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THE HOROFUNCTION BOUNDARY OF THE HEISENBERG GROUP: THE CARNOT-CARATHÉODORY METRIC

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Abstract. We find the horofunction boundary of the $(2n+1)$-dimensional Heisenberg group with the Carnot-Carathéodory distance and show that it is homeomorphic to a $2n$-dimensional disk and that the Busemann points correspond to the $(2n-1)$-sphere boundary of this disk. We also show that the compactified Heisenberg group is homeomorphic to a $(2n+1)$-dimensional sphere. As an application, we find the group of isometries of the Carnot-Carathéodory distance.

1. Introduction

Gromov [Gro81, 1.2] defines a boundary for a metric space (X, ρ) by means of the following construction. Let $C(X)$ be the space of continuous real-valued functions on X with the topology of uniform convergence on compact sets. Consider the quotient space $C_*(X) = C(X)/(\text{constant functions})$ and the map $i\colon X \to C_*(X)$ given by $i(x) =$ the equivalence class of the map $y \mapsto \rho(x, y)$. If (X, ρ) is *proper*, that is, closed balls in X are compact, then the map i is an embedding and the closure of $i(X)$ in $C_*(X)$ is compact. The topological boundary of $i(X)$ in $C_*(X)$, which we denote by $\partial_{\mathrm{h}}(X, \rho)$, is called the *horofunction boundary* of (X, ρ) and its elements are called *horofunctions*. The closure of $i(X)$ in $C_*(X)$, which we denote by $\mathrm{hc}(X, \rho)$, is called the *horofunction compactification* of (X, ρ). It is also known as the *Busemann compactification* of (X, ρ). An alternative construction of $\mathrm{hc}(X, \rho)$ using C^*-algebras is given by Rieffel [Rie02, 4.1]; he calls it the *metric compactification of* (X, ρ).

Rieffel defines the notion of an *almost geodesic ray* ([Rie02, Definition 4.3] and repeated in Definition 6.1), which serves as a substitute for a geodesic ray in a general metric space. A point on the horofunction boundary of a metric space is called a *Busemann point* if it is the limit of an almost geodesic ray. There are rather few examples of metric spaces where the horofunction boundary or Busemann points are explicitly known. The horofunction boundary of a proper CAT(0) space coincides with the "visual" boundary defined using equivalence classes of geodesic rays [BH99, Theorem 8.13] and all points in the horofunction boundary are Busemann

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points. Non-Busemann points may exist for other metric spaces; for example, this can occur for the metrics associated to certain norms on $\mathbb{R}^n$ (see [Wal07, Theorem 1.2] for a precise characterization of the Busemann points in such spaces). Lipschitz equivalent metrics can give horofunction boundaries that are not homotopy equivalent, [Rie02, Examples 5.1 and 5.2].

Although a horofunction on a metric space (X, ρ) was defined above as an equivalence class of functions, it is often convenient to choose a basepoint $x_0 \in X$ and identify $C_*(X)$ with $C(X)_{x_0}$, the subspace of $C(X)$ consisting of those functions that vanish at x_0, via the map: equivalence class of $h \mapsto h - h(x_0)$. Under this identification, $h \in C(X)_{x_0}$ is a horofunction if and only if there is a sequence $\{p_m\}$ in X such that $p_m \to \infty$ and the sequence of functions $y \mapsto \rho(p_m, y) - \rho(p_m, x_0)$ converges to h, in which case for brevity we say $\{p_m\}$ *converges to* h.

The $(2n+1)$-dimensional *Heisenberg group*, denoted $\mathbb{H}^n$, is the simply connected nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ with multiplication given by

$$(w, s)(z, t) = (w + z, s + t + 2\,\mathrm{Im}\langle w, z\rangle)$$

where $w, z \in \mathbb{C}^n$, $s, t \in \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product on $\mathbb{C}^n$. The identity element for this multiplication is $0 = (0, 0)$ and $(w, s)^{-1} = (-w, -s)$. We take $0 \in \mathbb{H}^n$ as a basepoint when considering horofunctions for various metrics on $\mathbb{H}^n$. Write global coordinates on $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ as $(x_1 + iy_1, \ldots, x_n + iy_n, t)$. The vector fields on $\mathbb{H}^n$ given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad j = 1, \ldots, n,$$

define a $2n$-dimensional horizontal distribution. The *Carnot-Carathéodory sub-Riemannian structure* on $\mathbb{H}^n$ is the sub-Riemannian structure for which the vector fields X_j, Y_j, $j = 1, \ldots, n$ are orthonormal ([CCG05, Section 2], [CDPT07, Section 2.1.2] where a different, but isomorphic, multiplication for $\mathbb{H}^n$ is used). The *Carnot-Carathéodory distance* between two points $a, b \in \mathbb{H}^n$, which we denote by $d_{\mathrm{cc}}(a, b)$, is the infimum of the lengths of all smooth horizontal curves joining a to b. The goal of this paper is to classify the horofunctions of $(\mathbb{H}^n, d_{\mathrm{cc}})$, find the Busemann points, and find the topological types of $\partial_{\mathrm{h}}(\mathbb{H}^n, d_{\mathrm{cc}})$ and $\mathrm{hc}(\mathbb{H}^n, d_{\mathrm{cc}})$.

It will be useful to compare d_{cc} with another metric on $\mathbb{H}^n$. The *Korányi gauge* [Kor85, 1.4] on $\mathbb{H}^n$ is the real-valued function $\|\cdot\|_K : \mathbb{H}^n \to \mathbb{R}$ defined by $\|(z, t)\|_K = \left(|z|^4 + t^2\right)^{1/4}$ where $|z| = \langle z, z\rangle^{1/2}$. The function $d_K(a, b) = \left\|a^{-1}b\right\|_K$ is a metric on $\mathbb{H}^n$, called the *Korányi metric*. Korányi showed that the *length metric* associated to d_K is precisely d_{cc}, that is, $d_{\mathrm{cc}}(a, b)$ is the infimum of the lengths, as measured using d_K, of all rectifiable curves joining a and b ([Kor85, 2.1]).

We give the following characterization of sequences in $\mathbb{H}^n$ that converge to a horofunction for d_{cc}.

Theorem (Theorem 5.16)**.** *Let $\{p_m = (w_m, s_m)\}$ be a sequence in $\mathbb{H}^n$ such that $p_m \to \infty$. Then $\{p_m\}$ converges to a horofunction, h, for d_{cc} if and only if one of the following (mutually exclusive) conditions is satisfied:*

1. *$w_m \to \infty$ and $\gamma_m = \exp\left(i\,\mu^{-1}\left(s_m/|w_m|^2\right)\right)(w_m/|w_m|)$ converges in $\mathbb{C}^n$ in which case $h(z, b) = \mathrm{Re}\langle u, z\rangle$ where $u = -\lim_{m\to\infty} \gamma_m$.*
2. *The sequence $\{w_m\}$ converges in $\mathbb{C}^n$ in which case $h(z, b) = |w| - |w - z|$ where $w = \lim_{m\to\infty} w_m$.*

Here, $\mu^{-1}\colon \mathbb{R} \to (-\pi, \pi)$ is the inverse of Gaveau's function (2.1). An analogous characterization of the sequences in $\mathbb{H}^n$ that converge to horofunctions for d_K is given by Theorem 3.3.

As a consequence of Theorem 5.16, we have the following classification of horofunctions for d_{cc} (Theorem 5.17):

$$\partial_{\mathrm{h}}(\mathbb{H}^n, d_{\mathrm{cc}}) = \big\{(z,b) \mapsto \mathrm{Re}\langle u, z\rangle \mid u \in S^{2n-1}\big\} \cup \{(z,b) \mapsto |w| - |w-z| \mid w \in \mathbb{C}^n\}$$

where $S^{2n-1} = \{z \in \mathbb{C}^n \mid |z| = 1\}$, the unit sphere in $\mathbb{C}^n$.

We identify the topological types of $\partial_{\mathrm{h}}(\mathbb{H}^n, d_{\mathrm{cc}})$ and the set of Busemann points and $\mathrm{hc}(\mathbb{H}^n, d_{\mathrm{cc}})$ as follows:

Theorem (Theorems 8.2 and 8.5)**.** *The horofunction boundary of* $(\mathbb{H}^n, d_{\mathrm{cc}})$ *is homeomorphic to a* $2n$*-disk, with the Busemann points corresponding to the* $(2n-1)$*-sphere boundary of such a disk. The horofunction compactification of* $(\mathbb{H}^n, d_{\mathrm{cc}})$ *is homeomorphic to a* $(2n+1)$*-sphere.*

Let $U(n)$ denote the group of unitary transformations of $\mathbb{C}^n$. For $T \in U(n)$, let $\widehat{T}\colon \mathbb{H}^n \to \mathbb{H}^n$ be given by $\widehat{T}(w,s) = (Tw, s)$. Also, let $C\colon \mathbb{H}^n \to \mathbb{H}^n$ be given by $C(w,s) = (\bar{w}, -s)$. These maps are isometries of both d_K and d_{cc} as well as Lie group automorphisms of $\mathbb{H}^n$. Let $\mathrm{Isom}(\mathbb{H}^n, \rho, 0)$, where $\rho = d_{\mathrm{cc}}$ or d_K, denote the group of isometries of ρ fixing 0. We prove the following theorem using horofunctions.

Theorem (Theorem 7.4)**.** *A map* $\tau\colon \mathbb{H}^n \to \mathbb{H}^n$ *is an isometry of* d_{cc} *fixing* 0 *if and only if it is of the form* $\widehat{T}$ *or* $C\widehat{T}$ *where* $T\colon \mathbb{C}^n \to \mathbb{C}^n$ *is a unitary transformation. Furthermore,* $\mathrm{Isom}(\mathbb{H}^n, d_{\mathrm{cc}}, 0) = \mathrm{Isom}(\mathbb{H}^n, d_K, 0) \cong U(n) \rtimes \mathbb{Z}/2\mathbb{Z}$.

Here $U(n) \rtimes \mathbb{Z}/2\mathbb{Z}$ denotes the semidirect product of $U(n)$ with the cyclic group of order 2. Kishimoto has shown ([Kis03, Theorem 4.2]) that an isometry of a Carnot group fixing the identity is a Lie group automorphism from which one can readily calculate $\mathrm{Isom}(\mathbb{H}^n, d_{\mathrm{cc}}, 0)$. Our method using horofunctions shows directly that an isometry of $\mathbb{H}^n$ (for d_K or d_{cc}) fixing 0 is of the form $\widehat{T}$ or $C\widehat{T}$.

For any left invariant metric ρ on $\mathbb{H}^n$, the left translation action of $\mathbb{H}^n$ on itself extends to a continuous action on $\mathrm{hc}(\mathbb{H}^n, \rho)$. In a previous paper ([KN09, Theorem 4.1]), we showed that $\partial_{\mathrm{h}}(\mathbb{H}^n, d_K)$ is homeomorphic to a $2n$-disk and $\mathrm{hc}(\mathbb{H}^n, d_K)$ is homeomorphic to a $(2n+1)$-sphere. However, there is no $\mathbb{H}^n$-*equivariant* homeomorphism $\partial_{\mathrm{h}}(\mathbb{H}^n, d_K) \to \partial_{\mathrm{h}}(\mathbb{H}^n, d_{\mathrm{cc}})$ because the $\mathbb{H}^n$-action on $\partial_{\mathrm{h}}(\mathbb{H}^n, d_K)$ is trivial while the $\mathbb{H}^n$-action on $\partial_{\mathrm{h}}(\mathbb{H}^n, d_{\mathrm{cc}})$ is non-trivial (Proposition 8.6).

2. The Carnot-Carathéodory distance

We discuss the properties of the Carnot-Carathéodory distance, d_{cc}, following the exposition of [CCG05] and relate it to the Korányi metric, d_K.

The metric d_{cc} is left invariant with respect to the left translation action of $\mathbb{H}^n$ on itself, that is, $d_{\mathrm{cc}}(ga, gb) = d_{\mathrm{cc}}(a, b)$ for all $g, a, b \in \mathbb{H}^n$. The *Carnot-Carathéodory gauge* on $\mathbb{H}^n$ is the real-valued function $\|\cdot\|_{\mathrm{cc}} : \mathbb{H}^n \to \mathbb{R}$ defined by $\|a\|_{\mathrm{cc}} = d_{\mathrm{cc}}(a, 0)$. Note that $d_{\mathrm{cc}}(a,b) = \left\|a^{-1}b\right\|_{\mathrm{cc}} = \left\|b^{-1}a\right\|_{\mathrm{cc}}$.

Gaveau's function, $\mu\colon (-\pi, \pi) \to \mathbb{R}$, is defined by

$$\mu(x) = \frac{x - \sin(x)\cos(x)}{\sin^2(x)} = \frac{x}{\sin^2(x)} - \cot(x). \tag{2.1}$$

The singularity at $x = 0$ is removable with $\mu(0) = 0$. Note that μ is an odd function and that it is strictly increasing on $(-\pi, \pi)$. Let $\mu^{-1}\colon \mathbb{R} \to (-\pi, \pi)$ denote the inverse of μ.

It is also convenient to define $\sigma\colon (-\pi, \pi) \to \mathbb{R}$ by $\sigma(x) = x/\sin(x)$. The singularity at $x = 0$ is removable with $\sigma(0) = 1$. Note that σ is an even function and that it is strictly increasing on $[0, \pi)$. In particular, $\sigma(x) \geq 1$ for $x \in (-\pi, \pi)$.

The geodesics for the Carnot-Carathéodory sub-Riemannian structure can be given explicitly, leading to the following formulas for the Carnot-Carathéodory gauge (see [CCG05, Theorems 5.14 and 5.15] for the case of $\mathbb{H}^1$; however, their analysis is also valid for $\mathbb{H}^n$, $n \geq 1$).

$$\|(z,t)\|_{\mathrm{cc}} = \begin{cases} \sigma\left(\mu^{-1}\left(t/|z|^2\right)\right)|z| & \text{if } z \neq 0, \\ \sqrt{\pi}\,|t|^{1/2} & \text{if } z = 0. \end{cases} \tag{2.2}$$

Define the function $\eta\colon \mathbb{R} \to \mathbb{R}$ by

$$\eta(x) = \left(\frac{x^4}{\sin^4(x) + (x - \sin(x)\cos(x))^2}\right)^{1/4}. \tag{2.3}$$

The singularity at $x = 0$ is removable with $\eta(0) = 1$. Note that η is an even function and that the derivative of η is positive on the interval $(0, \pi)$, from which it follows that $1 \leq \eta(x) \leq \sqrt{\pi}$ for $x \in [-\pi, \pi]$.

The Carnot-Carathéodory gauge and Korányi gauge, $\|(z,t)\|_K = \left(|z|^4 + t^2\right)^{1/4}$, are related as follows:

Proposition 2.1. *For $(z,t) \in \mathbb{H}^n$ with $z \neq 0$,*

$$\|(z,t)\|_{\mathrm{cc}} = \eta\left(\mu^{-1}\left(t/|z|^2\right)\right)\|(z,t)\|_K.$$

Proof. Let $\zeta = \mu^{-1}\left(t/|z|^2\right)$ where $z \neq 0$. Then

$$|z|^4 + t^2 = |z|^4 + |z|^4\mu^2(\zeta) = |z|^4\left(1 + \mu^2(\zeta)\right)$$

and so by (2.2),

$$\begin{aligned} \|(z,t)\|_{\mathrm{cc}}^4 = \left(\zeta^4/\sin^4(\zeta)\right)|z|^4 &= \left(\zeta^4/\sin^4(\zeta)\right)\left(1 + \mu^2(\zeta)\right)^{-1}\left(|z|^4 + t^2\right) \\ &= \eta^4(\zeta)\,\|(z,t)\|_K^4. \end{aligned}$$

Taking fourth roots, the conclusion follows. □

Note that the function $\mathbb{H}^n - \{0\} \times \mathbb{R} \to \mathbb{R}$ given by $(z,t) \mapsto \eta\left(\mu^{-1}\left(t/|z|^2\right)\right)$ extends continuously to $\mathbb{H}^n - \{0\}$ if we define its value at $(0,t)$, where $t \neq 0$, to be $\sqrt{\pi}$. With this interpretation, Proposition 2.1 is valid for all $(z,t) \neq 0$.

