CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 15, Pages 1–6 (January 24, 2011)
S 1088-4173(2010)00222-5

A NOTE ON THE HARMONIC MEASURE DOUBLING CONDITION

OLE J. BROCH, KARI HAG, AND STEFFEN JUNGE

ABSTRACT. We present a detailed and self-contained proof of the harmonic measure doubling characterization of bounded quasidisks due to Jerison and Kenig.

1. INTRODUCTION

Let $D \subset \overline{\mathbb{R}}^2$ be a Jordan domain and denote by $\omega(z, \gamma; D)$ the harmonic measure in D of a measurable set $\gamma \subset \partial D$ with respect to $z \in D$. To simplify the notation we write $\omega(\gamma) = \omega(z, \gamma; D)$ and $\omega^*(\gamma) = \omega(z^*, \gamma; D^*)$ for the harmonic measure in the complementary domain $D^* = \overline{\mathbb{R}}^2 \setminus \overline{D}$. We say that D satisfies the *harmonic measure doubling condition*, or that D is a *harmonic measure doubling domain* if there is a point $z_0 \in D$ and a constant $c \geq 1$ such that if $\gamma_1, \gamma_2 \subset \partial D$ are adjacent subarcs, then

$$\omega(z_0, \gamma_1; D) \leq c\, \omega(z_0, \gamma_2; D) \tag{1}$$

whenever $\operatorname{diam}(\gamma_1) \leq \operatorname{diam}(\gamma_2)$. That is to say: harmonic measure is doubling. Here the inequality for the diameters may be replaced by an equality.

Note that if inequality (1) holds for some $z_0 \in D$, then it holds for any other $z_1 \in D$, perhaps with a different constant [7]. Thus being a doubling domain is a property of the domain.

Clearly the unit disk $\mathbb{D}$ and its complementary domain $\mathbb{D}^*$ are harmonic doubling domains with constant $c = 1$ (and $z_0 = 0$ and $z_0^* = \infty$). Conversely, if a bounded Jordan domain D and its complementary domain D^* both are doubling with constant $c = 1$, D must be a disk. To see this, assume that D and D^* are harmonic doubling domains for $c = 1$ and the points $z_0 \in D$, $z_0^* \in D^*$. It is not difficult to see that then

$$\omega(z_0, \gamma_1; D) = \omega(z_0, \gamma_2; D) \quad \text{implies} \quad \operatorname{diam}(\gamma_1) = \operatorname{diam}(\gamma_2). \tag{2}$$

Of course the same conclusion holds in D^* as well. Now let $\phi \colon \mathbb{D} \to D$ and $\psi \colon \mathbb{D}^* \to D^*$ be conformal maps with $\phi(0) = z_0$ and $\psi(\infty) = z_0^*$. Let $I_1, I_2 \subset S^1 = \partial\mathbb{D}$ be adjacent arcs with $\operatorname{length}(I_1) = \operatorname{length}(I_2)$. Then $\omega(0, I_1; \mathbb{D}) = \omega(0, I_2; \mathbb{D})$, so that by conformal invariance $\omega(z_0, \phi(I_1); D) = \omega(z_0, \phi(I_2); D)$, and then (2) implies that

Received by the editors August 9, 2010.
2010 *Mathematics Subject Classification.* Primary 30C85, 30C62.
Key words and phrases. Harmonic measure, quasidisk.

$\operatorname{diam}(\phi(I_1)) = \operatorname{diam}(\phi(I_2))$. The doubling condition for $c = 1$ in $\mathbb{D}^*$ yields

$$\omega(z_0^*, \phi(I_1); D^*) = \omega(z_0^*, \phi(I_2); D^*),$$

so that by conformal invariance, once more we obtain

$$\omega(\infty, \psi^{-1}(\phi(I_1)); \mathbb{D}^*) = \omega(\infty, \psi^{-1}(\phi(I_2)); \mathbb{D}^*).$$

Let $f = \psi^{-1} \circ \phi \colon S^1 \to S^1$ denote the sewing homeomorphism associated with D (see e.g. [8] or [4]). We have proved that

$$\text{length}(I_1) = \text{length}(I_2) \quad \text{implies} \quad \text{length}(f(I_1)) = \text{length}(f(I_2)). \tag{3}$$

By performing two rotations, if necessary, we may assume that $f(1) = 1$. Then by continuity (3) implies that $f(-1) = -1$. Continuing bisecting arcs of the circle as in the proof of Characterization 2.3 in [5], we see that f fixes every point $e^{i\theta}$ on S^1 with $\theta = m\pi 2^{-n}$, $n = 1, 2, \ldots$ and $m = 1, \ldots, 2 \cdot 2^n$. By continuity, f is the identity mapping on S^1. Thus D must be a disk.

