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QUASI-ISOMETRIC CO-HOPFICITY OF NON-UNIFORM LATTICES IN RANK-ONE SEMI-SIMPLE LIE GROUPS

ILYA KAPOVICH AND ANTON LUKYANENKO

Abstract. We prove that if G is a non-uniform lattice in a rank-one semi-simple Lie group $\neq \mathrm{Isom}(\mathbb{H}^2_{\mathbb{R}})$, then G is quasi-isometrically co-Hopf. This means that every quasi-isometric embedding $G \to G$ is coarsely surjective and thus is a quasi-isometry.

1. Introduction

The notion of co-Hopficity plays an important role in group theory. Recall that a group G is said to be *co-Hopf* if G is not isomorphic to a proper subgroup of itself, that is, if every injective homomorphism $G \to G$ is surjective. A group G is *almost co-Hopf* if for every injective homomorphism $\phi : G \to G$ we have $[G : \phi(G)] < \infty$. Clearly, being co-Hopf implies being almost co-Hopf. The converse is not true: for example, for any $n \geq 1$, the free abelian group $\mathbb{Z}^n$ is almost co-Hopf but not co-Hopf.

It is easy to see that any freely decomposable group is not co-Hopf. In particular, a free group of rank at least 2 is not co-Hopf. It is also well known that finitely generated nilpotent groups are always almost co-Hopf and, under some additional restrictions, also co-Hopf [1]. An important result of Sela [17] states that a torsion-free non-elementary word-hyperbolic group G is co-Hopf if and only if G is freely indecomposable. Partial generalizations of this result are known for certain classes of relatively hyperbolic groups, by the work of Belegradek and Szczepański [2]. Co-Hopficity has also been extensively studied for 3-manifold groups and for Kleinian groups. Delzant and Potyagailo [9] gave a complete characterization of co-Hopfian groups among non-elementary geometrically finite Kleinian groups without 2-torsion.

A counterpart algebraic notion is that of Hopficity. A group G is said to be *Hopfian* if every surjective endomorphism $G \to G$ is necessarily injective, and hence is an automorphism of G. This notion is also extensively studied in geometric group theory. In particular, an important result of Sela [18] shows that every torsion-free word-hyperbolic group is Hopfian. The notion of Hopficity admits a number of interesting "virtual" variations. Thus a group G is called *cofinitely Hopfian* if every endomorphism of G whose image is of finite index in G, is an automorphism of G; see, for example [7].

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A key general theme in geometric group theory is the study of "large-scale" geometric properties of finitely generated groups. Recall that if (X, d_X) and (Y, d_Y) are metric spaces, a map $f : X \to Y$ is called a *coarse embedding* if there exist monotone non-decreasing functions $\alpha, \omega : [0, \infty) \to \mathbb{R}$ such that $\alpha(t) \le \omega(t)$, that $\lim_{t\to\infty} \alpha(t) = \infty$, and such that for all $x, x' \in X$ we have

$$(*) \qquad \alpha(d_X(x, x')) \le d_Y(f(x), f(x')) \le \omega(d_X(x, x')).$$

If d_X is a path metric, then for any coarse embedding $f : X \to Y$ the function $\omega(t)$ can be chosen to be affine, that is, of the form $\omega(t) = at + b$ for some $a, b \ge 0$.

A coarse map f is called a *coarse equivalence* if f is *coarsely surjective*, that is, if there is $C \ge 0$ such that for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \le C$. A map $f : X \to Y$ is called a *quasi-isometric embedding* if f is a coarse embedding and the functions $\alpha(t), \omega(t)$ in $(*)$ can be chosen to be affine, that is, of the form $\alpha(t) = \frac{1}{\lambda}t - \epsilon$, $\omega(t) = \lambda t + \epsilon$ where $\lambda \ge 1$, $\epsilon \ge 0$. Finally, a map $f : X \to Y$ is a *quasi-isometry* if f is a quasi-isometric embedding and f is coarsely surjective.

The notion of co-Hopficity has the following natural counterpart for metric spaces. We say that a metric space X is *quasi-isometrically co-Hopf* if every quasi-isometric embedding $X \to X$ is coarsely surjective, that is, if every quasi-isometric embedding $X \to X$ is a quasi-isometry. More generally, a metric space X is called *coarsely co-Hopf* if every coarse embedding $X \to X$ is coarsely surjective. Clearly, if X is coarsely co-Hopf, then X is quasi-isometrically co-Hopf. If G is a finitely generated group with a word metric d_G corresponding to some finite generating set of G, then every injective homomorphism $G \to G$ is a coarse embedding. This easily implies that if (G, d_G) is coarsely co-Hopf, then the group G is almost co-Hopf.

Example 1.1. The real line $\mathbb{R}$ is coarsely co-Hopf (and hence quasi-isometrically co-Hopf). This follows from the fact that any coarse embedding must send the ends of $\mathbb{R}$ to distinct ends. Since $\mathbb{R}$ has two ends, a coarse embedding induces a bijection on the set of ends of $\mathbb{R}$. It is then not hard to see that a coarse embedding from $\mathbb{R}$ to $\mathbb{R}$ must be coarsely surjective. See [6] for the formal definition of ends of a metric space.

Example 1.2. The rooted regular binary tree T_2 is not quasi-isometrically co-Hopf. We can identify the set of vertices of T_2 with the set of all finite binary sequences. The root of T_2 is the empty binary sequence ϵ and for a finite binary sequence x its left child is the sequence $0x$ and the right child is the sequence $1x$. Consider the map $f : T_2 \to T_2$ which maps T_2 isometrically to a copy of itself that "hangs below" the vertex 0. Thus $f(x) = 0x$ for every finite binary sequence x. Then f is an isometric embedding but the image $f(T_2)$ is not co-bounded in T_2 since it misses the entire infinite branch located below the vertex 1.

Example 1.3. Consider the free group $F_2 = F(a, b)$ on two generators. Then F_2 is not quasi-isometrically co-Hopf.

