CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 15, Pages 20–43 (March 1, 2011)
S 1088-4173(2011)00221-9

ON BRANSON'S Q-CURVATURE OF ORDER EIGHT

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Abstract. We prove universal recursive formulas for Branson's Q-curvature of order eight in terms of lower-order Q-curvatures, lower-order GJMS-operators and holographic coefficients. The results confirm a special case of a conjecture in [On conformally covariant powers of the Laplacian, `arXiv:0905.3992v3`].

Contents

1. Introduction and statement of results

It is well known that on any Riemannian manifold (M, g) of dimension $n \geq 2$, the second-order differential operator

$$P_2(g) = \Delta_g - \left(\frac{n}{2} - 1\right) \frac{\mathrm{scal}(g)}{2(n-1)} \tag{1.1}$$

is conformally covariant in the sense that

$$e^{(\frac{n}{2}+1)\varphi} P_2(e^{2\varphi} g)(u) = P_2(g)(e^{(\frac{n}{2}-1)\varphi} u)$$

for all $\varphi \in C^\infty(M)$ and all $u \in C^\infty(M)$. Here, Δ_g denotes the Laplace-Beltrami operator of the metric g and $\mathrm{scal}(g)$ is the scalar curvature of g. The operator P_2 is called the *conformal Laplacian* or *Yamabe operator*. More generally, in [GJMS92] Graham et al. proved that on any Riemannian manifold (M, g) of *even* dimension n, there exists a finite sequence $P_2(g), P_4(g), \ldots, P_n(g)$ of geometric differential operators of the form

$$\Delta_g^N + \text{lower order terms}$$

Received by the editors May 2, 2010.
2010 *Mathematics Subject Classification.* Primary 53B20 53B30; Secondary 53A30.
Key words and phrases. Q-curvature, conformally invariant powers of Laplacian, Poincaré-Einstein metrics, renormalized volume, universality.
This work was supported by SFB 647 "Space-Time-Matter" of DFG.

so that

$$e^{(\frac{n}{2}+N)\varphi}P_{2N}(e^{2\varphi}g)(u) = P_{2N}(g)(e^{(\frac{n}{2}-N)\varphi}u).$$

The operators $P_{2N}(g)$ are *geometric* in the sense that the lower order terms are completely determined by the metric and its curvature. On the flat space $\mathbb{R}^n$, there is no non-trivial curvature, and we have $P_{2N} = \Delta^N$. We shall follow common practice by referring to these operators as the GJMS-operators.

The constant terms of the GJMS-operators lead to the notion of Branson's Q-curvature (see [B95]). The critical GJMS-operator P_n is special in the sense that it has vanishing constant term; that is, $P_n(g)(1) = 0$. More generally, for $2N < n$, it is natural to write the constant term of P_{2N} in the form

$$P_{2N}(g)(1) = (-1)^N \left(\frac{n}{2} - N\right) Q_{2N}(g) \tag{1.2}$$

with a scalar Riemannian curvature invariant $Q_{2N}(g) \in C^\infty(M)$ of order $2N$. With this convention, the critical Q-curvature $Q_n(g)$ can be defined through Q_{2N}, $2N < n$, by continuation.[1]

Since the algorithmic definition in [GJMS92] is quite involved, a direct derivation of formulas for P_{2N} in terms of the metric is very complicated if possible at all. However, in the simplest cases $N = 1$ and $N = 2$ such evaluations are well known to yield the familiar Yamabe operator (1.1) and the Paneitz-operator

$$P_4 = \Delta^2 + \delta((n-2)\mathsf{J} - 4\mathsf{P})d + \left(\frac{n}{2} - 2\right)\left(\frac{n}{2}\mathsf{J}^2 - 2|\mathsf{P}|^2 - \Delta\mathsf{J}\right). \tag{1.3}$$

Here we use the notation

$$\mathsf{J} = \frac{\mathrm{scal}}{2(n-1)} \quad \text{and} \quad \mathsf{P} = \frac{1}{n-2}(\mathrm{Ric} - \mathsf{J}g).$$

P is called the Schouten tensor. In (1.3), it is regarded as an endomorphism of $\Omega^1(M)$. Equation (1.3) shows that, on manifolds of dimension $n \geq 4$,

$$Q_4 = \frac{n}{2}\mathsf{J}^2 - 2|\mathsf{P}|^2 - \Delta\mathsf{J}. \tag{1.4}$$

In particular, on manifolds of dimension four, the critical Q-curvature is given by

$$Q_4 = 2(\mathsf{J}^2 - |\mathsf{P}|^2) - \Delta\mathsf{J}. \tag{1.5}$$

It has often been said that the complexity of GJMS-operators and Q-curvatures increases exponentially with their order. It is tempting to compare this with the complexity of heat coefficients. Some explicit formulas for Q_6 and Q_8 in terms of the Schouten tensor P, the Weyl tensor W, and their covariant derivatives were derived in [GP03]. The enormous complexity of these formulas indicates that it is extremely hard to unveil the structure of high order Q-curvatures. A crucial part of the problem is to decide about the most natural way of stating the results.

In [J09a], we introduced and developed the idea to investigate Q-curvature from a conformal submanifold perspective. In particular, we introduced the notion of residue families $D^{res}_{2N}(g;\lambda)$. These are certain families of local operators which contain basic information on the structure of Q-curvatures and GJMS-operators. Besides motivations by representation theory, the approach builds on the interpretation of GJMS-operators as residues of the scattering operator of conformally compact Einstein metrics (see [GZ03]). The residue families satisfy systems of recursive relations which can be used to reveal the recursive structure of Q-curvatures

[1] The signs in (1.2) are required by the convention that $-\Delta$ is nonnegative.

and GJMS-operators. Along such lines, we found recursive formulas for the critical Q-curvatures Q_6 and Q_8, which express these quantities in terms of respective lower order GJMS-operators and lower order Q-curvatures. In [J09b] and [FJ09], these methods were further developed and led to the formulation of a number of conjectures.

In [J09a], the discussion of recursive formulas for Q_8 was dependent upon some technical assumptions. Here we remove these assumptions.

The formulation of the main results requires us to define one more ingredient. For a given metric g on the manifold M of even dimension n, let

$$g_+ = r^{-2}(dr^2 + g_r) \tag{1.6}$$

with

$$g_r = g + r^2 g_{(2)} + \cdots + r^{n-2} g_{(n-2)} + r^n (g_{(n)} + \log r \bar{g}_{(n)}) + \cdots \tag{1.7}$$

be a metric on $M \times (0, \varepsilon)$ so that the tensor $\mathrm{Ric}(g_+) + n g_+$ satisfies the Einstein condition

$$\mathrm{Ric}(g_+) + n g_+ = O(r^{n-2}) \tag{1.8}$$

together with a certain vanishing trace condition. These conditions uniquely determine the coefficients $g_{(2)}, \ldots, g_{(n-2)}$. They are given as polynomial formulas in terms of g, its inverse, the curvature tensor of g, and its covariant derivatives. The coefficient $\bar{g}_{(n)}$ and the quantity $\mathrm{tr}\, g_{(n)}$ are determined as well. Moreover, $\bar{g}_{(n)}$ is trace-free, and the trace-free part of $g_{(n)}$ is undetermined. A metric g_+ with these properties is called a Poincaré-Einstein metric with conformal infinity $[g]$. For full details see [FG07].

The volume form of g_+ can be written as

$$vol(g_+) = r^{-n-1} v(r) dr vol(g),$$

where

$$v(r) = vol(g_r)/vol(g) \in C^\infty(M).$$

The coefficients $v_0, \ldots, v_n$ in the Taylor series

$$v(r) = v_0 + v_2 r^2 + v_4 r^4 + \cdots + v_n r^n + \cdots$$

are known as the *renormalized volume* coefficients ([G00] and [G09]) or *holographic* coefficients ([J09a] and [BJ10]). The coefficient $v_{2j} \in C^\infty(M)$ is given by a local formula which involves at most $2j$ derivatives of the metric. Note also that v_n is uniquely determined by g since $\mathrm{tr}\, g_{(n)}$ is uniquely determined by g. It is called the holographic anomaly. Explicit formulas for the holographic coefficients v_2, v_4, v_6, and v_8 were derived in [G09].

The first main result describes the *critical* Q-curvature of order eight.

Theorem 1.1. *On manifolds of dimension* 8*, Branson's Q-curvature Q_8 is given by the formula*

$$\begin{aligned} Q_8 = &-3P_2(Q_6) - 3P_6(Q_2) + 9P_4(Q_4) \\ &+ 8P_2P_4(Q_2) - 12P_2^2(Q_4) + 12P_4P_2(Q_2) - 18P_2^3(Q_2) + 3!4!2^8 w_8, \end{aligned} \tag{1.9}$$

where w_8 is the coefficient of r^8 in the Taylor series of $\sqrt{v(r)}$.

In terms of holographic coefficients, the quantity w_8 can be expressed as

$$128w_8 = 64v_8 - 32v_6v_2 - 16v_4^2 + 24v_2^2v_4 - 5v_2^4 \tag{1.10}$$

(see Lemma 2.2).

A version of Theorem 1.1 was proved in Section 6.13 of [J09a] under the assumption that the polynomial $\mathcal{V}_8(\lambda)$ (see (2.13)) vanishes. In the present paper, we show that this assumption is vacuous (Proposition 2.2).

