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SUBGROUPS OF $PSL(3,\mathbb{C})$ WITH FOUR LINES IN GENERAL POSITION IN ITS LIMIT SET

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Abstract. In this article we provide an algebraic characterization of the subgroups of $PSL(3,\mathbb{C})$ for which the maximum number of complex lines in general position contained in its limit set, according to Kulkarni, is equal to four. Also, we give an explicit description of the discontinuity region, according to Kulkarni, of such groups.

Introduction

Classical Kleinian groups are divided into two classes: elementary and non-elementary groups. This classification is made according to the number of points in the limit set of the group. When the group is elementary, the number of points in the limit set is less than or equal to two. In the case when the group is non-elementary, the limit set is a perfect set (see for example [8]).

When we try to extend the notions of elementary and non-elementary groups to discrete subgroups of $PSL(3,\mathbb{C})$ acting on $\mathbb{P}^2_{\mathbb{C}}$, the first difficulty we encounter is the absence of a standard notion of limit set. However, in [1] it is proved, under certain assumptions on the discrete group, that Kulkarni's limit set agrees with the complement of the equicontinuity set. Moreover, Kulkarni's limit set is a union of complex projective lines. This leads us to think in elementary groups as those groups whose Kulkarni's limit set contains a finite number of lines. However, there are discrete subgroups of $PSL(3,\mathbb{C})$ whose limit set contains infinitely many complex lines, but the maximum number of complex lines in general position contained in this limit set is finite, for example the suspensions of classical Kleinian groups (see [2, 3]). It suggests that a notion of elementary group is given in terms of the maximum number of complex projective lines in general position contained in the limit set of the group.

We do not know a classification of those discrete subgroups of $PSL(3,\mathbb{C})$ such that the maximum number of complex lines in general position contained in its limit set is one, two or three.

In this paper we restrict our attention to subgroups of $PSL(3,\mathbb{C})$ such that the maximum number of complex projective lines in general position contained in Kulkarni's limit set is equal to four, and we obtain the following:

Theorem 0.1. *If $\Gamma \subset PSL(3,\mathbb{C})$ is a hyperbolic toral group, then the discontinuity region, in Kulkarni's sense, agrees with the equicontinuity set and it is projectively*

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equivalent to four disjoint copies of $\mathbb{H}\times\mathbb{H}$*, where* $\mathbb{H}\subset\mathbb{C}$ *denotes the upper half-plane.*

Theorem 0.2. *Let* $\Gamma\subset PSL(3,\mathbb{C})$ *be a discrete group. The maximum number of complex lines in general position contained in Kulkarni's limit set is equal to four, if and only if,* Γ *contains a hyperbolic toral group (see section* 2*) whose index is at most* 8*. Moreover, the limit set of this hyperbolic toral group is equal to the limit set of* Γ.

As a consequence of these results, the components of Kulkarni's discontinuity region are complete Kobayashi hyperbolic (compare to Theorem 1.3 in [1]).

Finally, we conjecture that in the case when the limit set of the discrete group $\Gamma\subset PSL(3,\mathbb{C})$ contains at least five complex lines in general position, then it contains an infinity of complex lines in general position.

This article is organized as follows: In section 1 we state some definitions and results about projective geometry and complex Kleinian groups. In particular, the concept of vertex is studied, and we prove that, under the hypothesis of Theorem 0.2, every family $\mathcal{L}\subset\Lambda(\Gamma)$ of complex lines, contains precisely two vertices.

In section 2, hyperbolic toral groups are defined, we compute Kulkarni's limit set for such groups and we prove Theorem 0.1. Moreover, we prove that Kulkarni's discontinuity region is not the maximal domain of discontinuity.

Finally, in section 3, we prove Theorem 0.2. The sketch of proof is the following: It suffices to prove the theorem for the subgroup which fixes both vertices. These vertices can be considered as the points $e_1=[1:0:0]$ and $e_2:=[0:1:0]$.

It is proved that there exists an element γ_L in the group which acts as a loxodromic element on both pencils of complex lines determined by e_1 and e_2 (see Lemma 3.4). Analogously, there exists an element γ_P in the group which acts as a parabolic element on both pencils of complex lines determined by e_1 and e_2 (see Lemma 3.5).

In consequence, the action of the group on each of these pencils of complex lines is not discrete. However, in each case, there is an equicontinuity set which consists of the complement of a circle of complex lines (see Lemma 3.7). Therefore, the group can be represented by matrices with real entries (see Lemma 3.8), and the equicontinuity set of the group acting on $\mathbb{P}^2_{\mathbb{C}}$ consists of four disjoint copies of $\mathbb{H}\times\mathbb{H}$ (see Proposition 3.9).

In order to find generators for the hyperbolic toral group, we use Proposition 3.11 together with Lemma 3.12 to prove the existence of a group of 2×2 submatrices which can be considered as a commutative discrete subgroup of $SL(2,\mathbb{Z})$ of rank at most two. Lemmas 3.4 and 3.6 show that such a group cannot be trivial. Since $SL(2,\mathbb{Z})$ does not contain copies of $\mathbb{Z}\oplus\mathbb{Z}$, such a subgroup must be of rank one and its generator is an hyperbolic toral automorphism. Finally, Lemma 3.10 shows the existence of two appropriated parabolic elements.

1. Preliminaries and notation

1.1. **Projective geometry.** We recall that the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ is defined as

$$\mathbb{P}^2_{\mathbb{C}}:=(\mathbb{C}^3\setminus\{0\})/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^3 \setminus \{0\}$ by the usual scalar multiplication. This is a compact connected complex 2-dimensional manifold. Let $[\ ] : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2_{\mathbb{C}}$ be the quotient map. If $\beta = \{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{C}^3$, we will write $[e_j] = e_j$ and if $w = (w_1, w_2, w_3) \in \mathbb{C}^3 \setminus \{0\}$, then we will write $[w] = [w_1 : w_2 : w_3]$. Also, $\ell \subset \mathbb{P}^2_{\mathbb{C}}$ is said to be a complex line if $[\ell]^{-1} \cup \{0\}$ is a complex linear subspace of dimension 2. Given $p, q \in \mathbb{P}^2_{\mathbb{C}}$ distinct points, there is a unique complex projective line passing through p and q, such complex projective line is called a *line*, for short, and it is denoted by $\overleftrightarrow{p,q}$.

Consider the action of $\mathbb{Z}_3$ (viewed as the cubic roots of the unity) on $SL(3,\mathbb{C})$ given by the usual scalar multiplication, then

$$PSL(3,\mathbb{C}) = SL(3,\mathbb{C})/\mathbb{Z}_3$$

is a Lie group whose elements are called projective transformations. Let $[[\ ]] : SL(3,\mathbb{C}) \to PSL(3,\mathbb{C})$ be the quotient map, $\gamma \in PSL(3,\mathbb{C})$ and $\widetilde{\gamma} \in GL(3,\mathbb{C})$, we will say that $\tilde{\gamma}$ is a lift of γ if there is a cubic root τ of $Det(\gamma)$ such that $[[\tau\widetilde{\gamma}]] = \gamma$; also, we will use the notation (γ_{ij}) to denote elements in $SL(3,\mathbb{C})$. One can show that $PSL(3,\mathbb{C})$ is a Lie group that acts transitively, effectively and by biholomorphisms on $\mathbb{P}^2_{\mathbb{C}}$ by $[[\gamma]]([w]) = [\gamma(w)]$, where $w \in \mathbb{C}^3 \setminus \{0\}$ and $\gamma \in SL_3(\mathbb{C})$.