The class of harmonic doubling domains is an interesting one. The following theorem, originally proved by Jerison and Kenig [6], characterizes harmonic doubling domains in terms of quasidisks. Recall that a quasidisk is the image of an Euclidean disk or half plane under a quasiconformal self mapping of $\overline{\mathbb{R}}^2$.

Theorem (Jerison-Kenig [6])**.** *Let Γ be a closed Jordan plane curve with complementary domains (relative to $\overline{\mathbb{R}}^2$) D and D^*. Then Γ is a quasicircle if and only if both D and D^* are harmonic measure doubling domains. The quasicircle constant and doubling constant depend only on each other.*

The theorem is not true in $\mathbb{H}$, and hence not in Jordan domains whose boundaries are not compactly contained in $\mathbb{R}^2$, since in this case boundary arcs of large Euclidean diameters need not carry much harmonic measure.

Jerison and Kenig's original proof of the sufficiency in Theorem 1 is based on higher dimensional methods and is not immediately accessible to those mostly familier with the planar theory. Garnett and Marshall's proof in [2, pp. 244–245] is not so detailed and contains misprints.

It is the purpose of the present note to present a detailed and self-contained proof relying only on well-known, planar methods.

2. Proof of the Theorem

For the sufficiency, suppose that D and D^* satisfy the harmonic doubling condition with respect to $z_0 \in D$ and $z_0^* \in D^*$, with the same constant b. We exhibit a constant $c = c(b)$ such that whenever two adjacent arcs γ_1 and γ_2 of Γ satisfy

$$0 < \omega(\gamma_1) = \omega(\gamma_2) \leq 1/2, \tag{4}$$

then $\omega^*(\gamma_2) \leq c\,\omega^*(\gamma_1)$. A result due to Krzyz [8] then says that Γ is a quasicircle, with the quasicircle constant depending only on c.

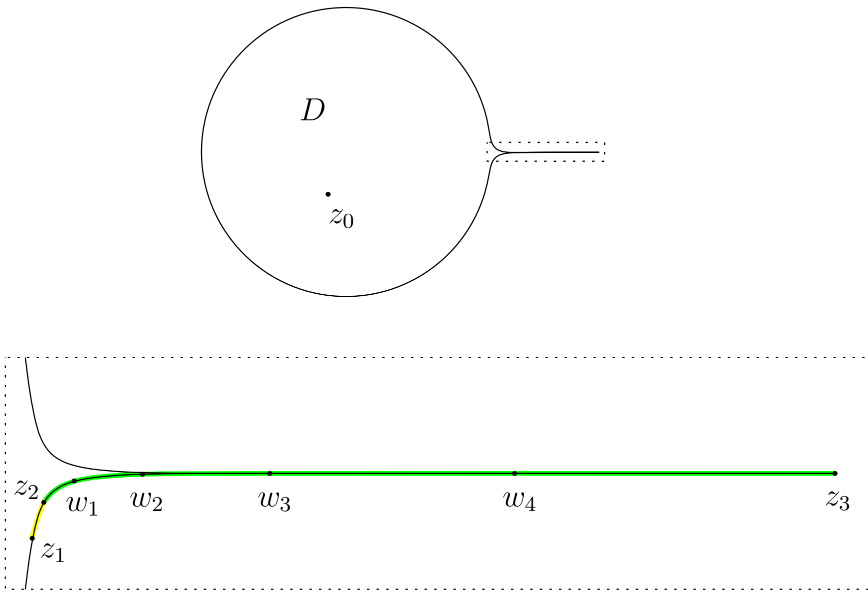

FIGURE 1. The figure illustrates how the sequence of w_k's for a given pair of γ_1 (yellow) and γ_2 (green) is defined. Moving z_1 and z_2 towards z_3 allows us to construct sequences of w_k's of arbitrary length. Hence, harmonic measure in D being doubling removes the possibility of outward cusps. Similarly D^* being a harmonic measure doubling domain excludes the possibility of inward cusps.