Theorem 1.1 confirms the special case $n = 8$ and $N = 4$ of a conjectural formula for all Q-curvatures Q_{2N} formulated in [J09b]. In connection with this conjecture, it is important to recognize that the coefficients in (1.9) have a uniform definition. In order to describe this, we introduce some notation. A sequence $I = (I_1, \dots, I_r)$ of integers $I_j \geq 1$ will be regarded as a composition of the sum $|I| = I_1 + I_2 + \cdots + I_r$, where two representations which contain the same summands but differ in the order of the summands are regarded as different. $|I|$ is called the size of I. For $I = (I_1, \dots, I_r)$, we set

$$P_{2I} = P_{2I_1} \circ \cdots \circ P_{2I_r}.$$

For any composition I, we define the multiplicity m_I by

$$m_I = -(-1)^r |I|!\,(|I|-1)! \prod_{j=1}^{r} \frac{1}{I_j!\,(I_j-1)!} \prod_{j=1}^{r-1} \frac{1}{I_j+I_{j+1}}. \tag{1.11}$$

Here, an empty product has to be interpreted as 1. Note that $m_{(N)} = 1$ for all $N \geq 1$. In these terms, the coefficient of the term

$$P_{2I}(Q_{2a}),\ |I| + a = 4$$

on the right-hand side of (1.9) is given by

$$-(-1)^a m_{(I,a)}, \tag{1.12}$$

and (1.9) can be stated as

$$\sum_{|I|+a=4} (-1)^a m_{(I,a)} P_{2I}(Q_{2a}) = 3!4!2^8 w_8. \tag{1.13}$$

This is a special case of Conjecture 9.2 in [J09b].

Of course, Theorem 1.1 does not yet provide an explicit formula for Q_8 in terms of the metric. Such a formula can be derived by combining it with formulas for the lower-order GJMS-operators P_2, P_4, and P_6 and the lower-order Q-curvatures Q_2, Q_4, and Q_6 in dimension $n = 8$. The relevant formulas will be discussed in Section 5. However, we emphasize that the resulting identities for Q_8 are structurally less natural than the description (1.9).

A second feature which distinguishes the formula (1.9) for Q_8 from other formulas is its universality in the dimension of the underlying space.

Theorem 1.2. *On any Riemannian manifold of dimension $n \geq 8$, Branson's Q-curvature Q_8 is given by the recursive formula* (1.9).

For a proof of Theorem 1.2 for the round spheres S^n see [J09b]. The results of [J09c] also cover the formula (1.9) for the conformally flat Möbius spheres $(S^q \times S^p, g_{S^q} - g_{S^p})$ with the round metrics on the factors.

Now we return to the critical case. A closer examination of (1.9) shows that in the sum on the right-hand side a substantial number of cancellations takes place. This leads to the following result.

Theorem 1.3. *On manifolds of dimension* 8*, Branson's* Q*-curvature* Q_8 *equals the sum of*

$$\begin{aligned}(1.14)\quad & -3P_2^0(Q_6) - 3P_6^0(Q_2) + 9P_4^0(Q_4) \\ & \qquad + 8P_2^0P_4(Q_2) - 12P_2^0P_2(Q_4) + 12P_4^0P_2(Q_2) - 18P_2^0P_2^2(Q_2),\end{aligned}$$

the divergence term

$$6\delta\left(c(2Q_4 + 3P_2(Q_2), Q_2)\right), \tag{1.15}$$

where $c(f,g) = fdg - gdf \in \Omega^1(M)$*, and*

$$3!4!2^7 v_8.$$

Here P_{2N}^0 *denotes the non-constant part of* P_{2N}*.*

The reader should note the tiny difference in the last terms in the formulas in Theorems 1.1 and 1.3: $2w_8$ is replaced by v_8. Note also that $P_2^0 = \Delta$.

Since the operators P_{2N}^0 are of the form $\delta(S_{2N}d)$ for some linear operators S_{2N} on $\Omega^1(M)$ (see [B95]), Theorem 1.3 reproves the following special case of a result of Graham and Zworski (see [GZ03]).

Corollary 1.1. *On closed manifolds* M *of dimension* 8*,*

$$\int_M Q_8 vol = 3!4!2^7 \int_M v_8 vol. \tag{1.16}$$

The present paper rests on the approach to Q-curvature developed in [J09a]. For the complete details we refer to this book and to Chapter 1 of [BJ10].[2]

The paper is organized as follows. Section 2 contains a proof of the recursive formula (1.9) for the critical Q-curvature Q_8. The key observation in this proof is the vanishing of the polynomial $\mathcal{V}_8(\lambda)$ (Proposition 2.2). Although the proof of Theorem 1.1 only applies the vanishing of the leading coefficient of $\mathcal{V}_8(\lambda)$, the vanishing result Proposition 2.2 is of independent interest. It shows that the vanishing property used here is related to other relations as e.g. the holographic formula for Q_8 (see [GJ07]). The reduced form (1.14) of (1.9) is derived in Section 3. In Section 4, we prove the universality of (1.9), i.e., Theorem 1.2. This proof also sheds new light on the proof in Section 2. In dimension $n \geq 8$, it is still true that an analog of the polynomial $\mathcal{V}_8(\lambda)$ has a vanishing leading coefficient although the polynomial itself does not vanish. This fact can be used to extend the arguments of Section 2. Here we give an alternative argument. The central point is to prove the formula in Proposition 4.1. This identity will be established as a consequence of a more general result (Theorem 4.1) which also provides a certain explanation of the appearance of the square root of $v(r)$ in Theorems 1.1 and 1.2. In Section 5, we discuss analogous descriptions of Q_4 and Q_6, and display universal formulas for the GJMS-operators P_4 and P_6. Section 6 contains comments on further developments. In particular, we describe the status of Conjecture 9.2 of [J09b].

2. Proof of Theorem 1.1

The basic idea of the proof of Theorem 1.1 is to compare two different evaluations of the leading coefficient of the Q-curvature polynomial $Q_8^{res}(\lambda)$.

[2] We also use the opportunity to correct some misprints in [J09a].

We first recall the notion of Q-curvature polynomials (or Q-polynomials for short) as introduced in [J09a]; see also Section 1.6 of [BJ10]. For this purpose, let

$$u \sim \sum_{N\geq 0} r^{\lambda+2N}\mathcal{T}_{2N}(g;\lambda)(f),\ \mathcal{T}_0(g;\lambda)(f)=f,\ r\to 0 \tag{2.1}$$

be the asymptotic expansions of an eigenfunction u of the Laplace-Beltrami operator of a Poincaré-Einstein metric g_+ as in (1.6) and (1.7):

$$-\Delta_{g_+}u=\lambda(n-\lambda)u.$$

In the asymptotic expansion (2.1), we suppress the analogous contributions of the form $\sum_{N\geq 0} r^{n-\lambda+2N}b_{2N}$. The coefficients $\mathcal{T}_{2N}(g;\lambda)$ are rational families (in λ) of differential operators of the form

$$\mathcal{T}_{2N}(g;\lambda)=\frac{1}{2^{2N}N!(\frac{n}{2}-\lambda-1)\cdots(\frac{n}{2}-\lambda-N)}P_{2N}(g;\lambda) \tag{2.2}$$

with respective polynomial families $P_{2N}(g;\lambda)=\Delta_g^N+LOT$. In particular, the poles of $\mathcal{T}_{2N}(\lambda)$ are contained in the set $\left\{\frac{n}{2}-1,\dots,\frac{n}{2}-N\right\}$. The families $P_{2N}(g;\lambda)$ contain the GJMS-operators for special parameters λ. More precisely, we have (see [GZ03])

$$P_{n-2N}(g;N)=P_{n-2N}(g)\quad \text{for } N=0,1,\dots,\tfrac{n}{2}. \tag{2.3}$$

Definition 2.1 (**Q-curvature polynomials).** For even $n\geq 2$ and $2\leq 2N\leq n$, the Nth Q-curvature polynomial is defined by

$$\begin{aligned} Q_{2N}^{res}(g;\lambda)=&-2^{2N}N!\left(\left(\lambda+\frac{n}{2}-2N+1\right)\cdots\left(\lambda+\frac{n}{2}-N\right)\right)\\ &\times\left[\mathcal{T}_{2N}^*(g;\lambda+n-2N)(v_0)+\cdots+\mathcal{T}_0^*(g;\lambda+n-2N)(v_{2N})\right]. \end{aligned} \tag{2.4}$$

We also set $Q_0^{res}(g;\lambda)=-1$.

In particular, the critical Q-curvature polynomial is given by the formula

$$\begin{aligned} Q_{n}^{res}(g;\lambda)=&-2^{n}\left(\frac{n}{2}\right)!\left(\left(\lambda-\frac{n}{2}+1\right)\cdots\lambda\right)\\ &\times\left[\mathcal{T}_{n}^*(g;\lambda)(v_0)+\cdots+\mathcal{T}_0^*(g;\lambda)(v_{n})\right]. \end{aligned} \tag{2.5}$$

$Q_{2N}^{res}(\lambda)$ is a polynomial of degree N. In particular, the critical Q-curvature polynomial $Q_n^{res}(\lambda)$ has degree $\frac{n}{2}$. It has vanishing constant term, i.e., $Q_n^{res}(0)=0$, and satisfies

$$\dot{Q}_n^{res}(0)=Q_n. \tag{2.6}$$

It is the latter property which motivates the name. For proofs of (2.6) see [GJ07], [BJ10] or [J09a].

The polynomials $Q_{2N}^{res}(\lambda)$ are proportional to the constant terms of the so-called residue families $D_{2N}^{res}(\lambda)$:

$$Q_{2N}^{res}(\lambda)=-(-1)^N D_{2N}^{res}(\lambda)(1).$$

One of the basic properties of the families $D_{2N}^{res}(\lambda)$ is that, for special values of the parameter λ, they factor into products of lower order residue families and GJMS-operators. These factorization identities allow us to express any of these families in terms of respective lower order families and GJMS-operators. As a consequence, we have the following.

Proposition 2.1. *The Q-curvature polynomials $Q^{res}_{2N}(\lambda)$ satisfy the factorization relations*

$$Q^{res}_{2N}\left(-\frac{n}{2}+2N-j\right)=(-1)^j P_{2j}\left(Q^{res}_{2N-2j}\left(-\frac{n}{2}+2N-j\right)\right)$$

for $j=1,\dots,N$.