Let $\lambda \geq 0$. *Non-isotropic dilation* by λ is the map $\delta_\lambda\colon \mathbb{H}^n \to \mathbb{H}^n$ given by $\delta_\lambda(z,s) = (\lambda z, \lambda^2 s)$. It is a homomorphism of $\mathbb{H}^n$ and satisfies $\delta_{\lambda_1} \circ \delta_{\lambda_2} = \delta_{\lambda_1\lambda_2}$ for $\lambda_1, \lambda_2 \geq 0$. Using the formulas for $\|\cdot\|_K$ and $\|\cdot\|_{\mathrm{cc}}$, it is straightforward to verify that for any $\lambda \geq 0$ and $a, b \in \mathbb{H}^n$ that $d_K(\delta_\lambda a, \delta_\lambda b) = \lambda\, d_K(a,b)$ and $d_{\mathrm{cc}}(\delta_\lambda a, \delta_\lambda b) = \lambda\, d_{\mathrm{cc}}(a,b)$.

3. The Horofunctions of $(\mathbb{H}^n, d_K)$

In this section we give a characterization of those sequences in $\mathbb{H}^n$ that converge to horofunctions for d_K (Theorem 3.3) and prove a proposition (Proposition 3.6) that will be used in our analysis of the horofunctions for d_{cc}.

We will use the following theorem to characterize the sequences in $\mathbb{H}^n$ that converge to horofunctions for d_K.

Theorem 3.1. *Let $\{p_m = (w_m, s_m)\}$ be a sequence in $\mathbb{H}^n$ such that $p_m \to \infty$. Then for all $(z,b) \in \mathbb{H}^n$,*

$$\lim_{m\to\infty} d_K(p_m,(z,b)) - \|p_m\|_K \; + \; \mathrm{Re}\left\langle \frac{|w_m|^2 + is_m}{(|w_m|^4 + s_m^2)^{3/4}} w_m, z \right\rangle = 0.$$

Proof. Given $q \in \mathbb{H}^n$, define $U_q = \{(p,t) \in \mathbb{H}^n \times \mathbb{R} \mid p \neq \delta_t q\}$ and define the function $f_q \colon U_q \to \mathbb{R}$ by $f_q(p,t) = \left\|p^{-1}\delta_t q\right\|_K$. Note that U_q is an open subset of $\mathbb{H}^n \times \mathbb{R}$ and that f_q is given by the formula

$$f_q(p,t) = \Big(|-w+tz|^4 + \big(-s + t^2 b - 2t\,\mathrm{Im}\langle w,z\rangle\big)^2\Big)^{1/4}$$

where $q = (z,b)$ and $p = (w,s)$. From this formula, we see that f_q is a smooth function on U_q.

The derivative with respect to t at 0 of the expression

$$\begin{aligned} &|-w+tz|^4 + \big(-s+t^2b-2t\,\mathrm{Im}\langle w,z\rangle\big)^2 \\ &= \big(|w|^2 - 2t\,\mathrm{Re}\langle w,z\rangle + t^2|z|^2\big)^2 + \big(-s+t^2b-2t\,\mathrm{Im}\langle w,z\rangle\big)^2 \end{aligned}$$

is $-4|w|^2\,\mathrm{Re}\langle w,z\rangle + 4s\,\mathrm{Im}\langle w,z\rangle$ and so for $p \neq 0$ (which is the case if and only $(p,0) \in U_q$) we have that

$$\begin{aligned} \frac{\partial f_q}{\partial t}(p,0) &= \tfrac{1}{4}\,\|p\|_K^{-3}\,\big(-4|w|^2\,\mathrm{Re}\langle w,z\rangle + 4s\,\mathrm{Im}\langle w,z\rangle\big) \\ &= -\,\mathrm{Re}\left\langle \frac{|w|^2+is}{(|w|^4+s^2)^{3/4}} w, z\right\rangle. \qquad (3.1) \end{aligned}$$

Observe that the function $(w,s) \mapsto \frac{|w|^2+is}{(|w|^4+s^2)^{3/4}} w$ is invariant under non-isotropic dilation by a positive constant and hence

$$\frac{\partial f_q}{\partial t}(\delta_\lambda p, 0) = \frac{\partial f_q}{\partial t}(p,0) \quad \text{for any } \lambda > 0. \qquad (3.2)$$

Define the function $u_q \colon U_q \to \mathbb{R}$ by

$$u_q(p,t) = \begin{cases} \left(\left\|p^{-1}\delta_t q\right\|_K - \|p\|_K\right)/t & \text{if } t \neq 0, \\[2ex] \dfrac{\partial f_q}{\partial t}(p,0) & \text{if } t = 0. \end{cases}$$

If $q = 0$, then u_q is identically zero. Assume $q \neq 0$. Let S_K^{2n} denote the unit sphere for d_K. Since f_q is continuously differentiable on U_q it follows that u_q is continuous on U_q and so the restriction of u_q to the compact set $S_K^{2n} \times [-1/(2\,\|q\|_K), 1/(2\,\|q\|_K)] \subset U_q$ is uniformly continuous. Hence given $\epsilon > 0$ there exists $\alpha > 0$ such that $0 < |t| < \alpha$ implies that for all $p \in S_K^{2n}$,

$$\left| \left(\left\|p^{-1}\delta_t q\right\|_K - \|p\|_K\right)/t - \frac{\partial f_q}{\partial t}(p,0)\right| = |u_q(p,t) - u_q(p,0)| < \epsilon.$$

Assume $\{p_m = (w_m, s_m)\}$ is a sequence in $\mathbb{H}^n$ such that $p_m \to \infty$. Let $t_m = \|p_m\|_K^{-1}$ and $\widehat{p}_m = \delta_{t_m} p_m$. We have that $\widehat{p}_m \in S_K^{2n}$ and

$$d_K(p_m, q) - \|p_m\|_K = \left(\left\|\widehat{p}_m^{-1}\delta_{t_m} q\right\|_K - 1\right)/t_m.$$

Since $\|p_m\|_K \to \infty$, we have that $t_m \to 0$ and so there exists N such that $m > N$ implies that $|t_m| < \alpha$ (for α as above). Hence $m > N$ implies

$$\left| d_K(p_m, q) - \|p_m\|_K - \frac{\partial f_q}{\partial t}(\widehat{p}_m, 0)\right| < \epsilon. \tag{3.3}$$

By (3.2), $\frac{\partial f_q}{\partial t}(\widehat{p}_m, 0) = \frac{\partial f_q}{\partial t}(p_m, 0)$ and so the conclusion follows from (3.1) and (3.3). □

Remark 3.2. Let U be an open subset of $\mathbb{H}^n$, let $F\colon U \to \mathbb{R}$ be a function, and let $x \in U$, $q \in \mathbb{H}^n$. The quantity $d/dt\ F(x\delta_t q)\big|_{t=0}$, if it exists, is called the *Pansu derivative* of F at x in the direction of q (see [CDPT07, Definition 6.3]). It is the Heisenberg group analog of a directional derivative. In particular, $\frac{\partial f_q}{\partial t}(p, 0)$ appearing in the proof of Theorem 3.1 is the Pansu derivative of $x \mapsto \|x\|_K$ at $x = p^{-1}$ in the direction of q.

A map $f\colon X \to X'$ between metric spaces (X, ρ) and (X', ρ') is C*-Lipschitz*, where $C > 0$, if $\rho'(f(x), f(y)) \le C\,\rho(x, y)$ for all $x, y \in X$. If a sequence of C-Lipschitz functions between metric spaces converges pointwise, then the sequence converges uniformly on compact subsets to a C-Lipschitz function. In any metric space (X, ρ) with basepoint x_0, and for any $a \in X$, the function $\rho_a : X \to \mathbb{R}$ defined by $\rho_a(x) = \rho(a, x) - \rho(a, x_0)$ is 1-Lipschitz. Therefore, if $\{a_m\}$ is a sequence in X and the sequence $\{\rho_{a_m}\}$ converges pointwise, then the sequence $\{\rho_{a_m}\}$ converges uniformly on compact subsets.

The following theorem is an immediate consequence of Theorem 3.1 and the above observation.

Theorem 3.3. *Let $\{p_m = (w_m, s_m)\}$ be a sequence in $\mathbb{H}^n$ such that $p_m \to \infty$. Then $\{p_m\}$ converges to a horofunction, h, for d_K if and only if $\gamma_m = \frac{|w_m|^2 + is_m}{(|w_m|^4 + s_m^2)^{3/4}} w_m$ converges in $\mathbb{C}^n$ in which case $h(z, b) = \operatorname{Re}\langle u, z\rangle$ where $u = -\lim_{m\to\infty}\gamma_m$.* □

Theorem 3.3 yields the following alternative proof of [KN09, Proposition 2.8] without making use of the conformal inversion on $(\mathbb{H}^n, d_K)$.

Proposition 3.4 ([KN09, Proposition 2.8])**.** *Let h be a horofunction for d_K. Then there exists $u \in \mathbb{C}^n$ with $|u| \le 1$ such that $h(z, b) = \operatorname{Re}\langle u, z\rangle$ for all $(z, b) \in \mathbb{H}^n$.*

Proof. Let $\{p_m = (w_m, s_m)\}$ be a sequence in $\mathbb{H}^n$ such that $p_m \to \infty$ and $d_K(p_m, q) - \|p_m\|_K \to h(q)$ for each $q \in \mathbb{H}^n$. Let $\gamma_m = \frac{|w_m|^2 + is_m}{(|w_m|^4 + s_m^2)^{3/4}} w_m$. By Theorem 3.3, $\{\gamma_m\}$ converges and $h(z, b) = \operatorname{Re}\langle u, z\rangle$ where $u = -\lim_{m\to\infty}\gamma_m$. Note that $|\gamma_m| = |w_m|/(|w_m|^4 + s_m^2)^{1/4} \le 1$ so $|u| \le 1$. □

Remark 3.5. Every function of the form $(z, b) \mapsto \operatorname{Re}\langle u, z\rangle$, where $|u| \le 1$, occurs as a horofunction for d_K by [KN09, Proposition 2.3]. This can also be readily deduced from Theorem 3.3 by considering sequences of the form $\{p_m = \delta_m(w, \lambda)\}$ where $\|(w, \lambda)\|_K = 1$.

We will make use of the following proposition in our analysis of the horofunctions for d_{cc} (see Theorem 5.12).

Proposition 3.6. *Let $\{p_m = (w_m, s_m)\}$ be a sequence in $\mathbb{H}^n$ such that $p_m \to \infty$ and $\lim_{m\in A} |s_m|/|w_m|^2 = \infty$ where $A = \{m \mid w_m \neq 0\}$. Then $\lim_{m\to\infty} d_K(p_m, q) - \|p_m\|_K = 0$ for all $q \in \mathbb{H}^n$.*

Proof. Let $\gamma_m = \frac{|w_m|^2 + is_m}{(|w_m|^4+s_m^2)^{3/4}} w_m$. We have $\gamma_m = 0$ if $m \notin A$ and

$$|\gamma_m| = \frac{|w_m|}{(|w_m|^4 + s_m^2)^{1/4}} = \left(1 + \left(|s_m|/|w_m|^2\right)^2\right)^{-1/4} \longrightarrow 0$$

as $m \to \infty$ and $m \in A$. Hence $\gamma_m \to 0$ and so by Theorem 3.1, $\lim_{m\to\infty} d_K(p_m, q) - \|p_m\|_K = 0$. ☐

4. Asymptotic estimates

We give asymptotic estimates for the inverse of Gaveau's function and some related functions. These estimates will be used in the next section in our analysis of the horofunctions for d_{cc}.

We use standard "big Oh" and "little oh" notation. For example, if $T \subset \mathbb{R}$ is such that $T \cap [0,\infty)$ is unbounded and if f and g are real-valued functions with domain T, then $f(t) = O(g(t))$ means that there exist $C, M > 0$ such that $|f(t)| \leq M|g(t)|$ for $t \in T$ and $t > C$, and $f(t) = o(g(t))$ means that for all $\epsilon > 0$ there exists $C > 0$ (depending on ϵ) such that $|f(t)| \leq \epsilon|g(t)|$ for $t \in T$ and $t > C$.

Let $\mu^{-1}\colon \mathbb{R} \to (-\pi, \pi)$ be the inverse of Gaveau's function (2.1).

Proposition 4.1. *We have*

$$\lim_{t\to\infty} t^{3/2}\left(\mu^{-1}(t) - \left(\pi - \sqrt{\pi}\, t^{-1/2}\right)\right) = -\pi^{3/2}/6.$$

Hence, $\mu^{-1}(t) = \pi - \sqrt{\pi}\, t^{-1/2} + O\left(t^{-3/2}\right)$.

Proof. The function μ extends to a meromorphic function on $\mathbb{C}$ and its Laurent expansion at $x = \pi$ is of the form

$$\mu(x) = \frac{\pi}{(x-\pi)^2} + g(x) \tag{4.1}$$

where $g(x)$ is an analytic function defined in a neighborhood of π such that $g(\pi) = \pi/3$.

Let $h(t) = g\left(\mu^{-1}(t)\right)$. Substituting $x = \mu^{-1}(t)$ into (4.1) and solving for $\mu^{-1}(t)$ yields:

$$\mu^{-1}(t) = \pi - \sqrt{\pi}\,\left(t - h(t)\right)^{-1/2}.$$

Hence

$$\begin{aligned}\mu^{-1}(t) - \left(\pi - \sqrt{\pi}\, t^{-1/2}\right) &= \sqrt{\pi}\left(t^{-1/2} - \left(t - h(t)\right)^{-1/2}\right)\\ &= -\sqrt{\pi}\,\frac{t^{1/2} - \left(t-h(t)\right)^{1/2}}{t^{1/2}\left(t-h(t)\right)^{1/2}}\\ &= \frac{-\sqrt{\pi}\, h(t)}{t^{1/2}\left(t-h(t)\right)^{1/2}\left(t^{1/2} + \left(t-h(t)\right)^{1/2}\right)}\end{aligned}$$

and so

$$t^{3/2}\left(\mu^{-1}(t) - \left(\pi - \sqrt{\pi}\, t^{-1/2}\right)\right) = \frac{-\sqrt{\pi}\, h(t)}{\left(1 - h(t)/t\right)^{1/2}\left(1 + \left(1-h(t)/t\right)^{1/2}\right)}. \tag{4.2}$$

Note that $\lim_{t\to\infty} \mu^{-1}(t) = \pi$ and so $\lim_{t\to\infty} h(t) = g\left(\lim_{t\to\infty} \mu^{-1}(t)\right) = \pi/3$. Hence the limit as $t \to \infty$ of the right side of (4.2) is $-\sqrt{\pi}(\pi/3)/2 = -\pi^{3/2}/6$. □

Corollary 4.2. *Let f be a function defined in a neighborhood of π such that $f(x) = a_0 + a_1(x-\pi) + a_2(x-\pi)^2 + O\left((x-\pi)^3\right)$. Then*

$$\lim_{t\to\infty} t\left(f\left(\mu^{-1}(t)\right) - \left(a_0 - a_1\sqrt{\pi}\,t^{-1/2}\right)\right) = \pi a_2.$$

Proof. The hypothesis on $f(x)$ implies

$$\lim_{x\to\pi} \frac{f(x) - (a_0 + a_1(x-\pi))}{(x-\pi)^2} = a_2.$$

Since $\lim_{t\to\infty} \mu^{-1}(t) = \pi$, it follows that

$$\lim_{t\to\infty} \frac{f\left(\mu^{-1}(t)\right) - \left(a_0 + a_1\left(\mu^{-1}(t) - \pi\right)\right)}{\left(\mu^{-1}(t) - \pi\right)^2} = a_2. \tag{4.3}$$

Proposition 4.1 implies that $\lim_{t\to\infty} \left(\mu^{-1}(t) - \pi\right)^2 / \left(\pi\, t^{-1}\right) = 1$. Combining this with (4.3) yields:

$$\lim_{t\to\infty} t\left(f\left(\mu^{-1}(t)\right) - \left(a_0 + a_1(\mu^{-1}(t) - \pi)\right)\right) = \pi a_2. \tag{4.4}$$

We have

$$t\left(f\left(\mu^{-1}(t)\right) - \left(a_0 - a_1\sqrt{\pi}\,t^{-1/2}\right)\right) = \tag{4.5}$$
$$t\left(f\left(\mu^{-1}(t)\right) - \left(a_0 + a_1\left(\mu^{-1}(t) - \pi\right)\right)\right) + a_1 t\left(\mu^{-1}(t) - \left(\pi - \sqrt{\pi}\,t^{-1/2}\right)\right).$$

Proposition 4.1 implies that $\lim_{t\to\infty} t\left(\mu^{-1}(t) - \left(\pi - \sqrt{\pi}\,t^{-1/2}\right)\right) = 0$. Combining this with (4.4) and (4.5) yields the conclusion of the corollary. □

Recall the function $\eta\colon \mathbb{R} \to \mathbb{R}$ given by (2.3).