The Cayley graph X of F_2 is a regular 4-valent tree with every edge of length 1. We may view X in the plane so that every vertex has one edge directed upward, and three downward. Picking a vertex v_0 of X, denote its left branch by X_1 and the remainder of the tree by X_2. We have $X_1 \cup X_2 = X$, and X_1 is a rooted ternary tree. Define a quasi-isometric embedding $f : X \to X$ by taking f to be a shift on X_1 (defined similarly to Example 1.2) and the identity on X_2. The map f is not coarsely surjective, but it is a quasi-isometric embedding. Moreover, for any vertices x, x' of X we have $|d(f(x), f(x')) - d(x, x')| \le 1$.

One can also see that $F_2 = F(a,b)$ is not quasi-isometrically co-Hopf for algebraic reasons. Let $u, v \in F(a,b)$ with $[u,v] \neq 1$. Then there is an injective homomorphism $h : F(a,b) \to F(a,b)$ such that $h(a) = u$ and $h(b) = v$. This homomorphism f is always a quasi-isometric embedding of $F(a,b)$ into itself.

If, in addition, u and v are chosen so that $\langle u, v\rangle \neq F(a,b)$, then $[F(a,b) : h(F(a,b))] = \infty$ and the image $h(F(a,b))$ is not co-bounded in $F(a,b)$.

Thus, the group F_2 is not almost co-Hopf and not quasi-isometrically co-Hopf.

Example 1.4. There do exist finitely generated groups that are algebraically co-Hopf but not quasi-isometrically co-Hopf. The simplest example of this kind is the solvable Baumslag-Solitar group $B(1,2) = \langle a, t | t^{-1}at = a^2\rangle$. It is well known that $B(1,2)$ is co-Hopf.

To see that $B(1,2)$ is not quasi-isometrically co-Hopf we use the fact that $B(1,2)$ admits an isometric properly discontinuous co-compact action on a proper geodesic metric space X that is "foliated" by copies of the hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$. We refer the reader to the paper of Farb and Mosher [12] for a detailed description of the space X, and will only briefly recall the properties of X here.

Topologically, X is homeomorphic to the product $\mathbb{R} \times T_3$ where T_3 is an infinite 3-regular tree (drawn upwards): there is a natural projection $p : X \to T_3$ whose fibers are homeomorphic to $\mathbb{R}$. The boundary of T_3 is decomposed into two sets: the "lower boundary" consisting of a single point u and the "upper boundary" $\partial_\delta X$ which is homeomorphic to the Cantor set (and can be identified with the set of dyadic rationals). For any bi-infinite geodesic ℓ in T_3 from u to a point of $\partial_\delta X$ the full-p-preimage of ℓ in X is a copy of the hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$ (in the upper-half plane model). The p-preimage of any vertex of T_3 is a horizontal horocycle in the $\mathbb{H}^2_{\mathbb{R}}$-"fibers". Any two $\mathbb{H}^2$-fibers intersect along a complement of a horoball in $\mathbb{H}^2_{\mathbb{R}}$.

Similar to the above example for $F(a,b)$, we can take a quasi-isometric embedding $f : T_3 \to T_3$ whose image misses an infinite subtree in T_3 and such that $|d(x,x') - d(f(x), f(x'))| \leq 1$ for any vertices x, x' of T_3. It is not hard to see that this map f can be extended along the p-fibers to a map $\widetilde{f} : X \to X$ such that $\widetilde{f}$ is a quasi-isometric embedding but not coarsely surjective. Since X is quasi-isometric to $B(1,2)$, it follows that $B(1,2)$ is not quasi-isometrically co-Hopf.

Example 1.5. Grigorchuk's group G of intermediate growth provides another intersting example of a group that is not quasi-isometrically co-Hopf. This group G is finitely generated and can be realized as a group of automorphisms of the regular binary rooted tree T_2. The group G has a number of unusual algebraic properties: it is an infinite 2-torsion group, it has intermediate growth, it is amenable but not elementary amenable and so on. See Chapter VIII in [8] for detailed background on the Grigorchuk group. It is known that there exists a subgroup K of index 16 in G such that $K \times K$ is isomorphic to a subgroup of index 64 in G. The map $K \to K \times K$, $k \mapsto (k, 1)$ is clearly a quasi-isometric embedding which is not coarsely surjective. Since both K and $K \times K$ are quasi-isometric to G, it follows that G is not quasi-isometrically co-Hopf.

For Gromov-hyperbolic groups and spaces, quasi-isometric co-Hopficity is closely related to the properties of their hyperbolic boundaries. We say that a compact metric space K is *topologically co-Hopf* if K is not homeomorphic to a proper subset of itself. We say that K is *quasi-symmetrically co-Hopf* if every quasi-symmetric map $K \to K$ is surjective. Note that for a compact metric space K being topologically co-Hopf obviously implies being quasi-symmetrically co-Hopf.

Example 1.6. A recent important result of Merenkov [15] shows that the converse implication does not hold. He constructed a round Sierpinski carpet $\mathbb{S}$ such that $\mathbb{S}$ is quasi-symmetrically co-Hopf. Since $\mathbb{S}$ is homeomorphic to the standard "square" Serpinski carpet, clearly $\mathbb{S}$ is not topologically co-Hopf.

It is well known (see, for example, [3]) that if X, Y are proper Gromov-hyperbolic geodesic metric spaces, then any quasi-isometric embedding $f : X \to Y$ induces a quasi-symmetric topological embedding $\partial f : \partial X \to \partial Y$ between their hyperbolic boundaries. It is then not hard to see that if G is a word-hyperbolic group whose hyperbolic boundary ∂G is quasi-symmetrically co-Hopf (e.g. if it is topologically co-Hopf), then G is quasi-isometrically co-Hopf. This applies, for example, to any word-hyperbolic groups whose boundary ∂G is homeomorphic to an n-sphere (with $n \geq 1$), such as fundamental groups of closed Riemannian manifolds with all sectional curvatures ≤ -1.