In particular, the critical Q-curvature polynomial $Q^{res}_8(\lambda)$ satisfies the relations

$$\begin{aligned}
Q^{res}_8(3)&=-P_2\left(Q^{res}_6(3)\right),\\
Q^{res}_8(2)&=P_4\left(Q^{res}_4(2)\right),\\
Q^{res}_8(1)&=-P_6\left(Q^{res}_2(1)\right),\\
Q^{res}_8(0)&=-P_8(1)=0.
\end{aligned}$$

Since $Q^{res}_8(\lambda)$ is a polynomial of degree 4, these relations together with

$$\dot{Q}^{res}_8(0)=Q_8 \tag{2.7}$$

imply that the coefficients of $Q^{res}_8(\lambda)$ can be written as linear combinations of

$$P_2\left(Q^{res}_6(3)\right),\ P_4\left(Q^{res}_4(2)\right),\ P_6\left(Q^{res}_2(1)\right)\quad\text{and}\quad Q_8.$$

Now we have $Q^{res}_2(\lambda)=\lambda Q_2$, and Theorem 6.11.8 in [J09a] yields the formulas

$$Q^{res}_4(\lambda)=-\lambda(\lambda+1)Q_4-\lambda(\lambda+2)P_2(Q_2) \tag{2.8}$$

and

$$\begin{aligned}
Q^{res}_6(\lambda)=\frac{1}{2}\lambda^2(\lambda-1)Q_6&\\
+\lambda^2(\lambda+1)P_2\left(Q_4+\frac{3}{2}P_2(Q_2)\right)&-\lambda(\lambda+1)(\lambda-1)P_4(Q_2).
\end{aligned} \tag{2.9}$$

It is easy to verify that $Q^{res}_4(\lambda)$ and $Q^{res}_6(\lambda)$ indeed satisfy the respective relations

$$\begin{aligned}
Q^{res}_6(1)&=-P_2\left(Q^{res}_4(1)\right),\\
Q^{res}_6(0)&=P_4\left(Q^{res}_2(0)\right),\\
Q^{res}_6(-1)&=P_6(1)=-Q_6,
\end{aligned}$$

and

$$\begin{aligned}
Q^{res}_4(-1)&=-P_2\left(Q^{res}_2(-1)\right),\\
Q^{res}_4(-2)&=-P_4(1)=-2Q_4.
\end{aligned}$$

Here some comments are in order. The above three factorization relations for the cubic polynomial $Q^{res}_6(\lambda)$ do not suffice for its characterization. However, in dimension $n\geq 10$, the analogous relations together with $Q^{res}_6(0)=0$ yield a characterization, and the above formula for $n=8$ follows by "analytic continuation" in the dimension. This argument differs from that in [J09a], where the proof of the analog of (2.9) for general dimensions rests on an explicit formula for Q_6, and the vanishing property $Q^{res}_6(0)=0$ appears as a consequence. Similarly, the two relations for the quadratic polynomial $Q^{res}_4(\lambda)$ together with the vanishing property $Q^{res}_4(0)=0$ characterize this polynomial. For a proof of $Q^{res}_{2N}(0)=0$ in full generality, see Section 1.6 of [BJ10].

Now the above formulas for $Q_2^{res}(\lambda)$, $Q_4^{res}(\lambda)$ and $Q_6^{res}(\lambda)$ together with (2.7) yield a formula for the quartic polynomial $Q_8^{res}(\lambda)$. The actual calculation shows that the coefficient of λ^4 in $-6Q_8^{res}(\lambda)$ is given by

$$\begin{aligned}(2.10)\quad & Q_8 + 3P_2(Q_6) + 3P_6(Q_2) - 9P_4(Q_4) \\ & \qquad - 8P_2P_4(Q_2) + 12P_2^2(Q_4) - 12P_4P_2(Q_2) + 18P_2^3(Q_2).\end{aligned}$$

We compare formula (2.10) with the result of a direct evaluation of the leading coefficient of $-6Q_8^{res}(\lambda)$. In this direction, we first observe that the definitions imply

$$\begin{aligned}(2.11)\quad -Q_8^{res}(\lambda) &= P_8^*(\lambda)(1) - 16\lambda P_6^*(\lambda)(v_2) + 2^6 3\lambda(\lambda-1)P_4^*(\lambda)(v_4) \\ &\quad - 2^9 3\lambda(\lambda-1)(\lambda-2)P_2^*(\lambda)(v_6) + 2^{10}3!\lambda(\lambda-1)(\lambda-2)(\lambda-3)v_8.\end{aligned}$$

We avoid a consideration of the complicated family $P_8(\lambda)$ by showing that the term $P_8^*(\lambda)(1)$ can be expressed as a linear combination of the other four terms in (2.11). For this purpose, we define for any manifold of even dimension n the polynomial

$$(2.12)\qquad \mathcal{V}_n(\lambda) = \Big[\lambda(\lambda-1)\cdots\Big(\lambda-\frac{n}{2}+1\Big)\Big]\sum_{j=0}^{\frac{n}{2}}(n+2j)\mathcal{T}_{2j}^*(\lambda)(v_{n-2j}).$$

The polynomial $\mathcal{V}_n$ has degree $\frac{n}{2}$. In [J09a], we formulated the conjecture that $\mathcal{V}_n(\lambda)$ vanishes (see Conjecture 6.11.2). The vanishing of $\mathcal{V}_n(\lambda)$ is equivalent to the system of $\frac{n}{2}+1$ relations which express the vanishing of its coefficients. Here we are interested in the quartic polynomial $\mathcal{V}_8(\lambda)$. Only the vanishing of its leading coefficient will be important in the sequel. We establish this vanishing as a consequence of the vanishing of $\mathcal{V}_8(\lambda)$. The proof of this property will *not* require us to make the family $P_8(\lambda)$ explicit.

For an alternative proof of the vanishing of the leading coefficient of an analog of $\mathcal{V}_8(\lambda)$ for manifolds of general dimensions we refer to Section 4.

The following result confirms Conjecture 6.11.2 of [J09a] for $\mathcal{V}_8(\lambda)$.

Proposition 2.2. *For any manifold of dimension $n = 8$, the quartic polynomial*

$$(2.13)\qquad \mathcal{V}_8(\lambda) = [\lambda(\lambda-1)(\lambda-2)(\lambda-3)]\sum_{j=0}^{4}(8+2j)\mathcal{T}_{2j}^*(\lambda)(v_{8-2j})$$

vanishes identically.

The strategy of the proof of Proposition 2.2 is the following. For general even n, the vanishing of $\mathcal{V}_n(\lambda)$ is equivalent to the conditions

$$(2.14)\qquad \mathcal{V}_n(0) = \mathcal{V}_n(1) = \cdots = \mathcal{V}_n\left(\frac{n}{2}-1\right) = 0 \quad\text{and}\quad \dot{\mathcal{V}}_n(0) = 0.$$

However, the conditions

$$(2.15)\qquad \mathcal{V}_n(0) = 0 \quad\text{and}\quad \dot{\mathcal{V}}_n(0) = 0$$

are known to be satisfied in full generality (Theorem 6.11.12 in [J09a]). We prove that the remaining conditions follow from a simpler system of conditions, and verify the latter ones for $\mathcal{V}_8(\lambda)$.

For the convenience of the reader, we also describe the arguments which prove (2.15). First, (2.2) shows that $\mathcal{V}_n(0) = 0$ is equivalent to $P_n^*(0)(1) = 0$. Thus,

the first assertion in (2.15) follows from $P_n(0) = P_n$ and the fact that P_n is a self-adjoint operator with vanishing constant term. The second condition in (2.15) is more subtle. (2.2) shows that the linear coefficient of $\mathcal{V}_n(\lambda)$ is given by

$$\frac{(-1)^{\frac{n}{2}}}{2^{n-1}(\frac{n}{2})!}(n\dot{P}_n^*(0)(v_0) - 2^{n-1}\left(\frac{n}{2}-1\right)!\left(\frac{n}{2}\right)!\sum_{j=0}^{\frac{n}{2}-1}(n+2j)\mathcal{T}_{2j}^*(0)(v_{n-2j})). \tag{2.16}$$

Now we combine $\dot{P}_n(0)(1) = (-1)^{\frac{n}{2}}Q_n$ [GZ03], the relation

$$n\dot{P}_n^*(0)(1) = n\dot{P}_n(0)(1) + 2^n\left(\frac{n}{2}\right)!\left(\frac{n}{2}-1\right)!\sum_{j=0}^{\frac{n}{2}-1}2j\mathcal{T}_{2j}^*(0)(v_{n-2j})$$

(see [GJ07], Proposition 2) and the *holographic formula*

$$n(-1)^{\frac{n}{2}}Q_n = 2^{n-1}\left(\frac{n}{2}\right)!\left(\frac{n}{2}-1\right)!\sum_{j=0}^{\frac{n}{2}-1}(n-2j)\mathcal{T}_{2j}^*(0)(v_{n-2j}) \tag{2.17}$$

for Q_n (see [GJ07], Theorem 1). It follows that

$$\begin{aligned} n\dot{P}_n^*(0)(1) = 2^{n-1}\left(\frac{n}{2}\right)!\left(\frac{n}{2}-1\right)!\sum_{j=0}^{\frac{n}{2}-1}(n-2j)\mathcal{T}_{2j}^*(0)(v_{n-2j}) \\ + 2^n\left(\frac{n}{2}\right)!\left(\frac{n}{2}-1\right)!\sum_{j=0}^{\frac{n}{2}-1}2j\mathcal{T}_{2j}^*(0)(v_{n-2j}), \end{aligned}$$

i.e., (2.16) vanishes. This proves the second assertion.

The following result provides a sufficient condition for the remaining vanishing properties in (2.14).

