_{C_i}\}_{i=1}^r \cup \mathcal{B}(K)$ and $\tilde{T} = \bigcup_{i=1}^r B_{C_i} \cup T$. Observe that the pair $(B, B \cap K)$, $B \in \mathcal{B}'(K)$ is isotopic to the trivial tangle. *We will call a $\tilde{T}$ a generalized increased pearl necklace.*

Let $\mathcal{B}_K = \{K \cap \mathrm{Int}(B_i) | B_i \in \mathcal{B}'(K)\}$. Next, we will construct the nerve of the closed coverings $\mathcal{B}_K$ and $\mathcal{B}'(K)$. For this purpose we will always consider open balls even if the collections consist of closed balls. Let $N_{\mathcal{B}'(K)}$ and $N_{\mathcal{B}_K}$ be the corresponding geometric realization of the nerves of $\mathcal{B}'(K)$ and $\mathcal{B}_K$, respectively. For simplicity, we will call $N_{\mathcal{B}'(K)}$ and $N_{\mathcal{B}_K}$ *the nerves* of $\mathcal{B}'(K)$ and $\mathcal{B}_K$, respectively.

Lemma 4.4. *Let $N_{\mathcal{B}'(K)}$, $N_{\mathcal{B}_K}$ be the nerves of $\mathcal{B}'(K)$ and $\mathcal{B}_K$, respectively. Then $N_{\mathcal{B}'(K)}$ is homeomorphic to $N_{\mathcal{B}_K}$ and homeomorphic to K.*

Proof. Recall from section 3.1 that the nerve in $\mathbb{R}^{n+2}$ of balls in $\mathcal{B}(K)$ is a collection of $(n+2)$-simplexes and $(n+1)$-dimensional diamonds with empty interior. By adding a ball B_C at a center C of a flower (which is in turn the center of a diamond) the effect on the nerve is to add the cone from C, thus filling the interior of the diamond. This is the nerve of the collection $\mathcal{B}'(K)$.

Let $S_K(v)$, $S_{\mathcal{B}'(K)}(v)$ and $S_{\mathcal{B}_K}(v)$ denote the star of a vertex v in the simplicial complexes K, $N_{\mathcal{B}'(K)}$ and $N_{\mathcal{B}_K}$ respectively. We will define an isotopy between these simplicial complexes locally on $S_K(v)$, leaving the boundary $Lk_K(v)$ of $S_K(v)$ fixed.

Let P be an n-hyperplane supporting the n-dimensional face of a cube $E \in \mathcal{C}_2$. Let v be a vertex in $K \cap P$. If $S_K(v)$ is contained in P, the centers of all the balls surrounding v lie on P, and therefore $S_K(v)$, $S_{\mathcal{B}'(K)}(v)$ and $S_{\mathcal{B}_K}(v)$ all coincide and are equal to the n-disk which is the geometric convex hull of the vertices surrounding v (see Figure 5).

Now take a vertex v which is a center of a flower on an $(n-1)$-dimensional cubic simplex S in which K turns. Then $S_K(v)$ is an n-dimensional disk whose boundary contains vertices $v_1, \ldots, v_r$ on different faces of a cube $E \in \mathcal{C}_2$. Observe that if $v_1, \ldots, v_r$ belong to more than two faces of E, then v has to be a vertex of the cube and, inversely, all the vertices of E are centers of flowers. By construction we did not add a ball with center v when we constructed $\mathcal{B}'(K)$. By proposition 4.3, only the balls forming a flower in v intersect. Therefore $N_{\mathcal{B}'(K)}$ is, locally around v, formed by n-dimensional faces of a diamond with vertices in $v_1, \ldots, v_r$, and these faces contain the boundary of $S_K(v)$ (see Figure 6).

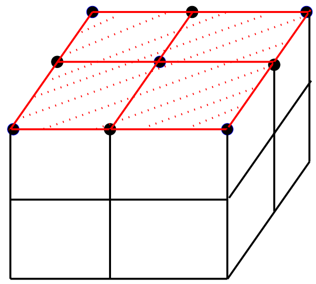

FIGURE 5. A schematic picture of the corresponding geometric nerve of balls centered at F.

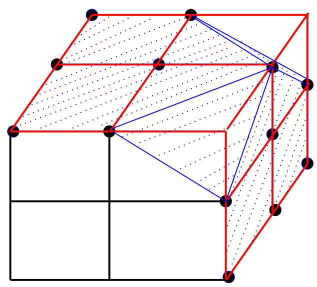

FIGURE 6. A schematic picture of the corresponding geometric nerve of balls centered at $F_1 \cup F_2$.

We can isotope $S_K(v)$ into $N_{\mathcal{B}'(K)}$ leaving this boundary fixed. After this isotopy, all of K, $N_{\mathcal{B}'(K)}$ and $N_{\mathcal{B}_K}$ coincide. □

We will use the following theorem to prove that $\tilde{T}$ is a regular neighborhood of K.