The main result of this paper is the following:

Theorem 1.7. *Let G be a non-uniform lattice in a rank-one semi-simple real Lie group other than Isom$(\mathbb{H}^2_{\mathbb{R}})$. Then G is quasi-isometrically co-Hopf.*

Thus, for example, if M is a complete finite volume non-compact hyperbolic manifold of dimension $n \geq 3$, then $\pi_1(M)$ is quasi-isometrically co-Hopf. Note that if G is a non-uniform lattice in Isom$(\mathbb{H}^2_{\mathbb{R}})$, then the conclusion of Theorem 1.7 does not hold since G is a virtually free group.

If G is a uniform lattice in a rank-one semi-simple real Lie group (including possibly a lattice in Isom$(\mathbb{H}^2_{\mathbb{R}})$), then G is Gromov-hyperbolic with the boundary ∂G being homeomorphic to $\mathbb{S}^n$ (for some $n \geq 1$). In this case it is easy to see that G is also quasi-isometrically co-Hopf since every topological embedding from $\mathbb{S}^n$ to itself is necessarily surjective.

Convention 1.8. *From now on and for the remainder of this paper let $X \neq \mathbb{H}^2_{\mathbb{R}}$ be a rank-one negatively curved symmetric space with metric d_X (or just d in most cases). Namely, X is isometric to a hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ (with $n \geq 3$), $\mathbb{H}^n_{\mathbb{C}}$ (with $n \geq 2$), $\mathbb{H}^n_H$ over the reals, complexes, or quaternions, or to the octonionic plane $\mathbb{H}^2_{\mathbb{O}}$.*

If G is as in Theorem 1.7, then G acts properly discontinuously (but with a non-compact quotient) by isometries on such a space X and there exists a G-invariant collection $\mathcal{B}$ of disjoint horoballs in X such that $(X \setminus \mathcal{B})/G$ is compact. The "truncated" space $\Omega = X \setminus \mathcal{B}$, endowed with the induced path-metric d_Ω is quasi-isometric to the group G by the Milnor-Schwartz Lemma. Thus it suffices to prove that (Ω, d_Ω) is quasi-isometrically co-Hopf.

Richard Schwartz [16] established quasi-isometric rigidity for non-uniform lattices in rank-one semi-simple Lie groups and we use his proof as a starting point.

First, using coarse cohomological methods (particularly techniques of Kapovich-Kleiner [14]), we prove that spaces homeomorphic to $\mathbb{R}^n$ with "reasonably nice" metrics are coarsely co-Hopf. This result applies to the Euclidean space $\mathbb{R}^n$ itself, to simply connected nilpotent Lie groups, to the rank-one symmetric spaces X mentioned above, as well as to the horospheres in X. Let $f : (\Omega, d_\Omega) \to (\Omega, d_\Omega)$

be a quasi-isometric embedding. Schwartz' work implies that for every peripheral horosphere σ in Ω there exists a unique peripheral horosphere σ' of X such that $f(\sigma)$ is contained in a bounded neighborhood of σ'. Using coarse co-Hopficity of horospheres, mentioned above, we conclude that f gives a quasi-isometry (with controlled constants) between σ and σ'. Then, following Schwartz, we extend the map f through each peripheral horosphere to the corresponding peripheral horoball B in X. We then argue that the extended map $\hat{f} : X \to X$ is a coarse embedding. Using coarse co-Hopficity of X, it follows that $\hat{f}$ is coarsely surjective, which implies that the original map $f : (\Omega, d_\Omega) \to (\Omega, d_\Omega)$ is coarsely surjective as well.

It seems likely that the proof of Theorem 1.7 generalizes to an appropriate subclass of relatively hyperbolic groups. However, a more intriguing question is to understand what happens for higher-rank lattices.

Problem 1.9. Let G be a non-uniform lattice in a semi-simple real Lie group of rank ≥ 2. Is G quasi-isometrically co-Hopf?

Unlike the groups considered in the present paper, higher-rank lattices are not relatively hyperbolic. Quasi-isometric rigidity for higher-rank lattices is known to hold, by the result of Eskin [11], but the proofs there are quite different from the proof of Schwartz in the rank-one case.

Another natural question is:

Problem 1.10. Let G be as in Theorem 1.7. Is G coarsely co-Hopf?

Our proof only yields quasi-isometric co-Hopficity, and it is possible that coarse co-Hopficity actually fails in this context.

The result of Merenkov (Example 1.6) produces the first example of a compact metric space K which is quasi-symmetrically co-Hopf but not topologically co-Hopf. Topologically, K is homeomorphic to the standard Sierpinkski carpet and there exists a word-hyperbolic group (in fact, a Kleinian group) with boundary homeomorphic to K. However, the metric structure on the Sierpinski carpet in Merenkov's example is not "group-like" and is not quasi-symmetric to the visual metric on the boundary of a word-hyperbolic group.

Problem 1.11. Does there exist a word-hyperbolic group G such that ∂G (with the visual metric) is quasi-symmetrically co-Hopf (and hence G is quasi-isometrically co-Hopf), but such that ∂G is not topologically co-Hopf? In particuar, do there exist examples of this kind where ∂G is homeomorphic to the Sierpinski carpet or the Menger curve?

The above question is particularly interesting for the family of hyperbolic buildings $I_{p,q}$ constructed by Bourdon and Pajot [5, 4]. In their examples $\partial I_{p,q}$ is homeomorphic to the Menger curve, and it turns out to be possible to precisely compute the conformal dimension of $\partial I_{p,q}$. Note that, similar to the Sierpinski carpet, the Menger curve is not topologically co-Hopf.

Problem 1.12. Are the Burdon-Pajot buildings $I_{p,q}$ quasi-isometrically co-Hopf? Equivalently, are their boundaries $\partial I_{p,q}$ quasi-symmetrically co-Hopf?