Lemma 2.1. *For $N = 1, \dots, \frac{n}{2} - 1$, the condition*

$$\mathcal{V}_n(N) = 0$$

follows from

$$\sum_{j=0}^{N}(n-j)\mathcal{T}_{2N-2j}^*(n-N)(v_{2j}) = 0.$$

Proof. By (2.2), the vanishing of $\mathcal{V}_n(N)$ is equivalent to

$$\begin{aligned} 2n\frac{(-1)^{\frac{n}{2}}}{2^n\left(\frac{n}{2}\right)!}P_n^*(N)(v_0) + (2n-2)\frac{(-1)^{\frac{n}{2}-1}}{2^{n-2}\left(\frac{n}{2}-1\right)!}NP_{n-2}^*(N)(v_2) \\ + \cdots + (2n-2N)\frac{(-1)^{\frac{n}{2}-N}}{2^{n-2N}\left(\frac{n}{2}-N\right)!}N!P_{n-2N}^*(N)(v_{2N}) = 0. \end{aligned} \tag{2.18}$$

Now we have the factorization relations

$$P_{n-2j}(N) = P_{2N-2j}(n-N)P_{n-2N},\ j = 0, \dots, N. \tag{2.19}$$

These follow from (2.3) and the fact that, for certain values of λ, the operators $\mathcal{T}_{2N}(\lambda)$ can be determined in stages, i.e., the map

$$fr^N \mapsto \mathcal{T}_{n-2N}(N)(f)r^{n-N} \mapsto \mathcal{T}_{2N-2j}(n-N)\left(\mathcal{T}_{n-2N}(N)(f)\right)r^{n-2j+N}$$

coincides with

$$fr^N \mapsto \mathcal{T}_{n-2j}(N)(f)r^{n-2j+N}.$$

Equation (2.19) shows that (2.18) follows from

$$2n\frac{(-1)^{\frac{n}{2}}}{2^n\left(\frac{n}{2}\right)!}P^*_{2N}(n-N)(v_0)+(2n-2)\frac{(-1)^{\frac{n}{2}-1}}{2^{n-2}\left(\frac{n}{2}-1\right)!}NP^*_{2N-2}(n-N)(v_2) + \cdots + (2n-2N)\frac{(-1)^{\frac{n}{2}-N}}{2^{n-2N}\left(\frac{n}{2}-N\right)!}N!v_{2N}=0. \tag{2.20}$$

However, (2.2) implies that this relation is equivalent to

$$2^{2N-n}\frac{N!}{(\frac{n}{2}-N)!}\left(2n\mathcal{T}^*_{2N}(n-N)(v_0)+\cdots+(2n-2N)\mathcal{T}^*_0(n-N)(v_{2N})\right)=0.$$

The proof is complete. □

Now we are ready to prove Proposition 2.2.

Proof of Proposition 2.2. It only remains to prove that $\mathcal{V}_8(1)=\mathcal{V}_8(2)=\mathcal{V}_8(3)=0$. In order to reduce the amount of numerical factors, we actually prove the more general result that

$$\mathcal{V}_n(1)=\mathcal{V}_n(2)=\mathcal{V}_n(3)=0$$

for all $n\geq 8$.[3] By Lemma 2.1, these relations follow from the identities

$$n\mathcal{T}^*_2(n-1)(v_0)+(n-1)v_2=0, \tag{2.21}$$

$$n\mathcal{T}^*_4(n-2)(v_0)+(n-1)\mathcal{T}^*_2(n-2)(v_2)+(n-2)v_4=0, \tag{2.22}$$

and

$$n\mathcal{T}^*_6(n-3)(v_0)+(n-1)\mathcal{T}^*_4(n-3)(v_2)+(n-2)\mathcal{T}^*_2(n-3)(v_4)+(n-3)v_6=0. \tag{2.23}$$

In the remainder of the proof we confirm these three relations. For this purpose, we apply the following explicit formulas for the quantities involved. First of all, the families $P_2(\lambda)$ and $P_4(\lambda)$ are given by

$$P_2(\lambda)=\Delta-\lambda\mathsf{J} \tag{2.24}$$

and

$$P_4(\lambda)=(\Delta-(\lambda+2)\mathsf{J})(\Delta-\lambda\mathsf{J})+\lambda(2\lambda-n+2)|\mathsf{P}|^2 + 2(2\lambda-n+2)\delta(\mathsf{P}d)+(2\lambda-n+2)(d\mathsf{J},d). \tag{2.25}$$

These two formulas are contained in Theorem 6.9.4 in [J09a]. Next, we have

$$v_2=-\frac{1}{2}\mathsf{J}\quad\text{and}\quad v_4=\frac{1}{8}(\mathsf{J}^2-|\mathsf{P}|^2). \tag{2.26}$$

Equation (2.21) easily follows from the definitions. Equation (2.22) is equivalent to

$$\frac{n}{8(n-2)n}\left(-n(\Delta-(n-2)\mathsf{J})\mathsf{J}+(n-2)^2|\mathsf{P}|^2-(n-2)\Delta\mathsf{J}\right) + \frac{1}{n-2}(\Delta\mathsf{J}-(n-2)\mathsf{J}^2)+\frac{n-2}{8}(\mathsf{J}^2-|\mathsf{P}|^2)=0.$$

[3] The arguments also prove that $\mathcal{V}_n(1)=\mathcal{V}_n(2)=0$ for $n\geq 6$.

It is straightforward to verify this relation. The proof of (2.23) requires some more work. We start by observing that (2.23) is equivalent to

$$\begin{aligned}(2.27)\quad & P_6^*(n-3)(v_0) - 6(n-1)P_4^*(n-3)(v_2) \\ & \qquad + 24(n-2)^2 P_2^*(n-3)(v_4) - 48(n-2)(n-3)(n-4)v_6 = 0.\end{aligned}$$

For the evaluation of (2.27), we apply a formula for $P_6(\lambda)$ which was derived as formula (6.10.2) in [J09a]. Its formulation requires us to introduce some notation. Let

$$g_t = g + tg_{(2)} + t^2 g_{(4)} = g - t\mathsf{P} + \frac{1}{4}t^2\left(\mathsf{P}^2 - \frac{\mathcal{B}}{n-4}\right),$$

where $\mathcal{B}$ is the Bach tensor (see (5.8)). Iterated derivatives with respect to t (at $t=0$) will be denoted by $'$. In particular, Δ' and Δ'' are the first and second metric variations of Δ for the variation of g defined by g_t. In these terms,

$$\begin{aligned}(2.28)\quad P_6(\lambda)u &= 4(n-4-2\lambda)(n-2-2\lambda)\left[\lambda(\log\det g)''' + \Delta''\right](u) \\ &\quad + 4(n-4-2\lambda)\left[(\lambda+2)(\log\det g)'' + \Delta'\right]P_2(\lambda)(u) \\ &\qquad + (\Delta-(\lambda+4)\mathsf{J})P_4(\lambda)u.\end{aligned}$$

For more details, see Section 4. Equation (2.28) implies

$$\begin{aligned}(2.29)\quad P_6^*(\lambda)(1) &= 4(n-4-2\lambda)(n-2-2\lambda)\left[\lambda(\log\det g)''' + \Delta''^*(1)\right] \\ &\quad + 4(n-4-2\lambda)P_2^*(\lambda)\left[(\lambda+2)(\log\det g)'' + \Delta'^*(1)\right] \\ &\qquad + P_4^*(\lambda)(\Delta-(\lambda+4)\mathsf{J})(1).\end{aligned}$$

In order to determine the quantity $\Delta''^*(1)$, we combine (2.29) with the relation

$$P_6\left(\frac{n}{2}-3\right) = P_6$$

and the fact that P_6 is self-adjoint. This yields the formula

$$4\left(\Delta'' - (\Delta'')^*\right)(1) = -\Delta|\mathsf{P}|^2 + 4\delta(\mathsf{P}d\mathsf{J}). \tag{2.30}$$

For $n=6$, the details of the calculation can be found in Section 6.10 of [J09a]. As to be expected, the result does not depend on the dimension. Since $\Delta''(1)=0$, we find

$$4(\Delta'')^*(1) = \Delta|\mathsf{P}|^2 - 4\delta(\mathsf{P}d\mathsf{J}). \tag{2.31}$$

Now, we evaluate (2.29) by using (2.31),

$$(\log\det g)'' = -\frac{1}{2}|\mathsf{P}|^2, \tag{2.32}$$

$$(\log\det g)''' = -\frac{1}{2(n-4)}(\mathcal{B},\mathsf{P}) - \frac{1}{2}\operatorname{tr}(\mathsf{P}^3), \tag{2.33}$$

and

$$(\Delta')^*(1) = \frac{1}{2}\Delta\mathsf{J}.$$

For the proofs of these results we refer to Section 6.10 of [J09a]. We obtain

$$\begin{aligned} P_6^*(\lambda)(1) &= (n-4-2\lambda)(n-2-2\lambda) \\ &\qquad \times \left[\Delta|\mathsf{P}|^2 - 4\delta(\mathsf{P}d\mathsf{J}) - \frac{2\lambda}{n-4}(\mathcal{B},\mathsf{P}) - 2\lambda\,\mathrm{tr}(\mathsf{P}^3)\right] \\ &\quad + 2(n-4-2\lambda)\left[-(\lambda+2)(\Delta-\lambda\mathsf{J})|\mathsf{P}|^2 + (\Delta-\lambda\mathsf{J})\Delta\mathsf{J}\right] - (\lambda+4)P_4^*(\lambda)\mathsf{J}. \end{aligned}$$

In particular, we find

$$\begin{aligned} P_6^*(n-3)(1) &= (n-2)(n-4) \\ &\qquad \times \left[\Delta|\mathsf{P}|^2 - 4\delta(\mathsf{P}d\mathsf{J}) - \frac{2(n-3)}{n-4}(\mathcal{B},\mathsf{P}) - 2(n-3)\,\mathrm{tr}(\mathsf{P}^3)\right] \\ &\quad - 2(n-2)\left[-(n-1)(\Delta-(n-3)\mathsf{J})|\mathsf{P}|^2 + (\Delta-(n-3)\mathsf{J})\Delta\mathsf{J}\right] \\ &\qquad\qquad - (n+1)P_4^*(n-3)\mathsf{J}. \end{aligned}$$