Theorem 4.5. *Let $\mathcal{B} := \{B_1, \cdots, B_r\}$ be a finite set of round open balls in either $\mathbb{R}^{n+2}$ with its euclidean metric or $\mathbb{S}^{n+2}$ with its spherical metric. Let v_i denote the center of B_i. Let $\bar{B}_i$ ($i = 1 \cdots r$) denote the corresponding closed balls and $\mathbb{S}_i^{n+1} := \partial\bar{B}_i$ their spherical boundaries. We will assume that if two balls are tangent, then there exists j such that B_j is centered at the point of tangency and that none of balls is contained in another. Suppose that the geometric nerve of $\mathcal{B}$ consisting of totally geodesic simplexes (with respect to the euclidean or spherical metric, respectively) defined before (see section 3.1) is an n-dimensional polyhedral sphere K in $\mathbb{R}^{n+2}$ or $\mathbb{S}^{n+2}$ such that for each closed ball $\bar{B}_i$, the pair $(\bar{B}_i, \bar{B}_i \cap K)$ is equivalent to the trivial tangle (in particular, K is locally flat). Then $\mathcal{N}(K) := \bigcup_{i=i}^{r} \bar{B}_i$ is a closed regular neighborhood of K and, in particular $\mathcal{N}(K) \cong \mathbb{D}^2 \times K$ since the normal bundle of an n-dimensional locally flat sphere in $\mathbb{R}^{n+2}$ or $\mathbb{S}^{n+2}$ is trivial.*

Proof. Let $V(K)$ be a regular neighborhood of K. Since K is locally flat, there exists a homeomorphism $\eta : \mathbb{D}^2 \times K \to V(K)$. For $0 < \delta \leq 1$, let $V_\delta(K) := \eta({\mathbb{D}^2}_\delta \times K)$, where ${\mathbb{D}^2}_\delta$ is the closed 2-disk of radius δ.

The proof will continue by induction on the dimension $m = n+2$. When $m = 2$, $\mathcal{B}$ consists of two balls B_1 and B_2 (in $\mathbb{R}^2$ or $\mathbb{S}^2$) such that $\bar{B}_1 \cap \bar{B}_2 = \emptyset$ and the result is obviously true. If $m = 3$, we have a pearl necklace in $\mathbb{R}^3$ or $\mathbb{S}^3$ [15] and the proof is contained there (see Figure 7).

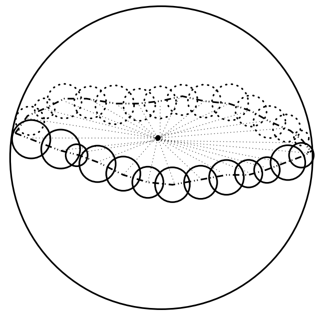

FIGURE 7. A schematic pearl necklace of dimension 3.

Let $m > 3$. Let $j \in \{1, \ldots, r\}$, and $\mathcal{B}_j := \{B_i \mid B_i \cap B_j \neq \emptyset\}$. The geometric realization of the nerve of $\mathcal{B}_j$, which we denote by S_j, is the star of the vertex corresponding to the center v_j of the ball B_j. The set of spherical $(n+1)$-balls $L_j := \{\bar{B}_k \cap \mathbb{S}_j^{n+1} \mid \bar{B}_k \cap \bar{B}_j \neq \emptyset\}$ satisfies the induction hypotheses of the theorem applied to the ambient space $\mathbb{S}_j^{n+1}$ and, in fact, the geometric realization of these $(n+1)$-balls in $\mathbb{S}_j^{n+1}$ is combinatorial equivalent (*i.e.*, PL-homeomorphic) to the link of the vertex v_j, *i.e.*, the boundary of the star S_j.

By hypothesis, the pair $(\bar{B}_j, \bar{B}_j \cap K)$ is equivalent to the trivial n-dimensional tangle. Let $M_j := \bigcup_{\bar{B}_k \in L_k} \bar{B}_k \cap \bar{B}_j$. For a fixed $j \in \{1, \ldots, r\}$, let $V_j := M_j \cup (V_\delta(K) \cap \bar{B}_j)$ (see Figure 8). Then for δ sufficiently small, V_j is a tubular neighborhood of $\bar{B}_j \cap K$ in $\bar{B}_j$. More precisely, the pair $(\bar{B}_j, V_j)$ is homeomorphic as a pair to $(I^{n+2}, I^n \times \frac{1}{2}I^2)$ where $I^{n+2} = [-1,1] \times \cdots \times [-1,1]$ ($(n+2)$-times) and $I^n \times \frac{1}{2}I^2 = [-1,1] \times \cdots \times [-1,1] \times [-\frac{1}{2}, \frac{1}{2}] \times [\frac{1}{2}, \frac{1}{2}]$ (the factor $[-1,1]$ occurs n-times). Since $\bar{B}_j - \mathrm{Int}\,(V_j) \cong I^{n+2} - \mathrm{Int}\,(I^n) \times \frac{1}{2}I^2 \cong \partial I^2 \times [-1,1] \times I^n$, one has that V_j is a strong deformation retract of $\bar{B}_j$.

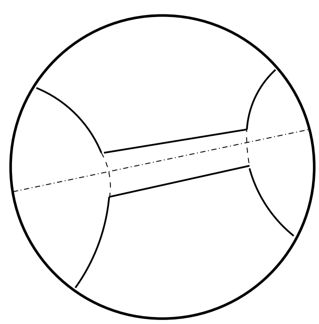

FIGURE 8. A schematic picture of the tubular neighborhood V_j.