It is also interesting to investigate quasi-isometric and coarse co-Hopficity for other natural classes of groups and metric spaces. In an ongoing work (in preparation), Jason Behrstock, Alessandro Sisto, and Harold Sultan study quasi-isometric co-Hopficity for mapping class groups and also characterize exactly when this property holds for fundamental groups of 3-manifolds.

2. Geometric objects

2.1. **Horoballs.** Recall that, by Convention 1.8, X is a rank one symmetric space different from $\mathbb{H}^2_{\mathbb{R}}$. Namely, X is isometric to a hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ (with $n \geq 3$), $\mathbb{H}^n_{\mathbb{C}}$ ($n \geq 2$), $\mathbb{H}^n_H$ over the reals, complexes, or quaternions, or to the octonionic plane $\mathbb{H}^2_{\mathbb{O}}$. We recall some properties of X; see [6], Chapter II.10, for details.

Definition 2.1. Let $0 \in X$ be a basepoint and γ a geodesic ray starting at 0. The associated function $b : X \to \mathbb{R}$ given by

$$b(x) = \lim_{s \to \infty} d(x, \gamma(s)) - s \tag{2.1}$$

is known as a Busemann function on X. A *horosphere* is a level set of a Busemann function. The set $b^{-1}[t_0, \infty) \subset X$ is a *horoball.* Up to the action of the isometry group on X, there is a unique Busemann function, horosphere, and horoball.

A Busemann function $b(x)$ provides a decomposition of X into *horospherical coordinates*, a generalization of the upper-halfspace model. Namely, let $\sigma = b^{-1}(0)$ and decompose $X = \sigma \times \mathbb{R}^+$ as follows: given $x \in X$, flow along the gradient of b for time $b(x)$ to reach a point $s \in \sigma$, and write $x = (s, e^{b(x)})$. In horospherical coordinates, the σ-fibers $\{s\} \times \mathbb{R}^+$ are geodesics, the $\mathbb{R}^+$-fibers $\sigma \times \{t_0\}$ are horospheres, and the sets $\sigma \times [t_0, \infty)$ are horoballs. Other horoballs appear as closed balls tangent to the boundary $\sigma \times \{0\}$.

If (M, d) is a metric space and $C \geq 0$, a path $\gamma : [a, b] \to M$, parameterized by arc-length, is called a *C-rough geodesic* in M, if for any $t_1, t_2 \in [a, b]$ we have

$$\big| d(\gamma(t_1), \gamma(t_2)) - |t_1 - t_2| \big| \leq C. \tag{2.2}$$

If Y, Y' are metric spaces, a map $f : Y \to Y'$ is *coarsely Lipschitz* if there exists $C > 0$ such that for any $y_1, y_2 \in Y$ we have $d_{Y'}(f(y_1), f(y_2)) \leq C d_Y(y_1, y_2)$. If Y is a path metric space, then it is easy to see that $f : Y \to Y'$ is coarsely Lipschitz if and only if there exist constants $C, C' > 0$ such that for any $y_1, y_2 \in Y$ with $d_Y(y_1, y_2) \leq C$ we have $d_{Y'}(f(y), f(y')) \leq C'$.

The following two lemmas appear to be well-known folklore facts:

Lemma 2.2. *There exists $C > 0$ with the following property: Let $\mathcal{B}$ be a horoball in X, $x_1 \in X \backslash \mathcal{B}$ and $x_2 \in \mathcal{B}$. Let b be the point in $\mathcal{B}$ closest to x_1. Then the piecewise geodesic $[x_1, b] \cup [b, x_2]$ is a C-rough geodesic.*

Proof. Acting by isometries of X, we may assume that $\mathcal{B}$ is a fixed horoball that is tangent to the boundary of X in the horospherical model. We may also assume that b is the top-most point of $\mathcal{B}$, so that x_1 lies in the vertical geodesic passing through b; see Figure 1.

Consider the "top" of $\mathcal{B}$, i.e. the maximal subset of $\partial\mathcal{B}$ that is a graph in horospherical coordinates. Considering the Riemannian metric on X in horospherical coordinates, one sees that the geodesic $[x_1, x_2]$ must pass through the top of $\mathcal{B}$. Setting C to be the radius of the top of $\mathcal{B}$, centered at b, completes the proof. $\square$

Lemma 2.3. *Let $\mathcal{B}_1, \mathcal{B}_2$ be disjoint horoballs, and $x_1 \in \mathcal{B}_1, x_2 \in \mathcal{B}_2$. Let $[b_1, b_2]$ be the minimal geodesic between $\mathcal{B}_1$ and $\mathcal{B}_2$. Then $[x_1, b_1] \cup [b_1, b_2] \cup [b_2, x_2]$ is a C-rough geodesic, for the value of C in Lemma 2.2.*

Proof. The proof is analagous to that of Lemma 2.2. We may normalize the horoballs $\mathcal{B}_1, \mathcal{B}_2$ as in Figure 2. The normalization depends only on the distance

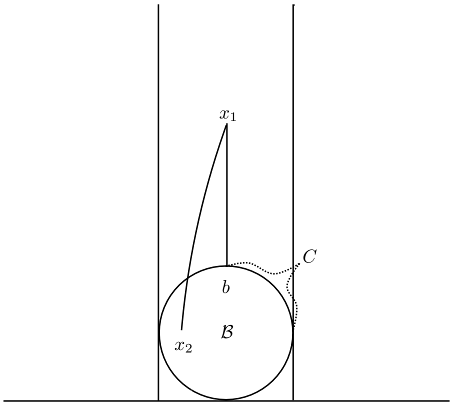

FIGURE 1. Lemma 2.2 for $X = \mathbb{H}^2_{\mathbb{R}}$.