Without loss of generality, we may assume that

$$\operatorname{diam}(\gamma_1) \leq \operatorname{diam}(\gamma_2). \tag{5}$$

Denote by z_1 and z_2 the end points of γ_1 and by z_2 and z_3 the end points of γ_2. For $z, w \in \gamma_1 \cup \gamma_2$ we let $\gamma(z, w)$ denote the arc of ∂D from z to w contained in $\gamma_1 \cup \gamma_2$ and order points along $\gamma_1 \cup \gamma_2$ in the direction from z_1 to z_3.

We choose $w_0, w_1, \ldots, w_n$ on γ_2 as follows. Let $w_0 = z_2$. Suppose that points $w_0, w_1, \ldots, w_k$ have been chosen along $\gamma_2 = \gamma(z_2, z_3)$ so that for all $0 \leq i \leq k-1$, w_{i+1} is the first point with

$$\operatorname{diam}(\gamma(w_i, w_{i+1})) = \operatorname{diam}(\gamma(z_1, w_i)).$$

If $\operatorname{diam}(\gamma(w_k, z_3)) < \operatorname{diam}(\gamma(w_k, z_1))$, then we stop and set $n = k$. Otherwise, continue. This process must stop with $n \leq b$, for by using the doubling condition in D we have that

$$\omega(\gamma_1) \leq \omega(\gamma(z_1, w_k)) \leq b\, \omega(\gamma(w_k, w_{k+1})).$$

This implies that

$$n\, \omega(\gamma_1) \leq b \sum_{k=0}^{n-1} \omega(\gamma(w_k, w_{k+1})) \leq b\, \omega(\gamma_2),$$

and by (4) we must conclude that $n \leq b$.

Next we will use the doubling condition in D^*. For this we choose $n \leq b$ as above so that $\operatorname{diam}(\gamma(w_n, z_3)) < \operatorname{diam}(\gamma(z_1, w_n))$ and $\omega^*(\gamma(w_n, z_3)) \leq b\,\omega^*(\gamma(z_1, w_n))$. Hence,

$$\begin{aligned} \omega^*(\gamma(z_2, z_3)) &= \omega^*(\gamma(z_2, w_n)) + \omega^*(\gamma(w_n, z_3)) \\ &\leq (1+b)\,\omega^*(\gamma(z_1, w_n)). \end{aligned} \tag{6}$$

Moreover, for $k = 0, 1, \ldots, n-1$ we have that

$$\operatorname{diam}(\gamma(w_k, w_{k+1})) = \operatorname{diam}(\gamma(z_1, w_k))$$

and

$$\omega^*(\gamma(w_k, w_{k+1})) \leq b\,\omega^*(\gamma(z_1, w_k)).$$

From this we see that

$$\begin{aligned} \omega^*(\gamma(z_1, w_{k+1})) &= \omega^*(\gamma(z_1, w_k)) + \omega^*(\gamma(w_k, w_{k+1})) \\ &\leq (1+b)\,\omega^*(\gamma(z_1, w_k)), \end{aligned}$$

for $k = 0, 1, \ldots, n-1$. Combining (6) with these n inequalities we get

$$\begin{aligned} \omega^*(\gamma(z_2, z_3)) &\leq (b+1)^{n+1}\,\omega^*(\gamma(z_1, z_2)) \\ &\leq (b+1)^{b+1}\,\omega^*(\gamma(z_1, z_2)). \end{aligned}$$

All in all we have that

$$\omega^*(\gamma_2) \leq (b+1)^{b+1}\omega^*(\gamma_1),$$

and Γ is a quasicircle by [8].

Conversely, assume that ∂D is a bounded quasicircle and let $\phi\colon D \to \mathbb{D}$ be a conformal mapping. Then ϕ has an extension to a quasiconformal self mapping of $\overline{\mathbb{R}}^2$ which we also denote by ϕ. By composing ϕ with a Möbius transformation, if necessary, we may assume that $\phi(\infty) = \infty$. Let γ_1 and γ_2 be adjacent arcs in ∂D with $\operatorname{diam}(\gamma_1) = \operatorname{diam}(\gamma_2)$ and let the pairs z_1, z_2 and z_2, z_3 be their end points, respectively. By Ahlfors' three point conditon [1] we have that

$$|z_3 - z_2| \leq \operatorname{diam}(\gamma_2) = \operatorname{diam}(\gamma_1) \leq a|z_2 - z_1|,$$