Proposition 4.3. *Let $\tilde{\eta} = \eta \circ \mu^{-1}$. Then $\tilde{\eta}(t) = \sqrt{\pi} - t^{-1/2} + o(t^{-1})$.*

Proof. The Taylor expansion of $\eta(x)$ at $x = \pi$ yields

$$\eta(x) = \sqrt{\pi} + \pi^{-1/2}(x-\pi) + O\left((x-\pi)^3\right).$$

The conclusion now follows from Corollary 4.2. □

Remark 4.4. The sharp asymptotic estimate $\tilde{\eta}(t) = \sqrt{\pi} - t^{-1/2} + O(t^{-3/2})$ is valid but we will not need this.

The following limit at infinity will be used in the proof of Theorem 5.12.

Proposition 4.5. *The derivative, $\tilde{\eta}'$, of $\tilde{\eta} = \eta \circ \mu^{-1}$ has the property*

$$\lim_{t\to\infty} 2\,t^{3/2}\,\tilde{\eta}'(t) = 1.$$

Proof. The Taylor expansion of $\eta'(x)/\mu'(x)$ at $x = \pi$ yields

$$\eta'(x)/\mu'(x) = -\tfrac{1}{2}\pi^{-3/2}(x-\pi)^3 + O\left((x-\pi)^4\right). \tag{4.6}$$

Proposition 4.1 implies that $\lim_{t\to\infty} t^{1/2}\left(\mu^{-1}(t) - \pi\right) = -\sqrt{\pi}$. Hence

$$\lim_{t\to\infty} t^{3/2}\left(\mu^{-1}(t) - \pi\right)^3 = -\pi^{3/2} \text{ and } \lim_{t\to\infty} t^{3/2}\left(\mu^{-1}(t) - \pi\right)^4 = 0. \tag{4.7}$$

Observing that $\tilde{\eta}'(t) = \eta'\left(\mu^{-1}(t)\right)/\mu'\left(\mu^{-1}(t)\right)$, the conclusion of the proposition follows from (4.6) and (4.7). □

5. The Horofunctions of $(\mathbb{H}^n, d_{cc})$

In this section we give a characterization of those sequences in $\mathbb{H}^n$ that converge to horofunctions for d_{cc} (Theorem 5.16). The resulting classification of horofunctions for d_{cc} is stated in Theorem 5.17.

Define the function $\omega\colon \mathbb{H}^n - \{0\} \longrightarrow [-\infty, \infty]$ by

$$\omega(w,s) = \begin{cases} s/|w|^2 & \text{if } w \neq 0, \\ \infty & \text{if } w = 0 \text{ and } s > 0, \\ -\infty & \text{if } w = 0 \text{ and } s < 0. \end{cases}$$

The function ω is continuous and its restriction to the open subset $\mathbb{H}^n - Z \subset \mathbb{H}^n$, where $Z = \{0\} \times \mathbb{R}$ (the center of the group $\mathbb{H}^n$), is smooth. Note that $\omega(\delta_\lambda p) = \omega(p)$ for all $\lambda > 0$.

We compute the Pansu derivative of ω (see Remark 3.2).

Lemma 5.1. *Let $p = (w,s) \in \mathbb{H}^n - Z$. Then for any $q = (z,b) \in \mathbb{H}^n$,*

$$\frac{d}{dt}\omega\left(p^{-1}\delta_t q\right)\Big|_{t=0} = -2|w|^{-4}\operatorname{Re}\left\langle \left(s - i\,|w|^2\right) w, z\right\rangle.$$

Proof. We have

$$\omega\left(p^{-1}\delta_t q\right) = \frac{-s + t^2 b - 2t\operatorname{Im}\langle w,z\rangle}{|-w+tz|^2} = \frac{-s + t^2 b - 2t\operatorname{Im}\langle w,z\rangle}{|w|^2 - 2t\operatorname{Re}\langle w,z\rangle + t^2|z|^2}.$$

Applying the quotient rule for differentiation yields

$$\frac{d}{dt}\omega\left(p^{-1}\delta_t q\right)\Big|_{t=0} = \frac{-2\operatorname{Im}\langle w,z\rangle|w|^2 - 2s\operatorname{Re}\langle w,z\rangle}{|w|^4} = -2|w|^{-4}\operatorname{Re}\left\langle \left(s - i\,|w|^2\right) w, z\right\rangle.$$

□

Given $q \in \mathbb{H}^n$, define $\Omega_q = \left\{(p,t) \in \mathbb{H}^n \times \mathbb{R} \mid p^{-1}\delta_t q \notin Z\right\}$ and define the function $e_q\colon \Omega_q \to \mathbb{R}$ by $e_q(p,t) = \left\|p^{-1}\delta_t q\right\|_{cc}$. Note that Ω_q is an open subset of $\mathbb{H}^n \times \mathbb{R}$ and that $(\mathbb{H}^n - Z) \times \{0\} \subset \Omega_q$. Using (2.2), we have the following formula for e_q:

$$e_q(p,t) = \sigma\left(\mu^{-1}\left(\omega\left(p^{-1}\delta_t q\right)\right)\right)|-w+tz| \tag{5.1}$$

where $p = (w,s)$, $q = (z,b)$, $\sigma(y) = y/\sin(y)$, and μ^{-1} is the inverse of Gaveau's function. Observe from (5.1) that e_q is smooth on Ω_q.

The next two lemmas will be used in the proof of Proposition 5.4.

Lemma 5.2. *Let $p = (w,s) \in \mathbb{H}^n - Z$. Then for any $\lambda > 0$,*

$$\frac{\partial e_q}{\partial t}(\delta_\lambda p, 0) = \frac{\partial e_q}{\partial t}(p, 0).$$

Proof. We have

$$e_q(\delta_\lambda p, t) = \left\|(\delta_\lambda p)^{-1}\delta_t q\right\|_{cc} = \lambda\left\|p^{-1}\delta_{t/\lambda} q\right\|_{cc} = \lambda\, e_q(p, t/\lambda).$$

Differentiating at $t = 0$ using the chain rule yields the conclusion. □

Lemma 5.3. *Let $p = (w,s) \in (\mathbb{H}^n - Z) \cap S^{2n}_{cc}$. Then*

$$|w|\cos\left(\mu^{-1}\left(s/|w|^2\right)\right) = 1 - s\,\mu^{-1}\left(s/|w|^2\right).$$

Proof. Recall that $\mu(y) = (y - \sin(y)\cos(y))/\sin^2(y)$. Substituting $y = \mu^{-1}(\alpha)$ where $\alpha = s/|w|^2$ yields

$$s/|w|^2 = \alpha = \left(\mu^{-1}(\alpha) - \sin\left(\mu^{-1}(\alpha)\right)\cos\left(\mu^{-1}(\alpha)\right)\right)/\sin^2\left(\mu^{-1}(\alpha)\right). \tag{5.2}$$

Since $p \in S^{2n}_{\rm cc}$ we have that $\|p\|_{\rm cc} = 1$. This condition is equivalent to $\mu^{-1}(\alpha)|w| = \sin\left(\mu^{-1}(\alpha)\right)$ by (2.2). Substituting this into (5.2) yields

$$s/|w|^2 = \left(\mu^{-1}(\alpha) - |w|\mu^{-1}(\alpha)\cos\left(\mu^{-1}(\alpha)\right)\right)/\left(|w|^2\left(\mu^{-1}(\alpha)\right)^2\right).$$

Solving for $|w|\cos\left(\mu^{-1}(\alpha)\right)$ gives the conclusion. □

The quantity $\frac{\partial e_q}{\partial t}(p,0)$ computed in the following proposition is the Pansu derivative of $x \mapsto \|x\|_{\rm cc}$ at $x = p^{-1}$ in the direction of q.

Proposition 5.4. *Let $p = (w,s) \in \mathbb{H}^n - Z$ and let $q = (z,b) \in \mathbb{H}^n$. Then*

$$\frac{\partial e_q}{\partial t}(p,0) = -\operatorname{Re}\left\langle \exp\left(i\,\mu^{-1}\left(s/|w|^2\right)\right)\frac{w}{|w|},\, z\right\rangle.$$

Proof. Observe that the function $(w,s) \mapsto -\operatorname{Re}\left\langle \exp\left(i\,\mu^{-1}\left(s/|w|^2\right)\right)(w/|w|), z\right\rangle$ is invariant under non-isotropic dilation by a positive constant and so by Lemma 5.2 we can assume $\|p\|_{\rm cc} = 1$ by replacing p with $\delta_{1/\|p\|_{\rm cc}}p$. By (5.1),

$$e_q(p,t) = \tilde{\sigma}\left(\omega\left(p^{-1}\delta_t q\right)\right)\,|-w + tz|,$$

where $p = (w,s)$ and $\tilde{\sigma} = \sigma \circ \mu^{-1}$,

$$\begin{aligned} \frac{d}{dt}\,|-w+tz|\,\bigg|_{t=0} &= \frac{d}{dt}\left(|w|^2 - 2t\operatorname{Re}\langle w,z\rangle + t^2|z|^2\right)^{1/2}\bigg|_{t=0} \\ &= -\operatorname{Re}\langle w/|w|, z\rangle. \end{aligned}$$

By Lemma 5.1,

$$\frac{d}{dt}\tilde{\sigma}\left(\omega\left(p^{-1}\delta_t q\right)\right)\bigg|_{t=0} = \tilde{\sigma}'\left(-s/|w|^2\right)\left(-2|w|^{-4}\operatorname{Re}\left\langle\left(s - i\,|w|^2\right)w, z\right\rangle\right).$$

Hence

$$\begin{aligned} \frac{\partial e_q}{\partial t}(p,0) &= \tilde{\sigma}'\left(-s/|w|^2\right)\left(-2|w|^{-4}\operatorname{Re}\left\langle\left(s - i\,|w|^2\right)w, z\right\rangle\right)|w| \\ &\quad + \tilde{\sigma}\left(-s/|w|^2\right)\left(-\operatorname{Re}\langle w/|w|, z\rangle\right) = \operatorname{Re}\langle\gamma\, w/|w|, z\rangle \end{aligned}$$

where

$$\begin{aligned} \gamma &= \tilde{\sigma}'\left(-s/|w|^2\right)\left(-2|w|^{-2}\left(s - i\,|w|^2\right)\right) - \tilde{\sigma}\left(-s/|w|^2\right) \\ &= \tilde{\sigma}'\left(-s/|w|^2\right)\left(-2s|w|^{-2}\right) - \tilde{\sigma}\left(-s/|w|^2\right) + i\,2\tilde{\sigma}'\left(-s/|w|^2\right). \end{aligned}$$

We have

$$\frac{\sigma'(x)}{\mu'(x)} = \frac{(\sin(x) - x\cos(x))/\sin^2(x)}{2(\sin(x) - x\cos(x))/\sin^3(x)} = \tfrac{1}{2}\sin(x)$$

and so $\tilde{\sigma}'(t) = \sigma'\left(\mu^{-1}(t)\right)/\mu'\left(\mu^{-1}(t)\right) = \frac{1}{2}\sin\left(\mu^{-1}(t)\right)$. Hence

$$\operatorname{Im}(\gamma) = 2\tilde{\sigma}'\left(-s/|w|^2\right) = 2\left(\tfrac{1}{2}\sin\left(\mu^{-1}\left(-s/|w|^2\right)\right)\right) = -\sin\left(\mu^{-1}\left(s/|w|^2\right)\right).$$

Since $\|p\|_{\mathrm{cc}} = 1$, $\tilde{\sigma}\left(s/|w|^2\right)|w| = 1$ and so $\mu^{-1}\left(s/|w|^2\right) = |w|^{-1}\sin\left(\mu^{-1}\left(s/|w|^2\right)\right)$. It follows that

$$\begin{aligned}\operatorname{Re}(\gamma) &= \tilde{\sigma}'\left(-s/|w|^2\right)\left(-2s|w|^{-2}\right) - \tilde{\sigma}\left(-s/|w|^2\right) \\ &= |w|^{-1}\sin\left(\mu^{-1}\left(s/|w|^2\right)\right)(s/|w|) - |w|^{-1} = |w|^{-1}\left(\mu^{-1}\left(s/|w|^2\right)s - 1\right) \\ &= -\cos\left(\mu^{-1}\left(s/|w|^2\right)\right) \quad \text{by Lemma 5.3.}\end{aligned}$$

Hence

$$\gamma = -\cos\left(\mu^{-1}\left(s/|w|^2\right)\right) - i\sin\left(\mu^{-1}\left(s/|w|^2\right)\right) = -\exp\left(i\,\mu^{-1}\left(s/|w|^2\right)\right),$$

completing the proof of the proposition. □

Theorem 5.5. *Let $\{p_m = (w_m, s_m)\}$ be a sequence in $\mathbb{H}^n - Z$ such that $p_m \to \infty$ and $\left\{|s_m|/|w_m|^2\right\}$ is bounded. Then for all $(z,b) \in \mathbb{H}^n$,*

$$\lim_{m\to\infty} d_{\mathrm{cc}}(p_m,(z,b)) - \|p_m\|_{\mathrm{cc}} + \operatorname{Re}\left\langle \exp\left(i\,\mu^{-1}\left(s_m/|w_m|^2\right)\right)\frac{w_m}{|w_m|},\, z\right\rangle = 0.$$

Proof. Define the function $v_q\colon \Omega_q \to \mathbb{R}$ by

$$v_q(p,t) = \begin{cases} \left(\left\|p^{-1}\delta_t q\right\|_{\mathrm{cc}} - \|p\|_{\mathrm{cc}}\right)/t & \text{if } t \neq 0, \\ \dfrac{\partial e_q}{\partial t}(p,0) & \text{if } t = 0. \end{cases}$$

Since e_q is continuously differentiable on Ω_q it follows that v_q is continuous on Ω_q.

Let S^{2n}_{cc} denote the unit sphere for d_{cc}. By hypothesis, there exists $C > 0$ such that $|s_m|/|w_m|^2 \leq C$ for all m. The set

$$K = S^{2n}_{\mathrm{cc}} \cap \left\{(w,s) \in \mathbb{H}^n - Z \mid |s|/|w|^2 \leq C\right\}$$

is a closed subset of S^{2n}_{cc} and thus compact. Let $\beta > 0$ be such that $K \times [-\beta,\beta] \subset \Omega_q$.