1.2. **Complex Kleinian groups.** Let $\Gamma \subset PSL(3,\mathbb{C})$ be a subgroup. The set $L_0(\Gamma)$ is defined as the closure of the points in $\mathbb{P}^2_{\mathbb{C}}$ with infinite isotropy group (see [7]). The set $L_1(\Gamma)$ is the closure of the set of cluster points of Γz where z runs over $\mathbb{P}^2_{\mathbb{C}} \setminus L_0(\Gamma)$. Recall that q is a cluster point for ΓK, where $K \subset \mathbb{P}^2_{\mathbb{C}}$ is a non-empty set, if there is a sequence $(k_m)_{m\in\mathbb{N}} \subset K$ and a sequence of distinct elements $(\gamma_m)_{m\in\mathbb{N}} \subset \Gamma$ such that $\gamma_m(k_m) \xrightarrow[m\to\infty]{} q$. The set $L_2(\Gamma)$ is defined as the closure of cluster points of ΓK where K runs over all the compact sets in $\mathbb{P}^2_{\mathbb{C}} \setminus (L_0(\Gamma) \cup L_1(\Gamma))$. The *Limit Set in the sense of Kulkarni* for Γ is defined as:

$$\Lambda(\Gamma) = L_0(\Gamma) \cup L_1(\Gamma) \cup L_2(\Gamma).$$

The *Discontinuity Region in the sense of Kulkarni* of Γ is defined as:

$$\Omega(\Gamma) = \mathbb{P}^2_{\mathbb{C}} \setminus \Lambda(\Gamma).$$

We will say that Γ is a *Complex Kleinian Group* if $\Omega(\Gamma) \neq \emptyset$.

The following two lemmas can be found in [2].

Lemma 1.1. *Let $\Gamma \subset PSL_3(\mathbb{C})$ be a subgroup, $p \in \mathbb{P}^2_{\mathbb{C}}$ such that $\Gamma p = p$ and ℓ a complex line not containing p. Define $\Pi = \Pi_{p,\ell} : \Gamma \longrightarrow Bihol(\ell)$ given by $\Pi(g)(x) = \pi(g(x))$ where $\pi = \pi_{p,\ell} : \mathbb{P}^2_{\mathbb{C}} - \{p\} \longrightarrow \ell$ is given by $\pi(x) = \overleftrightarrow{x,p} \cap \ell$, then:*

(i) *π is a holomorphic function.*
(ii) *Π is a group morphism.*
(iii) *If $Ker(\Pi)$ is finite and $\Pi(\Gamma)$ is discrete, then Γ acts discontinuously on $\Omega = (\bigcup_{z\in\Omega(\Pi(\Gamma))} \overleftrightarrow{z,p}) - \{p\}$. Here $\Omega(\Pi(\Gamma))$ denotes the discontinuity set of $\Pi(\Gamma)$.*
(iv) *If Γ is discrete, $\Pi(\Gamma)$ is non-discrete and ℓ is invariant, then Γ acts discontinuously on $\Omega = \bigcup_{z\in Eq(\Pi(\Gamma))} \overleftrightarrow{z,p} - (\ell \cup \{p\})$.*

Lemma 1.2. *Let $G \subset PSL(2,\mathbb{C})$ be a non-discrete group, then:*

(i) *The set $\mathbb{P}^1_{\mathbb{C}} \setminus Eq(G)$ is either empty, has one point, two points, a circle or $\mathbb{P}^1_{\mathbb{C}}$.*
(ii) *If $\mathcal{C}$ is an invariant closed set which contains at least 2 points, then $\mathbb{P}^1_{\mathbb{C}} \setminus Eq(G) \subset \mathcal{C}$.*
(iii) *The set $\mathbb{P}^1_{\mathbb{C}} \setminus Eq(G)$ is the closure of the loxodromic fixed points.*

1.3. **Counting lines.** In the paper [1] we described the limit set $\Lambda(\Gamma)$ as a union of complex lines, under certain conditions. There exist special points in $\Lambda(\Gamma)$ (called vertices) through which infinitely many complex lines contained in the limit set pass. The set of these points is important for the description of the group Γ and for the study of the action of Γ on its limit set. In the case when the maximum number of lines in general position contained in $\Lambda(\Gamma)$ is four, we prove that there exist precisely two vertices (see Proposition 1.4). In what follows, we give the formal definitions and results.

Definition 1.3. Let $\Gamma \subset PSL(3,\mathbb{C})$ be a discrete subgroup. Given a family $\mathcal{L}$ of complex lines contained in the limit set $\Lambda(\Gamma)$, we say that v is a *vertex* for $\mathcal{L}$, whenever it is an intersection point of two distinct lines in $\mathcal{L}$, and there are infinitely many complex lines contained in $\Lambda(\Gamma)$ and passing through v.

Proposition 1.4. *Let $\Gamma \subset PSL(3,\mathbb{C})$ be a discrete group. Assume that the maximum number of complex lines in general position in $\Lambda(\Gamma)$ is equal to four. If $\mathcal{L}$ is a family consisting of four complex lines in general position in $\Lambda(\Gamma)$, then:*

(i) *The family of lines $\mathcal{L}$ contains precisely two vertices.*
(ii) *For every vertex v of $\mathcal{L}$, the stabilizer subgroup, $Stab_\Gamma(v)$, is a subgroup of Γ with finite index.*

In order to prove Proposition 1.4 we state and prove the following lemmas.

Lemma 1.5. *Let $\Gamma \subset PSL(3,\mathbb{C})$ be a discrete subgroup. If the maximum number of lines in general position contained in $\Lambda(\Gamma)$ is four, then there are infinitely many complex lines contained in $\Lambda(\Gamma)$.*

Proof. Let $\{\ell_1,\dots,\ell_n\}$ $(n \geq 4)$ be the set of complex lines contained in $\Lambda(\Gamma)$. We consider the subgroup $\Gamma_0 := \bigcap_{i=1}^n Stab_\Gamma(\ell_i)$, which has finite index in Γ [it has finite index because any element $\gamma \in \Gamma$ acts as a permutation on the set of lines $\{\ell_1,\dots,\ell_n\}$ therefore $\gamma^n(\ell_i) = \ell_i$ for all $1 \leq i \leq n$]. If γ is any element in Γ_0, then there are four γ invariant lines in general position. Hence γ fixes four points in general position, so it is the identity element, therefore Γ is finite; a contradiction. □

Lemma 1.6. *If $\Gamma \subset PSL(3,\mathbb{C})$ is a discrete group such that the maximum number of complex lines in general position in $\Lambda(\Gamma)$ is greater than or equal to three, then $\Lambda(\Gamma)$ is a union of complex lines.*

Proof. Let $\mathcal{C} = \{\ell \in (\mathbb{P}^2_{\mathbb{C}})^* \,|\, \ell \subset \Lambda(\Gamma)\}$, then

$$C = \bigcup_{\ell \in \mathcal{C}} \ell \subset \Lambda(\Gamma)$$

is a Γ-invariant closed set and contains at least 3 lines in general position. Theorem 1.1 in [1] implies that $\mathbb{P}^2_{\mathbb{C}} \setminus C \subset Eq(\Gamma) = \Omega(\Gamma)$. In other words, $C \supset \Lambda(\Gamma)$. □

Lemma 1.7. *Let $\Gamma \subset PSL(3,\mathbb{C})$ be a discrete group. If the maximum number of complex lines in general position in $\Lambda(\Gamma)$ is equal to four, then the number of vertices for any family $\mathcal{L}$, consisting of four complex lines in general position in $\Lambda(\Gamma)$, is greater than one.*

Proof. Let $\mathcal{L} = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ be a family of four complex lines in general position in $\Lambda(\Gamma)$. If ℓ is any complex line contained in $\Lambda(\Gamma)$, then ℓ passes through an intersection point of complex lines in $\mathcal{L}$, because the maximum number of complex lines in general position in $\Lambda(\Gamma)$ is equal to four. By Lemma 1.5, there exists a vertex v for $\mathcal{L}$. Let us assume there is no other vertex for $\mathcal{L}$, then by Lemma 1.6

$$\Lambda(\Gamma) = \big(\bigcup_{\ell\in\mathcal{B}} \ell\big) \cup \big(\bigcup_{\ell\in\mathcal{A}} \ell\big),$$

where $\mathcal{B}$ is the set of complex lines in $\mathbb{P}^2_{\mathbb{C}}$ contained in $\Lambda(\Gamma)$ passing through v, and $\mathcal{A}$ is the set of complex lines contained in $\Lambda(\Gamma)$ not passing through the vertex v.