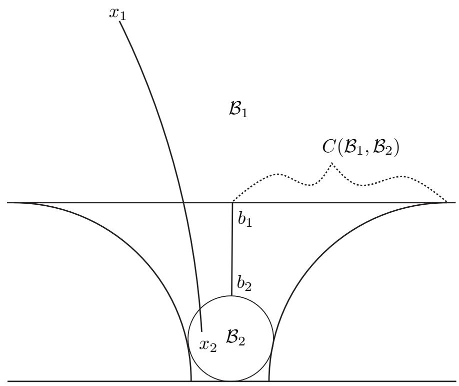

FIGURE 2. Lemma 2.3 for $X = \mathbb{H}^2_{\mathbb{R}}$.

$d(\mathcal{B}_1, \mathcal{B}_2)$. Any geodesic $[x_1, x_2]$ must then pass through compact regions near b_1 and b_2. Let $C(\mathcal{B}_1, \mathcal{B}_2)$ be the radius of this region in $\mathcal{B}_1$. Fixing $\mathcal{B}_1$ and varying $\mathcal{B}_2$, set $C = \sup C(\mathcal{B}_1, \mathcal{B}_2)$. The value $C(\mathcal{B}_1, \mathcal{B}_2)$ remains bounded if the distance between the horoballs goes to infinity (converging to the constant C in Lemma 2.2). Thus, the infimum is attained and $C < \infty$. This completes the proof. □

Lemma 2.4. *Let $\mathcal{B}_1, \mathcal{B}_2$ be disjoint horoballs, $x_1 \in \mathcal{B}_1$, $x_2 \in \mathcal{B}_2$. Denote the minimal geodesic between $\mathcal{B}_1$ and $\mathcal{B}_2$ by $[b_1, b_2]$. Then $d(x_1, b_1) \leq d(x_1, x_2)$.*

Proof. Fix $D > 0$ and allow $\mathcal{B}_1, \mathcal{B}_2, x_1 \in \mathcal{B}_1$, and $x_2 \in \mathcal{B}_2$ to vary with the restriction $d(x_1, x_2) = D$. Define a function f on the interval $[0, D]$ by

$$f(t) = \sup\{d(x_1, b_1) \ : \ d(\mathcal{B}_1, \mathcal{B}_2) = t\},$$

where the supremum is over all combinations of the variables with the restriction stated above, and b_1 denotes the closest point of $\mathcal{B}_1$ to $\mathcal{B}_2$. Then f is a decreasing function, since increasing t pushes the horoballs farther apart and forces x_1 closer to x_2. In particular, $f(D) = 0$ since necessarily $x_1 = b_1$. Conversely, $f(0) = D$, taking $x_2 = b_1 = b_2$. We then have, for any choice of disjoint $\mathcal{B}_1, \mathcal{B}_2$ and x_1, x_2 in the corresponding horoballs, that

$$d(x_1, b_1) \leq f(d(\mathcal{B}_1, \mathcal{B}_2)) \leq d(x_1, x_2) = D,$$

as desired. □

2.2. **Truncated spaces.**

Definition 2.5. Let $X \neq \mathbb{H}^2_{\mathbb{R}}$ be a negatively curved rank one symmetric space. A *truncated space* Ω is the complement in X of a set of disjoint open horoballs. A truncated space is *equivariant* if there is a (non-uniform) lattice $\Gamma \subset \mathrm{Isom}(X)$ that leaves Ω invariant, with Ω/Γ compact.

We will consider Ω with the induced path metric d_Ω from X. Under this metric, curvature remains negative in the interior of Ω. The curvature on the boundary need not be negative. For an extensive treatment of truncated spaces, see [16].

Remark 2.6. Note that truncated spaces are, in general, not uniquely geodesic. Specifically, if X is not a real hyperbolic space, then components of $\partial\Omega$ (which come from horospheres in X) are isometrically embedded in (Ω, d_Ω) copies of non-uniquely-geodesic Riemannian metrics on certain nilpotent groups. In particular, (Ω, d_Ω) is not necessarily a $CAT(0)$-space.

Remark 2.7. Let X be a negatively curved rank one symmetric space and $\Gamma \subset \mathrm{Isom}(X)$ a non-uniform lattice. Then X/Γ is a finite-volume manifold with cusps. In X, each cusp corresponds to a Γ-invariant family of horoballs. Removing the horoballs produces an equivariant truncated space Ω whose quotient Ω/Γ is the compact core of X/Γ.

Proposition 2.8. *Let X be a negatively curved rank one symmetric space and $\Omega \subset X$ an equivariant truncated space. Then the inclusion $\iota : (\Omega, d_\Omega) \hookrightarrow (X, d_X)$ is a coarse embedding.*

Proof. Since d_Ω and d_X are path metrics with the same line element, we have

$$d_X(x,y) \le d_\Omega(x,y). \tag{2.3}$$

To get the lower bound, define an auxilliary function

$$\beta(s) = \max\left\{d_\Omega(x,y) \;:\; x,y \in \Omega \text{ and } d_X(x,y) \le s\right\}. \tag{2.4}$$

Let K be a compact fundamental region for the action of Γ on Ω. Because Γ acts on Ω by isometries with respect to both metrics d_X and d_Ω, we may equivalently define $\beta(s)$ by

$$\beta(s) = \max\left\{d_\Omega(x,y) \;:\; x \in K, y \in \Omega \text{ and } d_X(x,y) \le s\right\}. \tag{2.5}$$

Because K is compact and the metrics d_X, d_Ω are complete, $\beta(s) \in (0,\infty)$ for $s \in (0,\infty)$. Furthermore, $\beta : [0,\infty] \to [0,\infty]$ is continuous and increasing, with $\beta(0) = 0$. Because horospheres have infinite diameter for both d_X and d_Ω (they are isometric to appropriate nilpotent Lie groups with left-invariant Riemannian metrics, see [16]), we also have $\beta(\infty) = \infty$.

