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ON DECOMPOSABLE RATIONAL MAPS

CARLOS CABRERA AND PETER MAKIENKO

ABSTRACT. If R is a rational map, the main result is a uniformization theorem for the space of decompositions of the iterates of R. Secondly, we show that Fatou conjecture holds for decomposable rational maps.

1. Introduction

This paper gives a dynamical approach to the algebraic problem of decomposition of rational maps. That is, to describe the set of decompositions of a rational map R, along with the decompositions of all its iterates R^n. We want to link geometric structures with the decomposition of rational maps. To this end, we construct a space which describes the space of decomposition of the cyclic semigroup generated by R.

We found that the fact that a map is decomposable can impose dynamical consequences. In particular, we show, using elementary arguments, that the Fatou conjecture is true for decomposable rational maps.

2. On stability of decomposable maps

Given a rational map R, the Julia set $J(R)$ is the smallest completely invariant closed set in the Riemann sphere $\bar{\mathbb{C}}$, with at least 3 points. The Fatou set $F(R)$ is the complement of the Julia set on $\bar{\mathbb{C}}$. By definition, the set $F(R)$ is open and completely invariant.

A map R is decomposable if there are maps R_1 and R_2, of degree at least 2, such that $R = R_1 \circ R_2$. In this section, we study stability properties for decomposable rational maps. The simple fact that the maps

$$R = R_1 \circ R_2 \text{ and } \tilde{R} = R_2 \circ R_1$$

are semiconjugated, provides arguments to show that J-stability implies hyperbolicity for decomposable maps. The Fatou conjecture, as restated in [7], states that all J-stable maps are hyperbolic.

First, we recall the definitions of J-stability; more details can be found in [7] and [9]. Let (X, d_1) and (Y, d_2) be metric spaces, a map $\phi : X \to Y$ is called K-quasiconformal, in Pesin's sense if, for every $x_0 \in X$,

$$\limsup_{r\to 0} \left\{ \frac{\sup\{|\phi(x_0) - \phi(x_1)| : |x_0 - x_1| < r\}}{\inf\{|\phi(x_0) - \phi(x_1)| : |x_0 - x_1| < r\}} \right\} \leq K.$$

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Let us recall that two rational maps R_1 and R_2 are *J-equivalent*, if there is a homeomorphism $h : J(R_1) \to J(R_2)$, which is quasiconformal in Pesin's sense and conjugates R_1 to R_2.

Given a family of maps $\{R_w\}$ depending holomorphically on a parameter $w \in W$, a map R_{w_0} in $\{R_w\}$ is called *J-stable* if there is a neighborhood V of w_0 such that R_w is J-equivalent to R_{w_0} for all $w \in V$, and the conjugating homeomorphisms depend holomorphically on w.

Theorem 2.1. *Let $R = R_1 \circ R_2$ and $\tilde{R} = R_2 \circ R_1$, such that $deg(R_i) > 1$ for $i = 1, 2$. If both maps, R and $\tilde{R}$, are J-stable. Then R and $\tilde{R}$ are hyperbolic.*

Let $Cr(R)$ denote the set of critical points of R. It is well known (see for instance [5]), that a J-stable map R is hyperbolic if and only if the set $Cr(R)$ belongs to $F(R)$.

Proof. Since R is J-stable, then R is in general position with respect to the Julia set; that is, R has no critical relations on $J(R)$. In particular, the local degree of each critical point of R is 2. To prove the claim, we will show that there are no critical points in $J(R)$. First notice that as a consequence of the Chain Rule, we have the equation

$$Cr(R) = Cr(R_1 \circ R_2) = R_2^{-1}(Cr(R_1)) \cup Cr(R_2).$$

Let x be a point in $R_2^{-1}(Cr(R_1)) \cap J(R)$, since $J(R)$ is completely invariant under R, every point in $R_2^{-1}(R_2(x))$ belongs to $J(R)$ and is a critical point of R. Also, because there are no critical relations in $J(R)$, the set $R_2^{-1}(R_2(x))$ consists of only one point. However, R_2 has degree at least 2, hence x is a critical point of R_2, but $R_2(x)$ is a critical point of R_1, which implies that the local degree of R in x is at least 4. This contradicts the fact that there are no critical relations in $J(R)$. Then $R_2^{-1}(Cr(R_1))$ belongs to the Fatou set $F(R)$.