The closed set $\bigcup_{\ell\in\mathcal{B}} \ell$ is Γ-invariant, because the vertex v is fixed by every element in Γ.

There are finitely many complex lines in $\mathcal{A}$, otherwise there would be another vertex for $\mathcal{L}$. Therefore, $\bigcup_{\ell\in\mathcal{A}} \ell$ is a Γ-invariant closed set, and the group $\Gamma_0 = \bigcap_{\ell\in\mathcal{A}} Stab_\Gamma(\ell)$ has finite index in Γ.

If ℓ_0 is any line in $\mathcal{A}$, then $D = (\bigcup_{\ell\in\mathcal{B}} \ell) \cup \ell_0$ is a Γ_0-invariant closed set and the maximum number of complex lines in general position contained in D is equal to three. Theorem 1.1 in [1] implies that

$$\mathbb{P}^2_{\mathbb{C}} \setminus D \subset Eq(\Gamma_0) = Eq(\Gamma) = \Omega(\Gamma) = \mathbb{P}^2_{\mathbb{C}} \setminus \Lambda(\Gamma).$$

Hence, $\Lambda(\Gamma) \subset D$; a contradiction to the hypothesis that the maximum number of complex lines in general position contained in $\Lambda(\Gamma)$ is four. □

Proof of Proposition 1.4. (i) By Lemma 1.7 there are at least two vertices for $\mathcal{L}$. If there were more than two vertices for $\mathcal{L}$, then there would be at least six lines in general position in $\Lambda(\Gamma)$, and it cannot happen. Therefore, there are precisely two vertices for $\mathcal{L}$.

(ii) If v is a vertex for $\mathcal{L}$, then for every $\gamma \in \Gamma$, $\gamma(v)$ is a vertex for $\mathcal{L}$. Hence, the set of vertices for $\mathcal{L}$ is Γ-invariant. By (i), there are precisely two vertices v_1 and v_2 for $\mathcal{L}$, then $Stab_\Gamma(v_1)$ and $Stab_\Gamma(v_2)$ have index at most two in Γ. □

2. Toral groups

An element $A \in SL(2,\mathbb{Z})$, is called a Hyperbolic Toral Automorphism if none of the eigenvalues of A lies on the unit circle (see [6]).

Definition 2.1. Any subgroup of $PSL(3,\mathbb{C})$ conjugated to the group

$$\Gamma_A = \left\{ \begin{pmatrix} A^k & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} \,\Big|\, \mathbf{b} \in M(2\times 1,\mathbb{Z}),\, k\in\mathbb{Z} \right\},$$

where $A \in SL(2,\mathbb{Z})$ is a Hyperbolic Toral Automorphism, is called a *Hyperbolic Toral Group.*

Proof of Theorem 0.1. It suffices to prove the theorem for the group Γ_A. First, we notice that Γ_A is the subgroup of $PSL(3,\mathbb{Z})$ generated by

$$\tilde{P}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \tilde{P}_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tilde{L} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.1}$$

therefore it is a discrete subgroup of $PSL(3,\mathbb{C})$.

Now, there is

$$\mathbf{t} = \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} \in GL(2,\mathbb{R}), \quad a, b \in \mathbb{R} \setminus \mathbb{Q}$$

such that

$$\mathbf{t}A\mathbf{t}^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}; \quad \alpha \in \mathbb{R} \setminus \mathbb{Q},\ |\alpha| > 1.$$

If we set

$$T = \begin{pmatrix} \mathbf{t} & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$L := T\tilde{L}T^{-1} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$P_1 := T\tilde{P}_1T^{-1} = \begin{pmatrix} 1 & 0 & y_0 \\ 0 & 1 & x_0 \\ 0 & 0 & 1 \end{pmatrix}, P_2 := T\tilde{P}_2T^{-1} = \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & z_0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $x_0 = -\frac{1}{ab-1}$, $\quad y_0 = \frac{a}{ab-1}$, $\quad z_0 = \frac{b}{ab-1}$.

We denote by $\widehat{\Gamma}_A$ the conjugated group $T\Gamma_A T^{-1}$. The general element in the group $\widehat{\Gamma}_A$ has the form:

$$\begin{pmatrix} \alpha^n & 0 & ky_0 + lx_0 \\ 0 & \alpha^{-n} & kx_0 + lz_0 \\ 0 & 0 & 1 \end{pmatrix}, \quad k, l, n \in \mathbb{Z}.$$

It is not hard to check the following: If $[z : 0 : 1]$ or $[0 : z : 1]$ lies on $L_0(\widehat{\Gamma}_A)$, then $z \in \mathbb{R}$.

Lemma 2.2. *If* $[z_1 : z_2 : 1]$ *lies on* $L_1(\widehat{\Gamma}_A)$*, then* $z_1, z_2 \in \mathbb{R}$.

Proof. Let us assume that $Im(z_1) \neq 0$. Thus there are $w = [u : v : 1]$ and a sequence

$$g_m = \begin{pmatrix} \alpha^{n_m} & 0 & k_m y_0 + l_m x_0 \\ 0 & \alpha^{-n_m} & k_m x_0 + l_m z_0 \\ 0 & 0 & 1 \end{pmatrix},$$

of **distinct** elements in $\widehat{\Gamma}_A$ such that

$$g_m(w) = \begin{bmatrix} \alpha^{n_m} u + k_m y_0 + l_m x_0 \\ \alpha^{-n_m} v + k_m x_0 + l_m z_0 \\ 1 \end{bmatrix} \xrightarrow[m\to\infty]{} \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}.$$

Hence, $\alpha^{n_m} u + k_m y_0 + l_m x_0 \xrightarrow[m\to\infty]{} z_1$ and $\alpha^{-n_m} v + k_m x_0 + l_m z_0 \xrightarrow[m\to\infty]{} z_2$. Since $Im(\alpha^{n_m} u + k_m y_0 + l_m x_0) = \alpha^{n_m} Im(u) \to Im(z_1) \neq 0$, we can assume that (n_m) is constant.

If either k_m or l_m is not bounded, then we can assume, without loss of generality, that $l_m \xrightarrow[m\to\infty]{} \infty$. Thus,

$$\frac{\alpha^{n_m}u + k_m y_0 + l_m x_0}{l_m} \xrightarrow[m\to\infty]{} 0 \quad \text{and} \quad \frac{\alpha^{-n_m}v + k_m x_0 + l_m z_0}{l_m} \xrightarrow[m\to\infty]{} 0,$$

then

$$\frac{k_m}{l_m} \xrightarrow[m\to\infty]{} -\frac{x_0}{y_0}, \quad \text{and} \quad \frac{k_m}{l_m} \xrightarrow[m\to\infty]{} -\frac{z_0}{x_0}.$$

It follows that $x_0^2 = y_0 z_0$, then $1 - ab = 0$; a contradiction. Hence, k_m and l_m are bounded, then there are only finitely many elements (g_m), $m \in \mathbb{N}$; a contradiction. Therefore $z_1 \in \mathbb{R}$, and similarly it is proved that $z_2 \in \mathbb{R}$. □

Lemma 2.2 implies that the points lying in $\overleftrightarrow{e_1, e_3} \cup \overleftrightarrow{e_2, e_3}$ and $L_0(\widehat{\Gamma}_A) \cup L_1(\widehat{\Gamma}_A)$ can be represented by points with real entries. Hence, by a similar argument to the one used in Lemma 4.8 in [9], the limit set $\Lambda(L) = \overleftrightarrow{e_1, e_3} \cup \overleftrightarrow{e_2, e_3}$ is contained in the limit set $\Lambda(\widehat{\Gamma}_A)$. Moreover, the limit set $\Lambda(P_1) = \overleftrightarrow{e_1, e_2}$ is contained in $L_0(\widehat{\Gamma}_A) \subset \Lambda(\widehat{\Gamma}_A)$. By Theorem 1.2 [1], we obtain $\Lambda(\widehat{\Gamma}_A) = \overline{\bigcup_{g\in\widehat{\Gamma}_A} \Lambda(g)}$.