Let β' be an increasing homeomorphism of $[0,\infty]$ with $\beta'(s) \ge \beta(s)$ for all s and consider its inverse $\alpha(t)$. For $x, y \in \Omega$ we then have

$$\begin{aligned} d_\Omega(x,y) &\le \beta\left(d_X(x,y)\right) \le \beta'\left(d_X(x,y)\right), \\ \alpha\left(d_\Omega(x,y)\right) &\le d_X(x,y). \end{aligned}$$

This concludes the proof. □

Remark 2.9. A more precise quantitative version of Proposition 2.8 can be obtained by studying geodesics in Ω; see [10].

2.3. **Mappings between truncated spaces.** For this section, let $\Omega \subset X$ be a truncated space, with $X \neq \mathbb{H}^2_{\mathbb{R}}$, and $f : \Omega \to \Omega$ a d_Ω-quasi-isometric embedding. To ease the exposition, we refer to the target truncated space as $\Omega' \subset X'$.

Lemma 2.10 (Schwartz [16]). *There exists $C > 0$ so that for every boundary horosphere σ of Ω, there exists a boundary horosphere σ' of Ω' such that $f(\sigma)$ is contained in a C-neighborhood of σ'.*

Using nearest-point projection, we may assume $f(\sigma) \subset \sigma'$.

Definition 2.11. Let $\mathcal{B}, \mathcal{B}'$ be horoballs with boundaries σ, σ'. A point in σ corresponds, in horospherical coordinates, to a geodesic ray in B. A map $\sigma \to \sigma'$ then extends to a map $\mathcal{B} \to \mathcal{B}'$ in the obvious fashion.

In view of Lemma 2.10, a d_Ω-quasi-isometric embedding $f : \Omega \to \Omega'$ likewise extends to a map $f : X \to X'$ by filling the map on each boundary horoball.

Lemma 2.12 (Schwartz [16])**.** *A quasi-isometry $f : \sigma \to \sigma'$ induces a quasi-isometry $\mathcal{B} \to \mathcal{B}'$, with uniform control on constants.*

Idea of proof. One considers the metric on the horospheres of $\mathcal{B}$ parallel to σ, or alternately fixes a model horosphere and varies the metric. One then shows that if f is a quasi-isometry with respect to one of the horospheres, it is also a quasi-isometry with respect to the horospheres at other horo-heights. One then decomposes the metric on $\mathcal{B}$ into a sum of the horosphere metric and the standard metric on $\mathbb{R}$, in horospherical coordinates. This replacement is coarsely Lipschitz, so the extended map is also coarsely Lipschitz. Taking the inverse of f completes the proof. □

3. Compactly supported cohomology

Definition 3.1. Let X be a simplicial complex and $K_i \subset X$ nested compacts with $\bigcup_i K_i = X$. Compactly supported cohomology $H_c^*(X)$ is defined by

$$H_c^*(X) = \varinjlim H^*(X, X \backslash K_i). \tag{3.1}$$

For a compact space X, $H_C^*(X) = H^*(X)$, but the two do not generally agree for unbounded spaces. We have $H_c^n(\mathbb{R}^n) = \mathbb{Z}$ and $H_c^n(\overline{\Omega}) = 0$ for a non-trivial truncated space Ω. In fact, one has the following lemma.

Lemma 3.2. *Let $Z \subset \mathbb{R}^n$ be a closed subset. Then $H_c^n(Z) \neq 0$ if and only if $Z = \mathbb{R}^n$.*

Proof. It is well known that the choice of nested compact sets does not affect $H_c^n(Z)$. Choose the sequence $K_i = \overline{B(0,i)} \cap Z$, the intersection of a closed ball and Z. With respect to the subset topology of Z, the boundary of K_i is given by $\partial_Z K_i := \partial K_i \cap \partial \overline{B(0,i)}$. We have by excision

$$H^n(Z, K_i) = H^n(K_i, \partial_Z K_i) = \widetilde{H}^n(K_i / \partial_Z K_i).$$

Note that $K_i \subset \overline{B(0,i)}$ and $\partial_Z K_i \subset \partial \overline{B(0,i)}$, so $K_i/\partial_Z K_i \subset \overline{B(0,i)}/\partial\overline{B(0,i)}$. Thus, if $K_i \neq B(0,i)$, then $K_i/\partial_Z K_i \subset S^n \backslash \{*\}$. That is, $K_i/\partial_Z K_i$ is a compact set in $\mathbb{R}^n$, and $\widetilde{H}^n(K_i/\partial_Z K_i) = 0$. Thus, if $Z = \mathbb{R}^n$, we have $H_c^n(Z) = \mathbb{Z}$. Otherwise, $H_c^n(Z) = 0$. □

Compactly supported cohomology is not invariant under quasi-isometries or uniform embeddings. The remainder of this section is distilled from [14], where compactly supported cohomology is generalized to a theory invariant under uniform embeddings. For our purposes, the basic ideas of this theory, made explicit below, are sufficient.

Definition 3.3. Let X be a simplicial complex with the standard metric assigning each edge length 1. Recall that a *chain* in X is a formal linear combination of simplices. The *support* of a chain is the union of the simplices that have non-zero coefficients in the chain. The *diameter* of a chain is the diameter of its support.

An acyclic metric simplicial complex X is *k-uniformly acyclic* if there exists a function α such that any closed chain with diameter d is the boundary of a $k+1$-chain of diameter at most $\alpha(d)$. If X is k-uniformly acyclic for all k, we say that it is *uniformly acyclic*.

Likewise, we say that a metric simplicial complex X is *k-uniformly contractible* if there exists a function α such that every continuous map $S^k \to X$ with image having diameter d extends to a map $B^{k+1} \to X$ with diameter at most $\alpha(d)$. If X is k-uniformly contractible for all k, we say it is *uniformly contractible*.

Remark 3.4. Rank one symmetric spaces and nilpotent Lie groups (with left-invariant Riemannian metrics) are uniformly contractible and uniformly acyclic.