The restriction of v_q to the compact set $K \times [-\beta,\beta]$ is uniformly continuous, so given $\epsilon > 0$ there exists $\alpha > 0$ such that $0 < |t| < \alpha$ implies that for all $p \in K$,

$$\left|\left(\left\|p^{-1}\delta_t q\right\|_{\mathrm{cc}} - \|p\|_{\mathrm{cc}}\right)/t - \frac{\partial e_q}{\partial t}(p,0)\right| = |v_q(p,t) - v_q(p,0)| < \epsilon.$$

Let $t_m = \|p_m\|_{\mathrm{cc}}^{-1}$ and let $\widehat{p}_m = \delta_{t_m} p_m$. We have that $\widehat{p}_m \in S^{2n}_{\mathrm{cc}}$ and also that $|\omega(\widehat{p}_m)| = |\omega(p_m)| = |s_m|/|w_m|^2 \leq C$, so $\widehat{p}_m \in K$. Observe that

$$d_{\mathrm{cc}}(p_m,q) - \|p_m\|_{\mathrm{cc}} = \left(\left\|\widehat{p}_m^{-1}\delta_{t_m} q\right\|_{\mathrm{cc}} - 1\right)/t_m.$$

Since $\|p_m\|_{\mathrm{cc}} \to \infty$, we have that $t_m \to 0$ and so there exists N such that $m > N$ implies that $|t_m| < \alpha$ (for α as above). Hence $m > N$ implies

$$\left|d_{\mathrm{cc}}(p_m,q) - \|p_m\|_{\mathrm{cc}} - \frac{\partial e_q}{\partial t}(\widehat{p}_m,0)\right| < \epsilon. \tag{5.3}$$

By Lemma 5.2, $\frac{\partial e_q}{\partial t}(\widehat{p}_m,0) = \frac{\partial e_q}{\partial t}(p_m,0)$ and so the conclusion follows from Proposition 5.4 and (5.3). □

The following corollary is an immediate consequence of Theorem 5.5.

Corollary 5.6. *Let $\{p_m = (w_m, s_m)\}$ be a sequence in $\mathbb{H}^n - Z$ such that $p_m \to \infty$ and $\left\{|s_m|/|w_m|^2\right\}$ is bounded. Then $\{p_m\}$ converges to a horofunction, h, for d_{cc} if and only if $\gamma_m = \exp\left(i\,\mu^{-1}\left(s_m/|w_m|^2\right)\right)(w_m/|w_m|)$ converges in $\mathbb{C}^n$ in which case $h(z,b) = \operatorname{Re}\langle u, z\rangle$ where $u = -\lim_{m\to\infty}\gamma_m$.* □

The next lemma will be used in the proof of Theorem 5.8.

Lemma 5.7. *For all* $(z,b) \in \mathbb{H}^n$*,*

$$\lim_{|t|\to\infty} d_{\rm cc}((0,t),(z,b)) - \|(0,t)\|_{\rm cc} = -|z|.$$

Proof. If $z = 0$, then by (2.2),

$$d_{\rm cc}((0,t),(z,b)) - \|(0,t)\|_{\rm cc} = \sqrt{\pi}\left(|t-b|^{1/2} - |t|^{1/2}\right).$$

For $|t| > |b|$ the absolute value of this expression is $\sqrt{\pi}|b|\left(|t-b|^{1/2} + |t|^{1/2}\right)^{-1}$, which converges to 0 as $|t| \to \infty$.

Assume that $z \neq 0$. In this case, by Proposition 2.1,

$$d_{\rm cc}((0,t),(z,b)) = \tilde{\eta}\left(|t-b|/|z|^2\right)\left(|z|^4 + (t-b)^2\right)^{1/4}$$

where $\tilde{\eta} = \eta \circ \mu^{-1}$. By Propositon 4.3,

$$\tilde{\eta}\left(|t-b|/|z|^2\right) = \sqrt{\pi} - |z|\,|t-b|^{-1/2} + o\left(|t|^{-1}\right).$$

Since $|t-b|^{-1/2} = |t|^{-1/2} + O\left(|t|^{-3/2}\right)$, it follows that

$$\tilde{\eta}\left(|t-b|/|z|^2\right) = \sqrt{\pi} - |z|\,|t|^{-1/2} + o\left(|t|^{-1}\right).$$

Also, $\left(|z|^4 + (t-b)^2\right)^{1/4} = |t|^{1/2} + o(1)$. Hence

$$\tilde{\eta}\left(|t-b|/|z|^2\right)\left(|z|^4 + (t-b)^2\right)^{1/4} = \sqrt{\pi}|t|^{1/2} - |z| + o(1)$$

and so

$$d_{\rm cc}((0,t),(z,b)) - \|(0,t)\|_{\rm cc} = \sqrt{\pi}|t|^{1/2} - |z| + o(1) - \sqrt{\pi}|t|^{1/2} = -|z| + o(1)$$

which is the conclusion of the lemma. □

Theorem 5.8. *Let* $\{p_m = (w_m, s_m)\}$ *be a sequence in* $\mathbb{H}^n$ *such that* $\{w_m\}$ *is bounded and* $|s_m| \to \infty$*. Then for all* $(z,b) \in \mathbb{H}^n$*,*

$$\lim_{m\to\infty} d_{\rm cc}(p_m,(z,b)) - \|p_m\|_{\rm cc} \;-\; (|w_m| - |w_m - z|) = 0.$$

Proof. Define functions $f_m\colon \mathbb{H}^n \to \mathbb{R}$, for $m = 1, 2, \ldots$, by

$$f_m(z,b) = d_{\rm cc}((0,s_m),(z,b)) - \sqrt{\pi}|s_m|^{1/2} + |z|.$$

The triangle inequality implies that for all (z,b), $(z',b') \in \mathbb{H}^n$,

$$|d_{\rm cc}((0,s_m),(z,b)) - d_{\rm cc}((0,s_m),(z',b'))| \le d_{\rm cc}((z,b),(z',b')),$$

and

$$|\,|z| - |z'|\,| \le |\,z - z'\,| \le d_K((z,b),(z',b')) \le d_{\rm cc}((z,b),(z',b')).$$

It follows that

$$|f_m(z,b) - f_m(z',b')| \le 2 d_{\rm cc}((z,b),(z',b')).$$

Hence each f_m is 2-Lipschitz.

By Lemma 5.7, the sequence $\{f_m\}$ converges pointwise to the zero function. Since each f_m is 2-Lipschitz, this convergence is necessarily uniform on compact subsets.

For a given $(z, b) \in \mathbb{H}^n$, the sequence $\{(-w_m, 0)(z, b)\}$ is bounded because $\|(-w_m, 0)(z, b)\|_{\text{cc}} \le |w_m| + \|(z, b)\|_{\text{cc}}$ and $\{w_m\}$ is bounded by hypothesis. In particular, the sequences $\{(-w_m, 0)\}$ and $\{(-w_m, 0)(z, b)\}$ lie in compact subsets of $\mathbb{H}^n$ and so

$$\lim_{m\to\infty} f_m(-w_m, 0) = 0 \quad \text{and} \quad \lim_{m\to\infty} f_m((-w_m, 0)(z, b)) = 0. \tag{5.4}$$

Since d_{cc} is left invariant, we have

$$d_{\text{cc}}((0, s_m), (-w_m, 0)(z, b)) = d_{\text{cc}}((w_m, s_m), (z, b)) = d_{\text{cc}}(p_m, (z, b))$$

and

$$d_{\text{cc}}((0, s_m), (-w_m, 0)) = d_{\text{cc}}((w_m, s_m), 0) = \|p_m\|_{\text{cc}}$$

and so

$$\begin{aligned} d_{\text{cc}}(p_m, (z, b)) - \|p_m\|_{\text{cc}} \; - \; (|w_m| - |w_m - z|) \; = \\ f_m((-w_m, 0)(z, b)) - f_m(-w_m, 0). \end{aligned} \tag{5.5}$$

By (5.4), the right side of the equality (5.5) converges to 0 as $m \to \infty$. □

The following corollary is an immediate consequence of Theorem 5.8 and Lemma 5.10.

Corollary 5.9. *Let $\{p_m = (w_m, s_m)\}$ be a sequence in $\mathbb{H}^n$ such that $\{w_m\}$ is bounded and $|s_m| \to \infty$. Then $\{p_m\}$ converges to a horofunction, h, for d_{cc} if and only if $\{w_m\}$ converges in $\mathbb{C}^n$ in which case $h(z, b) = |w| - |w - z|$ where $w = \lim_{m\to\infty} w_m$.* □

Lemma 5.10. *Let $\{w_m\}$ be a bounded sequence in $\mathbb{C}^n$. Then $\{|w_m| - |w_m - z|\}$ converges for every $z \in \mathbb{C}^n$ if and only if $\{w_m\}$ converges.*

Proof. If $\{w_m\}$ converges, then so does $\{|w_m| - |w_m - z|\}$ for any $z \in \mathbb{C}^n$ because the map $w \mapsto |w| - |w - z|$ is continuous.

Assume $\{|w_m| - |w_m - z|\}$ converges for every $z \in \mathbb{C}^n$. Since $\{w_m\}$ is bounded, its set of accumulation points is not empty. Let α and β be accumulation points of $\{w_m\}$. Then for all $z \in \mathbb{C}^n$,

$$\lim_{m\to\infty} |w_m| - |w_m - z| = |\alpha| - |\alpha - z| = |\beta| - |\beta - z|. \tag{5.6}$$

Letting $z = \alpha$ in (5.6) gives $|\beta - \alpha| = |\beta| - |\alpha|$ while letting $z = \beta$ in (5.6) gives $|\alpha - \beta| = |\alpha| - |\beta|$. Hence $|\alpha - \beta| = 0$ and so $\alpha = \beta$. It follows that $\{w_m\}$ converges. □

The next lemma is a key ingredient in the proof of Theorem 5.12.

Lemma 5.11. *Let $\{p_m = (w_m, s_m)\}$ be a sequence in $\mathbb{H}^n$ such that $w_m \to \infty$ and $|\omega(p_m)| = |s_m|/|w_m|^2 \to \infty$. Then for all $q = (z, b) \in \mathbb{H}^n$,*

$$\lim_{m\to\infty} \frac{|w_m|^3}{2s_m} \left(\omega\left(q^{-1}p_m\right) - \omega(p_m)\right) \ - \ (|w_m| - |w_m - z|) = 0.$$

Proof. We have $\omega\left(q^{-1}p_m\right) = (s_m + c_m)/|w_m - z|^2$ where $c_m = -b + 2\operatorname{Im}\langle w_m, z\rangle$.

The conditions $|w_m| \to \infty$ and $|s_m|/|w_m|^2 \to \infty$ imply that $|s_m| \to \infty$. Also note that $|w_m|/|s_m|^{1/2} \to 0$. Hence

$$c_m/|s_m|^{1/2} = -b/|s_m|^{1/2} + 2\operatorname{Im}\left\langle w_m/|s_m|^{1/2}, z\right\rangle$$

converges to 0 as $m \to \infty$.

Observe that

$$\begin{aligned}\frac{|w_m|^3}{2s_m}\left(\omega\left(q^{-1}p_m\right) - \omega(p_m)\right) &= \frac{|w_m|^3}{2s_m}\left(\frac{s_m + c_m}{|w_m - z|^2} - \frac{s_m}{|w_m|^2}\right)\\ &= \tfrac{1}{2}|w_m|^3\left(|w_m - z|^{-2} - |w_m|^{-2}\right) + d_m\end{aligned}$$

where $d_m = \left(|w_m|^3 c_m\right) / \left(2s_m|w_m - z|^2\right)$. We have that

$$|d_m| = \tfrac{1}{2}\left(|w_m|/|w_m - z|\right)^2 \left(|w_m|/|s_m|^{1/2}\right)\left(|c_m|/|s_m|^{1/2}\right).$$

Since $|w_m| \to \infty$, it follows that $|w_m|/|w_m - z| \to 1$; furthermore, as previously noted, $|w_m|/|s_m|^{1/2} \to 0$ and $c_m/|s_m|^{1/2} \to 0$. Hence $d_m \to 0$. Thus it suffices to show that

$$\ell_m = \tfrac{1}{2}|w_m|^3\left(|w_m - z|^{-2} - |w_m|^{-2}\right) - \left(|w_m| - |w_m - z|\right)$$

converges to 0 as $m \to \infty$. The identity

$$\begin{gathered}|w_m|^3\left(|w_m - z|^{-2} - |w_m|^{-2}\right) =\\ \left(|w_m|/|w_m - z| + 1\right)\left(|w_m|/|w_m - z|\right)\left(|w_m| - |w_m - z|\right)\end{gathered}$$

yields

$$\ell_m = \left(\tfrac{1}{2}\left(|w_m|/|w_m - z| + 1\right)\left(|w_m|/|w_m - z|\right) - 1\right)\left(|w_m| - |w_m - z|\right).$$

We have

$$\lim_{m\to\infty} \tfrac{1}{2}\left(|w_m|/|w_m - z| + 1\right)\left(|w_m|/|w_m - z|\right) - 1 = \tfrac{1}{2}(1 + 1) - 1 = 0.$$

Since $|\,|w_m| - |w_m - z|\,| \le |z|$, this implies that $\ell_m \to 0$. □

Theorem 5.12. *Let $\{p_m = (w_m, s_m)\}$ be a sequence in $\mathbb{H}^n$ such that $w_m \to \infty$ and $|\omega(p_m)| = |s_m|/|w_m|^2 \to \infty$. Then for all $(z, b) \in \mathbb{H}^n$,*

$$\lim_{m\to\infty} d_{\mathrm{cc}}(p_m, (z, b)) - \|p_m\|_{\mathrm{cc}} \ - \ (|w_m| - |w_m - z|) = 0.$$

Proof. Let $q = (z, b)$. Making use of Proposition 2.1, we have that

$$d_{\mathrm{cc}}(p_m, q) - \|p_m\|_{\mathrm{cc}} = \tilde{\eta}\left(\omega\left(q^{-1}p_m\right)\right) d_K(p_m, q) - \tilde{\eta}(\omega(p_m))\, \|p_m\|_K =$$
$$\tilde{\eta}\left(\omega\left(q^{-1}p_m\right)\right)\left(d_K(p_m, q) - \|p_m\|_K\right) \;+\; \left(\tilde{\eta}\left(\omega\left(q^{-1}p_m\right)\right) - \tilde{\eta}(\omega(p_m))\right)\|p_m\|_K .$$

By Proposition 3.6, $d_K(p_m, q) - \|p_m\|_K \to 0$. Since $\tilde{\eta}$ is bounded between 1 and $\sqrt{\pi}$, it follows that

$$\tilde{\eta}\left(\omega\left(q^{-1}p_m\right)\right)\left(d_K(p_m, q) - \|p_m\|_K\right) \longrightarrow 0.$$

Hence it suffices to show that

$$\tilde{\eta}\left(\omega\left(q^{-1}p_m\right) - \tilde{\eta}(\omega(p_m))\right)\|p_m\|_K \;-\; \left(|w_m| - |w_m - z|\right) \longrightarrow 0.$$

Let $r_m = \omega(p_m) = s_m/|w_m|^2$ and $r_m' = \omega\left(q^{-1}p_m\right) = (s_m + c_m)/|w_m - z|^2$ where $c_m = -b + 2\,\mathrm{Im}\langle w_m, z\rangle$. Consider the ratio

$$r_m'/r_m = \left(1 + c_m/s_m\right)\left(|w_m|/|w_m - z|\right)^2 .$$

As shown in the proof of Lemma 5.11, $|s_m| \to \infty$ and $c_m/|s_m|^{1/2} \to 0$. Hence $|c_m/s_m| = \left(|c_m|/|s_m|^{1/2}\right)|s_m|^{-1/2} \to 0$. Since $|w_m| \to \infty$, we have that $|w_m|/|w_m - z| \to 1$ and thus $r_m'/r_m \to 1$.

By the Mean Value Theorem, there exists a real number r_m'' lying between r_m' and r_m such that

$$\tilde{\eta}(r_m') - \tilde{\eta}(r_m) = \tilde{\eta}'(r_m'')(r_m' - r_m).$$

Since r_m'' lies between r_m' and r_m and $r_m'/r_m \to 1$, it follows that $r_m''/r_m \to 1$ and also that $|r_m''| \to \infty$ since $|r_m| \to \infty$.