Let $g \in \widehat{\Gamma}_A$ induced by the matrix

$$\begin{pmatrix} \alpha^n & 0 & ky_0 + lx_0 \\ 0 & \alpha^{-n} & kx_0 + lz_0 \\ 0 & 0 & 1 \end{pmatrix}, \quad k, l, n \in \mathbb{Z}.$$

If $n = 0$, then $\Lambda(g) = \overleftrightarrow{e_1, e_2}$. If $n \neq 0$, then g is a loxodromic element and $[-\frac{ky_0+lx_0}{\alpha^n-1} : -\frac{kx_0+lz_0}{\alpha^{-n}-1} : 1]$ is its saddle fixed point. The limit set $\Lambda(g)$ is equal to the union of the complex line determined by this saddle point and e_1, and the complex line determined by e_2 and this saddle point. If we set $p_{k,l,n} = [0 : -\frac{kx_0+lz_0}{\alpha^{-n}-1} : 1]$ and $q_{k,l,n} = [-\frac{ky_0+lx_0}{\alpha^n-1} : 0 : 1]$, then it is not hard to check that

$$\Lambda(g) = \overleftrightarrow{e_1, p_{k,l,n}} \cup \overleftrightarrow{e_2, q_{k,l,n}}.$$

Given that $y_0/x_0 = -a$ and $z_0/x_0 = -b$ are not rational numbers, the closure of the set

$$\bigcup_{g\in\widehat{\Gamma}_A} \Lambda(g)$$

is equal to

$$\overleftrightarrow{e_1, e_2} \cup \bigcup_{r\in\mathbb{R}} (\overleftrightarrow{e_1, [0 : r : 1]} \cup \overleftrightarrow{e_2, [r : 0 : 1]}).$$

Hence Kulkarni's discontinuity region for $\widehat{\Gamma}_A$ is biholomorphic to the set

$$\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \notin \mathbb{R} \text{ or } z_2 \notin \mathbb{R}\}.$$

Finally, $\Omega(\Gamma_A) = Eq(\Gamma_A)$ because $\Lambda(\Gamma_A)$ contains three lines in general position (see Theorem 1.1 in [1]). □

Remark 2.3. Kulkarni's discontinuity region for a hyperbolic toral group is not the maximal domain of discontinuity.

Proof. It suffices to prove the remark for the group $\widehat{\Gamma}_A$. Let $\mathcal{C}$ denote the closed $\widehat{\Gamma}_A$-invariant cone

$$\overleftrightarrow{e_1, e_2} \cup \bigcup_{r\in\mathbb{R}} \overleftrightarrow{e_1, [0 : r : 1]}.$$

It suffices to prove that $\widehat{\Gamma}_A$ acts properly and discontinuously on $\mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{C}$. Let us assume it is not the case, then there exists a sequence (k_m) of points in $\mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{C}$ such that $k_m \xrightarrow[m\to\infty]{} k \in \mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{C}$, and a sequence of distinct elements $\gamma_m \in \widehat{\Gamma}_A$ such that $\gamma_m(k_m) \xrightarrow[m\to\infty]{} z \in \mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{C}$. Since $Eq(\Pi_1(\widehat{\Gamma}_A)) = \overleftrightarrow{e_2, e_3} \setminus \{[0:r:1] \,|\, r \in \mathbb{R}\}$, we may assume that there is a neighborhood $U \subset \overleftrightarrow{e_2, e_3} \setminus \{[0:r:1] \,|\, r \in \mathbb{R}\}$ of $\pi_1(k)$, such that $\pi_1(k_m) \in U$, for all $m \in \mathbb{N}$, and $\Pi_1(\widehat{\Gamma}_A)$ is normal in U. Hence, there is a subsequence of $\Pi_1(\gamma_m)$, still denoted $\Pi_1(\gamma_m)$, that converges uniformly on compact subsets of U. Therefore $\lim_{m\to\infty} \Pi_1(\gamma_m)(\pi_1(k)) = \lim_{m\to\infty} \Pi_1(\gamma_m)(\pi_1(k_m)) = \lim_{m\to\infty} \pi_1(\gamma_m(k_m)) = \pi_1(z)$.

We notice that $\Pi_1(\gamma_m)$ can be identified with a Möbius transformation of the form $\zeta \mapsto \alpha^{n_m}\zeta + j_m x_0 + l_m y_0$, where n_m, j_m, l_m are integers, $x_0, y_0, \alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $|\alpha| > 1$. It follows that $\lim_{m\to\infty} \Pi_1(\gamma_m)(\pi_1(k)) = [0 : r_1 : r_2]$ for some $r_1, r_2 \in \mathbb{R}$, and we obtain that $\pi_1(z) = [0 : r_1 : r_2]$; a contradiction to the fact that $z \notin \mathcal{C}$.

□

As a consequence of the last remark, an Hyperbolic toral group cannot be conjugated to a subgroup of $PU(2,1)$ (see Cor. 4.13 in [9]).

Proposition 2.4. *If H is a finite index subgroup of the group G, then*

$$L_0(G) = L_0(H),$$
$$L_1(G) = \bigcup_{i=0}^{k-1} a_i(L_1(H)),$$

where $a_0H, \ldots, a_{k-1}H$ are the left cosets of H in G.

Proof. The inclusion $L_0(H) \subset L_0(G)$ is obtained by definition. Now, let (g_n) be a sequence of distinct elements of G such that $g_n(x) = x$ for all n. Since $[G:H] < \infty$, then we can assume that $g_1 H = g_n H$ for all $n > 1$. Thus, for each $n > 1$, there exists $h_n \in H$ such that $g_1 = g_n h_n$. Also, $h_m \neq h_n$ whenever $m \neq n$ (otherwise, $h_m = h_n$ and $g_m h_m = g_1 = g_n h_n$ implies that $g_m = g_n$). Moreover, $h_n = g_n^{-1} g_1$ implies that $h_n(x) = x$ for all $n > 1$. Therefore $L_0(G) \subset L_0(H)$.

Given that $L_0(H) = L_0(G)$, it follows that $L_1(H) \subset L_1(G)$. Since $L_1(G)$ is G invariant,

$$\bigcup_{i=0}^{k-1} a_i \cdot (L_1(H)) \subset L_1(G).$$

Let (g_n) be a sequence of distinct elements of G and $z \in \mathbb{P}^2_{\mathbb{C}} \setminus L_0(G) = \mathbb{P}^2_{\mathbb{C}} \setminus L_0(H)$ such that $g_n(z) \xrightarrow[m\to\infty]{} x$. Since $[G:H] < \infty$, we can assume that $g_n H = a_j H$ for all n. Thus, for each n, there exists $h_n \in H$ such that $a_j h_n = g_n$. Also, $h_m \neq h_n$ whenever $m \neq n$, and $h_n(z) \xrightarrow[m\to\infty]{} a_j^{-1}(x)$, so $a_j^{-1}(x) \in L_1(H)$, then $x \in a_j \cdot (L_1(H))$.

