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CONFORMAL METRICS ON THE UNIT BALL: THE GEHRING-HAYMAN PROPERTY AND THE VOLUME GROWTH

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Abstract. We continue the study of conformal metrics on the unit ball in Euclidean space. We assume that the density ρ associated with the metric satisfies a Harnack inequality and then consider how much we can relax the volume growth condition from that in [Proc. London Math. Soc. Vol. 77 (3) (1998), 635–664] so that the Gehring-Hayman property still holds along the radii, i.e., if a boundary point can be accessed via a path with ρ-length $M < \infty$, then the ρ-length of the corresponding radius is bounded by CM. It turns out that if the path is inside a Stolz cone, then this result holds irrespective of the volume growth condition. Moreover, even if the path is not inside a Stolz cone, we are able to relax the volume growth condition for large r, and still conclude that the corresponding radius is ρ-rectifiable. This observation leads to a new estimate on the size of the boundary set corresponding to the ρ-unrectifiable radii.

1. Introduction

Given a continuous density $\rho : \mathbb{B}^n \to \mathbb{R}_+$, we define a conformal metric d_ρ by setting

$$\text{length}_\rho(\gamma) = \int_\gamma \rho(z)|dz|$$

for a curve γ in $\mathbb{B}^n$, and

$$d_\rho(x, y) = \inf_\gamma \text{ length}_\rho(\gamma) \quad \text{for } x, y \in \mathbb{B}^n,$$

where the infimum is taken over all curves joining x and y in $\mathbb{B}^n$. We also define a measure μ_ρ by setting

$$\mu_\rho(E) = \int_E \rho^n dm_n \quad \text{for a Borel set } E \subset \mathbb{B}^n,$$

where m_n denotes the n-dimensional Lebesgue measure.

We assume that the density ρ satisfies a Harnack inequality, i.e., there exists a constant $A \geq 1$ so that

$$\frac{1}{A} \leq \frac{\rho(x)}{\rho(y)} \leq A$$

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whenever $x, y \in B(z, \frac{1}{2}(1-|z|))$ for some $z \in \mathbb{B}^n$. We also assume a growth condition on the *isodiametric profile* of $(\mathbb{B}^n, d_\rho)$ which we, following [2], define as a function $\eta_\rho : [0, \operatorname{diam}_\rho(\mathbb{B}^n)] \to [0, \infty]$,

$$\eta_\rho(r) = \sup\{\mu_\rho(D) \ : \ D \subset \mathbb{B}^n \ \text{ and } \ \operatorname{diam}_\rho(D) \le r\}.$$

Notice that the condition $\eta_\rho(r) \le Br^n$ for all $r > 0$ is equivalent to assuming the so-called *volume growth condition* [1],

$$\mu_\rho(B_\rho(x, r)) \le Br^n \quad \text{ for all } x \in \mathbb{B}^n \text{ and } r > 0. \tag{1.1}$$

The motivation for conformal metrics arises primarily from the theory of quasiconformal mappings. Recall that the average derivative

$$a_f(x) = \Big(\frac{1}{m_n(B_x)} \int_{B_x} J_f \ dm_n\Big)^{1/n}, \quad B_x = B(x, \frac{1}{2}(1-|x|)),$$

of a quasiconformal mapping $f : \mathbb{B}^n \to \Omega$ is a primary example of a density satisfying the above conditions. However, not all conformal densities arise from a quasiconformal mapping; see [1] for more information and examples.

The main question in this paper is whether it is possible to relax the condition (1.1) so that the *Gehring-Hayman property* still holds along radii, i.e., if it is possible to join $\xi \in \partial\mathbb{B}^n$ and 0 by a path $\tilde{\gamma}$ with $\operatorname{length}_\rho(\tilde{\gamma}) = M < \infty$, then $\operatorname{length}_\rho([0, \xi)) \le CM$ where $C \ge 1$ is a finite constant. In [1, Theorem 3.1] it is namely shown that assuming a Harnack inequality and (1.1) guarantees that the geodesic arc in $\mathbb{B}^n$ essentially is the shortest path with respect to ρ-distance between any x and y in $\overline{\mathbb{B}}^n$. Nevertheless, the authors of [1] do not comment in any way whether (1.1) is the best possible upper bound for the volume growth or not.