There are two semiconjugacies between R and $\tilde{R}$ as shown in the following diagram:

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{R} & \mathbb{C} \\
{\scriptstyle R_2}\big\downarrow & & \big\downarrow{\scriptstyle R_2} \\
\mathbb{C} & \xrightarrow{\tilde{R}} & \mathbb{C} \\
{\scriptstyle R_1}\big\downarrow & & \big\downarrow{\scriptstyle R_1} \\
\mathbb{C} & \xrightarrow{R} & \mathbb{C}
\end{array}$$

The first semiconjugacy, in fact R_2, sends $F(R)$ to $F(\tilde{R})$, hence the set $Cr(R_1) = R_2(R_2^{-1}(Cr(R_1)))$ belongs to $F(\tilde{R})$. By the same argument $Cr(R_2)$ is a subset of $F(R)$. Altogether $Cr(R)$ belongs to $F(R)$. Therefore, the map R is hyperbolic and, by the symmetry of the argument, the map $\tilde{R}$ is also hyperbolic. □

The previous theorem has the following corollary which was already noted in [8].

Corollary 2.2. *The following statements are equivalent:*

- *The map R is hyperbolic.*
- *There exist $n > 1$ such that R^n is J-stable in Rat_{d^n}.*
- *For every $n \geq 1$, the map R^n is J-stable in Rat_{d^n}.*

Intuitively, one could understand the facts above considering the dimension of the space of invariant line fields on the Julia set $J(R)$. This dimension is comparable with the number of critical values of R^n on $J(R)$, which grows linearly with respect to iteration. On the other hand, the condition of J-stability in $Rat_{d^n}(\mathbb{C})$ requires exponential growth of the dimensions with respect to iteration, but these dimensions should be comparable with the number of critical points of R^n on $J(R)$. The incompatibility of the rate of these growths is a contradiction to the existence of invariant line fields on $J(R)$.

Now let us consider for a given rational map R the Hurwitz space $H(R)$; that is, the set of all rational maps with the same combinatorics for the first iteration, equivalently

$$H(R) = \{Q \in Rat(\mathbb{C}) : \exists \phi \text{ and } \psi \in Homeo(\bar{\mathbb{C}}) \text{ such that } Q \circ \phi = \psi \circ R\}.$$

When R is a rational map in general position, with $\deg(R) = d$, the space $H(R)$ is open and dense in the space $Rat_d(\mathbb{C})$. Note that $H(R^n)$ consists of compositions of the form $R_1 \circ R_2 \circ ... \circ R_n$ with $R_i \in H(R)$. It is not clear if $H(R^n)$ consists of all compositions of this form. However,

$$\dim(H(R^n)) \leq n \dim H(R).$$

Now we can ask the analog of Corollary 2.2 for Hurwitz spaces:

(1) Assume that R^n is J-stable in $H(R^n)$, is it true that R^k is J-stable at $H(R^k)$ for $k \neq n$? We expect an affirmative answer for $k < n$.
(2) Assume that R^n is J-stable in $H(R^n)$ for all n, is it true that R is hyperbolic?

The second question is actually a modified version of Fatou conjecture. These questions make sense for entire and meromorphic transcendental maps.

The conditions of Theorem 2.1 are too strong, it is enough that one of the maps, say $R = R_1 \circ R_2$ is J-stable.

Proposition 2.3. *Let $R = R_1 \circ R_2$ and $\tilde{R} = R_2 \circ R_1$, such that $deg(R_i) > 1$ for $i = 1, 2$. If R is J-stable, then $\tilde{R}$ is J-stable.*

We will just sketch the proof of Proposition 2.3. Let us denote by $QC_J(R)$, the *J-stability component* of R. This is the path connected component of the J-equivalence class of R containing R. We need the following theorem which was proved in [9]; see also [7].