Thus, the left-hand side of (2.27) equals the sum of

$$\begin{aligned} &(n-2)(n-4)\left[\Delta|\mathsf{P}|^2 - 4\delta(\mathsf{P}d\mathsf{J}) - \frac{2(n-3)}{n-4}(\mathcal{B},\mathsf{P}) - 2(n-3)\,\mathrm{tr}(\mathsf{P}^3)\right] \\ &\quad - 2(n-2)\left[-(n-1)(\Delta-(n-3)\mathsf{J})|\mathsf{P}|^2 + (\Delta-(n-3)\mathsf{J})\Delta\mathsf{J}\right] \\ &\qquad\qquad + 2(n-2)P_4^*(n-3)\mathsf{J}, \end{aligned}$$

where

$$\begin{aligned} P_4^*(n-3)\mathsf{J} &= (\Delta-(n-3)\mathsf{J})(\Delta-(n-1)\mathsf{J})\mathsf{J} \\ &\qquad + (n-3)(n-4)|\mathsf{P}|^2\mathsf{J} + 2(n-4)\delta(\mathsf{P}d\mathsf{J}) + (n-4)\delta(\mathsf{J}d\mathsf{J}), \end{aligned}$$

and

$$\begin{aligned} &3(n-2)^2(\Delta-(n-3)\mathsf{J})(\mathsf{J}^2-|\mathsf{P}|^2) \\ &\qquad - 48(n-2)(n-3)(n-4)\left(-\frac{1}{8}\,\mathrm{tr}(\textstyle\bigwedge^3\mathsf{P}) - \frac{1}{24(n-4)}(\mathcal{B},\mathsf{P})\right). \end{aligned}$$

Here we made use of formula (5.7). Now a direct calculation using Newton's formula

$$6\,\mathrm{tr}(\textstyle\bigwedge^3\mathsf{P}) = \mathsf{J}^3 - 3\mathsf{J}|\mathsf{P}|^2 + 2\,\mathrm{tr}(\mathsf{P}^3) \tag{2.34}$$

shows that this sum vanishes. □

We continue with the following proof.

Proof of Theorem 1.1. The vanishing of $\mathcal{V}_8(\lambda)$ is equivalent to the identity[4]

$$\begin{aligned} P_8^*(\lambda)(1) &= 14\lambda P_6^*(\lambda)(v_2) - 2^4 9\lambda(\lambda-1)P_4^*(\lambda)(v_4) \\ &\quad + 2^6 15\lambda(\lambda-1)(\lambda-2)P_2^*(\lambda)(v_6) - 2^{10}3\lambda(\lambda-1)(\lambda-2)(\lambda-3)v_8. \end{aligned} \tag{2.35}$$

Combining this result with (2.11) yields

$$\begin{aligned} -Q_8^{res}(\lambda) &= -2\lambda P_6^*(\lambda)(v_2) + 2^4 3\lambda(\lambda-1)P_4^*(\lambda)(v_4) \\ &\quad - 2^6 3^2\lambda(\lambda-1)(\lambda-2)P_2^*(\lambda)(v_6) + 2^{10}3\lambda(\lambda-1)(\lambda-2)(\lambda-3)v_8. \end{aligned}$$

[4]The identities (2.11) and (2.35) can also be found in the proof of Theorem 6.13.1 in [J09a]. Here we correct misprints in the coefficients of v_8 in both formulas.

From this formula, we read-off the coefficient of λ^4 in $-6Q_8^{res}(\lambda)$ as

$$-12P_6^*(\lambda)^{[3]}(v_2) + 2^5 3^2 P_4^*(\lambda)^{[2]}(v_4) - 2^7 3^3 P_2^*(\lambda)^{[1]}(v_6) + 2^{11} 3^2 v_8, \tag{2.36}$$

where the superscripts indicate the coefficients of the respective powers of λ. Now we have

$$P_2^*(\lambda)^{[1]} = -\mathsf{J} = 2v_2, \quad P_4^*(\lambda)^{[2]} = \mathsf{J}^2 + 2|\mathsf{P}|^2 = -16v_4 + 12v_2^2$$

by (2.24) and (2.25), and

$$P_6^*(\lambda)^{[3]} = 16(\log\det g)''' + 8(\log\det g)''\mathsf{J} - \mathsf{J}(\mathsf{J}^2 + 2|\mathsf{P}|^2)$$

by (2.28). A calculation using (2.32), (2.33), (2.26), and (5.7) shows that

$$\begin{aligned}(\log\det g)'' &= 4v_4 - 2v_2^2,\\ (\log\det g)''' &= 12v_6 - 12v_2v_4 + 4v_2^3.\end{aligned}$$

Hence, we find

$$P_6^*(\lambda)^{[3]}(v_2) = 24(8v_2v_6 - 12v_2^2v_4 + 5v_2^4), \tag{2.37}$$

and it follows that (2.36) is given by the sum of

$$2^5 3^2 \left(-32v_2v_6 + 24v_2^2v_4 - 5v_2^4 - 16v_4^2\right) \tag{2.38}$$

and $3!4!2^7 v_8$. Now Lemma 2.2 shows that the coefficient of λ^4 in $-6Q_8^{res}(\lambda)$ equals

$$2^{12}3^2 w_8 = 3!4!2^8 w_8.$$

Comparing this result with (2.10) implies the assertion. □

The following elementary algebraic result was used in the proof of Theorem 1.1.

Lemma 2.2. *Let*

$$1 + w_2r^2 + w_4r^4 + w_6r^6 + w_8r^8 + \cdots$$

be the Taylor series of the function $w(r) = \sqrt{v(r)}$ *with*

$$v(r) = 1 + v_2r^2 + v_4r^4 + v_6r^6 + v_8r^8 + \cdots.$$

Then

$$\begin{aligned}2w_2 &= v_2,\\ 2w_4 &= \frac{1}{4}(4v_4 - v_2^2),\\ 2w_6 &= \frac{1}{8}(8v_6 - 4v_4v_2 + v_2^3),\\ 2w_8 &= \frac{1}{64}(64v_8 - 32v_6v_2 - 16v_4^2 + 24v_2^2v_4 - 5v_2^4).\end{aligned}$$

The assertion follows by squaring the Taylor series of w.

3. Proof of Theorem 1.3

We derive Theorem 1.3 from Theorem 1.1. The proof consists of two steps. In the first step we establish the following proposition.

Proposition 3.1. *On manifolds of dimension $n \geq 8$,*

$$\begin{aligned}(3.1)\quad & 3!4!2^8 w_8 - 3!4!2^7 v_8 \\ &= -12\left[Q_6 + 2P_2(Q_4) - 2P_4(Q_2) + 3P_2^2(Q_2)\right] Q_2 \\ &\qquad - 18\left[Q_4 + P_2(Q_2)\right]^2.\end{aligned}$$

Proof. By (1.10), the assertion is equivalent to

$$\begin{aligned}&48(32v_6v_2 + 16v_4^2 - 24v_2^2v_4 + 5v_2^4) \\ &\qquad = 2\left[Q_6 + 2P_2(Q_4) - 2P_4(Q_2) + 3P_2^2(Q_2)\right] Q_2 + 3\left[Q_4 + P_2(Q_2)\right]^2.\end{aligned}$$

For the proof of this identity we regard the relations

$$Q_2 = -2v_2, \quad Q_4 = -P_2(Q_2) - Q_2^2 + 16v_4$$

and

$$Q_6 = \left[-2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2)\right] - 6\left[Q_4 + P_2(Q_2)\right] Q_2 - 2!3!2^5 v_6$$

in dimension $n \geq 8$ (Propositions 5.1 and 5.2) as formulas for v_2, v_4, and v_6, and find

$$\begin{aligned}&48(32v_6v_2 + 16v_4^2 - 24v_2^2v_4 + 5v_2^4) \\ &\quad = 2\left[Q_6 + 2P_2(Q_4) - 2P_4(Q_2) + 3P_2^2(Q_2)\right] Q_2 + 12\left[Q_4 + P_2(Q_2)\right] Q_2^2 \\ &\qquad + 3\left[Q_4 + P_2(Q_2) + Q_2^2\right]^2 - 18\left[Q_4 + P_2(Q_2) + Q_2^2\right] Q_2^2 + 15Q_2^4.\end{aligned}$$

From here the assertion follows by simplification. □

We emphasize the important structural fact that the terms

$$Q_6 + 2P_2(Q_4) - 2P_4(Q_2) + 3P_2^2(Q_2) \quad \text{and} \quad Q_4 + P_2(Q_2)$$

on the right-hand side of (3.1) naturally appear also in the respective recursive formulas (5.1) and (5.4) for Q_4 and Q_6.

Now, Theorem 1.1 and Proposition 3.1 (in dimension $n = 8$) show that

$$\begin{aligned}Q_8 = &\left[-3P_2(Q_6) - 3P_6(Q_2) + 9P_4(Q_4)\right. \\ &\qquad \left. + 8P_2P_4(Q_2) - 12P_2^2(Q_4) + 12P_4P_2(Q_2) - 18P_2^3(Q_2)\right] \\ &\quad - 12\left[Q_6 + 2P_2(Q_4) - 2P_4(Q_2) + 3P_2^2(Q_2)\right] Q_2 - 18\left[Q_4 + P_2(Q_2)\right]^2 \\ &\qquad + 3!4!2^7 v_8.\end{aligned}$$

The decompositions

$$(3.2)\qquad P_2 = P_2^0 - 3Q_2, \quad P_4 = P_4^0 + 2Q_4 \quad \text{and} \quad P_6 = P_6^0 - Q_6$$

imply that this sum differs from

$$\begin{aligned}&\left[-3P_2^0(Q_6) - 3P_6^0(Q_2) + 9P_4^0(Q_4)\right. \\ &\qquad \left. + 8P_2^0P_4(Q_2) - 12P_2^0P_2(Q_4) + 12P_4^0P_2(Q_2) - 18P_2^0P_2^2(Q_2)\right] + 3!4!2^7 v_8\end{aligned}$$

by

$$\begin{aligned}&\big[9Q_2Q_6+3Q_6Q_2+18Q_4^2\\&\qquad-24Q_2P_4(Q_2)+36Q_2P_2(Q_4)+24Q_4P_2(Q_2)+54Q_2P_2^2(Q_2)\big]\\&\qquad\quad-12\left[Q_6+2P_2(Q_4)-2P_4(Q_2)+3P_2^2(Q_2)\right]Q_2-18\left[Q_4+P_2(Q_2)\right]^2.\end{aligned}$$

In the latter sum, we replace P_2 and P_4 by the decompositions in (3.2) and simplify. We find

$$\begin{aligned}12\left[\Delta(Q_4)Q_2-Q_4\Delta(Q_2)\right]+18\left[\Delta^2(Q_2)Q_2-\Delta(Q_2)\Delta(Q_2)\right]\\+54\left[\Delta(Q_2)Q_2^2-Q_2\Delta(Q_2^2)\right].\end{aligned}$$

Further simplification yields the formula in Theorem 1.3.