Note that if m is sufficiently large, then s_m and r_m'' have the same sign. Since $\tilde{\eta}'$ is an odd function, for sufficiently large m, we have $s_m\tilde{\eta}'(r_m'') = |s_m|\,\tilde{\eta}'(|r_m''|)$. Let

$$\alpha_m = 2s_m|w_m|^{-3}\,\tilde{\eta}'(r_m'')\,\|p_m\|_K .$$

Then for sufficiently large m,

$$\alpha_m = 2|s_m||w_m|^{-3}\,\tilde{\eta}'(|r_m''|)\,\|p_m\|_K = \left(\frac{\tilde{\eta}'(|r_m''|)}{\frac{1}{2}|r_m|^{-3/2}}\right)\left(\frac{\|p_m\|_K}{|s_m|^{1/2}}\right)$$
$$= \left(\frac{\tilde{\eta}'(|r_m''|)}{\frac{1}{2}|r_m''|^{-3/2}}\right)\left(|r_m''/r_m|^{-3/2}\right)\left(1 + \left(|w_m|^2/|s_m|\right)^2\right)^{1/4} .$$

We have that $|r_m''/r_m|^{-3/2} \to 1$ and $\left(1 + \left(|w_m|^2/|s_m|\right)^2\right)^{1/4} \to 1$. By Proposition 4.5 we have that $\tilde{\eta}'(|r_m''|)/\left(\frac{1}{2}|r_m''|^{-3/2}\right) \to 1$ and so it follows that $\alpha_m \to 1$. Observe that

$$\left(\tilde{\eta}\left(\omega\left(q^{-1}p_m\right)\right) - \tilde{\eta}(\omega(p_m))\right)\|p_m\|_K = \alpha_m \frac{|w_m|^3}{2s_m}\left(r_m' - r_m\right).$$

Hence

$$\left(\tilde{\eta}\left(\omega\left(q^{-1}p_m\right)\right) - \tilde{\eta}(\omega(p_m))\right)\|p_m\|_K \;-\; \left(|w_m| - |w_m - z|\right) \tag{5.7}$$
$$= \alpha_m\left(\frac{|w_m|^3}{2s_m}\left(r_m' - r_m\right) - \left(|w_m| - |w_m - z|\right)\right)$$
$$+(1 - \alpha_m)\left(|w_m| - |w_m - z|\right).$$

By Lemma 5.11, $\left(|w_m|^3/2s_m\right)\left(r_m' - r_m\right) - \left(|w_m| - |w_m - z|\right) \to 0$. Since $\alpha_m \to 1$ and $\left|\,|w_m| - |w_m - z|\,\right| \le |z|$, we have that $(1 - \alpha_m)\left(|w_m| - |w_m - z|\right) \to 0$. Hence the right side of the equality (5.7) converges to 0, completing the proof. □

The following corollary is an immediate consequence of Theorem 5.12 and Lemma 5.14.

Corollary 5.13. *Let $\{p_m = (w_m, s_m)\}$ be a sequence in $\mathbb{H}^n$ such that $w_m \to \infty$ and $|\omega(p_m)| = |s_m|/|w_m|^2 \to \infty$. Then $\{p_m\}$ converges to a horofunction, h, for d_{cc} if and only if $\{w_m/|w_m|\}$ converges in $\mathbb{C}^n$ in which case $h(z,b) = \mathrm{Re}\langle u, z\rangle$ where $u = \lim_{m\to\infty} w_m/|w_m|$.* □

Lemma 5.14. *Let $\{w_m\}$ be a sequence in $\mathbb{C}^n$ such that $w_m \to \infty$. Then $\{|w_m| - |w_m - z|\}$ converges for every $z \in \mathbb{C}^n$ if and only if $\{w_m/|w_m|\}$ converges in which case $\lim_{m\to\infty} |w_m| - |w_m - z| = \mathrm{Re}\langle u, z\rangle$ where $u = \lim_{m\to\infty} w_m/|w_m|$.*

Proof. We have

$$\begin{aligned} |w_m| - |w_m - z| &= \left(|w_m|^2 - |w_m - z|^2\right) / \left(|w_m| + |w_m - z|\right) \\ &= \left(2\,\mathrm{Re}\langle w_m, z\rangle - |z|^2\right) / \left(|w_m| + |w_m - z|\right) \\ &= \left(2\,\mathrm{Re}\langle w_m/|w_m|, z\rangle - |z|^2/|w_m|\right) / \left(1 + |w_m/|w_m| - z/|w_m|\,|\right). \end{aligned} \tag{5.8}$$

Since $|w_m| \to \infty$, we have that $|z|^2/|w_m| \to 0$ and $1 + |w_m/|w_m| - z/|w_m|\,| \to 2$ and so by (5.8), $|w_m| - |w_m - z|$ converges if and only if $\mathrm{Re}\langle w_m/|w_m|, z\rangle$ converges in which case $\lim_{m\to\infty} |w_m| - |w_m - z| = \lim_{m\to\infty} \mathrm{Re}\langle w_m/|w_m|, z\rangle$. Since $\mathrm{Re}\langle\,\cdot\,,\,\cdot\,\rangle$ is an inner product on a finite dimensional vector space, $\mathrm{Re}\langle w_m/|w_m|, z\rangle$ converges for every $z \in \mathbb{C}^n$ if and only if $w_m/|w_m|$ converges in $\mathbb{C}^n$ in which case $\lim_{m\to\infty} \mathrm{Re}\langle w_m/|w_m|, z\rangle = \mathrm{Re}\left\langle \lim_{m\to\infty} w_m/|w_m|, z\right\rangle$. □

Corollaries 5.6 and 5.13 combine to give the following characterization of sequences $\{(w_m, s_m)\}$ in $\mathbb{H}^n$ that converge to horofunctions for d_{cc} in the case $w_m \to \infty$.

Proposition 5.15. *Let $\{p_m = (w_m, s_m)\}$ be a sequence in $\mathbb{H}^n$ such that $w_m \to \infty$. Then $\{p_m\}$ converges to a horofunction, h, for d_{cc} if and only if $\gamma_m = \exp\left(i\,\mu^{-1}\left(s_m/|w_m|^2\right)\right)(w_m/|w_m|)$ converges in $\mathbb{C}^n$ in which case $h(z,b) = \mathrm{Re}\langle u, z\rangle$ where $u = -\lim_{m\to\infty} \gamma_m$.*

Proof. Assume $\{(w_m, s_m)\}$ converges to a horofunction, h, for d_{cc}. Note that $\gamma_m \in S^{2n-1} = \{z \in \mathbb{C}^n \mid |z| = 1\}$. Since S^{2n-1} is compact, the set of accumulation points of $\{\gamma_m\}$ is not empty. Let v and v' be accumulation points of $\{\gamma_m\}$. Then there exist subsequences $\{\gamma_{m_j}\}$ and $\{\gamma_{k_j}\}$ of $\{\gamma_m\}$ such that $\gamma_{m_j} \to v$ and $\gamma_{k_j} \to v'$. By further passage to a subsequence, we can assume that either $|s_{m_j}|/|w_{m_j}|^2$ is bounded or that $|s_{m_j}|/|w_{m_j}|^2 \to \infty$. Note that any subsequence of $\{(w_m, s_m)\}$ converges to the *same* horofunction, h. In the case $|s_{m_j}|/|w_{m_j}|^2$ is bounded, Corollary 5.6 implies that $h(z,b) = \mathrm{Re}\langle -v, z\rangle$ for all $(z,b) \in \mathbb{H}^n$. In the case $|s_{m_j}|/|w_{m_j}|^2 \to \infty$, note that $\exp\left(i\,\mu^{-1}\left(s_{m_j}/|w_{m_j}|^2\right)\right) \to -1$ and so $-v = -\lim_{j\to\infty} \gamma_{m_j} = \lim_{j\to\infty} w_{m_j}/|w_{m_j}|$; hence, by Corollary 5.13 we have that $h(z,b) = \mathrm{Re}\langle -v, z\rangle$ for all $(z,b) \in \mathbb{H}^n$. The same argument applied to $\{\gamma_{k_j}\}$ yields $h(z,b) = \mathrm{Re}\langle -v', z\rangle$ for all $(z,b) \in \mathbb{H}^n$. Hence $\mathrm{Re}\langle -v, z\rangle = \mathrm{Re}\langle -v', z\rangle$ for all $z \in \mathbb{C}^n$ from which it follows that $v = v'$. This shows that $\{\gamma_m\}$ converges and that $h(z,b) = \mathrm{Re}\langle -v, z\rangle$ for all $(z,b) \in \mathbb{H}^n$.

We now prove the converse. Assume $\{\gamma_m\}$ converges. Let $u = -\lim_{m\to\infty} \gamma_m$. Note that any subsequence of $\{-\gamma_m\}$ also converges to u. Let $(z,b) \in \mathbb{H}^n$ and let $\ell_m = d_{\mathrm{cc}}((w_m, s_m), (z,b)) - \|(w_m, s_m)\|_{\mathrm{cc}}$. By the triangle inequality, $|\ell_m| \le \|(z,b)\|_{\mathrm{cc}}$. Hence the sequence $\{\ell_m\}$ is bounded and so its set of accumulation points is nonempty. Let α and β be accumulation points. Then there exist subsequences

$\{(w_{m_j}, s_{m_j})\}$ and $\{(w_{k_j}, s_{k_j})\}$ of $\{(w_m, s_m)\}$ such that $\ell_{m_j} \to \alpha$ and $\ell_{k_j} \to \beta$. By further passage to a subsequence, we can assume that either $|s_{m_j}|/|w_{m_j}|^2$ is bounded or that $|s_{m_j}|/|w_{m_j}|^2 \to \infty$. In the case $|s_{m_j}|/|w_{m_j}|^2$ is bounded, Corollary 5.6 implies that $\alpha = \mathrm{Re}\langle u, z\rangle$. In the case $|s_{m_j}|/|w_{m_j}|^2 \to \infty$, note that $\exp\left(i\,\mu^{-1}\left(s_{m_j}/|w_{m_j}|^2\right)\right) \to -1$ and so $u = -\lim_{j\to\infty}\gamma_{m_j} = \lim_{j\to\infty} w_{m_j}/|w_{m_j}|$; hence, by Corollary 5.13 we have that $\alpha = \mathrm{Re}\langle u, z\rangle$. The same argument applied to $\{\ell_{k_j}\}$ shows that $\beta = \mathrm{Re}\langle u, z\rangle$. Hence $\alpha = \beta$. It follows that $\{\ell_m\}$ converges. Since this is valid for every $(z, b) \in \mathbb{H}^n$, the sequence $\{(w_m, s_m)\}$ converges to a horofunction for d_{cc}. □

The following theorem characterizes those sequences in $\mathbb{H}^n$ that converge to horofunctions for d_{cc}.

Theorem 5.16. *Let $\{p_m = (w_m, s_m)\}$ be a sequence in $\mathbb{H}^n$ such that $p_m \to \infty$. Then $\{p_m\}$ converges to a horofunction, h, for d_{cc} if and only if one the following (mutually exclusive) conditions is satisfied:*

(1) *$w_m \to \infty$ and $\gamma_m = \exp\left(i\,\mu^{-1}\left(s_m/|w_m|^2\right)\right)(w_m/|w_m|)$ converges in $\mathbb{C}^n$ in which case $h(z, b) = \mathrm{Re}\langle u, z\rangle$ where $u = -\lim_{m\to\infty}\gamma_m$.*
(2) *The sequence $\{w_m\}$ converges in $\mathbb{C}^n$ in which case $h(z, b) = |w| - |w - z|$ where $w = \lim_{m\to\infty} w_m$.*

Proof. The conclusion of the theorem follows from Corollary 5.9, Proposition 5.15, and the following claim.

Claim. If $\{(w_m, s_m)\}$ converges to a horofunction, h, for d_{cc}, then either $\{w_m\}$ is bounded or $w_m \to \infty$.

Suppose that the assertions "$\{w_m\}$ is bounded" and "$w_m \to \infty$" are both false. Then $\{w_m\}$ has subsequences $\{w_{m_j}\}$ and $\{w_{k_j}\}$ such that $\{w_{m_j}\}$ is bounded and $w_{k_j} \to \infty$. Recall that any subsequence of $\{(w_m, s_m)\}$ converges to the same horofunction, h. By Corollary 5.9, $\{w_{m_j}\}$ converges and for all $(z, b) \in \mathbb{H}^n$, $h(z, b) = |w| - |w - z|$ where $w = \lim_{j\to\infty} w_{m_j}$. By Proposition 5.15, for all $(z, b) \in \mathbb{H}^n$, $h(z, b) = \mathrm{Re}\langle u, z\rangle$ where $u = -\lim_{j\to\infty}\exp\left(i\,\mu^{-1}\left(s_{k_j}/|w_{k_j}|^2\right)\right)(w_{k_j}/|w_{k_j}|)$. Hence, for all $z \in \mathbb{C}^n$, we have $|w| - |w - z| = \mathrm{Re}\langle u, z\rangle$ which is impossible because $z \mapsto \mathrm{Re}\langle u, z\rangle$ is linear whereas $z \mapsto |w| - |w - z|$ is not linear. □

Theorem 5.16 precisely identifies the functions in $C(\mathbb{H}^n)_0$ that occur as horofunctions for d_{cc} as follows.

Theorem 5.17 (Classification of horofunctions for d_{cc})**.** *Let h be a horofunction for d_{cc}. Then either there exists $u \in S^{2n-1}$ such that $h(z, b) = \mathrm{Re}\langle u, z\rangle$ for all $(z, b) \in \mathbb{H}^n$ or there exists $w \in \mathbb{C}^n$ such that $h(z, b) = |w| - |w - z|$ for all $(z, b) \in \mathbb{H}^n$. All functions of the form $(z, b) \mapsto \mathrm{Re}\langle u, z\rangle$ with $u \in S^{2n-1}$ or of the form $(z, b) \mapsto |w| - |w - z|$ with $w \in \mathbb{C}^n$ occur as horofunctions for d_{cc}. Thus*

$$\partial_h(\mathbb{H}^n, d_{\mathrm{cc}}) = \left\{(z, b) \mapsto \mathrm{Re}\langle u, z\rangle \mid u \in S^{2n-1}\right\} \cup \left\{(z, b) \mapsto |w| - |w - z| \mid w \in \mathbb{C}^n\right\}.$$

Proof. By Theorem 5.16, all horofunctions for d_{cc} have the indicated form; furthermore, for any $w \in \mathbb{C}^n$ the sequence $\{p_m = (w, m)\}$ in $\mathbb{H}^n$ converges to the horofunction $h(z, b) = |w| - |w - z|$ and for any $u \in S^{2n-1}$ the sequence $\{p_m = (-m\,u, 0)\}$ in $\mathbb{H}^n$ converges to the horofunction $h(z, b) = \mathrm{Re}\langle u, z\rangle$. □

6. Busemann points

We recall Rieffel's notion of an "almost geodesic ray" and of a Busemann point in a metric space.

Definition 6.1 ([Rie02, Definition 4.3]). Let (X, ρ) be a metric space and let $T \subseteq [0, \infty)$ be an unbounded subset with $0 \in T$. Let $\gamma \colon T \to X$ be a function.

(a) γ is a *geodesic ray* if $\rho(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in T$.
(b) γ is an *almost geodesic ray* if for every $\epsilon > 0$ there is an integer N such that $t, s \in T$ and $t \geq s \geq N$ implies
$$|\rho(\gamma(t), \gamma(s)) + \rho(\gamma(s), \gamma(0)) - t| < \epsilon.$$
(c) γ is a *weakly geodesic ray* if for every $y \in X$ and every $\epsilon > 0$ there is an integer N such that if $s, t \geq N$, then
$$|\rho(\gamma(t), \gamma(0)) - t| < \epsilon$$
and
$$|\rho(\gamma(t), y) - \rho(\gamma(s), y) - (t - s)| < \epsilon.$$

A geodesic ray is an almost geodesic ray and an almost geodesic ray is a weakly geodesic ray [Rie02, Lemma 4.5]; furthermore, every horofunction is the limit of a weakly geodesic ray [Rie02, Theorem 4.7] when X is proper and has a countable basis.

Definition 6.2 ([Rie02, Definition 4.8]). In a metric space (X, ρ), a horofunction, h, is a *Busemann point* if it is the limit of an almost geodesic ray, that is, there is an almost geodesic ray $\gamma \colon T \to X$ such that for all $x \in X$, $\lim_{t\to\infty} \rho(\gamma(t), x) - \rho(\gamma(t), x_0) = h(x) - h(x_0)$, where x_0 is a basepoint for X.

The following proposition can be used to detect non-Busemann points in the horofunction boundary of a metric space (see Theorem 6.5).