Theorem 2.4 (McMullen-Sullivan)**.** *On every analytic family H, the set of J-stable maps is open and dense. Moreover, the set of structurally stable maps is also dense in H.*

Let H be an analytic family. By Theorem 2.4, the set $U = QC_J(R) \cap H$ is an open set. Every holomorphically embedded disk D in U containing R, depending on a complex parameter t, is equivalent to a family of Beltrami coefficients μ_t, whose associated quasiconformal maps f_t conjugate R to R_t along D. The maps f_t form a holomorphic motion of $J(R)$, using Słodkowski's Extended λ-Lemma, the maps f_t can be extended to a neighborhood of $J(R)$ for every t. Moreover, the extension can be taken to preserve the dynamics (see [13, Theorem 1.7]). Now consider the push-forward operator $(R_2)_*$ which sends the family μ_t to the family of Beltrami differentials $(R_2)_*\mu_t$ defined on a neighborhood W of $\tilde{R}$. The complementary components of W can be taken to be simply connected. With this choice, we can extend the maps $(R_2)_*\mu_t$ to the whole sphere by attaching, with surgery, Blaschke maps on each complementary component. Solving the Beltrami equation for the resulting Beltrami coefficients will induce a family of rational maps, J-equivalent to $\tilde{R}$. Thus $\tilde{R}$ is J-stable.

The heart of the proof lies in the fact that the semiconjugacy defines a bijective operator in the space of invariant line fields on the Julia set.

3. A semigroup associated to a rational map R

In this section, for every rational map R we construct a suitable semigroup S_R, such that the space of analytic equivalences of S_R uniformizes the space of virtual decompositions of R. The semigroup S_R will be a semigroup of correspondences on the affine part $\mathcal{A}_R$ of R, as defined by M. Lyubich and Y. Minsky in [6].

Let us recall first Lyubich and Minsky's construction, given a rational map R defined in the Riemann sphere $\bar{\mathbb{C}}$, consider the inverse limit

$$\mathcal{N}_R = \{\hat{z} = (z_1, z_2, ...) : R(z_{n+1}) = z_n\}.$$

The natural extension of R is the map $\hat{R} : \mathcal{N}_R \to \mathcal{N}_R$ given in coordinates by $\hat{R}(\hat{z})_n = R(z_n)$. There is a family of maps $\pi_n : \mathcal{N}_R \to \bar{\mathbb{C}}$, the coordinate projections, defined by $\pi_n(\hat{z}) = z_n$, which semiconjugates the action of $\hat{R}$ with R, that is $\pi_n \circ \hat{R} = R \circ \pi_n$. By endowing $\mathcal{N}_R$ with the topology of the restriction of Tychonoff topology, one can show that the map $\hat{R}$ is a homeomorphism. The regular part $\mathcal{R}_R$ is the maximal subset of $\mathcal{N}_R$ which admits a Riemannian structure, of complex dimension one, compatible with the coordinate projections π_n. A leaf is a path connected component of the regular part. A theorem by Lyubich and Minsky (see [6, Lemma 3.3]) states that, besides leaves associated to Herman rings, all leaves are simply connected. The affine part $\mathcal{A}_R$ consists of the regular points whose leaves are conformally isomorphic to the complex plane $\mathbb{C}$. Let $\mathcal{C} = \{a_1, a_2, ..., a_n\}$ be a repelling periodic cycle for R. An invariant lift of $\mathcal{C}$ is the set of points $\hat{a}$ in $\mathcal{N}_R$ such that all the coordinates of $\hat{a}$ belong to $\mathcal{C}$. Invariant lifts of periodic repelling points belong to the affine part. Moreover, the uniformizing function of the leaves containing these invariant leaves is a Poincaré function associated to $\mathcal{C}$. Since there are infinitely many repelling periodic cycles, the affine part consists of an infinite number of leaves.

Let us remind that a holomorphic correspondence K is a subset of a product of complex spaces $B \times C$ such that K is the union of countably many analytic varieties; the projections are holomorphic and the projection of K to the first coordinate is surjective.