It appears that the answer to our question depends fundamentally on whether we allow $\tilde{\gamma}$ to be an arbitrary path in the unit ball or do we restrict it, for example, in a cone-shape neighborhood of ξ. Indeed, for a *Stolz cone* at $\xi \in \partial\mathbb{B}^n$,

$$\operatorname{Cone}(\xi, \lambda, h) = \bigcup\{B(t\xi, \lambda(1-t)) : 1-h \le t < 1\},$$

where $\lambda \in (0, 1)$ and $h \in (0, 1]$, we have the following theorem.

Theorem 1.1. *Let $\tilde{\gamma} \subset Cone(\xi, \lambda, 1)$ be a curve joining 0 and ξ so that $length_\rho(\tilde{\gamma}) = M < \infty$. Then $length_\rho([0, \xi)) \le CM$ where $C = C(A, \lambda, n) \ge 1$, i.e., C is a finite constant depending only on A, λ and n and nothing else.*

In other words, Theorem 1.1 holds independently of the volume growth. On the other hand, for arbitrary paths joining ξ and 0 in the unit ball, the situation is quite different. The only known relevant method, which has been used, e.g., in [1], is based on estimating the modulus of the families of paths and, as evaluation of examples like $\eta_\rho(r) = Br^n(\log(1 + 1/r))^p$, $p > 0$, reveals, it does not allow us to relax exceedingly the volume growth condition for small r. However, if we replace (1.1) with the condition

$$\eta_\rho(r) \le Br^n\beta(r) \quad \text{ for all } \ r > 0 \tag{1.2}$$

where β is a positive increasing function on $(0, \infty)$ with $\lim_{r\to 0} \beta(r) = 1$, we have the following theorem.

Theorem 1.2. *Let $\tilde{\gamma} \subset \mathbb{B}^n$ be a curve joining 0 and ξ so that $length_\rho(\tilde{\gamma}) = M < \infty$. If the density ρ satisfies (1.2), then $length_\rho([0, \xi)) \le CM$ where $C = C(A, B, n, \xi)$ ≥ 1.*

Since the constant C in Theorem 1.2 depends also on ξ, we cannot conclude the Gehring-Hayman theorem under volume growth (1.2) in general; nevertheless, we have the following corollary.

Corollary 1.3. *Let $\tilde{\gamma} \subset \mathbb{B}^n$ be a curve joining 0 and ξ so that $\text{length}_\rho(\tilde{\gamma}) < \infty$. If the density ρ satisfies* (1.2)*, then also $\text{length}_\rho([0,\xi)) < \infty$.*

Our final theorem is an extension of the Radial Limit Theorem [1, Theorem 4.4]. This result also relates to [3, Remark 1.3] where the size of the boundary set $E \subset \partial\mathbb{B}^n$ where the conformal deformation mapping can "blow up" was estimated.

Theorem 1.4. *Let $E = \{\xi \in \partial\mathbb{B}^n : \text{ length}_\rho([0,\xi)) = \infty\}$. If ρ satisfies*

$$\eta_\rho(r) \le Br^n(\log(e+r))^p$$

for some $B > 0$ and $0 \le p < n-1$, then $\text{cap}_n(E) = 0$.

Here, cap_n denotes the conformal n-capacity. Recall that $\text{cap}_n(E) = 0$ for a Borel set E if and only if

$$\inf\Big\{\int_{\mathbb{R}^n} |\nabla u|^n dm_n : \ u \in C^\infty(\mathbb{R}^n),\ u|K \ge 1 \text{ and } u|(\mathbb{R}^n \setminus B(0,2) \equiv 0)\Big\} = 0$$

for all compact sets $K \subset E$.

As a final remark we note that our Theorem 1.1 is an analogue of the inequality (4.1) in [1] (whose proof can be found, e.g., in [4, Lemma 1.3]); the difference is that the inequality (4.1) gives an upper bound for the ρ-diameter of the Stolz cone at ξ in terms of the ρ-diameter of the arc $[(1-h)\xi, \xi)$. This inequality plays an important role, for example, in proving many results of [1] and [4].