□

Proposition 2.5. *If* Γ *contains a Hyperbolic Toral Group* Γ_A *of finite index, then* $\Lambda(\Gamma) = \Lambda(\Gamma_A)$.

Proof. We may assume that Γ_A is generated by the parabolic elements P_1, P_2 and the loxodromic element L (see the proof of Theorem 0.1), where

$$L = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, P_1 = \begin{pmatrix} 1 & 0 & y_0 \\ 0 & 1 & x_0 \\ 0 & 0 & 1 \end{pmatrix}, P_2 = \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & z_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, we prove that $\Lambda(\Gamma)$ contains three complex lines in general position.

Let ℓ_1, ℓ_2, ℓ_3 denote the complex lines $\overleftrightarrow{e_2, e_3}$, $\overleftrightarrow{e_1, e_3}$, $\overleftrightarrow{e_1, e_2}$, respectively. We notice that $\Lambda(P_1) = \ell_3 \subset L_0(\Gamma_A) \subset L_0(\Gamma) \subset \Lambda(\Gamma)$. Thus, it suffices to show that $\ell_1 \cup \ell_2 = \Lambda(L) \subset \Lambda(\Gamma)$, and for this, it is enough to prove that

$$\ell_1 \cap (L_0(\Gamma) \cup L_1(\Gamma)) \text{ and } \ell_2 \cap (L_0(\Gamma) \cup L_1(\Gamma))$$

have empty interior in ℓ_1 and ℓ_2, respectively.

By Proposition 2.4, $\ell_2 \cap L_0(\Gamma) = \ell_2 \cap L_0(\Gamma_A)$ and by the proof of Theorem 0.1, the set $\ell_2 \cap L_0(\Gamma_A)$ has empty interior in ℓ_2. Similarly $\ell_1 \cap L_0(\Gamma)$ has empty interior in ℓ_1.

We remark that $\ell_3 \neq \gamma(\ell_2)$ for all $\gamma \in \Gamma$. [Otherwise, there is $\gamma \in \Gamma$ such that $\gamma^{-1}(\ell_3) = \ell_2$. Since $\ell_3 \subset L_0(\Gamma_A) = L_0(\Gamma)$, then $\ell_2 = \gamma^{-1}(\ell_3) \subset \gamma^{-1}(L_0(\Gamma)) = L_0(\Gamma) = L_0(\Gamma_A)$; a contradiction]. Similarly, $\ell_3 \neq \gamma(\ell_1)$ for all $\gamma \in \Gamma$.

By Lemma 2.2, $a_i^{-1}(\ell_2) \cap L_1(\Gamma_A)$ is contained (except, possibly, for one point) in the set $a_i^{-1}(\ell_2) \cap \mathbb{P}^2_{\mathbb{R}}$, for all $i = 0, \ldots, k-1$. Since $a_i^{-1}(\ell_2) \cap \mathbb{P}^2_{\mathbb{R}}$ has empty interior in $a_i^{-1}(\ell_2)$, it follows that $a_i^{-1}(\ell_2) \cap L_1(\Gamma_A)$ has empty interior in $a_i^{-1}(\ell_2)$ for all $i = 0, \ldots, k-1$. Hence $\ell_2 \cap a_i(L_1(\Gamma_A))$ has empty interior in ℓ_2 for all $i = 0, \ldots, k-1$. Therefore,

$$\ell_2 \cap (L_1(\Gamma)) = \bigcup_{i=0}^{k-1} (a_i(L_1(\Gamma_A)) \cap \ell_2)$$

has empty interior in ℓ_2. Analogously $\ell_1 \cap (L_1(\Gamma))$ has empty interior in ℓ_1.

Given that $\Lambda(\Gamma_A)$, and $\Lambda(\Gamma)$ contain at least three lines in general position, Theorem 1.1 in [1] implies that $\Omega(\Gamma_A) = Eq(\Gamma_A)$ and $\Omega(\Gamma) = Eq(\Gamma)$. Since $[\Gamma : \Gamma_A] < \infty$, it follows that $Eq(\Gamma_A) = Eq(\Gamma)$. Hence, $\Lambda(\Gamma_A) = \Lambda(\Gamma)$. □

3. Four-line groups

Through this section $\Gamma \subset PSL(3, \mathbb{C})$ is a discrete group such that the maximum number of complex lines in general position lying in $\Lambda(\Gamma)$ is equal to four. Let $\mathcal{L}$ denote a family of four complex lines in general position in $\Lambda(\Gamma)$. The points $e_1 = [1 : 0 : 0]$ and $e_2 = [0 : 1 : 0]$, are the vertices for $\mathcal{L}$, and the complex lines $\ell_1 = \overleftrightarrow{e_2, e_3}$, $\ell_2 = \overleftrightarrow{e_1, e_3}$, belong to $\mathcal{L}$. The subgroup $Stab_\Gamma(e_1) \cap Stab_\Gamma(e_2)$ is denoted by Γ_0. The projections $\pi_{e_i,\ell_i} : \mathbb{P}^2_{\mathbb{C}} \setminus \{e_i\} \to \ell_i$ and $\Pi_{e_i,\ell_i} : \Gamma \to Bihol(\ell_i)$, $i = 1, 2$, are respectively denoted by π_i and Π_i, $i = 1, 2$.

Lemma 3.1. *Either $\Pi_1(\Gamma_0)$ or $\Pi_2(\Gamma_0)$ contains loxodromic elements.*

Proof. On the contrary, let us assume that $\Pi_1(\Gamma_0)$ and $\Pi_2(\Gamma_0)$ do not contain loxodromic elements. Thus, each element $\gamma \in \Gamma_0$ has a lift $\tilde{\gamma} \in GL(3,\mathbb{C})$ which is given by:

$$\tilde{\gamma} = \begin{pmatrix} \gamma_{11} & 0 & \gamma_{13} \\ 0 & \gamma_{22} & \gamma_{23} \\ 0 & 0 & 1 \end{pmatrix},$$

where $|\gamma_{11}| = |\gamma_{22}| = 1$. Given that Γ_0 has finite index in Γ, we have that $Eq(\Gamma) = Eq(\Gamma_0)$. By Theorem 1.1 in [1], $\Omega(\Gamma) = Eq(\Gamma) = Eq(\Gamma_0)$.

Claim: *The set $Eq(\Gamma_0)$ is equal to $\mathbb{P}^2_{\mathbb{C}} \setminus \overleftrightarrow{e_1, e_2}$.*

Let

$$g_n = \begin{pmatrix} \gamma_{11}^{(n)} & 0 & \gamma_{13}^{(n)} \\ 0 & \gamma_{22}^{(n)} & \gamma_{23}^{(n)} \\ 0 & 0 & 1 \end{pmatrix}, |\gamma_{11}^{(n)}| = |\gamma_{22}^{(n)}| = 1, \quad n \in \mathbb{N},$$

be a sequence of distinct elements in Γ_0. The sequences $\gamma_{13}^{(n)}$ and $\gamma_{23}^{(n)}$ cannot be simultaneously bounded. Otherwise, there exists a subsequence of g_n, still denoted g_n, such that $g_n \xrightarrow[m\to\infty]{} g \in PGL(3,\mathbb{C})$; a contradiction to the fact that Γ is discrete. Hence we have the following cases:

Case 1. Precisely one of the sequences $\gamma_{13}^{(n)}$ and $\gamma_{23}^{(n)}$ is not bounded. By Lemma 3.2 in [1], there is a subsequence of g_n, still denoted g_n, and a pseudo-projective map S such that $g_n \xrightarrow[m\to\infty]{} S$ and the proof of the same lemma implies that S is induced by one of the matrices:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

In any situation, $Ker(S) = \overleftrightarrow{e_1, e_2}$.