In a fiber F, of the form $\pi_n^{-1}(x)$ for $x \in \bar{\mathbb{C}}$, we can define the set of deck transformations, or dual monodromies, which are given by the correspondences $\pi_n^{-1} \circ \pi_n$. The fact that the conformal structure on leaves is compatible with the projections π_n, means that the leaf admits a conformal structure such that, in this structure, the maps are holomorphic. In particular, given two leaf saturated sets $\mathcal{B}$ and $\mathcal{C}$ in $\mathcal{A}_R$. If the cardinality of leaves in $\mathcal{B}$ is at most countable, then $(\pi_n|_{\mathcal{B}})^{-1} \circ \pi_n|_{\mathcal{C}}$ is a holomorphic correspondence in $\mathcal{B} \times \mathcal{C}$, here $\pi_n|_D$ denotes the restriction of the map π_n to the set D. When $\mathcal{B} = \mathcal{B}$, then $(\pi_n|_{\mathcal{B}})^{-1} \circ \pi_n|_{\mathcal{B}}$ is a semigroup.

If L and L' are two leaves in $\mathcal{A}_R$, then the map $\hat{R}$ sends $(\pi_n|_{L'})^{-1} \circ \pi_n|_L$ to $(\pi_n|_{R(L')})^{-1} \circ \pi_n|_{R(L)}$. In particular, if $\mathcal{B}$ is an $\hat{R}$-invariant, leaf saturated, set in $\mathcal{A}_R$, the action of $\hat{R}$ commutes with the action of $(\pi_n|_{\mathcal{B}})^{-1} \circ \pi_n|_{\mathcal{B}}$ in $\mathcal{A}_R$.

A leaf L in $\mathcal{A}_R$ is called periodic if there exist some n such that L is invariant under $\hat{R}^n$. Let $\mathcal{C}(R)$ be the set of all periodic leaves in $\mathcal{A}_R$. The semigroup of deck correspondences is the holomorphic correspondence $(\pi_n|_{\mathcal{C}(R)})^{-1} \circ \pi_n|_{\mathcal{C}(R)}$ and will be denoted by $\pi_n^{-1} \circ \pi_n$.

Now, let us define the semigroup S_R as the semigroup $\langle \mathcal{C}(R), \hat{R}, \pi_n^{-1} \circ \pi_n \rangle$, generated by the constant maps on the set $\mathcal{C}(R)$, the dynamics of $\hat{R}$ and $\pi_n^{-1} \circ \pi_n$. We refer to $\mathcal{C}(R)$ as the set of constants of S_R and the dynamical part of S_R will be the semigroup generated by $\hat{R}$ and $\{\pi_n^{-1} \circ \pi_n\}$.

Definition 3.1. A marked monomorphism $\rho : S_R \to S_{R_1}$ is a monomorphism that sends constants to constants, maps analytically leaves to leaves, and sends the dynamical part of S_R to the dynamical part of S_{R_1}. That is, the map ρ sends the semigroup generated by $\hat{R}$ to the semigroup generated by $\hat{R}_1$ and the action of deck transformations to the action of deck transformations.

By definition, a marked monomorphism also sends fibers, of the family of projections π_n, to fibers. An analytical isomorphism is a marked monomorphism ρ whose inverse is also a marked monomorphism.

Theorem 3.2. *If S_1 and S_2 are semigroups associated to R_1 and R_2, and let $\psi : S_1 \to S_2$ be a marked monomorphism of semigroups, then up to Möbius conjugacy of the maps R_1 and R_2, there exist $\Psi : \mathbb{C} \to \mathbb{C}$ such that the following diagram commutes:*

$$\begin{array}{ccc} S_1 & \xrightarrow{\psi} & S_2 \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ C & \xrightarrow{\Psi} & C \end{array}$$

Where C and C' denote either the sphere $\bar{\mathbb{C}}$, the plane $\mathbb{C}$, or the puncture plane $\mathbb{C}^*$ whenever the exceptional sets of R_1, R_2 have 0, 1 or 2 points respectively. Moreover, the map Ψ is Möbius, if and only if ψ is an analytic isomorphism.