2. Proofs of the results

Proof of Theorem 1.1. Fix any $\lambda \in (1/2, 1)$. It is easy to check that if the density ρ satisfies the Harnack inequality with the constant $A \ge 1$ for all balls $B(x, \frac{1}{2}(1-|x|))$, then it satisfies the inequality with another constant $A' = A'(A,\lambda,n) \ge A$ for all balls $B(x, \lambda(1-|x|))$, too.

Let $\xi \in \partial\mathbb{B}^n$ be so that there is a curve $\tilde{\gamma} \subset \text{Cone}(\xi,\lambda,1)$ with endpoints 0 and ξ so that $\text{length}_\rho(\tilde{\gamma}) = M < \infty$. Denote by γ the radius $[0,\xi)$.

For $\tilde{\gamma}_j$, a closed subcurve of $\tilde{\gamma}$ which connects the two boundary components of the annulus

$$A_j = \overline{B}(\xi, 2^{-j}) \setminus B(\xi, 2^{-j-1})$$

in $A_j \cap \text{Cone}(\xi,\lambda,1)$, and for $\gamma_j = \gamma \cap A_j$, it holds that

$$\text{length}(\tilde{\gamma}_j) \ge \text{length}(\gamma_j) = 2^{-j-1}.$$

Since every point of $A_j \cap \text{Cone}(\xi,\lambda,1)$ can be joined to γ_j with a finite number (depending only on n) of balls of type $B(x, \lambda(1-|x|))$, the Harnack inequality implies that there is a constant $C = C(A')$ so that

$$C\text{length}_\rho(\tilde{\gamma}_j) \ge \text{length}_\rho(\gamma_j).$$

The claim follows now from summing over j:

$$\text{length}_\rho(\gamma) = \sum_j \text{length}_\rho(\gamma_j) \le \sum_j C\text{length}_\rho(\tilde{\gamma}_j) \le C\text{length}_\rho(\tilde{\gamma}) = CM. \qquad \square$$

The proof of our main result, Theorem 1.2, is similar in spirit to the proof of the classical Gehring-Hayman theorem in [1]. Let us first recall that the modulus $\mathrm{mod}\Gamma \in [0,\infty]$ of a family Γ of curves in $\mathbb{B}^n$ is defined as

$$\mathrm{mod}\,\Gamma = \inf_{\tilde\rho} \int_{\mathbb{B}^n} \tilde\rho^n \, dm_n.$$

Here the infimum is taken over all Borel measurable densities $\tilde\rho : \mathbb{B}^n \to [0,\infty]$ which are admissible. A density $\tilde\rho$ is admissible, if $\mathrm{length}_{\tilde\rho}(\gamma) \geq 1$ for all $\gamma \in \Gamma$.

If E and F are subsets of $\mathbb{B}^n$, then we denote by $\mathrm{mod}\,(E,F;\mathbb{B}^n)$ the modulus of the family of all rectifiable curves in $\mathbb{B}^n$ connecting E and F. Let us recall a well-known lower bound for $\mathrm{mod}\,(E,F;\mathbb{B}^n)$: If E and F are disjoint continua in $\mathbb{B}^n$, then

$$\phi_n(t) \leq \mathrm{mod}(E,F;\mathbb{B}^n), \tag{2.1}$$

where

$$t = \frac{\mathrm{dist}(E,F)}{\min\{\mathrm{diam}(E),\mathrm{diam}(F)\}}$$

and

$$\phi_n(t) = \frac{\omega_{n-1}}{2[\log(\lambda_n(1+t))]^{n-1}}.$$

Here ω_{n-1} is the surface area of $\partial\mathbb{B}^n$ and $\lambda_n \geq 1$ is a constant that depends only on n.

Let us also recall the fact that $\mathrm{mod}(E,F;\mathbb{B}^n) = \infty$ whenever E and F are connected non-degenerate sets in $\overline{\mathbb{B}^n}$ whose closures have non-empty intersection. See, for example, [5] for the proof of these elementary properties of the modulus.

Our next lemma extends [1, Lemma 3.2].