The previous arguments imply that the set $\bigcup_{j=1}^r V_j$ is a regular neighborhood of K and furthermore, since $\bigcup_{j=1}^r V_j$ is a strong deformation retract of $\mathcal{N}(K) = \bigcup_{j=1}^r \bar{B}_j$, it follows that $\mathcal{N}(K)$ is also a regular neighborhood of K. $\square$

Corollary 4.6. *Let $\tilde{T}$ be a generalized increased pearl necklace subordinate to K. Then $\tilde{T}$ is isotopically equivalent to a closed regular neighborhood $\mathcal{N}$ of K.*

4.3. **Description of the limit set.** Let K be a smooth n-knot. Let $T = \bigcup_{i=1}^{r} B_i$ be a generalized pearl necklace subordinate to K and $\tilde{T}$ be the corresponding generalized increased pearl necklace. Let Γ be the group generated by reflections I_j through ∂B_j, $B_j \in T$. To guarantee that the group Γ is Kleinian we will use the Poincaré Polyhedron Theorem (see [8], [18], and [22]). This theorem establishes conditions for the group to be discrete. In practice these conditions are very hard to verify, but in our case all of them are satisfied automatically from the construction, since the balls B_i, $B_j \in T$ are either disjoint, tangent or their boundaries meet orthogonally.

This theorem also gives us a presentation for the group Γ. In our case, the dihedral angles n_{ij} between the faces F_i, F_j are $\frac{\pi}{2}$, if the faces are adjacent or 0 otherwise (by definition). Therefore, we have the following presentation of Γ:

$$\Gamma = \langle I_j, j = 1, \ldots, n | \ \ (I_j)^2 = 1, \ \ (I_i I_j)^{n_{ij}} = 1 \rangle.$$

The fundamental domain for Γ is $D = \mathbb{S}^{n+2} - T$ and Γ is geometrically finite.

The natural question for the Kleinian group Γ is: What is its limit set? Recall that to find the limit set of Γ, we need to find all the accumulation points of its orbits. We will do this by stages.

Stage I. We will apply induction on the number of reflections.

(1) First step: We reflect with respect to ∂B_1, *i.e.*, we apply $I_1 \in \Gamma$. Notice that $D_1 = D \cup I_1(D) = (\mathbb{S}^{n+2} - T) \cup (B_1 - (I_1(T))$ is a fundamental domain of an index two subgroup Γ_1 in Γ.

   Claim. The set $D_1^{'} = (\tilde{T} - B_1) \cup I_1(\tilde{T})$ is a regular neighborhood of the geometric realization of its nerve, which is isotopic to the connected sum of K with its mirror image $-K$.

   In fact, observe that the reflection map I_1 replaces the trivial tangle $(B_1, B_1 \cap K)$ by a new tangle $C_1 = (B_{C_1}, K_{C_1})$ which is isotopic to an orientation-reversing copy of the tangle

$$C = (\mathbb{S}^{n+2} - \mathrm{Int}(B_1), K - \mathrm{Int}(B_1 \cap K)) = (B_C, K_C),$$

   which is homeomorphic to the canonical tangle associated to K up to isotopy. Then we apply theorem 4.5.

(2) Second step: We reflect with respect to ∂B_2, *i.e.*, we apply $I_2 \in \Gamma$. Notice that $D_2 = D_1 \cup I_2(D_1) = (\mathbb{S}^{n+2} - T) \cup (B_1 - I_1(T)) \cup (B_2 - I_2(D_1))$ is a fundamental domain of an index two subgroup Γ_2 in Γ_1.

   The set $D_2^{'} = (\tilde{T} - \bigcup_{i=1}^{2} B_i) \cup I_1(\tilde{T}) \cup I_2(D_1^{'})$ is a regular neighborhood of the geometric realization of its nerve, which is isotopic to the connected sum $K\#(-K)\#K\#(-K)$.

(3) r^{th}-step: We reflect with respect to ∂B_r, *i.e.*, we apply $I_r \in \Gamma$. Notice that $D_r = D_{r-1} \cup I_r(D_{r-1})$ is a fundamental domain of an index two subgroup Γ_r in Γ_{r-1}.

   The set $D_r^{'} = I_1(\tilde{T}) \cup I_2(D_1^{'}) \cup I_3(D_2^{'}) \cup \ldots \cup I_r(D_{r-1}^{'})$ is a regular neighborhood of the geometric realization of its nerve, which is isotopic to the connected sum of 2^{r-1} copies of K and 2^{r-1} copies of $-K$.

At the end of the r^{th}-step, we obtain a regular neighborhood $\tilde{T}_1$ of a new tame knot K_1, which is isotopic to the connected sum of 2^{r-1} copies of K and 2^{r-1} copies of $-K$. Notice that $\tilde{T}_1 \subset \mathrm{Int}(\tilde{T})$.

Stage II.

Repeat k-times Stage I. At this stage we obtain a regular neighborhood $\tilde{T}_k$ of a new tame knot K_k which is isotopic to the connected sum of 2^{kr-1} copies of K and 2^{kr-1} copies of $-K$. By construction, $\tilde{T}_k \subset \mathrm{Int}(\tilde{T}_{k-1})$.