Proof. Since ψ conjugates the action of the deck transformations $\pi_n^{-1} \circ \pi_n$, it sends fibers of π on fibers of π, hence induces a map Ψ defined on the image of the projections π_n. The Riemannian structure on $\mathcal{A}_R$ is consistent with the projections π_n; because ψ preserves the leaf structure on $\mathcal{C}(R)$, the map Ψ is also analytic. If ψ is an isomorphism, the map ψ has an inverse which descends to an analytic inverse of Ψ; hence, the map Ψ is Möbius. $\square$

Let us remind that two decompositions (see for example [11]), $R_1 \circ R_2 \circ ... \circ R_m$ and $P_1 \circ P_2 \circ ... \circ P_m$, are called equivalent if there are Möbius transformations γ_i, for $i = 1, ..., m-1$, such that

$$P_1 = R_1 \circ \gamma_1,\ P_i = \gamma_{i-1}^{-1} \circ R_i \circ \gamma_i,\ \text{for } 1 < i < m,\ \text{and } P_m = \gamma_{m-1}^{-1} \circ R_m.$$

Definition 3.3. A rational map R is called prime, indecomposable, if whenever we have $R = P \circ Q$, where P and Q are rational maps, then either P or Q belong to $PSL(2, \mathbb{C})$. A decomposition of $R = R_1 \circ R_2 \circ ... \circ R_n$ is called a prime decomposition if, and only if, each R_i is prime of degree at least 2 for all i.

The rational map R is called virtually decomposable if there exists a number $n > 0$ and prime rational maps R_1,...,R_m such that $R_1 \circ R_2 \circ ... \circ R_m$ is a decomposition of R^n nonequivalent to R^n.

Every decomposable rational map is virtually decomposable. As a consequence of Ritt's theorems for polynomials, every virtually decomposable polynomial is decomposable. Surprisingly for rational maps this statement is false. M. Zieve constructed the following counterexample.

Example 3.4. Let $R(z) = \frac{(z-1)^2}{(z+1)^2}$, then one can check that

$$R^2(z) = \frac{4z}{(z+1)^2} \circ z^2.$$

Which is nonequivalent to $R \circ R$. It is remarkable that this example appears already on rational maps of degree 2.

Let R be a virtually decomposable rational map, such that R^n has other decompositions. Hence, R^{kn} also has other decompositions for every k. Is it true that eventually there are no new decompositions? In other words, whether the list of decompositions for the iterates R^m is finitely generated, let $g(R)$ be the number of generators of this list.

Let $\alpha(R)$ be the supremum of the numbers m for which new decompositions of R^m appear. In general, the number $\alpha(R)$ can be infinite. Clearly the finiteness of $g(R)$ implies the finiteness of $\alpha(R)$. It is a problem to determine that the reciprocal also holds. This is equivalent to the nonexistence of infinitely many decompositions for a given rational map.

In [10], P. Müller and M. Zieve proved that if P is a polynomial of degree $d \geq 2$, and not associated to parabolic orbifolds (see the definition below), then $\alpha(P)$ is bounded by $\log_2 d$. Moreover, this bound is sharp, and for some exceptions both $\alpha(R)$ and $g(R)$ can be infinite.

Michael Zieve suggested the conjecture that, with the exception of rational maps associated to parabolic orbifolds, the number $\alpha(R)$ is bounded in terms of the degree. We believe that, with the same exceptions, the number $G(R)$ is bounded and is comparable with $\alpha(R)\dim(H(R)/Aff(\mathbb{C}))$, where $Aff(\mathbb{C})$ acts on $H(R)$ by conjugation.

Let us remind that a parabolic orbifold is a Thurston orbifold $\mathcal{O}$ with nonnegative Euler characteristic. When the map R is postcritically finite, the only maps associated to parabolic orbifolds are maps that are Möbius conjugated to maps of the form

$$z \mapsto z^n, \text{ Chebychev polynomials and Lattès maps.} \tag{3.1}$$

In the paper [12], J. Ritt gave the description of all the solutions of the equation of the form

$$R_1 \circ R_2 = R_2 \circ R_1.$$

In [3], A. Eremenko reformulated Ritt's theorem in dynamical terms.