4. Proof of Theorem 1.2

Let $n\geq 8$. Then the Q-curvature polynomial $Q_8^{res}(\lambda)$ can be written in the form

$$\begin{aligned}Q_8^{res}(\lambda)=&-\lambda\prod_{k=1}^{3}\left(\frac{\lambda+\frac{n}{2}-8+k}{k}\right)Q_8\\&+\lambda\sum_{j=1}^{3}(-1)^j\prod_{\substack{k=1\\k\neq j}}^{4}\left(\frac{\lambda+\frac{n}{2}-8+k}{k-j}\right)P_{2j}\left(\mathcal{Q}_{8-2j}^{res}\left(-\frac{n}{2}+8-j\right)\right),\end{aligned}\tag{4.1}$$

where the polynomials $\mathcal{Q}_{2j}^{res}(\lambda)$ are determined by

$$\lambda\mathcal{Q}_{2j}^{res}(\lambda)=Q_{2j}^{res}(\lambda),\quad j=1,2,3.$$

We recall that $\mathcal{Q}_{2j}^{res}(\lambda)$ is well defined since by Theorem 1.6.6 in [BJ10],

$$Q_{2j}^{res}(0)=0,\quad j=1,\dots,4.$$

The formula in (4.1) follows from the fact that the 4 factorization identities in Proposition 2.1 (for $N=4$) and the vanishing property $Q_8^{res}(0)=0$ characterizes this polynomial. Combining (4.1) with the analogous formulas

$$\begin{aligned}\mathcal{Q}_6^{res}(\lambda)=&\prod_{k=1}^{2}\left(\frac{\lambda+\frac{n}{2}-6+k}{k}\right)Q_6\\&+\sum_{j=1}^{2}(-1)^j\prod_{\substack{k=1\\k\neq j}}^{3}\left(\frac{\lambda+\frac{n}{2}-6+k}{k-j}\right)P_{2j}\left(\mathcal{Q}_{6-2j}^{res}\left(-\frac{n}{2}+6-j\right)\right)\end{aligned}\tag{4.2}$$

and

$$\mathcal{Q}_4^{res}(\lambda)=-\left(\lambda+\frac{n}{2}-3\right)Q_4-\left(\lambda+\frac{n}{2}-2\right)P_2(Q_2)\tag{4.3}$$

we find that, for general n, the leading coefficient of $-6Q_8^{res}(\lambda)$ is given by the *same* sum as in (2.10). Thus, for the proof of Theorem 1.2 it suffices to verify the following proposition.

Proposition 4.1. $-Q_8^{res}(\lambda)^{[4]}=4!2^8w_8$.

Proposition 4.1 is a consequence of the following general description of the leading coefficients of the families $P_{2N}(\lambda)$.

Theorem 4.1. *For even $n \geq 2$ and $2N \leq n$, the leading coefficient of the degree N polynomial*

$$\left(\lambda-\frac{n}{2}+1\right)\cdots\left(\lambda-\frac{n}{2}+N\right)\mathcal{T}_{2N}(\lambda)$$

is the multiplication operator by the function

$$\left(v^{-\frac{1}{2}}\right)^{[2N]}. \tag{4.4}$$

Here we use the notation $f^{[N]}$ for the coefficient of r^N in the Taylor series of f. In particular, $v^{[2N]} = v_{2N}$. We illustrate Theorem 4.1 by two examples.

Example 4.1. A calculation shows that

$$\left(v^{-\frac{1}{2}}\right)^{[4]} = -\frac{1}{2}\left(v_4 + \frac{3}{4}v_2^2\right). \tag{4.5}$$

Hence,

$$\begin{aligned} P_4(\lambda)^{[2]} &= 2^4 2!\left(\left(\lambda-\frac{n}{2}+1\right)\left(\lambda-\frac{n}{2}+2\right)\mathcal{T}_4(\lambda)\right)^{[2]} && \text{(by definition)}\\ &= 2^4 2!\left(v^{-\frac{1}{2}}\right)^{[4]} && \text{(by Theorem 4.1)}\\ &= -16v_4 + 12v_2^2 && \text{(by (4.5))}\\ &= \mathsf{J}^2 + 2|\mathsf{P}|^2. \end{aligned}$$

This result fits with (2.25).

Example 4.2. A calculation shows that

$$\left(v^{-\frac{1}{2}}\right)^{[6]} = -\frac{1}{2}\left(v_6 - \frac{3}{2}v_2v_4 + \frac{5}{8}v_2^3\right). \tag{4.6}$$

Hence,

$$\begin{aligned} P_6(\lambda)^{[3]} &= -2^6 3!\left(\left(\lambda-\frac{n}{2}+1\right)\cdots\left(\lambda-\frac{n}{2}+3\right)\mathcal{T}_6(\lambda)\right)^{[3]} && \text{(by definition)}\\ &= -2^6 3!\left(v^{-\frac{1}{2}}\right)^{[6]} && \text{(by Theorem 4.1)}\\ &= 192\left(v_6 - \frac{3}{2}v_2v_4 + \frac{5}{8}v_2^3\right) && \text{(by (4.6)).} \end{aligned}$$

This result fits with (2.37).

Now, for general N, combining (2.4) with Theorem 4.1 yields

$$Q_{2N}^{res}(\lambda)^{[N]} = -2^{2N}N!\left[(v^{-\frac{1}{2}})^{[2N]}v_0 + \cdots + (v^{-\frac{1}{2}})^{[0]}v_{2N}\right]. \tag{4.7}$$

However, the right-hand side of (4.7) coincides with

$$-2^{2N}N!\left(v^{-\frac{1}{2}}v\right)^{[2N]} = -2^{2N}N!(v^{\frac{1}{2}})^{[2N]} = -2^{2N}N!w_{2N}.$$

Hence, we have proved the following proposition.

Proposition 4.2. *For even $n \geq 2$ and $2N \leq n$,*

$$Q_{2N}^{res}(\lambda)^{[N]} = -2^{2N}N!w_{2N}.$$

In particular, this result proves Conjecture 1.6.2 in [BJ10].

For $N = 4$, we obtain Proposition 4.1.

We continue with the following proof.

Proof of Theorem 4.1. A straightforward calculation shows that the Laplace operator of the metric $g_+ = r^{-2}(dr^2 + g_r)$ is given by the formula

$$\Delta_{g_+} = r^2 \frac{\partial^2}{\partial r^2} - (n-1) r \frac{\partial}{\partial r} + \frac{1}{2} r^2 \frac{\partial}{\partial r} (\log \det g_r) \frac{\partial}{\partial r} + r^2 \Delta_{g_r}.$$

We write the Taylor series of the even function $D(r) = \log \det g_r$ in the form

$$D(r) = D^{(0)} + r^2 D^{(1)} + \frac{r^4}{2!} D^{(2)} + \cdots + \frac{r^n}{(\frac{n}{2})!} D^{(\frac{n}{2})} + \cdots$$

and expand

$$\Delta_{g_r} = \Delta^{(0)} + r^2 \Delta^{(1)} + \frac{r^4}{2!} \Delta^{(2)} + \cdots + \frac{r^n}{(\frac{n}{2})!} \Delta^{(\frac{n}{2})} + \cdots .$$

The ansatz

$$u \sim \sum_{N \geq 0} r^{\lambda + 2N} \mathcal{T}_{2N}(\lambda)(f), \quad \mathcal{T}_0(\lambda)(f) = f$$

for solutions of the eigen-equation

$$-\Delta_{g_+} u = \lambda(n - \lambda) u$$

leads to the recursive relations

$$\begin{aligned} (\Delta^{(0)} + (2N-2+\lambda) D^{(1)}) \mathcal{T}_{2N-2}(\lambda)(f) \\ + \cdots + \frac{1}{(N-1)!} (\Delta^{(N-1)} + \lambda D^{(N)}) \mathcal{T}_0(\lambda)(f) \\ = -2N(2\lambda - n + 2N) \mathcal{T}_{2N}(\lambda)(f). \end{aligned} \tag{4.8}$$

For $N = 3$, the latter formula yields (2.28). Let ω_{2N} be the leading coefficient of the polynomial

$$\left(\lambda - \frac{n}{2} + N\right) \cdots \left(\lambda - \frac{n}{2} + 1\right) \mathcal{T}_{2N}(\lambda)(f).$$

We also set $\omega_0 = f$. Equation (4.8) shows that the coefficients ω_{2N} are recursively determined by the system of relations

$$D^{(1)} \omega_{2N-2} + \cdots + \frac{1}{(N-1)!} D^{(N)} \omega_0 = -4N \omega_{2N}, \ N \geq 1. \tag{4.9}$$

Now

$$-\sum_{N \geq 1} 4N \omega_{2N} r^{2N-2} = -2 \frac{1}{r} \frac{\partial}{\partial r} \left(\sum_{N \geq 0} \omega_{2N} r^{2N} \right)$$

and

$$\sum_{N\geq 1}\left(D^{(1)}\omega_{2N-2}+\cdots+\frac{1}{(N-1)!}D^{(N)}\omega_0\right)r^{2N-2}$$
$$=\frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r}\left(D^{(0)}+r^2D^{(1)}+\cdots+\frac{r^n}{(\frac{n}{2})!}D^{(\frac{n}{2})}+\cdots\right)\left(\sum_{N\geq 0}\omega_{2N}r^{2N}\right)$$
$$=\frac{1}{r}\frac{\partial}{\partial r}\log v(r)\left(\sum_{N\geq 0}\omega_{2N}r^{2N}\right).$$

Since for general metrics g the quantities in these identities are only well defined for $2N\leq n$, they are to be interpreted as those for appropriate finite sums. Now the assertion follows from the fact that the function $\psi=v^{-\frac{1}{2}}$ satisfies the differential equation

$$-2\frac{\partial}{\partial r}\psi=\frac{\partial}{\partial r}(\log v)\psi.$$

The proof is complete. □

Theorem 4.1 also implies the following result. Let n be even and $2N\leq n$.