Lemma 2.1. *Suppose that ρ satisfies (1.2) and let E be a non-empty subset of $\mathbb{B}^n$ and let $L \geq \delta > 0$. Assume that $\mathrm{diam}_\rho(E) \leq \delta$ and that Γ is a family of curves in $\mathbb{B}^n$ so that every curve $\gamma \in \Gamma$ has one endpoint in E and $\mathrm{length}_\rho(\gamma) \geq L$. Then*

$$\mathit{mod}\,\Gamma \leq \frac{C\beta(4L)}{[\log(1+L/\delta)]^{n-1}}$$

where $C = C(B,n) \geq 1$.

Proof. Define $\tilde\rho : \mathbb{B} \to [0,\infty)$ by

$$\tilde\rho(x) = \begin{cases} \frac{\rho(x)}{(\log(1+L/\delta))(\delta+\mathrm{dist}_\rho(x,E))} & \text{for } x \in \mathbb{B}^n,\ \mathrm{dist}_\rho(x,E) < L, \\ 0 & \text{elsewhere.} \end{cases}$$

This function is clearly Borel measurable and we claim that $\tilde\rho$ is an admissible density for the curve family Γ.

Let $\gamma \in \Gamma$. We may assume that $\gamma : I \to \mathbb{B}^n$ has an arc-length parametrization with $I = [0,\mathrm{length}(\gamma)]$ and $\gamma(0) \in E$. For $s \in I$, let $t(s)$ be the ρ-length of $\gamma|[0,s]$, i.e.,

$$t(s) = \int_0^s \rho(\gamma(u))\, du.$$

Obviously, $\mathrm{dist}_\rho(\gamma(s),E) \leq t(s)$ for all $s \in I$. Moreover, since $t(s)$ is a continuous increasing function of s and $t(\mathrm{length}(\gamma)) = \mathrm{length}_\rho(\gamma) \geq L$, there exists a smallest number $s_0 \in I$ with $t(s_0) = L$. It now follows from the definition of $\tilde{\rho}$ that

$$\begin{aligned}\text{length}_\rho(\gamma) &\geq \frac{1}{\log(1+L/\delta)} \int_0^{s_0} \frac{\rho(\gamma(s))}{\delta + \text{dist}_\rho(\gamma(s), E)}\, ds \\ &\geq \frac{1}{\log(1+L/\delta)} \int_0^{L} \frac{dt}{\delta + t} = 1.\end{aligned}$$

Thus $\tilde{\rho}$ is admissible.

Select a point $x_0 \in E$ and let k be the smallest positive integer with $\delta + L \leq 2^{k+1}\delta$. Since $\delta \leq L$, taking logarithms gives

$$1 \leq k \leq \frac{\log(1+L/\delta)}{\log 2}. \tag{2.2}$$

Since $\text{diam}_\rho(E) \leq \delta$, we have that $\{x \in \mathbb{B}^n : \text{dist}_\rho(x, E) < L\} \subset B_\rho(x_0, 2^{k+1}\delta)$.

Let $B_j = B_\rho(x_0, 2^j\delta)$ for $j \in \mathbb{N}$. Then $\tilde{\rho}$ vanishes outside B_{k+1} and so

$$\text{mod}\,\Gamma \leq \int_{\mathbb{B}^n} \tilde{\rho}^n\, dm_n = \int_{B_{k+1}} \tilde{\rho}^n\, dm_n = \int_{B_1} \tilde{\rho}^n\, dm_n + \sum_{j=1}^{k} \int_{B_{j+1}\setminus B_j} \tilde{\rho}^n\, dm_n.$$

Since $\text{dist}_\rho(x, E) \geq 2^j\delta - \delta$ for $x \in B_{j+1} \setminus B_j$ and $j \in \mathbb{N}$, we obtain

$$\begin{aligned}\text{mod}\,\Gamma &\leq \frac{1}{[\log(1+L/\delta)]^n}\Big(\int_{B_1} \frac{\rho^n}{\delta^n}\, dm_n + \sum_{j=1}^{k} \int_{B_{j+1}\setminus B_j} \frac{\rho^n}{2^{jn}\delta^n}\, dm_n\Big) \\ &\leq \frac{1}{[\log(1+L/\delta)]^n}\Big(\frac{\mu_\rho(B_1)}{\delta^n} + \sum_{j=1}^{k} \frac{1}{2^{jn}\delta^n}\mu_\rho(B_{j+1})\Big).\end{aligned}$$