Theorem 3.5 (Eremenko)**.** *Let R_1 and R_2 be a pair of commuting rational maps. Then either there exists a pair of numbers, n and m, such that $R_1^n = R_2^m$ or, there is a parabolic orbifold $\mathcal{O}$ such that maps R_1 and R_2 are covering maps from $\mathcal{O}$ to $\mathcal{O}$.*

Maps R associated to parabolic orbifolds have affine laminations $\mathcal{A}_R$ with special geometry as it is shown in the following theorem due to Lyubich and Kaimanovich (see [4]).

Theorem 3.6 (Kaimanovich-Lyubich)**.** *The affine lamination $\mathcal{A}_R$ admits a continuously varying Euclidean structure on leaves if and only if the map admits a parabolic orbifold.*

As a consequence of Theorem 3.6, we have the following proposition.

Proposition 3.7. *If a map R admits a parabolic orbifold, then the semigroup generated by the restrictions of deck $\langle \pi_n^{-1} \circ \pi_n \rangle$ to leaves is a group of mappings.*

Proof. By Theorem 3.6, the leaves admit an Euclidean structure compatible with projections. This implies that, under a suitable uniformization for all leaves in $\mathcal{C}(R)$, the deck transformations act on leaves as a group of translations. □

Let us define now the space of analytic equivalences $A(S_R)$.

Definition 3.8. The space of analytic deformations of S_R is the space of triples $(S_{R_1}, \rho_1, \rho_2)$, where $R \neq R_1$ and $\rho_1 : S_R \to S_{R_1}$ and $\rho_2 : S_{R_1} \to S_R$ are marked monomorphisms.

We say that $(S_{R_2}, \rho_1, \rho_2)$ and $(S_{R_3}, \phi_1, \phi_2)$ are analytically equivalent if and only if there is an isomorphism γ from S_{R_2} and S_{R_3}. Let $A(S_R)$ denote the space of analytic equivalences of S_R.

If (S_{R_1}, h, g) belongs to $A(S_R)$, then $h \circ g$ and $g \circ h$ commutes with $\hat{R}$ and $\hat{R}_1$, respectively. The next theorem shows the correspondence of the space $A(S_R)$ with the number of virtual decompositions of the iterates of R.

Theorem 3.9. *The map R is virtually decomposable if and only if*

$$card(A(S_{R^n})) > 1.$$

Moreover, the number of virtual decompositions of R is in one-to-one correspondence with the points of $A(S_R)$.

Proof. Assume that R is virtually decomposable, then there exists n such that R^n has a decomposition $Q_1 \circ Q_2$ such that Q_i is not equivalent to R^j for some $j \leq n$. Then the semigroup associated to $Q_2 \circ Q_1$ is analytically equivalent to S_R but not Möbius equivalent, therefore $card(A(S_{R^n})) > 1$. Now let us assume that there exists a number $n > 0$, such that there is more than one analytic equivalence for S_{R^n}. Then there are analytic equivalences q_1 and q_2 between S_{R^n} and a semigroup S_Q, associated to a rational map Q. By Theorem 3.2, the map $q_1 \circ q_2$ descends to an analytic map $Q_1 \circ Q_2$, defined on the Riemann sphere with at most 2 punctures; hence, $Q_1 \circ Q_2$ is a rational map, such that $Q_1 \circ Q_2$ commutes with R. By the same reasoning, $Q_2 \circ Q_1$ commutes with Q. Let us assume that R and $Q_1 \circ Q_2$ are associated to a parabolic orbifold $\mathcal{O}$, then the semiconjugacies

$$Q_1 \circ R = Q \circ Q_1,$$
$$Q_2 \circ Q = R \circ Q_2,$$

imply that Q is also associated to the same parabolic orbifold $\mathcal{O}$. Hence, $S_Q = S_R$ which contradicts the definition of analytic deformation of S_R.