Proposition 4.3. *On any manifold of even dimension n, the leading coefficient of the degree N polynomial*

$$\mathcal{V}_{2N}(\lambda)=\left[\left(\lambda-\frac{n}{2}+1\right)\cdots\left(\lambda-\frac{n}{2}+N\right)\right]\sum_{j=0}^{N}(2N+2j)\mathcal{T}^*_{2j}(\lambda)(v_{2N-2j}) \tag{4.10}$$

vanishes, i.e.,

$$\mathcal{V}_{2N}(\lambda)^{[N]}=0. \tag{4.11}$$

Proof. As above, the following identities are to be interpreted as relations for terminating sums (if necessary). By Theorem 4.1, the assertion is equivalent to

$$\sum_{N\geq 0}\left(\sum_{j=0}^{N}(2N+2j)\left(v^{-\frac{1}{2}}\right)^{[2j]}v^{[2N-2j]}\right)r^{2N}=0. \tag{4.12}$$

Now we have

$$\sum_{N\geq 0}2N\left(\sum_{j=0}^{N}\left(v^{-\frac{1}{2}}\right)^{[2j]}v^{[2N-2j]}\right)r^{2N}=r\frac{\partial}{\partial r}(v^{-\frac{1}{2}}v)=\frac{1}{2}rv^{-\frac{1}{2}}\frac{\partial v}{\partial r}$$

and

$$\sum_{N\geq 0}\left(\sum_{j=0}^{N}2j\left(v^{-\frac{1}{2}}\right)^{[2j]}v^{[2N-2j]}\right)r^{2N}=r\frac{\partial}{\partial r}(v^{-\frac{1}{2}})v=-\frac{1}{2}rv^{-\frac{1}{2}}\frac{\partial v}{\partial r}.$$

Summing both relations proves (4.12). □

Similarly as in Section 2, one can use the vanishing result

$$\mathcal{V}_8(\lambda)^{[4]}=0$$

to express $Q_8^{res}(\lambda)^{[4]}$ in terms of $P_2(\lambda)$, $P_4(\lambda)$, $P_6(\lambda)$ and $v_2,\ldots,v_8$. This gives another proof of Theorem 1.2.

By Proposition 4.3, the degree of the polynomial $\mathcal{V}_{2N}(\lambda)$ is $N-1$. In contrast to the critical case, the polynomial $\mathcal{V}_{2N}(\lambda)$ does not vanish identically if $2N < n$. In the following example, we make $\mathcal{V}_2$ and $\mathcal{V}_4$ explicit. For a further discussion of $\mathcal{V}_{2N}(\lambda)$, see Section 6.

Example 4.3. We have $\mathcal{V}_2(\lambda) = \left(\frac{n}{2}-1\right) Q_2$, and the linear polynomial $\mathcal{V}_4(\lambda)$ equals

$$\mathcal{V}_4(\lambda) = \frac{1}{4}\left(\frac{n}{2}-2\right)\left[-\left(\lambda-\frac{n}{2}+2\right)(Q_4+P_2(Q_2))+Q_4\right].$$

These formulas show that

$$\mathcal{V}_2(\lambda) = \left(\frac{n}{2}-1\right)\mathcal{Q}_2^{res}(\lambda-n+2) \quad \text{and} \quad \mathcal{V}_4(\lambda) = \frac{1}{4}\left(\frac{n}{2}-2\right)\mathcal{Q}_4^{res}(\lambda-n+4)$$

using $\mathcal{Q}_2^{res}(\lambda) = Q_2$ and (4.3).

5. Recursive formulas for Q_4, Q_6 and P_4, P_6

Theorems 1.1 – 1.3 are analogs of similar results for the Q-curvatures Q_4 and Q_6. Here we sketch proofs of these formulas. In addition, we display universal recursive formulas for P_4 and P_6. In combination with Theorems 1.1 – 1.3, these results can be used to derive more explicit formulas for Q_8.

We start with the discussion of Q_4 and Q_6.

Proposition 5.1. *On manifolds of dimension $n \geq 4$,*

$$Q_4 = -P_2(Q_2) - Q_2^2 + 2!2^3 v_4. \tag{5.1}$$

This formula is equivalent to

$$Q_4 = -P_2(Q_2) + 2!2^4 w_4 \tag{5.2}$$

with

$$8w_4 = 4v_4 - v_2^2.$$

In dimension $n = 4$, the reduced form of (5.1) *reads*

$$Q_4 = -P_2^0(Q_2) + 2!2^3 v_4. \tag{5.3}$$

In particular,

$$\int_{M^4} Q_4 vol = 2!2^3 \int_{M^4} v_4 vol.$$

We recall that $8v_4 = \mathsf{J}^2 - |\mathsf{P}|^2$ (see (2.26)). The assertions are simple consequences of (1.1) and (1.4).

Proposition 5.2. *On manifolds of dimension $n \geq 6$,*

$$Q_6 = \left[-2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2)\right] - 6\left[Q_4 + P_2(Q_2)\right]Q_2 - 2!3!2^5 v_6. \tag{5.4}$$

This formula is equivalent to

$$Q_6 = \left[-2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2)\right] - 2!3!2^6 w_6 \tag{5.5}$$

with

$$16w_6 = 8v_6 - 4v_4 v_2 + v_2^3.$$

In dimension $n = 6$, the reduced form of (5.4) *reads*

$$Q_6 = \left[-2P_2^0(Q_4) + 2P_4^0(Q_2) - 3P_2^0 P_2(Q_2)\right] - 2!3!2^5 v_6. \tag{5.6}$$

In particular,

$$\int_{M^6} Q_6 vol = 2!3!2^5 \int_{M^6} v_6 vol.$$

We recall that

$$8v_6 = -\operatorname{tr}(\bigwedge^3 \mathsf{P}) - \frac{1}{3(n-4)}(\mathcal{B}, \mathsf{P}), \tag{5.7}$$

where

$$\mathcal{B}_{ij} = \Delta(\mathsf{P})_{ij} - \nabla^k\nabla_j(\mathsf{P})_{ik} + \mathsf{P}^{kl}\mathsf{W}_{kijl} \tag{5.8}$$

generalizes the Bach tensor. For a proof of (5.7), see Theorem 6.9.2 in [J09a].

Proof. Let $n = 6$. We first sketch a proof of (5.4) along the same lines as the arguments in Section 2. By Theorem 6.11.9 in [J09a], we have

$$Q_6^{res}(\lambda) = \frac{1}{2}\lambda(\lambda-1)(\lambda-2)Q_6 + \lambda^2(\lambda-1)P_2\left(Q_4 + \frac{3}{2}P_2(Q_2)\right) - \lambda^2(\lambda-2)P_4(Q_2).$$

This formula is a consequence of the 3 factorizations

$$Q_6^{res}(2) = -P_2(Q_4^{res}(2)),\ Q_6^{res}(1) = P_4(Q_2^{res}(1)),\ Q_6^{res}(0) = P_6(1) = 0$$

and $\dot{Q}_6^{res}(0) = Q_6$. It follows that the coefficient of λ^3 in $2Q_6^{res}(\lambda)$ is given by

$$Q_6 + 2P_2(Q_4) - 2P_4(Q_2) + 3P_2^2(Q_2). \tag{5.9}$$

On the other hand, by definition, $Q_6^{res}(\lambda)$ equals

$$P_6^*(\lambda)(v_0) - 12\lambda P_4^*(\lambda)(v_2) + 2^4 3!\lambda(\lambda-1)P_2^*(\lambda)(v_4) - 2^6 3!\lambda(\lambda-1)(\lambda-2)v_6. \tag{5.10}$$

In this sum, the term $P_6^*(\lambda)(1)$ can be written as a linear combination of the other three terms. We express this fact as the vanishing result

$$\mathcal{V}_6(\lambda) = 0. \tag{5.11}$$

In fact, (5.11) follows from

$$\mathcal{V}_6(0) = \mathcal{V}_6(1) = \mathcal{V}_6(2) = 0 \quad \text{and} \quad \dot{\mathcal{V}}_6(0) = 0.$$

In Section 2, we have seen that the first and the last properties are special cases of a general result, and we proved $\mathcal{V}_6(1) = \mathcal{V}_6(2) = 0$ (for $n = 6$).[5] Now (5.11) is equivalent to the identity

$$P_6^*(\lambda)(1) = 10\lambda P_4^*(\lambda)(v_2) - 2^6\lambda(\lambda-1)P_2^*(\lambda)(v_4) + 2^6 3\lambda(\lambda-1)(\lambda-2)v_6. \tag{5.12}$$

Thus, (5.10) implies

$$Q_6^{res}(\lambda) = -2\lambda P_4^*(\lambda)(v_2) + 2^5\lambda(\lambda-1)P_2^*(\lambda)(v_4) - 2^6 3\lambda(\lambda-1)(\lambda-2)v_6, \tag{5.13}$$

and using (2.25) we find

$$2Q_6^{res}(\lambda)^{[3]} = -6\mathsf{J}^3 + 12\mathsf{J}|\mathsf{P}|^2 - 2^6 3!v_6. \tag{5.14}$$

Now a comparison of (5.9) and (5.14) implies

$$Q_6 + 2P_2(Q_4) - 2P_4(Q_2) + 3P_2^2(Q_2) = -6(\mathsf{J}^2 - 2|\mathsf{P}|^2)\mathsf{J} - 2^6 3!v_6.$$

We complete the proof of (5.4) by rewriting this identity using (5.1) (in dimension $n = 6$).