The conditions (1.2) and (2.2) and the definition of k now imply that

$$\begin{aligned}\text{mod}\,\Gamma &\leq \frac{1}{[\log(1+L/\delta)]^n}\Big(2^n B\beta(2\delta) + \sum_{j=1}^{k} 2^n B\beta(2^{j+1}\delta)\Big) \\ &= \frac{2^n B(k+1)\beta(4L)}{[\log(1+L/\delta)]^n} \\ &\leq \frac{2^n B/\log 2}{[\log(1+L/\delta)]^{n-1}}\ \beta(4L).\end{aligned}$$

Lemma 2.1 is proven. □

Proof of Theorem 1.2. Let $\xi \in \partial\mathbb{B}^n$ be so that there is a curve $\tilde{\gamma} \subset \mathbb{B}^n$ with endpoints 0 and ξ so that $\text{length}_\rho(\tilde{\gamma}) = M < \infty$. Denote by γ the radius $[0, \xi)$.

Again, for each $j \in \mathbb{N}$, consider the annulus

$$A_j = \overline{B}(\xi, 2^{-j}) \setminus B(\xi, 2^{-j-1})$$

and let $\tilde{\gamma}_j$ be a closed subcurve of $\tilde{\gamma}$ which connects the two boundary components of A_j in $A_j \cap \mathbb{B}^n$ and $\gamma_j = \gamma \cap A_j$.

Observe first that

$$\text{mod}\,(\gamma_j, \tilde{\gamma}_j; \mathbb{B}^n) \geq c_1(n) > 0. \tag{2.3}$$

This observation follows simply from the estimate (2.1) and that $\text{dist}(\gamma_j, \tilde{\gamma}_j) \leq 2^{-j} \leq 2\min\{\text{diam}(\gamma_j), \text{diam}(\tilde{\gamma}_j)\}$.

On the other hand, if $\text{length}_\rho(\alpha) \geq c_2 \text{length}_\rho(\tilde{\gamma}_j)$ for all curves α that join γ_j and $\tilde{\gamma}_j$ in $\mathbb{B}^n$, then also $\text{length}_\rho(\alpha) \geq c_2 \text{diam}_\rho(\tilde{\gamma}_j)$, and Lemma 2.1 and the facts that $\text{length}_\rho(\tilde{\gamma})$ is finite and $\lim_{r\to 0}\beta(r) = 1$, and therefore $\beta(r) \le 10$ for small $r > 0$, hence imply that there is a finite j_0 (depending on ξ) so that

$$\text{mod}\,(\gamma_j, \tilde{\gamma}_j; \mathbb{B}^n) \le \frac{C'\beta(4c_2\text{length}_\rho(\tilde{\gamma}_j))}{[\log(1+c_2)]^{n-1}} \le \frac{10C'}{[\log(1+c_2)]^{n-1}} \tag{2.4}$$

for all $j \ge j_0$. Here C' is the constant in Lemma 2.1. But by the lower bound estimate (2.3) this is impossible if the constant c_2 is sufficiently large depending on B and n. Hence we deduce that for each $j \ge j_0$ there is a curve α_j connecting the sets γ_j and $\tilde{\gamma}_j$ so that

$$\text{length}_\rho(\alpha_j) \le c_2\text{length}_\rho(\tilde{\gamma}_j) \tag{2.5}$$

where $c_2 = c_2(B, n)$.

Let us next consider two cases according to whether the euclidean length of α_j is or is not substantially smaller than the euclidean length of γ_j.

Suppose first that $\text{length}(\alpha_j) \le \frac{1}{10}\text{length}(\gamma_j)$. Since α_j connects γ_j and $\tilde{\gamma}_j$, we have that $\text{dist}(\gamma_j, \tilde{\gamma}_j) \le \frac{1}{10}\text{length}(\gamma_j)$. However, we also have $\text{length}(\gamma_j) \le \text{length}(\tilde{\gamma}_j)$ which means that $\tilde{\gamma}_j$ must have a subcurve of euclidean length comparable to $\text{length}(\gamma_j)$ near the Whitney ball containing γ_j. It now follows from the Harnack inequality that $\text{length}_\rho(\gamma_j) \le c_3\text{length}_\rho(\tilde{\gamma}_j)$ for some c_3 depending only on A.