Proposition 6.3. *Let (X, ρ) be a metric space and let $x_0 \in X$ be a basepoint. Assume the horofunction h is a Busemann point and that $\gamma \colon T \to X$ is an almost geodesic ray converging to h. Then $\lim_{s\to\infty} h(\gamma(s)) - h(x_0) + \rho(\gamma(s), x_0) = 0$.*

Proof. Since γ is an almost geodesic ray, given $\epsilon > 0$ there is an integer N such that $t, s \in T$ and $t \geq s \geq N$ implies
$$|\rho(\gamma(t), \gamma(s)) + \rho(\gamma(s), \gamma(0)) - t| < \epsilon. \tag{6.1}$$

Setting $s = t$ in (6.1), we have for $t \in T$ and $t \geq N$ that
$$|\rho(\gamma(t), \gamma(0)) - t| < \epsilon. \tag{6.2}$$

Since γ converges to h, there exists an integer N' such that $t \in T$ and $t \geq N'$ implies
$$|\rho(\gamma(t), \gamma(0)) - \rho(\gamma(t), x_0) - h(\gamma(0)) + h(x_0)| < \epsilon. \tag{6.3}$$

Observe that
$$\begin{gathered}\rho(\gamma(t), \gamma(s)) - \rho(\gamma(t), x_0) + \rho(\gamma(s), x_0) = \\ (\rho(\gamma(t), \gamma(s)) + \rho(\gamma(s), \gamma(0)) - t) \;-\; (\rho(\gamma(t), \gamma(0)) - t) \\ + \rho(\gamma(t), \gamma(0)) - \rho(\gamma(t), x_0) - h(\gamma(0)) + h(x_0) \\ - (\rho(\gamma(s), \gamma(0)) - \rho(\gamma(s), x_0) - h(\gamma(0)) + h(x_0)).\end{gathered}$$

So combining (6.1), (6.2), and (6.3) we have for $t, s \in T$ and $t \geq s \geq \max(N, N')$ that

$$|\rho(\gamma(t), \gamma(s)) - \rho(\gamma(t), x_0) + \rho(\gamma(s), x_0)| < 4\epsilon. \tag{6.4}$$

Keeping s fixed and letting $t \to \infty$ in (6.4) shows that $s \geq \max(N, N')$ implies

$$|h(\gamma(s)) - h(x_0) + \rho(\gamma(s), x_0)| \leq 4\epsilon.$$

Hence $\lim_{s\to\infty} h(\gamma(s)) - h(x_0) + \rho(\gamma(s), x_0) = 0$. □

An *isometry* of a metric space (X, ρ) is a surjective function $f : X \to X$ such that $\rho(f(x), f(y)) = \rho(x, y)$ for all $x, y \in X$. When X is a (finite dimensional) connected topological manifold, the condition that $f : X \to X$ be surjective is redundant. Knowledge of the horofunctions of (X, ρ) can assist in the effort to find all the isometries of (X, ρ) by means of the following proposition.

Proposition 6.4. *Let (X, ρ) be a metric space and let h be a horofunction. If $\tau : X \to X$ is an isometry, then $h \circ \tau$ is also a horofunction. If h is a Busemann point, then $h \circ \tau$ is also a Busemann point.*

Proof. Let $x_0 \in X$ be a basepoint and let $\{p_m\}$ be a sequence in X such that $\rho(p_m, x_0) \to \infty$ and for all $x \in X$,

$$\lim_{m\to\infty} \rho(p_m, x) - \rho(p_m, x_0) = h(x) - h(x_0).$$

Since τ is an isometry,

$$\begin{aligned} \rho\left(\tau^{-1}p_m, x\right) - \rho\left(\tau^{-1}p_m, x_0\right) &= \rho(p_m, \tau x) - \rho(p_m, \tau x_0) \\ &= \rho(p_m, \tau x) - \rho(p_m, x_0) - (\rho(p_m, \tau x_0) - \rho(p_m, x_0)). \end{aligned} \tag{6.5}$$

The limit as $m \to \infty$ of the right side of the last equality in (6.5) is $h\circ\tau(x) - h\circ\tau(x_0)$. Hence $h \circ \tau$ is a horofunction.

Assume h is a Busemann point and $\gamma : T \to X$ is an almost geodesic ray converging to the horofunction h. Since τ^{-1} is an isometry, for all $s, t \in T$,

$$\begin{aligned} &\rho(\gamma(t), \gamma(s)) + \rho(\gamma(s), \gamma(0)) - t \\ &= \rho\left(\tau^{-1} \circ \gamma(t), \tau^{-1} \circ \gamma(s)\right) + \rho\left(\tau^{-1} \circ \gamma(s), \tau^{-1} \circ \gamma(0)\right) - t, \end{aligned}$$

and so it is clear from Definition 6.1(b) that $\tau^{-1} \circ \gamma$ is also an almost geodesic ray. Then using (6.5) with $\{\gamma(t) \mid t \in T\}$ replacing $\{p_m\}$, we see that $\tau^{-1} \circ \gamma$ converges to $h \circ \tau$. Hence $h \circ \tau$ is a Busemann point. □

Let $u \in S^{2n-1}$. By direct computation, the map $\gamma : [0, \infty) \to \mathbb{H}^n$ given by $\gamma(t) = (-tu, 0)$ is a geodesic ray for d_{cc} and by Theorem 5.16(1), γ converges to the horofunction $(z, b) \mapsto \text{Re}\langle u, z\rangle$. Thus such functions are Busemann points; indeed, they are the only Busemann points.

Theorem 6.5. *A horofunction, h, for d_{cc} is a Busemann point if and only if it is of the form $h(z, b) = \text{Re}\langle u, z\rangle$, for some $u \in S^{2n-1}$.*

Proof. By Theorem 5.17, the horofunction boundary of $\mathbb{H}^n$ for d_{cc} consists of the set of functions

$$\left\{(z, b) \mapsto \text{Re}\langle u, z\rangle \mid u \in S^{2n-1}\right\} \cup \{(z, b) \mapsto |w| - |w - z| \mid w \in \mathbb{C}^n\}.$$

We already showed that horofunctions of the form $(z, b) \mapsto \text{Re}\langle u, z\rangle$ with $u \in S^{2n-1}$ are Busemann points. Let h be a horofunction of the form $h(z, b) = |w| - |w - z|$

where $w \in \mathbb{C}^n$. Suppose that there exists an almost geodesic ray $\gamma\colon T \to \mathbb{H}^n$ converging to h. Let $\{t_m\} \subset T \subset [0,\infty)$ be such that $t_m \to \infty$ and let $(w_m, s_m) = \gamma(t_m)$. By Proposition 6.3, $\lim_{m\to\infty} |w| - |w - w_m| + \|(w_m, s_m)\|_{\mathrm{cc}} = 0$. Since $\|(w_m, s_m)\|_{\mathrm{cc}} \to \infty$, this implies that $w_m \to \infty$ and so by Theorem 5.16(1), there exists $u \in S^{2n-1}$ such that $h(z,b) = \mathrm{Re}\langle u, z\rangle$ for all $(z,b) \in \mathbb{H}^n$. Thus $\mathrm{Re}\langle u, z\rangle = |w| - |w-z|$ for all $z \in \mathbb{C}^n$, which is impossible. We conclude that no such γ exists. □

7. Isometries of $(\mathbb{H}^n, d_K)$ and $(\mathbb{H}^n, d_{\mathrm{cc}})$

In this section we use horofunctions to compute the isometry groups of $(\mathbb{H}^n, d_K)$ and $(\mathbb{H}^n, d_{\mathrm{cc}})$; see Theorems 7.1, 7.2, and 7.4.

We begin by constructing some interesting isometries of d_K fixing $0 \in \mathbb{H}^n$.

Let $T\colon \mathbb{C}^n \to \mathbb{C}^n$ be a unitary transformation; that is, $\langle Tu, Tv\rangle = \langle u, v\rangle$ for all $u, v \in \mathbb{C}^n$. Define $\widehat{T}\colon \mathbb{H}^n \to \mathbb{H}^n$ by $\widehat{T}(z,t) = (Tz, t)$. We have

$$\begin{aligned}\widehat{T}((w,s)(z,t)) &= \widehat{T}(w+z,\ s+t+2\,\mathrm{Im}\langle w,z\rangle)\\ = (T(w+z),\ s+t+2\,\mathrm{Im}\langle w,z\rangle) &= (Tw+Tz,\ s+t+2\,\mathrm{Im}\langle Tw,Tz\rangle)\\ &= \widehat{T}(w,s)\widehat{T}(z,t).\end{aligned}$$

Hence $\widehat{T}$ is a Lie group homomorphism, indeed, an isomorphism since $\widehat{T}^{-1} = \widehat{T^{-1}}$.

For $p = (w,s) \in \mathbb{H}^n$,

$$\left\|\widehat{T}(p)\right\|_K = \|(Tw, s)\|_K = \left(|Tw|^4 + s^2\right)^{1/4} = \left(|w|^4 + s^2\right)^{1/4} = \|p\|_K .$$

It follows that for all $p, q \in \mathbb{H}^n$,

$$d_K\left(\widehat{T}(p), \widehat{T}(q)\right) = \left\|\widehat{T}(p)^{-1}\widehat{T}(q)\right\|_K = \left\|\widehat{T}\left(p^{-1}q\right)\right\|_K = \left\|p^{-1}q\right\|_K = d_K(p,q),$$

which shows that $\widehat{T}$ is an isometry of d_K.

Define $C\colon \mathbb{H}^n \to \mathbb{H}^n$ by $C(w,s) = (\bar{w}, -s)$, where $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_n)$ for $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ and $\bar{w}_j$ is the conjugate of w_j. We have

$$\begin{aligned}C((w,s)(z,t)) &= C(w+z,\ s+t+2\,\mathrm{Im}\langle w,z\rangle)\\ = (\overline{w+z},\ -s-t-2\,\mathrm{Im}\langle w,z\rangle) &= (\bar{w}+\bar{z},\ -s-t+2\,\mathrm{Im}\langle \bar{w},\bar{z}\rangle)\\ &= (\bar{w},-s)(\bar{z},-t) = C(w,s)C(z,t).\end{aligned}$$

Hence C is a Lie group homomorphism, indeed, an isomorphism since $C^{-1} = C$.

For $p = (w,s) \in \mathbb{H}^n$,

$$\|C(p)\|_K = \|(\bar{w}, -s)\|_K = \left(|\bar{w}|^4 + (-s)^2\right)^{1/4} = \left(|w|^4 + s^2\right)^{1/4} = \|p\|_K .$$

It follows that for all $p, q \in \mathbb{H}^n$,

$$d_K(C(p), C(q)) = \left\|C(p)^{-1}C(q)\right\|_K = \left\|C\left(p^{-1}q\right)\right\|_K = \left\|p^{-1}q\right\|_K = d_K(p,q),$$

which shows that C is an isometry of d_K.

Theorem 7.1. *A map $\tau\colon \mathbb{H}^n \to \mathbb{H}^n$ is an isometry of d_K fixing 0 if and only if it is of the form $\widehat{T}$ or $C\widehat{T}$ where $T\colon \mathbb{C}^n \to \mathbb{C}^n$ is a unitary transformation.*

Proof. We have already demonstrated that maps of the form $\widehat{T}$ or $C\widehat{T}$ are isometries of d_K fixing 0. Assume $\tau\colon \mathbb{H}^n \to \mathbb{H}^n$ is an isometry of d_K with $\tau(0) = 0$. Write $\tau(z,t) = (\tau_1(z,t),\ \tau_2(z,t)) \in \mathbb{C}^n \times \mathbb{R}$.

Claim. $\tau_1(z,t)$ is independent of t and $\mathbb{R}$-linear in z.

Let $u \in \mathbb{C}^n$ with $|u| \le 1$. The function $\phi_u(z,t) = \operatorname{Re}\langle u, z\rangle$ is a horofunction for d_K (Remark 3.5) and all horofunctions have this form (Proposition 3.4). By Proposition 6.4, $\phi_u \circ \tau$ is a horofunction for d_K and hence there exists $v \in \mathbb{C}^n$ with $|v| \le 1$ such that $\phi_u \circ \tau = \phi_v$. It follows that $\operatorname{Re}\langle u, \tau_1(z,t)\rangle = \operatorname{Re}\langle v, z\rangle$ for all $(z,t) \in \mathbb{H}^n$. In particular, $\operatorname{Re}\langle u, \tau_1(z,t) - \tau_1(z,0)\rangle = 0$ for all $|u| \le 1$ and $(z,t) \in \mathbb{H}^n$. It follows that $\tau_1(z,t) = \tau_1(z,0)$ for all $(z,t) \in \mathbb{H}^n$. Define $T\colon \mathbb{C}^n \to \mathbb{C}^n$ by $Tz = \tau_1(z,0)$. For $\lambda \in \mathbb{R}$ and $w, z \in \mathbb{C}^n$,

$$\begin{aligned}\operatorname{Re}\langle u, T(\lambda w + z)\rangle &= \operatorname{Re}\langle v, \lambda w + z\rangle = \lambda \operatorname{Re}\langle v, w\rangle + \operatorname{Re}\langle v, z\rangle \\ &= \lambda \operatorname{Re}\langle u, Tw\rangle + \operatorname{Re}\langle u, Tz\rangle = \operatorname{Re}\langle u, \lambda Tw + Tz\rangle.\end{aligned}$$

Since this is valid for all $|u| \le 1$, we conclude $T(\lambda w + z) = \lambda Tw + Tz$ thus verifying the claim.

For all $z \in \mathbb{C}^n$ and $s, t \in \mathbb{R}$,

$$d_K((z,s),(z,t))^2 \;=\; d_K(\tau(z,s),\tau(z,t))^2.$$

The left side of this expression is $|-s+t|$ and the right side is $|-\tau_2(z,s) + \tau_2(z,t)|$ yielding:

$$|-s+t| = |-\tau_2(z,s) + \tau_2(z,t)|.$$

Thus, for each $z \in \mathbb{C}^n$, $s \mapsto \tau_2(z,s)$ is an isometry of $\mathbb{R}$ with the Euclidean metric. The isometries of $\mathbb{R}$ are of the form $s \mapsto \pm s + s_0$ and so, for each z, either $\tau_2(z,s) = s + \tau_2(z,0)$ for all s or $\tau_2(z,s) = -s + \tau_2(z,0)$ for all s. Since the function $z \mapsto \tau_2(z,1) - \tau_2(z,0) \in \{-1,1\}$ is continuous and $\mathbb{C}^n$ is connected, it must be constant. It follows that either

Case 1. For all $(z,s) \in \mathbb{H}^n$, $\tau_2(z,s) = s + \tau_2(z,0)$, or

Case 2. For all $(z,s) \in \mathbb{H}^n$, $\tau_2(z,s) = -s + \tau_2(z,0)$.

In Case 1, the identity $\|\tau(z,s)\|_K^4 = \|(z,s)\|_K^4$ yields

$$|Tz|^4 + (s + \tau_2(z,0))^2 = |z|^4 + s^2$$

which in turn gives

$$|Tz|^4 + 2s\;\tau_2(z,0) + \tau_2(z,0)^2 = |z|^4. \tag{7.1}$$

Since (7.1) holds for all s, it follows that $\tau_2(z,0) = 0$ and so $\tau_2(z,s) = s$ and $|Tz| = |z|$. Similarly, in Case 2 we conclude $\tau_2(z,s) = -s$ and $|Tz| = |z|$.

In either case, we have the condition $|Tz| = |z|$ for all $z \in \mathbb{C}^n$ which implies $\operatorname{Re}\langle Tw, Tz\rangle = \operatorname{Re}\langle w, z\rangle$ for all $w, z \in \mathbb{C}^n$ by the real polarization identity.