Case 2. Both sequences $\gamma_{13}^{(n)}$ and $\gamma_{23}^{(n)}$ are not bounded. By Lemma 3.2 in [1], there is a subsequence of g_n, still denoted g_n, and a pseudo-projective map S such that $g_n \xrightarrow[m\to\infty]{} S$ and S is induced by one of the matrices:

$$\begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \mu \\ 0 & 0 & 0 \end{pmatrix}, |\lambda| \leq 1, |\mu| \leq 1.$$

In any situation $Ker(S) = \overleftrightarrow{e_1, e_2}$.

Finally, applying Lemmas 3.2 and 3.3 in [1], we prove the claim.

Hence, $\Lambda(\Gamma) = \overleftrightarrow{e_1, e_2}$; a contradiction to the hypothesis that the maximum number of complex lines in general position in $\Lambda(\Gamma)$ is equal to four. □

Lemma 3.2. *If $\Pi_2(\Gamma_0)$ contains a loxodromic element, then $\bigcap_{\tau\in\Pi_2(\Gamma_0)} Fix(\tau)$ contains a single point. An analogous statement holds in the case when $\Pi_1(\Gamma_0)$ contains a loxodromic element.*

Proof. On the contrary, let us assume that $F_2 = \bigcap_{\tau\in\Pi_2(\Gamma_0)} Fix(\tau)$ contains more than one point, then $F_2 = \{e_1, z\}$ for some $z \in \ell_2 \setminus \{e_1\}$. By conjugating by a projective transformation, we may assume that $z = e_3$. Thus each element $\gamma \in \Gamma_0$ has a lift $\tilde{\gamma} \in SL(3,\mathbb{C})$ which is given by:

$$\tilde{\gamma} = \begin{pmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{22} & \gamma_{23} \\ 0 & 0 & 1 \end{pmatrix},$$

where $\gamma_{11}\gamma_{22} = 1$. In consequence $\ell_1 = \overleftrightarrow{e_2, e_3}$, $\ell_3 = \overleftrightarrow{e_1, e_2}$ and e_1 are invariant under the action of Γ_0. Moreover, the closure of the Γ_0-orbit of the complex line ℓ_2, denoted $\overline{\Gamma_0 \cdot \ell_2}$, is a closed proper subset of $\mathbb{P}^2_{\mathbb{C}}$. Hence,

$$U = \mathbb{P}^2_{\mathbb{C}} \setminus (\ell_1 \cup \ell_3 \cup (\overline{\Gamma_0 \cdot \ell_2})),$$

is an open, Γ_0-invariant subset of $\mathbb{P}^2_{\mathbb{C}}$, and the maximum number of complex lines in general position lying in its complement is *three*. By Theorem 1.1 in [1], $U \subset Eq(\Gamma_0) = Eq(\Gamma) = \Omega(\Gamma)$, then $\Lambda(\Gamma) \subset \ell_1 \cup \ell_3 \cup (\overline{\Gamma_0 \cdot \ell_2})$; a contradiction to the hypothesis that the maximum number of complex lines in general position in $\Lambda(\Gamma)$ is equal to four. □

Lemma 3.3. *The groups* $\Pi_1(\Gamma_0)$ *and* $\Pi_2(\Gamma_0)$ *contain loxodromic elements.*

Proof. By Lemma 3.1, either $\Pi_1(\Gamma_0)$ or $\Pi_2(\Gamma_0)$ contain loxodromic elements. Without loss of generality let us assume that $\Pi_1(\Gamma_0)$ contains a loxodromic element. If $\Pi_2(\Gamma_0)$ does not contain loxodromic elements, every element $\gamma \in \Gamma_0$ has a lift in $GL(3,\mathbb{C})$ which is given by:

$$\begin{pmatrix} a & 0 & b \\ 0 & c & d \\ 0 & 0 & 1 \end{pmatrix},$$

where $|a| = 1$. In consequence, there exist

$$\gamma = \begin{pmatrix} a & 0 & b \\ 0 & c & d \\ 0 & 0 & 1 \end{pmatrix}, \text{ where } |c| < 1, \text{ and } \tau = \begin{pmatrix} a' & 0 & b' \\ 0 & c' & d' \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma_0,$$

such that $\Pi_1(\gamma)$ is loxodromic and $Fix(\Pi_1(\gamma)) \neq Fix(\Pi_1(\tau))$. Now, $\kappa = \tau\gamma\tau^{-1}\gamma^{-1}$ is induced by the matrix

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x = -b - ab' + a'b + b'$, $y = -d - cd' + c'd + d'$. We note that $y \neq 0$ because $Fix(\Pi_1(\gamma)) \neq Fix(\Pi_1(\tau))$. The sequence

$$\gamma^m \kappa \gamma^{-m} = \begin{pmatrix} 1 & 0 & a^m x \\ 0 & 1 & c^m y \\ 0 & 0 & 1 \end{pmatrix}, \quad |a| = 1, |c| < 1,$$

has a subsequence that tends to a matrix of the form:

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

as $m \to \infty$; a contradiction to the hypothesis that Γ is discrete. □

Lemma 3.4. *There is an element $\gamma_L \in \Gamma_0$ such that $\Pi_1(\gamma_L)$ and $\Pi_2(\gamma_L)$ are loxodromic.*

Proof. By Lemma 3.3, there are $\gamma_1, \gamma_2 \in \Gamma_0$ such that $\Pi_1(\gamma_1), \Pi_2(\gamma_2)$ are loxodromic. If $\Pi_2(\gamma_1), \Pi_1(\gamma_2)$ are not loxodromic, then straightforward computations show that $\Pi_1(\gamma_1\gamma_2)$ and $\Pi_2(\gamma_1\gamma_2)$ are loxodromic. $\square$

From now on, γ_L denotes a fixed element in Γ_0 such that $\Pi_1(\gamma_L)$ and $\Pi_2(\gamma_L)$ are loxodromic. Also, by conjugating with a projective transformation, we may assume that γ_L has a lift $\tilde{\gamma}_L = (\gamma_{Lij})$ which is a diagonal matrix.

Lemma 3.5. *There is an element $\tau \in \Gamma_0$ such that $\Pi_1(\tau)$ and $\Pi_2(\tau)$ are parabolic elements.*

Proof. By Lemma 3.2, there exist $\gamma_1, \gamma_2 \in \Gamma_0$ such that $Fix(\Pi_1(\gamma_1)) \neq Fix(\Pi_1(\gamma_L))$ and $Fix(\Pi_2(\gamma_2)) \neq Fix(\Pi_2(\gamma_L))$.

Set $\kappa_j = \gamma_L\gamma_j\gamma_L^{-1}\gamma_j^{-1}$, $j = 1, 2$, then $\Pi_i(\kappa_j)$ is parabolic whenever $i = j$ and it is either the identity or parabolic in the other case. Thus, the only interesting case is $\Pi_1(\kappa_2) = Id$ and $\Pi_2(\kappa_1) = Id$, but in this case, a straightforward computation shows that $\Pi_1(\kappa_1\kappa_2)$ and $\Pi_2(\kappa_1\kappa_2)$ are parabolic. $\square$

In what follows, γ_P denotes a fixed element in Γ_0, with a lift (γ_{Pij}), such that $\Pi_1(\gamma_P)$ and $\Pi_2(\gamma_P)$ are parabolic.