where $a = a(D)$. By an elementary distortion theorem for quasiconformal mappings (Corollary 2.7 in [3]) we obtain

$$|\phi(z_3) - \phi(z_2)| \leq c|\phi(z_2) - \phi(z_1)|,$$

$c = c(a)$. In the following, we assume without loss of generality that the arc $\phi(\gamma_2)$ is the larger arc. If this arc subtends an angle of less than π at the origin, then

$$\omega(0, \phi(\gamma_2); \mathbb{D}) \leq c\,\frac{\pi}{2}\,\omega(0, \phi(\gamma_1); \mathbb{D}).$$

In the case when $\phi(\gamma_2)$ subtends an angle larger than π, we look at the subarc of $\phi(\gamma_2)$ from $\phi(z_2)$ to $\phi(z_3')$ of length π and apply Ahlfors' three point condition to $\gamma_2' = \gamma(z_2, z_3')$ and $\gamma_1' = \gamma(z_1', z_2)$ of the same diameter and proceed as before (Figure 2). In this case we have that

$$\omega(0, \phi(\gamma_2); \mathbb{D}) \leq \pi c\,\omega(0, \phi(\gamma_1); \mathbb{D}),$$

and we conclude that D is doubling. In the same way we prove that D^* is doubling.

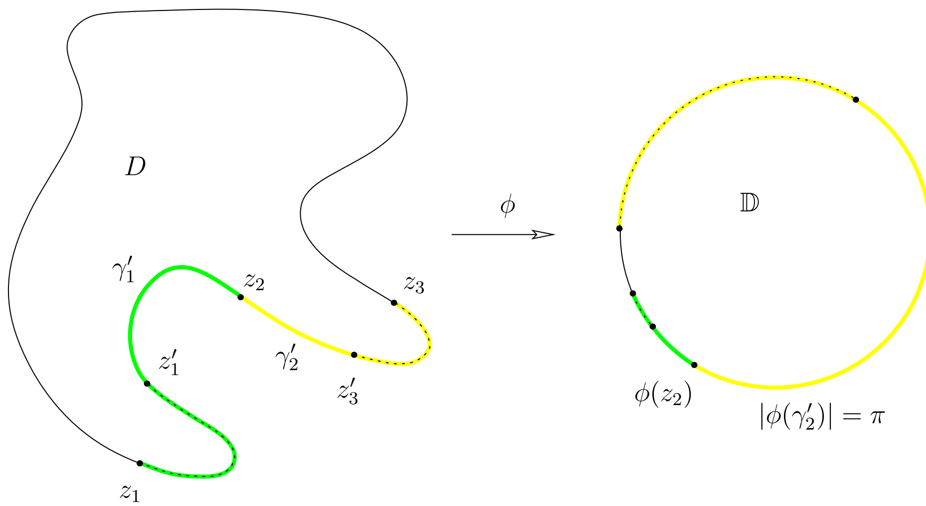

FIGURE 2. If $\omega(0, \phi(\gamma_2); \mathbb{D}) > 1/2$, the dotted subarcs γ_1' and γ_2' and their images under ϕ are removed from the argument with the remaining green and yellow subarcs satisfying the conditions from the first case. The first case guarantees that the remaining green part on the right hand side cannot get too small.

3. Remarks

In our definition of the harmonic doubling condition, we demand that $\text{diam}(\gamma_1) \le \text{diam}(\gamma_2)$ should imply (1). However, in Jerison and Kenig's original proof of the Theorem [6], it is stated that it is enough to assume that $\text{diam}(\gamma_1) \le k\, \text{diam}(\gamma_2)$, for some constant $k \ge 1$.

The harmonic doubling condition in D only is not enough to guarantee that D is a quasidisk [6]. Indeed, a bounded Jordan domain is doubling if and only if it is a *John domain* (Kim and Langmeyer [7]).

There still remains the question of whether the doubling conditon with $c = 1$ in D *only* is enough to guarantee that D is a disk.

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SINTEF Fisheries and aquaculture, NO 7465 Trondheim, Norway
E-mail address: ole.jacob.broch@sintef.no

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO 7491 Trondheim, Norway
E-mail address: kari@math.ntnu.no

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO 7491 Trondheim, Norway
E-mail address: steffen.junge@math.ntnu.no