Lemma 3.5. *Let X, Y be uniformly contractible and geometrically finite metric simplicial complexes and $f : X \to Y$ a uniform embedding. Then there exists an iterated barycentric subdivision of X and $R > 0$ depending only on the uniformity constants of f, X, and Y such that f is approximated by a continuous simplicial map with additive error of at most R.*

Proof. We first approximate f by a continuous (but not simplicial) map by working on the skeleta of X. Starting with the 0-skeleton, adjust the image of each vertex by distance at most 1 so that the image of each vertex of X is a vertex of Y. Next, assuming inductively that f is continuous on each k-simplex of X, we now extend to the $k+1$ skeleton using the uniform contractibility of Y. Since error was bounded on the k-simplices, it remains bounded on the $k+1$-skeleton.

Now that f has been approximated by a continous map, a standard simplicial approximation theorem replaces f by a continuous simplicial map, with bounded error depending only on the geometry of X and Y (see for example the proof of Theorem 2C.1 of [13]). □

Lemma 3.6. *Let X and Y be uniformly acyclic simplicial complexes and $f : X \to Y$ a uniform embedding. Suppose furthermore that f is a continuous simplicial map. Then if $H^n_c(X) \cong H^n_c(fX)$.*

Proof. We first construct a left inverse ρ of the map $f_* : C_*(X) \to C_*(fX)$ induced by f on the chain complex of X, up to a chain homotopy P. That is, P will be a map $C_*(X) \to C_{*+1}(X)$ satisfying, for each $c \in C_*(X)$, the homotopy condition

$$\partial Pc = c - \rho f_* c - P\partial c \tag{3.2}$$

and furthermore with diameter of Pc controlled uniformly by the diameter of c.

We start with the 0-skeleton. Each vertex $v' \in fX$ is the image of some vertex $v \in X$ (not necessarily unique). Set $\rho(v') = v$, and extend by linearity to $\rho : C_0(fX) \to C_0(X)$. To define P, let v be an arbitrary vertex in X and note that $\partial v = 0$. We have to satisfy $\partial Pv = v - \rho f_* v$. Since X is acyclic, there exists a 1-chain Pv satisfying this condition. Furthermore, note that $\rho f_* v$ is, by construction, a vertex such that $f(\rho f_* v) = f(v)$. Since f is a uniform embedding, $d(\rho f_* v, v)$ is uniformly bounded above. Thus, Pv may be chosen using *uniform* acyclicity so that its diameter is also uniformly bounded above.

Assume next that ρ and P are defined for all $i < k$ with uniform control on diameters. Let σ be a k-simplex in X. Then $\partial \rho f_* \sigma$ is a chain in X whose diameter is bounded independently of σ. Then, by uniform acyclicity there is a chain σ' with $\partial \sigma' = \partial \rho f_* \sigma$. We define $\rho(\sigma) = \sigma'$. As before, we need to link σ' back to σ. We have

$$\partial(\sigma - \sigma' - P\partial\sigma) = \partial\sigma - \rho f_* \partial\sigma - \partial P \partial\sigma. \tag{3.3}$$

By the homotopy condition (3.2), we further have

$$\partial(\sigma - \sigma' - P\partial\sigma) = \partial\sigma - \rho f_* \partial\sigma - (\partial\sigma - \rho f_* \partial\sigma - P\partial\partial\sigma) = 0. \tag{3.4}$$

Thus, by bounded acyclicity there is a $k+1$ chain $P\sigma$ such that

$$\partial P\sigma = \sigma - \sigma' - P\partial\sigma, \tag{3.5}$$

as desired. We extend both ρ and P by linearity to all of $C_k(fX)$ and $C_k(X)$, respectively.

To conclude the argument, let K be a compact subcomplex of X and consider the complex $X/(X\backslash K) = K/\partial K$. The maps P and $\rho \circ f_*$ on $C_*(X)$ induce maps on $C_*(K/\partial K)$, and the condition $\partial Pc + P\partial c = c - \rho f_* c$ remains true for the induced maps and chains.

Because chain-homotopic maps on C_* induce the same maps on homology, we have, for $h \in H_*(K/\partial K)$, $h = \rho f_* h$. Conversely, $f_*\rho$ is the identity on cell complexes, so still the identity on homology. Thus, $H_*(K/\partial K) \cong H_*(fK/\partial fK)$. By duality, $H^*(fK/\partial fK) \cong H^*(K/\partial K)$.

Taking K_i to be an exhaustion of X by compact subcomplexes and taking a direct limit, we conclude that $H^*_c(X) \cong H^*_c(fX)$. □

Corollary 3.7. *Let X and Y be uniformly acyclic simplicial complexes and $f : X \to Y$ a uniform embedding. There exists an $R > 0$ depending only on the uniformity constants of f, X, and Y so that $H^n_c(N_R(fX)) \cong H^n_c(X)$.*

Proof. Lemma 3.5 approximates f by a continuous simplicial map, within uniform additive error. Lemma 3.6 shows that the resulting approximation induces an isomorphism on compactly supported cohomology. □

Theorem 3.8 (Coarse co-Hopficity)**.** *Let (X, d_X) be a manifold homeomorphic to $\mathbb{R}^n$, with d_X a path metric that is uniformly acyclic and uniformly contractible. For each pair of non-decreasing functions $\alpha, \omega : [0,\infty) \to \mathbb{R}$ with $\alpha(t) < \omega(t)$ and $\lim_{t\to\infty} \alpha(t) = \infty$, there exists a C' such that any (α, ω)-coarse embedding $f : X \to X$ is C'-coarsely surjective.*

Proof. By Corollary 3.7, there is a uniform $R > 0$ such that $H^n_c(N_R(fX)) \cong H^n_c(X) \cong \mathbb{Z}$. By Lemma 3.2, $N_R(fX) = X$. Taking $C' = C + 2R$ completes the proof. □

4. Main result

Theorem 4.1 (Quasi-Isometric co-Hopficity)**.** *Let $\Omega \subset X$ and $\Omega' \subset X'$ be equivariant truncated spaces and $f : (\Omega, d_\Omega) \to (\Omega', d_{\Omega'})$ a quasi-isometric embedding. Then f is coarsely surjective with respect to the truncated metric d_Ω.*

Proof. By Lemma 2.10, we may assume that f maps boundary horospheres of Ω to boundary horospheres of Ω'. By Theorem 3.8, f is a surjection up to a constant independent of the boundary horosphere in question. We then have an extension $F : X \to X'$, as in Definition 2.11.