Lemma 3.6. *If $\mid \gamma_{L11} \mid < \mid \gamma_{L33} \mid$, then $\mid \gamma_{L22} \mid > \mid \gamma_{L33} \mid$.*

Proof. On the contrary, let us assume that $|\gamma_{L22}| < |\gamma_{L33}|$. By straightforward computations, the matrix

$$\begin{pmatrix} 1 & 0 & \gamma_{L11}^m\gamma_{P13}\gamma_{L33}^{-m} \\ 0 & 1 & \gamma_{L22}^m\gamma_{P23}\gamma_{L33}^{-m} \\ 0 & 0 & 1 \end{pmatrix}$$

is a lift of $\gamma_L^{-m}\gamma_P\gamma_L^m$. Hence, $\gamma_L^{-m}\gamma_P\gamma_L^m \xrightarrow[m\to\infty]{} Id$; a contradiction to the fact that Γ_0 is discrete. $\square$

Lemma 3.7. *The sets $\ell_1 \setminus Eq(\Pi_1(\Gamma_0))$ and $\ell_2 \setminus Eq(\Pi_2(\Gamma_0))$ are circles.*

Proof. The set $\ell_1 \setminus Eq(\Pi_1(\Gamma_0))$ is not empty nor consists of one single point because there exists a loxodromic element in $\Pi_1(\Gamma_0)$. Moreover, it cannot consist of two points because there exist a loxodromic element and a parabolic element in $\Pi_1(\Gamma_0)$ with precisely one fixed point in common. Furthermore, $\ell_1 \setminus Eq(\Pi_1(\Gamma_0))$ is not equal to ℓ_1, because the set

$$\Pi_1\left(\bigcup_{\ell\in\mathcal{C}} \ell \setminus \{e_1\}\right),$$

where $\mathcal{C}$ denotes the set of complex lines contained in $\Lambda(\Gamma_0)$ passing through e_1, is a $\Pi_1(\Gamma_0)$-invariant closed and proper subset of ℓ_1. Thus Lemma 1.2 yields the result. The proof for $\ell_2 \setminus Eq(\Pi_2(\Gamma_0))$ is analogous. $\square$

Lemma 3.8. *The group Γ_0, up to conjugation, leaves $\mathbb{P}^2_{\mathbb{R}}$ invariant.*

Proof. By Lemma 3.4, there is an element $\gamma_L \in \Gamma_0$ such that $\Pi_1(\gamma_L)$ and $\Pi_2(\gamma_L)$ are loxodromic elements. Thus, after conjugating with a projective transformation, we may assume that $Fix(\Pi_1(\gamma_L)) = \{[e_2],[e_3]\}$ and $Fix(\Pi_2(\gamma_L)) = \{[e_1],[e_3]\}$. In consequence, γ_L has a lift $\tilde{\gamma}_L \in SL(3,\mathbb{C})$ given by:

$$\tilde{\gamma}_L = \begin{pmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{pmatrix},$$

where $\gamma_{11}\gamma_{22}\gamma_{33} = 1$. Thus, there are $\alpha_1, \alpha_2 \in \mathbb{C}^*$ such that:

$$\ell_1 \setminus Eq(\Pi_1(\Gamma_0)) = [\{r\alpha_1 e_2 + s e_3 \mid r, s \in \mathbb{R}\} \setminus \{0\}];$$
$$\ell_2 \setminus Eq(\Pi_2(\Gamma_0)) = [\{r\alpha_2 e_1 + s e_3 \mid r, s \in \mathbb{R}\} \setminus \{0\}].$$

Let $\eta \in PSL(3,\mathbb{C})$ be the element induced by the linear map:

$$\tilde{\eta} = \begin{pmatrix} \alpha_2 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By straightforward computations, we obtain that

$$\Pi_1(\eta^{-1}\Gamma_0\eta)[\{re_2 + se_3 \,|\, r, s \in \mathbb{R}\} \setminus \{0\}] = [\{re_2 + se_3 \,|\, r, s \in \mathbb{R}\} \setminus \{0\}];$$
$$\Pi_2(\eta^{-1}\Gamma_0\eta)[\{re_1 + se_3 \,|\, r, s \in \mathbb{R}\} \setminus \{0\}] = [\{re_1 + se_3 \,|\, r, s \in \mathbb{R}\} \setminus \{0\}].$$

In consequence, $\mathbb{P}^2_{\mathbb{R}}$ is $\eta^{-1}\Gamma_0\eta$-invariant. $\square$

In what follows, we assume that $\mathbb{P}^2_{\mathbb{R}}$ is Γ_0-invariant.

Proposition 3.9. *The equicontinuity set of Γ is projectively equivalent to four disjoint copies of $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H} \subset \mathbb{C}$ denotes the upper half-plane.*

Proof. Let us set

$$\mathbb{R}(\ell_1) = [\{re_2 + se_3 | r, s \in \mathbb{R}\} \setminus \{0\}],$$
$$\mathbb{R}(\ell_2) = [\{re_1 + se_3 | r, s \in \mathbb{R}\} \setminus \{0\}].$$

The open set given by

$$U = \mathbb{P}^2_{\mathbb{C}} \setminus \Big(\bigcup_{j=1}^{2} \bigcup_{p \in \mathbb{R}(\ell_j)} \overleftrightarrow{e_j, p}\Big),$$

is equal to four disjoint copies of $\mathbb{H} \times \mathbb{H}$. Moreover, it is Γ_0-invariant and its complement contains four complex lines in general position. By Theorem 1.1 in [1], $U \subset Eq(\Gamma_0)$. On the other hand, if ℓ is a complex line such that $e_1 \in \ell \subset \Lambda(\Gamma)$, then

$$\bigcup_{p \in \mathbb{R}(\ell_1)} \overleftrightarrow{e_1, p} \subset \overline{\bigcup_{\gamma \in \Gamma_0} \overleftrightarrow{e_1, x_\gamma}} = \overline{\Gamma_0 \ell} \subset \Lambda(\Gamma), \text{ where } x_\gamma = \Pi_1(\gamma)(\pi_1(\ell \setminus \{e_1\})).$$

Analogously, if ℓ is a complex line such that $e_2 \in \ell \subset \Lambda(\Gamma)$, then

$$\bigcup_{p \in \mathbb{R}(\ell_2)} \overleftrightarrow{e_2, p} \subset \overline{\bigcup_{\gamma \in \Gamma_0} \overleftrightarrow{e_2, y_\gamma}} = \overline{\Gamma_0 \ell} \subset \Lambda(\Gamma), \text{ where } y_\gamma = \Pi_2(\gamma)(\pi_2(\ell \setminus \{e_2\})).$$

In consequence $Eq(\Gamma) = \Omega(\Gamma) \subset U$. Finally, since Γ_0 is a subgroup of finite index of Γ, we see that $Eq(\Gamma_0) = Eq(\Gamma)$. Therefore, $Eq(\Gamma) = U$. $\square$

In what follows, we denote by $Par(\Gamma_0)$ the set

$$Par(\Gamma_0) = \{\gamma \in \Gamma_0 \,|\, \Pi_1(\gamma) \text{ and } \Pi_2(\gamma) \text{ is either parabolic or the identity}\}.$$

Lemma 3.10. *The set $Par(\Gamma_0)$ is a group isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

Proof. If we denote by H the subgroup of all matrices in $SL(3,\mathbb{C})$ of the form

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b \in \mathbb{R},$$

then it is not hard to check that every element $\gamma \in Par(\Gamma_0)$ has a lift $\tilde{\gamma} \in H$. It follows, by straightforward computations, that $Par(\Gamma_0)$ is a group. We denote by $\widetilde{Par(\Gamma_0)}$ the subgroup of H consisting of those matrices that are lifts of elements in $Par(\Gamma_0)$.