Let $x \in \bigcap_{l=1}^{\infty} \tilde{T}_l$. We shall prove that x is a limit point. Indeed, there exists a sequence of closed balls $\{B_m\}$ with $B_m \subset \tilde{T}_m$ such that $x \in B_m$ for each m. We can find a $z \in \mathbb{S}^{n+2} - \tilde{T}$ and a sequence $\{w_m\}$ of distinct elements of Γ, such that $w_m(z) \in B_m$. Since $diam(B_m) \to 0$ it follows that $w_m(z)$ converges to x. Hence $\bigcap_{l=1}^{\infty} \tilde{T}_l \subset \Lambda(\Gamma)$. The other inclusion $\Lambda(\Gamma) \subset \bigcap_{l=1}^{\infty} \tilde{T}_l$ clearly holds. Therefore, the limit set is given by

$$\Lambda(\Gamma) = \varprojlim_{l} \tilde{T}_l = \bigcap_{l=1}^{\infty} \tilde{T}_l.$$

Remark 4.7. This description is similar to the one which appears in the work of Peter Scott on [31]: Suppose that G is a right-angled reflection group with the fundamental domain P in $\mathbb{H}^n$. Given a facet F of P, let P_F denote the union of P and its reflection in F. Then P_F is a fundamental domain of an index two subgroup in G. Now, define inductively index two subgroups $G = G_0 > G_1 > G_2 > \ldots$, where G_i has the fundamental domain $P_i = PF_{-1}$. Then $\bigcap_i G_i = \{1\}$. In particular, if we apply this to our construction, the union of the fundamental domains is the entire discontinuity set.

Theorem 4.8. *Let T be a generalized pearl necklace of a non-trivial tame knot K of dimension n consisting of closed round balls in $\mathcal{B}$. Let $\tilde{T}$ be the corresponding generalized increased pearl necklace. Let Γ be the group generated by reflections on the boundary of each ball of T. Let $\Lambda(\Gamma)$ be the corresponding limit set. Then $\Lambda(\Gamma)$ is homeomorphic to $\mathbb{S}^n$.*

Proof. The proof makes use of an infinite process similar to the one used to prove that the Whitehead manifold cross $\mathbb{R}$ is homeomorphic to $\mathbb{R}^4$ ([24] and [26]).

Each regular neighborhood $\tilde{T}_i$ of the n-dimensional knot K_i is homeomorphic to $K_i \times \mathbb{D}^2$ and satisfies $\tilde{T}_i \subset \tilde{T}_{i-1}$, for each i. Note that K_i is knotted in $\tilde{T}_{i-1}$, in the sense that $K_{i-1} \times 0$ is not isotopic in $\tilde{T}_{i-1}$ to K_i.

Consider $\tilde{T}_{i-1} \times I$, $I = [0,1]$. Since K_i is of codimension 3 in $\tilde{T}_{i-1} \times I$, then it is not *topologically* knotted in $\tilde{T}_{i-1} \times I$ ([3] and [32]), *i.e.*, it is isotopic, in $\tilde{T}_{i-1} \times I$, to $K_{i-1} \times 0$.

The fact that K_i is not topologically knotted in $\tilde{T}_{i-1} \times I$ implies that $\tilde{T}_i \times I$ can be deformed isotopically inside $\tilde{T}_{i-1} \times I$ onto a small tubular neighborhood of K_{i-1}.

The above facts imply that the pair $\tilde{T}_{m+1} \times \frac{1}{2^{m+1}} I \subset \tilde{T}_m \times \frac{1}{2^m} I$ is topologically equivalent to $\mathbb{S}^n \times \frac{1}{2^{m+1}} \mathbb{D}^3 \subset \mathbb{S}^n \times \frac{1}{2^m} \mathbb{D}^3$, *i.e.*, we have the following commutative diagram:

$$\begin{array}{ccccccc}
\tilde{T}_1 \times I & \longleftarrow & \tilde{T}_2 \times \frac{1}{2} I & \longleftarrow & \cdots \tilde{T}_m \times \frac{1}{2^m} I & \longleftarrow & \cdots \Lambda \\
\sim\downarrow & & \sim\downarrow & & \sim\downarrow & & \downarrow \\
\mathbb{S}^n \times \mathbb{D}^3 & \longleftarrow & \mathbb{S}^n \times \frac{1}{2} \mathbb{D}^3 & \longleftarrow & \cdots \mathbb{S}^n \times \frac{1}{2^m} \mathbb{D}^3 & \longleftarrow & \cdots \mathbb{S}^n
\end{array}$$

Therefore, by the universal property of the inverse limits, there exists a homeomorphism from Λ to $\mathbb{S}^n$ associated to this sequence of maps. $\square$