By Lemma 2.12, for each boundary horoball $\mathcal{B}$, the restriction $F|_{\mathcal{B}}$ is a quasi-isometry. By assumption, $F|_\Omega$ is a d_Ω-quasi-isometry, so $F|_\Omega$ is a d-uniform embedding by Proposition 2.8. Since X is a path metric space, F is then coarsely Lipschitz on all of X.

We now show that F is a uniform embedding by establishing a lower bound for distances between image points. Recall that all distances are measured with respect to $d = d_X$ unless another metric is explicitly mentioned.

Let $L \gg 2$ so that F is coarsely L-Lipschitz and $F|_{\mathcal{B}}$ is coarsely L-co-Lipschitz for every boundary horoball $\mathcal{B}$. Let α, ω be increasing proper functions so that f is an (α, ω)-uniform embedding.

Let $x_1, x_2 \in X$ with $d(x_1, x_2) \gg 0$. We need to provide a lower bound for $d(Fx_1, Fx_2)$ in terms of $d(x_1, x_2)$. Clearly, the lower bound will go to ∞ since F is an isometry along vertical geodesics in horoballs. There are four cases to consider; in all cases we can ignore additive noise by working with sufficiently large $d(x_1, x_2)$ and slightly increasing L.

(1) Let $x_1, x_2 \in \mathcal{B}$ for the same horoball $\mathcal{B}$. Then $d(fx_1, fx_2) > d(x_1, fx_2)/L$.
(2) Let $x_1, x_2 \in \Omega$. This case is controlled by the uniform embeddings $\Omega \hookrightarrow X$ and $\Omega' \hookrightarrow X'$ (Proposition 2.8) and the d_Ω-quasi-isometry constants of f.
(3) Let $x_1 \in \Omega, x_2 \in \mathcal{B}$ for a horoball $\mathcal{B}$. Let $b \in B$ be the closest point to x_1. Then by Lemma 2.2, $[x_1, b] \cup [b, x_2]$ is a C-quasi-geodesic for a universal C depending only on X and X' (see also Figure 1). We consider two sub-cases:
 Suppose that $d(x_1, b) > d(x_1, x_2)/L^3$. Let $b' \in f\mathcal{B}$ be the closest point to fx_1. Then by definition of b, we have

$$d(f^{-1}b', x_1) \geq d(b, x_1) \geq d(x_1, x_2)/L^3.$$

 Using Lemma 2.4, we conclude

$$d(fx_1, fx_2) \geq d(b', fx_2) \geq \alpha(d(f^{-1}, x_2)) \geq \alpha(d(x_1, x_2)/L^3).$$

 Suppose, instead, that $d(x_1, b) \leq d(x_1, x_2)/L^3$. Then we have the estimate $d(fx_1, fb) \leq d(x_1, x_2)/L^2$. We also have $d(x_2, b) \approx d(x_1, x_2)$, so $d(fx_2, fb) \geq d(x_1, x_2)/L$. Consider now $b' \in \mathcal{B}$, the closest point to fx_1. By Lemma 2.2, $d(fb, fb') \leq d(fb, fx_1)$. Thus,

$$d(fx_1, fx_2) \geq d(x_1, x_2)/L - d(x_1, x_2)/L^2.$$

(4) Let $x_1 \in \mathcal{B}_1, x_2 \in \mathcal{B}_2$ be in disjoint horoballs. This case is identical to the previous one, except one uses Lemma 2.3 rather than Lemma 2.2.

We have then provided a lower bound for $d(Fx_1, Fx_2)$ for any pair of points $x_1, x_2 \in X$. Thus, the extended map F is a coarse embedding. By Theorem 3.8, F is then coarsely surjective. Namely, there exists $R > 0$ so that $N_R(F(X)) = X'$ (the neighborhood is taken with respect to d).

We now show that the coarse surjectivity of F with respect to d implies the coarse surjectivity of f with respect to d_Ω.

Let $\omega' \in \Omega'$ be an arbitrary point. Since F is coarsely surjective, there exists $x \in X$ so that $d_{X'}(f(x), \omega') \le R$. If $x \in \Omega$, then we have shown that $\omega' \in N_R(f(\Omega))$. Otherwise, x is contained in a horoball associated with Ω. In appropriate horospherical coordinates, the horoball is given by $S \times (t_0, \infty)$ and x can be written as (s_1, t_1), with $t_1 > t_0$. Likewise, $f(x)$ has coordinates (s'_1, t'_1), with $(t'_1 > t'_0)$. Furthermore, we have $f(s_1, t_0) = (s'_1, t_0)$. Now, $\omega' \in \Omega'$, so it has horospherical coordinates (s'_2, t'_2) with $t'_2 < t'_0$. It is easy to see that

$$\begin{aligned} R &\ge d_{X'}(\omega', (s'_1, t'_1)) \ge d_{X'}(\omega', (s'_1, t'_0)) \\ &= d_{X'}(\omega', f(s_1, t_0)) \ge d_{X'}(\omega', f(\Omega)). \end{aligned} \tag{4.1}$$

Thus, for an arbitrary $\omega' \in \Omega'$ we have $d_{X'}(\omega', f(\Omega)) \le R$. Because $\Omega' \hookrightarrow X'$ is a uniform embedding, this implies that $f : \Omega \to \Omega'$ is coarsely surjective. □

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Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, Illinois 61801
URL: http://www.math.uiuc.edu/~kapovich/
E-mail address: kapovich@math.uiuc.edu

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, Illinois 61801
URL: http://lukyanenko.net
E-mail address: anton@lukyanenko.net