Then, by Theorem 3.5, the maps R and $Q_1 \circ Q_2$ have a common iterate. That is, there are numbers n_1 and n_2 such that $R^{n_1} = (Q_1 \circ Q_2)^{n_2}$. By construction, the map $Q_1 \circ Q_2$ is not Möbius equivalent to R^i; hence, the map R is virtually decomposable. □

3.1. **Decomposition graphs.** Let $\mathcal{D}$ be the semigroup of decomposable rational maps. We construct a directed graph $\mathcal{G}$ associated to $\mathcal{D}$, where the vertices are the elements of $\mathcal{D}$ and there is a directed edge, from R to $\tilde{R}$, if there are two rational maps R_1 and R_2 such that $R = R_1 \circ R_2$ and $\tilde{R} = R_2 \circ R_1$. Given a map $R \in \mathcal{D}$, let $G(R)$ be the connected component of $\mathcal{G}$ containing R. We call $G(R)$, the *graph based at* R.

Example 3.10. Consider a map that has a decomposition $R = R_1 \circ R_2 \circ R_3$. Then the graph based on R contains, at least, a triangle with vertices R, $R_2 \circ R_3 \circ R_1$ and $R_3 \circ R_1 \circ R_2$. Other decorations may appear from other decompositions of the map R as it is shown next.

Let us remind of the first Ritt theorem for the decomposition of polynomials.

Theorem 3.11 (First Ritt's theorem)**.** *Let $P_1 \circ ... \circ P_m$ and $Q_1 \circ ... \circ Q_n$ be two primes decompositions of a polynomial P, then $n = m$.*

In [1], an erratum of the paper [2], W. Bergweiler wrote the following counterexample (which is due to M. Zieve) to the First Ritt's theorem for rational maps.

$$R(z) := z^3 \circ \frac{z^2-4}{z-1} \circ \frac{z^2+2}{z+1} = \frac{z\,(z-8)^3}{(z+1)^3} \circ z^3.$$

One can check that each factor is prime. The graph $G(R)$ contains the triangle above together with a segment, based on R, connecting R with $\frac{z(z-8)^3}{(z+1)^3} \circ z^3$.

The graphs $G(R)$ give a topological realization of the decomposition structure of R. That is, two maps R_1 and R_2 have the same decomposable set if, and only if, the graphs $G(R_1)$ and $G(R_2)$ are isomorphic. However, the graphs $G(R)$, as defined so far, are very big. For every $\gamma \in PSL(2,\mathbb{C})$, we have $R = (R_1 \circ \gamma^{-1}) \circ (\gamma \circ R_2)$. Hence, in the graph based on R, the point R is connected to the maps $\gamma \circ \tilde{R} \circ \gamma^{-1}$. To refine the information in $G(R)$, we consider a quotient of $\mathcal{D}$ by the conjugacy action of $PSL(2,\mathbb{C})$. Under this quotient, the graphs $G(R)$ become finite and it makes sense to consider their fundamental groups $\pi_1(G(R), R)$. These groups provide invariants for the decomposition structure of rational maps.

It is possible to simplify even more the information in $G(R)$ by considering a CW completion of the graph. Namely, complete every triangle induced by $R_1 \circ R_2 \circ R_3$ by a 3-simplex, to every tetrahedron induced by $R_1 \circ R_2 \circ R_3 \circ R_4$ by a 4-simplex and, so on. In this setting, the cohomology groups of this CW-complex give other sets of invariants.

We finally note that given a map R, there is a correspondence between the vertices in $G(R)$ and the elements in $A(R)$, such that the edges in $G(R)$ correspond to marked monomorphisms.

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Instituto de Matemáticas, Unidad Cuernavaca, University Nacional Autonoma de Mexico, Universidad s/n Col Lomas de Chamilpa, 62100 Cuernavaca, Mexico

E-mail address: carlos@matcuer.unam.mx

Instituto de Matemáticas, Unidad Cuernavaca, University Nacional Autonoma de Mexico, Universidad s/n Col Lomas de Chamilpa, 62100 Cuernavaca, Mexico

E-mail address: makienko@matcuer.unam.mx