5. Dynamically-defined fibered wild knots

In 1925 Emil Artin described two methods for constructing knotted spheres of dimension n in $\mathbb{S}^{n+2}$ from knots in $\mathbb{S}^{n+1}$. One of these methods is called *spinning* and uses the rotation process. A way to visualize it is the following. We can consider $\mathbb{S}^2$ as an $\mathbb{S}^1$-family of half-equators (meridians=$\mathbb{D}^1$) such that the respective points of their boundaries are identified to obtain the poles. Then the formula $Spin(\mathbb{D}^1) = \mathbb{S}^2$ means to send homeomorphically the unit interval $\mathbb{D}^1$ to a meridian of $\mathbb{S}^2$ such that $\partial\mathbb{S}^1 = \{0, 1\}$ is mapped to the poles and, multiply the interior of $\mathbb{D}^1$ by $\mathbb{S}^1$. In other words, one spins the meridian with respect to the poles to obtain $\mathbb{S}^2$. Similarly, consider $\mathbb{S}^{n+1}$ as an $\mathbb{S}^1$-family of half-equators ($\mathbb{D}^n$) where boundaries are respectively identified, hence $Spin(\mathbb{D}^n) = \mathbb{S}^{n+1}$ means to send homeomorphically $\mathbb{D}^n$ to a meridian of $\mathbb{S}^{n+1}$ and keeping $\partial\mathbb{D}^n$ fixed, multiply the interior of $\mathbb{D}^n$ by $\mathbb{S}^1$. In particular, if we start with a fibered tame knot K, then $Spin(K)$ also fibers (see [37]).

We recall that a knot K in $\mathbb{S}^{n+2}$ is *fibered* if there exists a locally trivial fibration $f : (\mathbb{S}^{n+2} - K) \to \mathbb{S}^1$. We require further that f be well-behaved near K; that is, that it has a neighborhood framed as $\mathbb{S}^n \times \mathbb{D}^2$, with $K \cong \mathbb{S}^n \times \{0\}$, in such a way that the restriction of f to $\mathbb{S}^n \times (\mathbb{D}^2 - \{0\})$ is the map into $\mathbb{S}^1$ given by $(x, y) \to \frac{y}{|y|}$. It follows that each $f^{-1}(x) \cup K$, $x \in \mathbb{S}^1$, is an $(n+1)$-manifold with boundary K: in fact a Seifert (hyper-) surface for K (see [29], page 323).

Examples 5.1.
1. The right-handed trefoil knot and the figure-eight knot are fibered knots with fiber the punctured torus.
2. The unknot $\mathbb{S}^n \hookrightarrow \mathbb{S}^{n+2}$ is fibered by the projection map $(\mathbb{S}^n * \mathbb{S}^1) - \mathbb{S}^n \to \mathbb{S}^1$. Fibers are $(n+1)$-disks.
3. Let $K \subset \mathbb{S}^3$ be a fibered tame knot. Then $Spin(K)$ also fibers. Again, using the Spin process, we can get fibered knots in any dimension. Thus, there exist non-trivial fibered knots in any dimension $n \geq 3$ (compare [3]).

Next, we will apply our construction to a non-trivial fibered tame knot K of dimension n, with fiber S. Let $T = \bigcup_{i=1}^{r} B_i$ be a generalized pearl necklace subordinate to K and $\tilde{T}$ be the corresponding generalized increased pearl necklace. Let Γ be the group generated by reflections I_j on the boundary of balls $B_j \in T$, $j = 1, \ldots, r$. Observe that the fundamental domain for the group Γ is $D = \mathbb{S}^{n+2} - \mathrm{Int}(T)$ (see definition 2.4) and in this case D is homeomorphic to $\Omega(\Gamma)/\Gamma$ (see section 2).

Lemma 5.2. *Let $T = \bigcup_{i=1}^{r} B_i$ be a generalized pearl necklace subordinate to a non-trivial fibered tame knot K of dimension n, with fiber S. Then $\Omega(\Gamma)/\Gamma$ fibers over the circle with fiber S^*, diffeomorphic to the closure of S in $\mathbb{S}^{n+2}$.*

Proof. Let $\widetilde{P} : (\mathbb{S}^{n+2} - K) \to \mathbb{S}^1$ be the given fibration with fiber the manifold S. Observe that $\widetilde{P}\,|_{\mathbb{S}^{n+2}-\mathrm{Int}(T)} \stackrel{\mathrm{def}}{=} P$ is a fibration and, after modifying $\widetilde{P}$ by isotopy if necessary, we can consider that the fiber S cuts the boundary of each ball $B_i \in T$ transversely, in n-disks (see Figure 9).

Hence the space D fibers over the circle with fiber the $(n+1)$-manifold S^*, which is the closure of the manifold S in $\mathbb{S}^{n+2}$ (see section 2). Since $\Omega(\Gamma)/\Gamma \cong D$, the result follows. $\square$

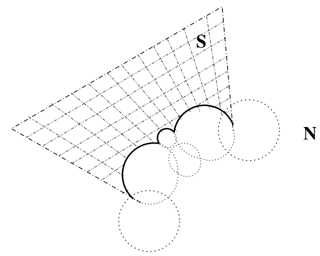

FIGURE 9. A schematic picture of the fiber intersecting each ball in an n-disk.

By the above lemma, in order to describe completely the orbit space $(\mathbb{S}^{n+2} - \Lambda(\Gamma))/\Gamma$ in the case that the original knot is fibered, we only need to determine its monodromy, which is precisely the monodromy of the knot.