For all $w, z \in \mathbb{C}^n$,

$$d_K((w,0),(z,0))^4 \;=\; d_K(\tau(w,0),\tau(z,0))^4.$$

The left side of this expression is $|-w+z|^4 + (-2\operatorname{Im}\langle w, z\rangle)^2$ and the right side is $|-Tw+Tz|^4 + (-2\operatorname{Im}\langle Tw, Tz\rangle)^2$. Since $|-Tw+Tz| = |T(-w+z)| = |-w+z|$, we obtain

$$(\operatorname{Im}\langle w, z\rangle)^2 = (\operatorname{Im}\langle Tw, Tz\rangle)^2. \tag{7.2}$$

Since $\operatorname{Im}\langle w, z\rangle$ and $\operatorname{Im}\langle Tw, Tz\rangle$ are $\mathbb{R}$-bilinear functions of (w,z), (7.2) implies that either

Case 1. For all $w, z \in \mathbb{C}^n$, $\operatorname{Im}\langle w, z\rangle = \operatorname{Im}\langle Tw, Tz\rangle$, or

Case 2. For all $w, z \in \mathbb{C}^n$, $\operatorname{Im}\langle w, z\rangle = -\operatorname{Im}\langle Tw, Tz\rangle$.

In Case 1, T is unitary (recall that $\mathrm{Re}\langle Tw, Tz\rangle = \mathrm{Re}\langle w, z\rangle$ also holds) and in Case 2, the map $\overline{T}$ given by $\overline{T}(z) = \overline{Tz}$ is unitary.

In order to complete the proof, observe that for T unitary, $(z, s) \mapsto (Tz, s)$ (this is $\widehat{T}$) and $(z, s) \mapsto (\overline{T}z, -s)$ (this is $C\widehat{T}$) are isometries (as previously noted) whereas $(z, s) \mapsto (Tz, -s)$ and $(z, s) \mapsto (\overline{T}z, s)$ are not isometries (because $(z, s) \mapsto (z, -s)$ is not an isometry). □

Let $U(n)$ denote the group of unitary transformations of $\mathbb{C}^n$ and let $\mathbb{Z}/2\mathbb{Z}$ denote the cyclic group of order 2 with generator g and identity element e. Let $U(n) \rtimes \mathbb{Z}/2\mathbb{Z}$ denote the semidirect product of $U(n)$ with $\mathbb{Z}/2\mathbb{Z}$ where the action of the generator of $\mathbb{Z}/2\mathbb{Z}$ on $U(n)$ is given by $(gT)(z) = \overline{T\bar{z}}$ for $T \in U(n)$ and $z \in \mathbb{C}^n$.

Theorem 7.2. *The map $\theta \colon U(n) \rtimes \mathbb{Z}/2\mathbb{Z} \to \mathrm{Isom}(\mathbb{H}^n, d_K, 0)$ given by $\theta(T, e) = \widehat{T}$ and $\theta(T, g) = C\widehat{T}$ is an isomorphism of topological groups.*

Proof. If $S, T \in U(n)$, then it is straightforward to show that $\widehat{ST} = \widehat{S}\widehat{T}$ and $C\widehat{T}C = \widehat{T'}$ where $T'(z) = \overline{T}(\bar{z})$. Hence θ is a homomorphism. It is clearly continuous and injective. By Theorem 7.1, it is surjective. Since $U(n) \rtimes \mathbb{Z}/2\mathbb{Z}$ is compact, θ is also a homeomorphism. □

Remark 7.3. We showed that C and $\widehat{T}$, for $T \in U(n)$, are Lie group automorphisms of $\mathbb{H}^n$. Hence $\mathrm{Isom}(\mathbb{H}^n, d_K, 0)$ is a subgroup of the group of Lie group automorphisms of $\mathbb{H}^n$. The full group of isometries $\mathrm{Isom}(\mathbb{H}^n, d_K)$ is isomorphic to the semidirect product $\mathbb{H}^m \rtimes \mathrm{Isom}(\mathbb{H}^n, d_K, 0)$, where $\mathrm{Isom}(\mathbb{H}^n, d_K, 0)$ acts on $\mathbb{H}^n$ by $\tau \cdot p = \tau(p)$. The isomorphism $\mathbb{H}^m \rtimes \mathrm{Isom}(\mathbb{H}^n, d_K, 0) \to \mathrm{Isom}(\mathbb{H}^n, d_K)$ is given by $(p, \tau) \mapsto L_p\tau$ where L_p is left translation by p.

The maps C and $\widehat{T}$, for $T \in U(n)$, are also isometries of d_{cc}. This can be deduced from the fact that d_{cc} is the length metric associated to d_K (see [Kor85, 2.1]), which implies that any isometry of d_K is also an isometry of d_{cc}, or by direct computation as follows. For $w \neq 0$,

$$\omega\left(\widehat{T}(w, s)\right) = s/|Tw|^2 = s/|w|^2 = \omega(w, s)$$

and

$$\omega(C(w, s)) = -s/|\bar{w}|^2 = -s/|w|^2 = -\omega(w, s).$$

Hence $\omega \circ \tau = \pm\omega$ for $\tau = C$ or $\widehat{T}$. For such τ,

$$d_{\mathrm{cc}}(\tau(p), \tau(q)) = \tilde{\eta}\left(\omega\left(\tau(p)^{-1}\tau(q)\right)\right) d_K(\tau(p), \tau(q))$$
$$= \tilde{\eta}\left(\omega\left(\tau\left(p^{-1}q\right)\right)\right) d_K(p, q) = \tilde{\eta}\left(\omega\left(p^{-1}q\right)\right) d_K(p, q) = d_{\mathrm{cc}}(p, q).$$

Theorem 7.4. *A map $\tau \colon \mathbb{H}^n \to \mathbb{H}^n$ is an isometry of d_{cc} fixing $0 \in \mathbb{H}^n$ if and only if it is of the form $\widehat{T}$ or $C\widehat{T}$ where $T \colon \mathbb{C}^n \to \mathbb{C}^n$ is a unitary transformation. Furthermore,* $\mathrm{Isom}(\mathbb{H}^n, d_{\mathrm{cc}}, 0) = \mathrm{Isom}(\mathbb{H}^n, d_K, 0) \cong U(n) \rtimes \mathbb{Z}/2\mathbb{Z}$.

Proof. We have already demonstrated that maps of the form $\widehat{T}$ or $C\widehat{T}$ are isometries of d_{cc} fixing 0. Assume $\tau \colon \mathbb{H}^n \to \mathbb{H}^n$ is an isometry of d_{cc} with $\tau(0) = 0$. Write $\tau(z, t) = (\tau_1(z, t), \tau_2(z, t)) \in \mathbb{C}^n \times \mathbb{R}$.

Let $u \in S^{2n-1}$ and let $h : \mathbb{H}^n \to \mathbb{R}$ be the function $h(z, t) = \mathrm{Re}\langle u, z\rangle$. By Theorem 6.5, h is a Busemann point in the horofunction boundary of $(\mathbb{H}, d_{\mathrm{cc}})$ and all Busemann points have this form. By Proposition 6.4, $h \circ \tau$ is a Busemann point and so there exists $v \in S^{2n-1}$ such that $\mathrm{Re}\langle u, \tau_1(z, t)\rangle = \mathrm{Re}\langle v, z\rangle$ for all $(z, t) \in \mathbb{H}^n$.

As in the proof of Theorem 7.1, this implies that $\tau_1(z,t)$ is independent of t and $\mathbb{R}$-linear in z. Let $T\colon \mathbb{C}^n \to \mathbb{C}^n$ be given by $Tz = \tau_1(z,0)$.

For all $z \in \mathbb{C}^n$ and $s,t \in \mathbb{R}$,

$$d_{\mathrm{cc}}((z,s),(z,t))^2 \;=\; d_{\mathrm{cc}}(\tau(z,s),\tau(z,t))^2.$$

The left side of this expression is $\pi|-s+t|$ and the right side is $\pi|-\tau_2(z,s)+\tau_2(z,t)|$, yielding:

$$|-s+t| = |-\tau_2(z,s)+\tau_2(z,t)|.$$

As in the proof of Theorem 7.1, it follows that either:

Case 1. For all $(z,s) \in \mathbb{H}^n$, $\tau_2(z,s) = s + \tau_2(z,0)$, or

Case 2. For all $(z,s) \in \mathbb{H}^n$, $\tau_2(z,s) = -s + \tau_2(z,0)$.

Write $f(z) = \tau_2(z,0)$. For all $(z,s) \in \mathbb{H}^n$ with $z \neq 0$,

$$\begin{aligned}(7.3)\qquad \tilde{\eta}^4\left(s/|z|^2\right)\left(|z|^4+s^2\right) &= \|(z,s)\|_{\mathrm{cc}}^4 = \|\tau(z,s)\|_{\mathrm{cc}}^4 \\ &= \tilde{\eta}^4\left((\alpha s+f(z))/|Tz|^2\right)\left(|Tz|^4+(\alpha s+f(z))^2\right)\end{aligned}$$

where $\alpha = 1$ in Case 1 and $\alpha = -1$ in Case 2.

Letting $s = 0$ in (7.3) gives

$$(7.4)\qquad |z|^4 = c\left(|Tz|^4 + f(z)^2\right) \text{ where } c = \tilde{\eta}^4\left(f(z)/|Tz|^2\right).$$

Since $\tilde{\eta} \geq 1$, we have $c \geq 1$ and so (7.4) implies that $|z|^4 \geq |Tz|^4$.

Letting $s = -\alpha f(z)$ in (7.3) gives

$$(7.5)\qquad |Tz|^4 = d\left(|z|^4 + f(z)^2\right) \text{ where } d = \tilde{\eta}^4\left(f(z)/|Tz|^2\right).$$

Since $\tilde{\eta} \geq 1$, we have $d \geq 1$ and so (7.5) implies that $|Tz|^4 \geq |z|^4$. Hence $|Tz|^4 = |z|^4$ and so $|Tz| = |z|$. Substituting $|Tz| = |z|$ into (7.4) gives $|z|^4 = c\left(|z|^4 + f(z)^2\right)$. Since $c \geq 1$ this can only occur if $c = 1$ and $f(z) = 0$. Hence τ is of the form $\tau(z,s) = (Tz,s)$ for all $(z,s) \in \mathbb{H}^n$ or of the form $\tau(z,s) = (Tz,-s)$ for all $(z,s) \in \mathbb{H}^n$.

Assume $w \neq z$. Then

$$\begin{aligned}(7.6)\qquad & \tilde{\eta}^4\left(\frac{-s-2\operatorname{Im}\langle w,z\rangle}{|-w+z|^2}\right)\left(|-w+z|^4+(-s-2\operatorname{Im}\langle w,z\rangle)^2\right) \\ &= d_{\mathrm{cc}}((w,s),(z,0))^4 \;=\; d_{\mathrm{cc}}(\tau(w,s),\tau(z,0))^4 = \; d_{\mathrm{cc}}((Tw,\alpha s),(Tz,0))^4 \\ &= \tilde{\eta}^4\left(\frac{-\alpha s-2\operatorname{Im}\langle Tw,Tz\rangle}{|-Tw+Tz|^2}\right)\left(|-Tw+Tz|^4+(-\alpha s-2\operatorname{Im}\langle Tw,Tz\rangle)^2\right) \\ &= \tilde{\eta}^4\left(\frac{-\alpha s-2\operatorname{Im}\langle Tw,Tz\rangle}{|-w+z|^2}\right)\left(|-w+z|^4+(-\alpha s-2\operatorname{Im}\langle Tw,Tz\rangle)^2\right)\end{aligned}$$

where $\alpha = 1$ in Case 1 and $\alpha = -1$ in Case 2.

Letting $s = -2\operatorname{Im}\langle w,z\rangle$ in (7.6) gives

$$(7.7)\qquad |-w+z|^4 = e\left(|-w+z|^4 + (2\alpha\operatorname{Im}\langle w,z\rangle - 2\operatorname{Im}\langle Tw,Tz\rangle)^2\right)$$

where $e = \tilde{\eta}^4\left((2\alpha\operatorname{Im}\langle w,z\rangle - 2\operatorname{Im}\langle Tw,Tz\rangle)/|-w+z|^2\right)$. Since $\tilde{\eta} \geq 1$, we have $e \geq 1$ and so (7.7) implies that $e = 1$ and $(2\alpha\operatorname{Im}\langle w,z\rangle - 2\operatorname{Im}\langle Tw,Tz\rangle)^2 = 0$. Hence $\alpha\operatorname{Im}\langle w,z\rangle = \operatorname{Im}\langle Tw,Tz\rangle$.

As in the proof of Theorem 7.1, we conclude that if $\alpha = 1$, then T is unitary and if $\alpha = -1$, then $\overline{T}$ is unitary. Thus the isometries of d_{cc} are of the form $\tau(z,s) = (Tz,s)$ with T unitary or of the form $\tau(z,s) = (Tz,-s)$ with $\overline{T}$ unitary.

By Theorem 7.1, the isometries of d_K are also of this form and so $\mathrm{Isom}(\mathbb{H}^n, d_{\mathrm{cc}}, 0) = \mathrm{Isom}(\mathbb{H}^n, d_K, 0)$; furthermore, by Theorem 7.2, $\mathrm{Isom}(\mathbb{H}^n, d_K, 0) \cong U(n) \rtimes \mathbb{Z}/2\mathbb{Z}$. □

Remark 7.5. We have that $\mathrm{Isom}(\mathbb{H}^n, d_{\mathrm{cc}}, 0) = \mathrm{Isom}(\mathbb{H}^n, d_K, 0)$ is a subgroup of the group of Lie group automorphisms of $\mathbb{H}^n$. The full groups of isometries of d_{cc} and d_K also coincide; that is, $\mathrm{Isom}(\mathbb{H}^n, d_{\mathrm{cc}}) = \mathrm{Isom}(\mathbb{H}^n, d_K)$ (see Remark 7.3).

Kishimoto has shown ([Kis03, Theorem 4.2]) by means of a careful study of the geodesics of 2-step Carnot groups and using results of Hamenstädt [Ham90] that an isometry of a Carnot group fixing the identity is a Lie group automorphism. In particular, this applies to $(\mathbb{H}^n, d_{\mathrm{cc}})$ and the calculation of $\mathrm{Isom}(\mathbb{H}^n, d_{\mathrm{cc}}, 0)$ is then readily obtained. Our method using horofunctions shows directly that an isometry of $\mathbb{H}^n$ (for d_K or d_{cc}) fixing 0 is of the form $\widehat{T}$ or $C\widehat{T}$.

8. The topological types of the horofunction boundary and compactification

In this section we show that the horofunction boundary of $(\mathbb{H}^n, d_{\mathrm{cc}})$ is homeomorphic to a $2n$-disk (Theorem 8.2) and that the corresponding horofunction compactification is homeomorphic to a $(2n+1)$-sphere (Theorem 8.5). Finally, we compare the horofunction compactifications of $(\mathbb{H}^n, d_K)$ and $(\mathbb{H}^n, d_{\mathrm{cc}})$.

Let $D^{2n} = \{z \in \mathbb{C}^n \mid |z| \le 1\}$, the closed unit disk in $\mathbb{C}^n$, and let

$$\Sigma^{2n} = \partial(D^{2n} \times [-1,1]) = S^{2n-1} \times [-1,1] \ \cup\ D^{2n} \times \{-1,1\} \ \subset\ \mathbb{H}^n.$$

Note that Σ^{2n} is homeomorphic to a $2n$-sphere since it is the boundary of a $(2n+1)$-cell.