We denote by $Lat : H \to \mathbb{R}^2$ the group morphism given by $Lat((\gamma_{ij})) = (\gamma_{13}, \gamma_{23})$. In order to prove that $Par(\Gamma_0)$ is isomorphic to a lattice in $\mathbb{R}^2$, it suffices to show that there are two elements in $Lat(\widetilde{Par(\Gamma_0)})$ which are $\mathbb{R}$-linearly independent. The matrix

$$\kappa_1 = \begin{pmatrix} 1 & 0 & \gamma_{L_{11}}\gamma_{P_{13}}\gamma_{L_{33}}^{-1} \\ 0 & 1 & \gamma_{L_{22}}\gamma_{P_{23}}\gamma_{L_{33}}^{-1} \\ 0 & 0 & 1 \end{pmatrix}$$

is the lift in $\widetilde{Par(\Gamma_0)}$ of $\gamma_L^{-1}\gamma_P\gamma_L$. Finally, we observe that the system of linear equations

$$rLat(\kappa_1) + sLat(\gamma_P) = 0$$

has determinant $\gamma_{P_{23}}\gamma_{P_{13}}\gamma_{L_{33}}^{-1}(\gamma_{L_{11}} - \gamma_{L_{22}}) \neq 0.$ □

We denote by $\widetilde{Par(\Gamma_0)}$ the subgroup of $SL(3,\mathbb{C})$, consisting of those matrices of the form

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b \in \mathbb{R},$$

which are lifts of elements in $Par(\Gamma_0)$. The function $Lat : \widetilde{Par(\Gamma_0)} \to \mathbb{R}^2$ is the group morphism given by $Lat(\gamma_{ij}) = (\gamma_{13}, \gamma_{23})$.

We define $\check{\Gamma}$ as the intersection of the stabilizers, in Γ, of every component of $Eq(\Gamma)$. It is not hard to check that $\check{\Gamma}$ is a subgroup of Γ of index at most 4, which contains $Par(\Gamma_0)$. Moreover, $\check{\Gamma}$ can be lifted to a subgroup $\tilde{\Gamma}$ of $GL(3,\mathbb{R})$ where each element has the form:

$$\begin{pmatrix} a & 0 & c \\ 0 & b & d \\ 0 & 0 & 1 \end{pmatrix},$$

where $a, b > 0$ and $c, d \in \mathbb{R}$.

As a consequence, $\Pi_1(\check{\Gamma})$ and $\Pi_2(\check{\Gamma})$ do not contain elliptic elements.

In what follows, $eLat : \tilde{\Gamma} \to GL(2,\mathbb{R}^+)$ denotes the function given by

$$eLat(\gamma_{ij}) = \begin{pmatrix} \gamma_{11} & 0 \\ 0 & \gamma_{22} \end{pmatrix}.$$

We remark that $eLat$ is a group morphism whose kernel is $\widetilde{Par(\Gamma_0)}$.

Proposition 3.11. *$Lat(\widetilde{Par(\Gamma_0)})$ is invariant, as a set of $\mathbb{R}^2$, under the action of the group $eLat(\tilde\Gamma)$.*

Proof. Let $(a,b) \in Lat(\widetilde{Par(\Gamma_0)})$ and $(\gamma_{ij}) \in eLat(\tilde\Gamma)$. Thus, there is $\tau \in \widetilde{Par(\Gamma_0)}$ and $\gamma \in \tilde\Gamma$ such that $Lat(\tau) = (a,b)$ and $eLat(\gamma) = (\gamma_{ij})$. Hence, $\kappa = \gamma\tau\gamma^{-1} \in \widetilde{Par(\Gamma_0)}$ and $Lat(\kappa) = (\gamma_{11}a, \gamma_{22}b)$. □

Lemma 3.12. *$eLat(\tilde\Gamma)$ is a commutative discrete group with at most 2 generators.*

Proof. Since the map $\rho : eLat(\tilde\Gamma) \to \mathbb{R}^2$,

$$\rho\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = (Log(a), Log(b)),$$

is a group isomorphism, it suffices to show that $eLat(\tilde\Gamma)$ is discrete. On the contrary, there are sequences of distinct elements $(\alpha_m), (\beta_m) \in \mathbb{R}^+$ such that $\begin{pmatrix} \alpha_m & 0 \\ 0 & \beta_m \end{pmatrix} \in eLat(\tilde\Gamma)$ and $\alpha_m, \beta_m \xrightarrow[m\to\infty]{} 1$. Let $\gamma_m = (\gamma_{i,j}^{(m)}) \in \tilde\Gamma$ such that

$$eLat(\gamma_m) = \begin{pmatrix} \alpha_m & 0 \\ 0 & \beta_m \end{pmatrix}.$$

Since $Lat(\widetilde{Par(\Gamma_0)})$ is a lattice of rank 2, there is a sequence $(\tau_m) \in Lat(\widetilde{Par(\Gamma_0)})$ such that $(Lat(\tau_m) + (\gamma_{13}^{(m)}, \gamma_{23}^{(m)}))$ is a bounded sequence. Thus we can assume that there are $c, d \in \mathbb{R}$ such that $(Lat(\tau_m) + (\gamma_{13}^{(m)}, \gamma_{23}^{(m)})) \xrightarrow[m\to\infty]{} (c,d)$. Now, it is not hard to check that

$$\gamma_m\tau_m \xrightarrow[m\to\infty]{} \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1; \end{pmatrix};$$

a contradiction to the fact that $\tilde\Gamma$ is discrete. □

Proof of Theorem 0.2. If Γ contains a hyperbolic toral group Γ_A of index at most eight, then Proposition 2.5 implies that $\Lambda(\Gamma) = \Lambda(\Gamma_A)$. By the proof of Theorem 0.1, the maximum number of complex lines in general position in $\Lambda(\Gamma_A)$ is equal to four.

In order to prove the converse, it suffices to prove that $\check\Gamma$ is a hyperbolic toral group.

Let σ be a subset of $\check\Gamma$ such that the number of elements in σ is equal to the rank of the discrete commutative group $eLat(\tilde\Gamma)$ and its lifts to $\tilde\Gamma$ generate $eLat(\tilde\Gamma)$ as a commutative discrete group. Let us fix an element $\tau_0 \in \sigma$, by conjugating with a projective transformation, we may assume that τ_0 has a lift $\tilde\tau_0 \in \tilde\Gamma$ which is a diagonal matrix. Finally, let γ_1, γ_2 be two generators of the group $Par(\Gamma_0)$.

We claim that $\sigma \cup \{\gamma_1, \gamma_2\}$ is a set of generators for $\check\Gamma$. In fact, let $\gamma \in \check\Gamma$ and $\tilde\gamma \in \tilde\Gamma$ be a lift. There is an element u in the group generated by σ with a lift $\tilde u \in \tilde\Gamma$ such that $eLat(\tilde\gamma) = eLat(\tilde u)$. Hence, $\gamma u^{-1} \in Ker(eLat) = Par(\Gamma_0)$, so the claim is proved.

Now, let $v = (a, c), w = (b, d)$ be two linearly independent vectors in $\mathbb{R}^2$ such that $Lat(\widetilde{Par}(\Gamma_0)) = \mathbb{Z}v + \mathbb{Z}w$. Also, set

$$\mathbf{t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; T = \begin{pmatrix} \mathbf{t} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, some computations show that:

$$T^{-1}\gamma_j T = \begin{pmatrix} I & e_j \\ 0 & 1 \end{pmatrix} \text{ for } j = 1, 2.$$

Now, for each $h \in \tilde{\sigma}$, we have

$$T^{-1}hT = \begin{pmatrix} \mathbf{t}^{-1}\mathbf{ht} & \nu_h \\ 0 & 1 \end{pmatrix},$$

and $\nu_h = 0$, whenever $h = eLat(\tilde{\tau}_0)$. Proposition 3.11 yields that the group $G = \langle \{\mathbf{t}^{-1}\mathbf{ht} : h \in \tilde{\sigma}\} \rangle$ is an infinite commutative subgroup of $SL(2, \mathbb{Z})$. It is a known fact (see [4, 5]) that $SL(2, \mathbb{Z})$ is a word hyperbolic group, and word hyperbolic groups do not contain copies of $\mathbb{Z} \oplus \mathbb{Z}$. Hence, G has rank one or zero.

Finally, G cannot be trivial by Lemmas 3.4 and 3.6; therefore, its rank is one and its generator is an hyperbolic toral automorphism. □

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