Consider the orientation-preserving index two subgroup $\widetilde{\Gamma} \subset \Gamma$. The fundamental domain for $\widetilde{\Gamma}$ is

$$\widetilde{D} = D \cup I_j(D) = (\mathbb{S}^{n+2} - \mathrm{Int}(T)) \cup (B_j - \mathrm{Int}(I_j(T))),$$

for some $I_j \in \Gamma$ and $B_j \in T$. Observe that $I_j(D) \cap D \cong \mathbb{S}^{n-1} \times \mathbb{D}^2$.

Lemma 5.3. *Let $T = \bigcup_{i=1}^r B_i$ be a generalized pearl necklace subordinate to a non-trivial fibered tame knot K of dimension n, with fiber S. Then $\Omega(\widetilde{\Gamma})/\widetilde{\Gamma}$ fibers over the circle with fiber an $(n+1)$-manifold S^{**}, which is homeomorphic to the $(n+1)$-manifold S^* joined along an n-disk to a copy of itself in $\mathbb{S}^{n+2}$ modulo $\widetilde{\Gamma}$.*

Proof. First, we will prove that the fundamental domain $\widetilde{D}$ for $\widetilde{\Gamma}$ fibers over the circle.

Let $\widetilde{P} : (\mathbb{S}^{n+2} - K) \to \mathbb{S}^1$ be the given fibration with fiber the manifold S and let $P : (\mathbb{S}^{n+2} - T) \to \mathbb{S}^1$ be its corresponding restriction. Observe that the canonical tangle K_T associated to K also fibers over the circle with fiber S, hence $B_j - I_j(K)$ fibers over the circle with fiber S_j which is homeomorphic to S. By the same argument of the above lemma, $I_j(D) = B_j - I_j(T)$ fibers over the circle with fiber S_j^*, which is homeomorphic to S^*, the closure of S, via the fibration P_j.

Given $\theta \in \mathbb{S}^1$, let $P^{-1}(\theta) = S_\theta^*$ and $P_j^{-1}(\theta) = S_{j\theta}^*$ be the corresponding fibers. Notice that $I_j(S_\theta^*) = S_{j\theta}^*$, hence $S_\theta^* \cap \partial B_j = S_{j\theta}^* \cap \partial B_j$. Therefore $\widetilde{D}$ fibers over the circle with fiber an $(n+1)$-manifold $\hat{S}_\theta$, which is homeomorphic to the $(n+1)$-manifold S^* joined along an n-disk to a copy of itself in $\mathbb{S}^{n+2}$.

Since $\Omega(\widetilde{\Gamma})/\widetilde{\Gamma}$ is homeomorphic to $\widetilde{D}/\widetilde{\Gamma}$ (see section 2), and $\widetilde{D}/\widetilde{\Gamma}$ fibers over the circle with fiber $S^{**} = \hat{S}/\sim$, where $\partial B_K \sim I_j(\partial B_k)$, $k \neq j$, via $I_k I_j^{-1} \in \widetilde{\Gamma}$. The result follows. □

Since $\widetilde{\Gamma}$ is a normal subgroup of Γ, it follows by Lemma 8.1.3 in [33] that $\widetilde{\Gamma}$ has the same limit set than Γ. Therefore $\mathbb{S}^{n+2} - \Lambda(\Gamma) = \mathbb{S}^{n+2} - \Lambda(\widetilde{\Gamma})$.

Theorem 5.4. *Let $T = \bigcup_{i=1}^r B_i$ be a generalized pearl necklace subordinate to a non-trivial fibered tame knot K of dimension n, with fiber S . Let Γ be the group* *generated by reflections on each ball of T and let $\widetilde{\Gamma}$ be the orientation-preserving index two subgroup of Γ. Let $\Lambda(\Gamma) = \Lambda(\widetilde{\Gamma})$ be the corresponding limit set. Then:*

(1) *There exists a locally trivial fibration $\psi : (\mathbb{S}^{n+2} - \Lambda(\Gamma)) \to \mathbb{S}^1$, where the fiber $\Sigma_\theta = \psi^{-1}(\theta)$ is an orientable $(n+1)$-manifold with one end, which is homeomorphic to the connected sum along n-disks of an infinite number of copies of S.*

(2) *$\overline{\Sigma_\theta} - \Sigma_\theta = \Lambda(\Gamma)$, where $\overline{\Sigma_\theta}$ is the closure of Σ_θ in $\mathbb{S}^{n+2}$.*

Proof. We will first prove that $\mathbb{S}^{n+2} - \Lambda(\Gamma)$ fibers over the circle. We know that $\zeta : \Omega(\widetilde{\Gamma}) \to \Omega(\widetilde{\Gamma})/\widetilde{\Gamma}$ is an infinite-fold covering since $\widetilde{\Gamma}$ acts freely on $\Omega(\widetilde{\Gamma})$. By the previous lemma, there exists a locally trivial fibration $\phi : \Omega(\widetilde{\Gamma})/\widetilde{\Gamma} \to \mathbb{S}^1$ with fiber S^{**}. Then $\psi = \phi \circ \zeta : \Omega(\widetilde{\Gamma}) \to \mathbb{S}^1$ is a locally trivial fibration. The fiber is $\Gamma(S^*)$, *i.e.*, the orbit of the fiber.