Define the map $\check{\mu}\colon [-1,1] \to [-\pi,\pi]$ by

$$\check{\mu}(s) = \begin{cases} \mu^{-1}(s/(1-|s|)) & \text{if } |s| < 1, \\ \pi & \text{if } s = 1, \\ -\pi & \text{if } s = -1, \end{cases}$$

where μ^{-1} is the inverse of Gaveau's function. The map $\check{\mu}$ is continuous, strictly increasing, and an odd function. Define the map $\psi\colon \Sigma^{2n} \to C(\mathbb{H}^n)_0$ by

$$\psi(w,s)(z,b) = \begin{cases} -\operatorname{Re}\langle \exp(i\,\check{\mu}(s))w, z\rangle & \text{if } |w| = 1 \text{ and } s \le 1, \\ \\ |w|/(1-|w|) - |\,w/(1-|w|) - z| & \text{if } |w| < 1 \text{ and } s = \pm 1. \end{cases}$$

Lemma 8.1. *The map ψ is continuous and its image is $\partial_{\mathrm{h}}(\mathbb{H}^n, d_{\mathrm{cc}})$.*

Proof. The restrictions of ψ to $S^{2n-1} \times [-1,1]$ and to $\operatorname{int}\big(D^{2n}\big) \times \{-1,1\}$ are clearly continuous. Hence it suffices to show that if $\{w_m\}$ is a sequence in D^{2n} such that $|w_m| < 1$ and $w_m \to w$ with $|w| = 1$, then for all $(z,b) \in \mathbb{H}^n$,

$$\psi(w_m, \pm 1)(z,b) = |w_m|/(1-|w_m|) - |\,w_m/(1-|w_m|) - z| \longrightarrow \operatorname{Re}\langle w, z\rangle. \tag{8.1}$$

Letting $w_m' = w_m/(1-|w_m|)$, we have $w_m' \to \infty$ and $w_m'/|w_m'| = w_m/|w_m| \to w$ and so (8.1) follows from Lemma 5.14.

Note that $\psi(-w,0)(z,b) = \operatorname{Re}\langle w, z\rangle$ for $|w| = 1$ and that the map $\operatorname{int}\big(D^{2n}\big) \to \mathbb{C}^n$ given by $w \mapsto w/(1-|w|)$ is a homeomorphism and so Theorem 5.17 implies that the image of ψ is the horofunction boundary of $(\mathbb{H}^n, d_{\mathrm{cc}})$. □

Define the map $p\colon \Sigma^{2n} \to D^{2n}$ by $p(w,s) = -\exp(i\,\check{\mu}(s))w$. Note that $p(w,\pm 1) = w$. Clearly, p is continuous and surjective; furthermore, since Σ^{2n} is compact, p is an identification map. Observe that ψ factors as $\psi = \bar{\psi}\circ p$ where $\bar{\psi}\colon D^{2n} \to \partial_{\mathrm{h}}(\mathbb{H}^n, d_{\mathrm{cc}})$ is given by

$$\bar{\psi}(u)(z,b) = \begin{cases} \operatorname{Re}\langle u, z\rangle & \text{if } |u| = 1,\\ |u|/(1-|u|) - |\,u/(1-|u|) - z| & \text{if } |u| < 1.\end{cases}$$

Theorem 8.2. *The map $\bar{\psi}\colon D^{2n} \to \partial_{\mathrm{h}}(\mathbb{H}^n, d_{\mathrm{cc}})$ is a homeomorphism and $\bar{\psi}\left(S^{2n-1}\right)$ is the set of Busemann points.*

Proof. Since $\psi = \bar{\psi}\circ p$ and p is an identification map, $\bar{\psi}$ is continuous. By Lemma 8.1, $\bar{\psi}$ is surjective. Assume $\bar{\psi}(u) = \bar{\psi}(v)$. Then either $|u| = |v| = 1$ or $|u| < 1$ and $|v| < 1$ because $\bar{\psi}(u)(z,b)$ is linear in z when $|u| = 1$ and not linear in z when $|u| < 1$. In the case $|u| = |v| = 1$ we have $\operatorname{Re}\langle u, z\rangle = \operatorname{Re}\langle v, z\rangle$ for all $z \in \mathbb{C}^n$, which implies $u = v$. In the case $|u| < 1$ and $|v| < 1$, letting $\bar{u} = |u|/(1-|u|)$ and $\bar{v} = |v|/(1-|v|)$, we have $|\bar{u}| - |\bar{u} - z| = |\bar{v}| - |\bar{v} - z|$ for all $z \in \mathbb{C}^n$. As in the proof of Lemma 5.10, this implies $\bar{u} = \bar{v}$. Since the map $\operatorname{int}(D^{2n}) \to \mathbb{C}^n$ given by $w \mapsto w/(1-|w|)$ is injective, it follows that $u = v$ which shows that $\bar{\psi}$ is injective. Hence $\bar{\psi}$ is a continuous bijection and thus a homeomorphism since D^{2n} is compact.

The assertion that $\bar{\psi}\left(S^{2n-1}\right)$ is the set of Busemann points is a direct consequence of Theorem 6.5. □

Define the map $\Phi\colon \operatorname{int}\left(D^{2n}\right) \times (-1,1) \to \mathbb{H}^n$ by

$$\Phi(w,s) = \left(w/(1-|w|), s/\left((1-|s|)(1-|w|)^2\right)\right).$$

Then Φ is a homeomorphism whose inverse is given by

$$\Phi^{-1}(y,t) = \left(y/(1+|y|), t/\left((1+|y|)^2 + |t|\right)\right).$$

Define the map $\Psi\colon D^{2n} \times [-1,1] \to C(\mathbb{H}^n)_0$ by

$$\Psi(w,s)(q) = \begin{cases} \psi(w,s)(q) & \text{if } (w,s) \in \Sigma^{2n} = \partial\left(D^{2n} \times [-1,1]\right),\\[2ex] d_{\mathrm{cc}}(\Phi(w,s),q) - \|\Phi(w,s)\|_{\mathrm{cc}} & \text{if } (w,s) \in \operatorname{int}\left(D^{2n}\right) \times (-1,1).\end{cases}$$

Lemma 8.3. *The map Ψ is continuous and its image is* $\operatorname{hc}(\mathbb{H}^n, d_{\mathrm{cc}})$.

Proof. The restrictions of Ψ to Σ^{2n} and to $\operatorname{int}\left(D^{2n}\right)\times(-1,1)$ are clearly continuous. Hence, it suffices to show that if $\{(w_m, s_m)\}$ is a sequence in $\operatorname{int}\left(D^{2n}\right) \times (-1,1)$ that converges to $(w,s) \in \Sigma^{2n}$, then for all $(z,b) \in \mathbb{H}^n$,

$$d_{\mathrm{cc}}(\Phi(w_m,s_m),(z,b)) - \|\Phi(w_m,s_m)\|_{\mathrm{cc}} \;\longrightarrow\; \psi(w,s)(z,b). \tag{8.2}$$

There are two cases to consider: Case 1, with $(w,s) \in S^{2n-1} \times [-1,1]$, and Case 2, with $(w,s) \in \operatorname{int}(D^{2n}) \times \{-1,1\}$. Let $(w'_m, s'_m) = \Phi(w_m, s_m)$. In Case 1, we have $|w_m| \to 1$ and $w'_m \to \infty$ and

$$\exp\left(i\mu^{-1}\left(s'_m/|w'_m|^2\right)\right) w'_m/|w'_m| = \exp\left(i\mu^{-1}\left(s_m/\left((1-|s_m|)|w_m|^2\right)\right)\right) w_m/|w_m|$$

converges to $\exp\left(i\mu^{-1}\left(s/(1-|s|)\right)\right) w$ if $s \neq \pm 1$ and to $-w$ if $s = \pm 1$. By Theorem 5.16(1), we have that (8.2) holds. In Case 2, we have $|w| < 1$ and so $\{w'_m\}$ converges to $w/(1-|w|)$ and we conclude from Theorem 5.16(2) that (8.2) again holds.

Since Ψ is continuous we have that $\Psi(D^{2n} \times [-1,1])$ is compact and therefore closed in $C(\mathbb{H}^n)_0$. Also $\Psi\left(\operatorname{int}\left(D^{2n}\right) \times (-1,1)\right)$ is dense in $\Psi\left(D^{2n} \times [-1,1]\right)$, so it follows that $\Psi\left(D^{2n} \times [-1,1]\right)$ is the horofunction compactification of $(\mathbb{H}^n, d_{\mathrm{cc}})$. □

Given a space X, a subspace $A \subset X$ and a map $f\colon A \to Y$, let X/f denote the quotient space of X obtained by identifying two points x, y in A whenever $f(x) = f(y)$. Let $\Pi(f)\colon X \to X/f$ denote the corresponding quotient map. We apply this construction to $D^{2n} \times [-1,1]$ and the map $p\colon \Sigma^{2n} \to D^{2n}$. Define the map $\overline{\Psi}\colon D^{2n} \times [-1,1]/p \to \mathrm{hc}(\mathbb{H}^n, d_{\mathrm{cc}})$ by

$$\overline{\Psi}(\Pi(p)(w,s)) = \begin{cases} \bar{\psi}(p(w,s)) & \text{if } (w,s) \in \Sigma^{2n}, \\ \Psi(w,s) & \text{if } (w,s) \in \operatorname{int}\left(D^{2n}\right) \times (-1,1). \end{cases}$$

Proposition 8.4. *The map* $\overline{\Psi}\colon D^{2n} \times [-1,1]/p \to \mathrm{hc}(\mathbb{H}^n, d_{\mathrm{cc}})$ *is a homeomorphism.*

Proof. The map Ψ factors as $\Psi = \overline{\Psi} \circ \Pi(p)$. It follows from Lemma 8.3 that $\overline{\Psi}$ is continuous and that its image is $\mathrm{hc}(\mathbb{H}^n, d_{\mathrm{cc}})$. Let $A = \Pi(p)\left(\Sigma^{2n}\right)$ and $B = \Pi(p)\left(\operatorname{int}\left(D^{2n}\right) \times (-1,1)\right)$. Note that $\overline{\Psi}(A) \cap \overline{\Psi}(B) = \emptyset$. The restriction of $\overline{\Psi}$ to B is injective because the restriction of Ψ to $\operatorname{int}\left(D^{2n}\right) \times (-1,1)$ is injective. For $q \in \Sigma^{2n}$ we have that $\overline{\Psi}(\Pi(p)(q)) = \bar{\psi}(p(q))$ and so by Theorem 8.2, the restriction of $\overline{\Psi}$ to A is injective. Consequently, $\overline{\Psi}$ is injective. Hence $\bar{\Psi}$ is a continuous bijection and thus a homeomorphism since $D^{2n} \times [-1,1]/p$ is compact. □

Theorem 8.5. *The horofunction compactification of* $(\mathbb{H}^n, d_{\mathrm{cc}})$ *is homeomorphic to a* $(2n+1)$*-sphere.*

Proof. By Proposition 8.4, it suffices to show that the space $D^{2n} \times [-1,1]/p$ is homeomorphic to a $(2n+1)$-sphere. Define the map $\theta\colon D^{2n} \times [-1,1] \to D^{2n} \times [-1,1]$ by $\theta(w,s) = (-\exp(i\,\check{\mu}(s))w, s)$. We have that θ is a homeomorphism whose inverse is given by $\theta^{-1}(w,s) = (-\exp(-i\,\check{\mu}(s))w, s)$. Define the map $p_1\colon \Sigma^{2n} \to D^{2n}$ by $p_1(w,s) = w$. Note that $p = p_1 \circ \theta|_{\Sigma^{2n}}$. The induced map $\hat{\theta}\colon D^{2n} \times [-1,1]/p_1 \to D^{2n} \times [-1,1]/p$ given by $\hat{\theta}(\Pi(p_1)(x)) = \Pi(p)(\theta(x))$ is a homeomorphism. Let $B_1 = \Pi(p_1)\left(D^{2n} \times [-1,0]\right)$ and let $B_2 = \Pi(p_1)\left(D^{2n} \times [0,1]\right)$. Observe that B_1 and B_2 are homeomorphic to $(2n+1)$-disks and that $\partial B_1 = \partial B_2 = B_1 \cap B_2$. Hence $D^{2n} \times [-1,1]/p_1 = B_1 \cup B_2$ is homeomorphic to a $(2n+1)$-sphere. □

A property shared by the metric spaces $(\mathbb{H}^n, d_K)$ and $(\mathbb{H}^n, d_{\mathrm{cc}})$ is that their horofunction compactifications are homeomorphic to a $(2n+1)$-sphere and their horofunction boundaries are homeomorphic to a $2n$-disk with the Busemann points corresponding to the boundary of such a $2n$-disk (for $(\mathbb{H}^n, d_K)$, see [KN09, Theorem 4.1]). Nevertheless, $(\mathbb{H}^n, d_K)$ and $(\mathbb{H}^n, d_{\mathrm{cc}})$ are not isometric. This can be proved using horofunctions as follows. Let $\tau\colon (\mathbb{H}^n, d_K) \to (\mathbb{H}^n, d_{\mathrm{cc}})$ be an isometry. Consider the sequence $\{p_m = (0,m)\}$ in $\mathbb{H}^n$. By Theorem 3.3, for all $q \in \mathbb{H}^n$, $d_K(p_m, q) - \|p_m\|_K$ converges to 0. Let $\{p_{m_j}\}$ be a subsequence of $\{p_m\}$ such that $\{\tau(p_{m_j})\}$ converges to a horofunction, h, for d_{cc}. We have, for all $q \in \mathbb{H}^n$,

$$d_{\mathrm{cc}}(\tau(p_{m_j}), \tau(q)) - \left\|\tau(p_{m_j})\right\|_{\mathrm{cc}} = d_K(p_{m_j}, q) - \left\|p_{m_j}\right\|_K$$

and so

$$h(\tau(q)) = \lim_{j\to\infty} d_{\mathrm{cc}}(\tau(p_{m_j}), \tau(q)) - \left\|\tau(p_{m_j})\right\|_{\mathrm{cc}} = \lim_{j\to\infty} d_K(p_{m_j}, q) - \left\|p_{m_j}\right\|_K = 0.$$

Hence h is the zero function; however, by the classification of horofunctions for d_{cc} (Theorem 5.17) the zero function is not a horofunction for d_{cc}. Thus no such isometry τ exists.

Given any left invariant metric ρ on $\mathbb{H}^n$, the left translation action of $\mathbb{H}^n$ on itself extends to a continuous action on $\mathrm{hc}(\mathbb{H}^n, \rho)$ via the formula $(g \cdot f)(x) = f(g^{-1}x) - f(g^{-1})$ where $g, x \in \mathbb{H}^n$ and $f \in C(\mathbb{H}^n)_0$. The horofunction compactifications of $(\mathbb{H}^n, d_K)$ and $(\mathbb{H}^n, d_{\mathrm{cc}})$ differ in the manner in which the left translation action extends to the horofunction boundary.

Proposition 8.6. *The $\mathbb{H}^n$-action on $\partial_{\mathrm{h}}(\mathbb{H}^n, d_K)$ is trivial and the $\mathbb{H}^n$-action on $\partial_{\mathrm{h}}(\mathbb{H}^n, d_{\mathrm{cc}})$ is not trivial. Consequently, there does not exist an $\mathbb{H}^n$-equivariant homeomorphism $\partial_{\mathrm{h}}(\mathbb{H}^n, d_K) \to \partial_{\mathrm{h}}(\mathbb{H}^n, d_{\mathrm{cc}})$.*

Proof. A horofunction for d_K has the form $h(z, b) = \mathrm{Re}\langle u, z\rangle$ where $|u| \le 1$ (Proposition 3.4) and so if $g = (w, s) \in \mathbb{H}^n$, then

$$\begin{aligned}(g \cdot h)(z, b) &= h(-w + z, -s + b - 2\,\mathrm{Im}\langle w, z\rangle) - h(-w, -s)\\ &= \mathrm{Re}\langle u, -w + z\rangle - \mathrm{Re}\langle u, -w\rangle = \mathrm{Re}\langle u, z\rangle = h(z, b).\end{aligned}$$

Hence the $\mathbb{H}^n$-action on $\partial_{\mathrm{h}}(\mathbb{H}^n, d_K)$ is trivial.

A horofunction for d_{cc} is either of the form $h(z, b) = \mathrm{Re}\langle u, z\rangle$ where $|u| = 1$ or of the form $h(z, b) = |v| - |v - z|$ where $v \in \mathbb{C}^n$ (Theorem 5.17). As shown above, $g \cdot h = h$ if $h(z, b) = \mathrm{Re}\langle u, z\rangle$. If $h(z, b) = |v| - |v - z|$, then $(w, s) \cdot h(z, b) = |w + v| - |w + v - z|$. Hence, for such h, we have $(w, s) \cdot h = h$ if and only if $w = 0$. In particular, the $\mathbb{H}^n$-action on $\partial_{\mathrm{h}}(\mathbb{H}^n, d_{\mathrm{cc}})$ is *not* trivial. □

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