The identities (5.5) and (5.6) follow from (5.4) by straightforward calculations.

[5] For an alternative proof of the identity (5.12) we refer to Lemma 6.11.10 in [J09a].

In [J09a], we gave a proof that (5.4) remains valid in all dimensions $n \geq 6$. It rests on an explicit formula for Q_6 which follows from a combination of the relation

$$P_6\left(\frac{n}{2}-3\right)(1) = -\left(\frac{n}{2}-3\right)Q_6$$

with the formula (2.28). Next, we present an alternative proof of the universality of (5.5) along the lines of Section 4. For $n \geq 6$, (4.2) and (4.3) show that the leading coefficient of $2Q_6^{res}(\lambda)$ is still given by the linear combination (5.9). On the other hand, Proposition 4.2 implies

$$2Q_6^{res}(\lambda)^{[3]} = -2!3!2^6 w_6.$$

The comparison of both results proves the universality of (5.5). Now (5.4) follows by direct calculation using Proposition 5.1. Note that the latter arguments do not require that we know explicit expressions for the quantities involved. □

We stress that the multiplicities in (5.1) and (5.4) again are given by the general rule (1.12).

The following results describe the non-constant parts of the GJMS-operators P_4 and P_6. Of course, their constant terms are given by the corresponding Q-curvatures.

Proposition 5.3. *On manifolds of dimension $n \geq 4$,*

$$P_4^0 = (P_2^2)^0 - 4\delta(\mathsf{P}d). \tag{5.15}$$

The result follows by a calculation from (1.3).

Proposition 5.4. *On manifolds of dimension $n \geq 6$,*

$$P_6^0 = \left[2P_2P_4 + 2P_4P_2 - 3P_2^3\right]^0 - 48\delta(\mathsf{P}^2 d) - \frac{16}{n-4}\delta(\mathcal{B}d). \tag{5.16}$$

Proposition 5.4 follows from (5.4) by infinitesimal conformal variation. For the details we refer to Section 6.12 of [J09a].

6. Final comments

Among all Q-curvatures of a manifold of even dimension n, the critical Q-curvature Q_n is distinguished by the property that its behaviour under conformal changes of the metric is governed by the *linear* differential operator P_n. More precisely, the pair (P_n, Q_n) satisfies the fundamental identity

$$e^{n\varphi}Q_n(e^{2\varphi}g) = Q_n(g) + (-1)^{\frac{n}{2}}P_n(g)(\varphi)$$

for all $\varphi \in C^\infty(M)$. Proposition 5.4 shows that, up to a second-order operator, the critical P_6 can be written as a linear combination of compositions of lower order GJMS-operators. Moreover, the multiplicities of the compositions in that sum are related to the multiplicities of corresponding terms in the recursive formula (5.4). Along the same line, Theorem 1.1 suggests that we expect an analogous recursive formula for the critical GJMS-operator P_8. Theorem 11.1 in [J09b] establishes the conformal covariance of such a natural candidate for the GJMS-operator P_8. It remains an open problem to prove that this operator actually coincides with P_8.

It is well known ([B95], Corollary 1.5) that the contribution to Q_{2N} which involves the maximal number of derivatives is given by

$$(-1)^{n-1}\Delta^{N-1}\mathsf{J}.$$

This is obvious for Q_4 and can be reproduced for Q_6 and Q_8 by using (5.4) and (1.9), respectively. Indeed, the latter facts are special cases of the summation formula

$$\sum_{|I|=N} m_I = 0$$

(see Lemma 2.1 in [J09b]).

A full comparison of (1.9) (in general dimensions) with the formula of Gover and Peterson for Q_8 (in general dimensions) (see Figure 5 in [GP03]) remains a challenge. As an example, we consider the contribution of $(\Delta\mathsf{P}, \Delta\mathsf{P})$ to Q_8. By [GP03], this term has the coefficient

$$-12\frac{n^2-4n+8}{(n-4)^2} = -12\left(1+\frac{4}{n-4}+\frac{8}{(n-4)^2}\right). \tag{6.1}$$

The result is confirmed by (1.9). In fact, the term $(\Delta\mathsf{P}, \Delta\mathsf{P})$ contributes to Q_8 only through

$$-12P_2^2(Q_4), \quad -9P_4(Q_4), \quad -3P_2(Q_6) \quad \text{and} \quad 3!4!2^7 v_8.$$

The first two terms yield

$$48(\Delta\mathsf{P}, \Delta\mathsf{P}) \quad \text{and} \quad -36(\Delta\mathsf{P}, \Delta\mathsf{P}).$$

(5.4) and (5.7) show that the third term contributes by

$$-24(\Delta\mathsf{P}, \Delta\mathsf{P}) - \frac{48}{n-4}(\Delta\mathsf{P}, \Delta\mathsf{P}).$$

Finally, Graham's formula for v_8 and (5.8) show that the last term contributes by[6]

$$-\frac{96}{(n-4)^2}(\Delta\mathsf{P}, \Delta\mathsf{P}).$$

Summarizing, we find (6.1).

A version of (4.1) holds true for all Q-curvature polynomials $Q_{2N}^{res}(\lambda)$ with $2 \leq 2N \leq n$. It follows that the leading coefficient of $Q_{2N}^{res}(\lambda)$ can be written as the product of $(-1)^{N-1}(N-1)!$ and a linear combination of the form

$$\sum_{|I|+a=N} \mu_{(I,a)} P_{2I}(Q_{2a}) = Q_{2N} + \text{terms involving lower order } Q\text{-curvatures}$$

with certain coefficients $\mu_I \in \mathbb{R}$, $|I| = N$. On the other hand, Proposition 4.2 implies that this sum coincides with $-2^{2N}N!w_{2N}$. Hence we obtain an identity of the form

$$\sum_{|I|+a=N} \mu_{(I,a)} P_{2I}(Q_{2a}) = (-1)^N N!(N-1)!2^{2N} w_{2N}.$$

This proves Conjecture 9.2 in [J09b], up to the algebraic problem to establish the identifications

$$(-1)^a m_{(I,a)} = \mu_{(I,a)} \quad \text{for all } (I,a). \tag{6.2}$$

For small $|I|+a$, the relations (6.2) follow by an evaluation of the algorithm which generates the formulas for Q-curvature polynomials in terms of Q-curvatures and GJMS-operators.

Although these arguments suffice to prove Conjecture 9.2 of [J09b] also for, say, Q_{10}, in the present paper we have restricted the attention to Q_8 since only in that

[6] Here we correct a misprint in equation (2.23) of [G09]: the term $\mathrm{tr}(\Omega^{(1)})^2$ is to be replaced by $\mathrm{tr}((\Omega^{(1)})^2)$. The tensor $\Omega^{(1)} = \frac{\mathcal{B}}{4-n}$ is the first extended obstruction tensor.

case we gain a complete understanding of Q_8 in terms of the metric. In fact, a fully explicit formula in terms of P and Graham's first two extended obstruction tensors $\Omega^{(1)}$, $\Omega^{(2)}$ (see [G09]) follows by combining (1.9) with the formulas displayed in Section 5. Future applications will show to which extent such explicit versions are of interest.

The polynomial $\mathcal{V}_{2N}(\lambda)$ (see (4.10)) seems to be related to the Q-curvature polynomial $Q^{res}_{2N}(\lambda)$ by the formula

$$2^{2N-2}(N-1)!\mathcal{V}_{2N}(\lambda) = \left(\frac{n}{2}-N\right)\mathcal{Q}^{res}_{2N}(\lambda-n+2N). \tag{6.3}$$

In more explicit terms, (6.3) states the equality

$$(\lambda-n+2N)\sum_{j=0}^{N}(2N+2j)\mathcal{T}^*_{2j}(\lambda)(v_{2N-2j}) = -2N(n-2N)\sum_{j=0}^{N}\mathcal{T}^*_{2j}(\lambda)(v_{2N-2j})$$

of rational functions in λ. The special cases $N=1$ and $N=2$ of (6.3) appear in Example 4.3. For $N=3$ and $N=4$, the relation (6.3) can be proved by direct calculations, too. In fact, by Proposition 4.3 the polynomial $\mathcal{V}_{2N}(\lambda)$ has degree $N-1$, and thus it suffices to verify that $\mathcal{V}_{2N}(\lambda)$ satisfies the N factorization identities which correspond to those of $Q^{res}_{2N}(\lambda)$ by Proposition 2.1. In particular, for $\mathcal{V}_6(\lambda)$, these state that

$$\begin{aligned} 32\mathcal{V}_6\left(\frac{n}{2}-3\right) &= -P_6(1),\\ 32(n-2)\mathcal{V}_6\left(\frac{n}{2}-2\right) &= (n-6)P_4\mathcal{V}_2\left(\frac{n}{2}+2\right),\\ 32(n-4)\mathcal{V}_6\left(\frac{n}{2}-1\right) &= (n-6)P_2\mathcal{V}_4\left(\frac{n}{2}+1\right). \end{aligned}$$

In the critical case $2N=n$, (6.3) would imply that $\mathcal{V}_n(\lambda)=0$. This is the assertion of Conjecture 6.11.2 in [J09a]. Since $\mathcal{Q}^{res}_{2N}(\lambda)$ has degree $N-1$, (6.3) would also imply that $\mathcal{V}_{2N}(\lambda)^{[N]}=0$, i.e., Proposition 4.3.

Finally, we note that alternative universal recursive formulas for Q-curvatures can be derived by using some of the additional identities which are satisfied by the Q-curvature polynomials (see [FJ09]).

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