On the other hand, if $\text{length}(\alpha_j) > \frac{1}{10}\text{length}(\gamma_j)$, then it again follows from the Harnack inequality that $\text{length}_\rho(\gamma_j) \le c_4\text{length}_\rho(\alpha_j)$ for some c_4 depending only on A. By combining this with (2.5) we arrive at $\text{length}_\rho(\gamma_j) \le c_2c_4\text{length}_\rho(\tilde{\gamma}_j)$. We conclude that for all $j \ge j_0$ we have

$$\text{length}_\rho(\gamma_j) \le c_5\text{length}_\rho(\tilde{\gamma}_j)$$

with a constant $c_5 = c_2c_4$ depending on A, B and n.

Finally, we have to deal also with the subcurves γ_j and $\tilde{\gamma}_j$ for $j < j_0$. Since 0 is in the closure of both γ and $\tilde{\gamma}$, and $\text{length}(\gamma_j) \le \text{length}(\tilde{\gamma}_j)$ for all j, it follows from the Harnack inequality that there is a constant c_6 depending only on A, n, and j_0 so that

$$\text{length}_\rho\Big(\bigcup_{j<j_0}\gamma_j\Big) \le c_6\text{length}_\rho\Big(\bigcup_{j<j_0}\tilde{\gamma}_j\Big).$$

Since j_0 depends only on ξ, it follows that there is a constant $C = \max\{c_5, c_6\}$ depending only on A, B, n and ξ so that

$$\text{length}_\rho(\gamma) = \sum_j \text{length}_\rho(\gamma_j) \le \sum_j C\text{length}_\rho(\tilde{\gamma}_j) \le C\text{length}_\rho(\tilde{\gamma}).$$

Theorem 1.2 now follows. □

Proof of Theorem 1.4. First we have to show that E is a Borel set. For $k \in \mathbb{N}$, let $E_k = \{\xi \in \partial\mathbb{B}^n : \ \text{length}_\rho([0,\xi)) \le k\}$. Let (ξ_i) be a sequence in some E_k with $(\xi_i) \to \xi$. Then, for any $\lambda \in [0,1)$, we have that

$$\text{length}_\rho([0,\lambda\xi]) = \int_0^\lambda \rho(t\xi)\ dt = \lim_{i\to\infty}\int_0^\lambda \rho(t\xi_i)\ dt \le k.$$

This implies that

$$\text{length}_\rho([0,\xi)) = \lim_{i\to\infty}\text{length}_\rho([0,\lambda\xi)) \le k$$

and so $\xi \in E_k$. It follows that E_k is closed and therefore the set $E = \partial\mathbb{B}^n \setminus \left(\bigcup_{k\in\mathbb{N}} E_k\right)$ is a Borel set.

Let $F = \overline{B}(0, \frac{1}{2})$ and consider the family Γ of curves in $\mathbb{B}^n$ connecting E and F. Assume towards a contradiction that $\mathrm{cap}_n(E) > 0$. Then also $\mathrm{mod}\,\Gamma > 0$. On the other hand, Lemma 2.1 provides us an upper bound

$$\mathrm{mod}\,\Gamma \leq \frac{C(\log(e + 4L))^p}{[\log(1 + L/\delta)]^{n-1}}.$$

Now, if $\mathrm{length}_\rho(\alpha) = \infty$ for all $\alpha \in \Gamma$, then we may choose $\delta = 1$ and L arbitrarily large in Lemma 2.1, which implies $\mathrm{mod}\,\Gamma = 0$ which is a contradiction. Therefore there exists a curve $\alpha \in \Gamma$ with $\mathrm{length}_\rho(\alpha) < \infty$.

Let $\xi \in E$ and $x \in F$ be the endpoints of α. Since x can be connected to 0 by a curve of finite ρ-length, there exists a curve $\tilde{\gamma}$ in $\mathbb{B}^n$ with endpoints 0 and ξ and $\mathrm{length}_\rho(\tilde{\gamma}) < \infty$. It follows from Corollary 1.3 that $\mathrm{length}_\rho([0, \xi)) < \infty$ which is a contradiction with the definition of E. □

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