We now describe $\Sigma_\theta = \Gamma(S^*)$ in detail. Let $\widetilde{P} : (\mathbb{S}^{n+2} - K) \to \mathbb{S}^1$ be the given fibration. The fibration $\widetilde{P}\ |_{\mathbb{S}^{n+2}-\mathrm{Int}(T)} \stackrel{\mathrm{def}}{=} P$ has been chosen as in the lemmas above. The fiber $\widetilde{P}^{-1}(\theta) = P^{-1}(\theta)$ is a Seifert surface S^* of K, for each $\theta \in \mathbb{S}^1$. We suppose S^* is oriented. Recall that the boundary of S^* cuts each ball B_j in an n-disk a_j.

The reflection I_j maps both a copy of $T - B_j$ (called T^j) and a copy of S^* (called S_1^{*j}) into the ball B_j, for $j = 1, 2, \ldots, r$. Observe that both T^j and S_1^{*j} have opposite orientation and that S^* and S_1^{*j} are joined by the n-disk a_j (see Figure 10) which, in both manifolds, has the same orientation.

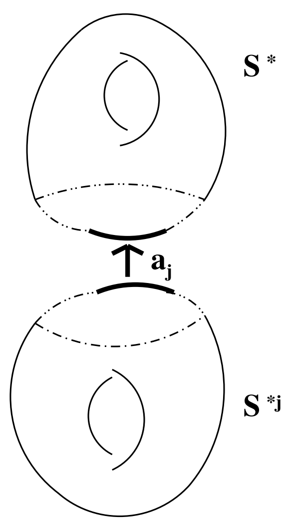

FIGURE 10. A schematic picture of the sum of two Seifert surfaces S^* and S^{*j} along the n-disk a_j.

Recall the description of the limit set Λ of Γ in section 4. At the end of the first stage, we have a new regular neighborhood $\tilde{T}_1$ of the knot K_1. By the same argument as above, its complement fibers over the circle with fiber the Seifert surface S_1^*, which is in turn homeomorphic to the sum of N_1 copies of S^* along the respective n-disks.

Continuing this process, at the end we obtain that $\Lambda(\Gamma)$ fibers over the circle with fiber Σ_θ which is homeomorphic to the connected sum along n-disks of an infinite number of copies of S. Notice that the diameter of the n-disks tends to zero.

Next, we will describe its set of ends. Consider the Fuchsian model (see [22]). In this case, we are considering the trivial knot. Then its limit set is the unknotted sphere $\mathbb{S}^n$ and its complement fibers over $\mathbb{S}^1$ with fiber the disk $\mathbb{D}^{n+1}$.

In each step we are adding copies of S to this disk in such a way that they accumulate on the boundary. If we intersect this disk with any compact set, we have just one connected component. Hence it has only one end. Therefore, our Seifert surface has one end (see Figure 11).

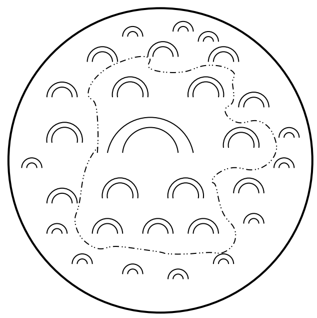

FIGURE 11. A schematic picture of a disk with copies of S intersected with a compact set.

The first part of the theorem has been proved. For the second part, observe that the closure of the fiber in $\mathbb{S}^{n+2}$ is the fiber union its boundary. Therefore $\overline{\Sigma_\theta} - \Sigma_\theta = \Lambda(\Gamma)$. □

Remark 5.5. This theorem gives an open book decomposition ([35], [29], pages 340–341) of $\mathbb{S}^{n+2} - \Lambda(\Gamma)$, where the "binding" is the knot $\Lambda(\Gamma)$, and each "page" is an orientable $(n+1)$-manifold with one end (the fiber).

Indeed, this decomposition can be thought of in the following way. By the above theorem, $\mathbb{S}^{n+2} - \Lambda(\Gamma)$ is $\Sigma_\theta \times [0,1]$ modulo the identification of the top with the bottom through a characteristic homeomorphism. Consider $\overline{\Sigma_\theta} \times [0,1]$ and identify the top with the bottom. This is equivalent to keeping $\partial\overline{\Sigma_\theta}$ fixed and spinning $\Sigma_\theta \times \{0\}$ with respect to $\partial\overline{\Sigma_\theta}$ until it is glued with $\Sigma_\theta \times \{1\}$. Removing $\partial\overline{\Sigma_\theta}$ we obtain the open book decomposition.

6. Monodromy

Let K be a non-trivial tame fibered n-knot and let S be the fiber. Since $\mathbb{S}^{n+2} - K$ fibers over the circle, we know that $\mathbb{S}^{n+2} - K$ is a mapping torus equal to $S \times [0,1]$ modulo a characteristic homeomorphism $\psi : S \to S$ that glues $S \times \{0\}$ to $S \times \{1\}$. This homeomorphism induces a homomorphism

$$\psi_\# : \Pi_1(S) \to \Pi_1(S)$$

called *the monodromy of the fibration*.

Another way to understand the monodromy is through *Poincaré's first return map* of a flow, defined as follows. Let M be a connected, compact manifold and let f_t be a flow that possesses a transverse section η. It follows that if $x \in \eta$, then there exists a continuous function $t(x) > 0$ such that $f_t \in \eta$. We may define Poincaré's first return map $F : \eta \to \eta$ as $F(x) = f_{t(x)}(x)$. This map is a diffeomorphism and induces a homomorphism of Π_1 called *the monodromy* (see [34], chapter 5).

For the manifold $\mathbb{S}^{n+2} - K$, the flow that defines Poincaré's first return map Φ is the flow that cuts transversely each page of its open book decomposition.

Consider a generalized pearl necklace T subordinate to K. As we have observed during the reflecting process, K and S are copied in each reflection. So the flow Φ is also copied. Hence, Poincaré's map can be extended at each stage, giving us in the end a homeomorphism $\psi : \Sigma_\theta \to \Sigma_\theta$ that identifies $\Sigma_\theta \times \{0\}$ with $\Sigma_\theta \times \{1\}$. This homeomorphism induces the monodromy of the limit n-knot.

By the long exact sequence associated to a fibration, we have

$$0 \to \Pi_1(\Sigma_\theta) \to \Pi_1(\mathbb{S}^{n+2} - \Lambda(\Gamma)) \overset{\Psi}{\underset{\to}{\leftarrow}} \mathbb{Z} \to 0, \tag{1}$$

which has a homomorphism section $\Psi : \mathbb{Z} \to \Pi_1(\mathbb{S}^{n+2} - \Lambda(\Gamma))$. Therefore (1) splits. As a consequence $\Pi_1(\mathbb{S}^{n+2} - \Lambda(\Gamma))$ is the semi-direct product of $\mathbb{Z}$ with $\Pi_1(\Sigma_\theta)$. This gives a method for computing the fundamental group of a limit n-knot whose complement fibers over the circle.

Notice that there is only one homomorphism from $\Pi_1(\mathbb{S}^{n+2} - \Lambda(\Gamma))$ onto $\mathbb{Z}$, since by Alexander duality $H^1(\mathbb{S}^{n+2} - \Lambda(\Gamma)) \cong \mathbb{Z}$. Therefore, the monodromy of the limit knot $\Lambda(\Gamma)$ is completely determined by the monodromy of the knot K.

Remark 6.1. The monodromy provides us with a way to distinguish between two limit fibered knots Λ_1 and Λ_2.

Theorem 6.2. *Let T be a generalized pearl necklace of a non-trivial tame fibered n-knot $K = Spin^{n-1}(K')$, where $K' \subset \mathbb{S}^3$ is a non-trivial tame fibered knot of dimension one and $n = 1, \ldots, 5$. Let Γ be the group generated by reflections on each ball of T. Let $\Lambda(\Gamma)$ be the corresponding limit set. Then $\Lambda(\Gamma)$ is wildly embedded in $\mathbb{S}^{n+2}$.*

Proof. Let K' be a non-trivial tame fibered 1-knot. Then the fiber S is a Seifert surface of genus g whose boundary is K'. The fundamental group of S is the free group in $2g$ generators $\{a_i, b_i : i = 1, \ldots, g\}$, hence the fundamental group of the fiber Σ_θ of the limit n-knot is the free group with generators $\{a_i^j, b_i^j : i = 1, \ldots, g;\ j \in \mathbb{N}\}$. Since $\Pi_1(\mathbb{S}^{n+2} - K) \cong \Pi_1(\mathbb{S}^3 - K')$ (see [29]), it follows that the monodromy maps, in both cases, coincide. Let $\psi_\#$ be the monodromy of K. Then the monodromy of the limit n-knot $\widetilde{\psi_\#} : \Pi_1(\Sigma_\theta) \to \Pi_1(\Sigma_\theta)$ sends $a_i^j \mapsto (\psi_\#(a_i))^j$ and $b_i^j \mapsto (\psi_\#(b_i))^j$.

Thus,

$$\begin{aligned} \Pi_1(\mathbb{S}^{n+2} - \Lambda(\Gamma)) &\cong \Pi_1(\mathbb{S}^1) \ltimes_{\widetilde{\psi_\#}} \Pi_1(\Sigma_\theta) \\ &= \{a_i^j, b_i^j, c : a_i^j * c = (\psi_\#(a_i))^j,\ \ b_i^j * c = (\psi_\#(b_i))^j\}. \end{aligned}$$

Therefore, the fundamental group $\Pi_1(\mathbb{S}^{n+2} - \Lambda(\Gamma))$ is infinitely generated. This implies that $\Lambda(\Gamma)$ is wildly embedded. □

Corollary 6.3. *There exist infinitely many inequivalent wild n-knots in $\mathbb{R}^{n+2}$.*

Corollary 6.4. *Let T be a generalized pearl-necklace whose template is a non-trivial tame fibered n-knot K. Then $\Pi_1(\Omega(\Gamma)/\Gamma) \cong \mathbb{Z} \ltimes_{\psi_\#} \Pi_1(\Sigma_\theta)$.*

As a consequence of Theorem 4.8, Theorem 6.2 and Corollary 6.3, we have:

Theorem 4.1. *There exist infinitely many non-equivalent knots $\psi : \mathbb{S}^n \to \mathbb{S}^{n+2}$ wildly embedded as limit sets of geometrically finite Kleinian groups, for $n = 1, 2, 3, 4, 5$.*

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