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CANONICAL THURSTON OBSTRUCTIONS FOR SUB-HYPERBOLIC SEMI-RATIONAL BRANCHED COVERINGS

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Abstract. We prove that the canonical Thurston obstruction for a sub-hyperbolic semi-rational branched covering exists if the branched covering is not CLH-equivalent to a rational map.

1. Introduction

Let S^2 be the two-sphere. We use $\widehat{\mathbb{C}}$ to denote the Riemann sphere which is S^2 equipped with the standard complex structure. All maps in this paper are orientation-preserving.

Let $f : S^2 \to S^2$ be a branched covering of degree $d \geq 2$. Let

$$C_f = \{x \in S^2 \mid \deg_x f \geq 2\}$$

denote the set of the critical points of f and

$$P_f = \overline{\bigcup_{k\geq 1} f^k(C_f)}$$

denote the post-critical set of f.

We say f is *critically finite* if $\sharp P_f$ is finite. We say f is *geometrically finite* if $\sharp P_f$ is infinite but the accumulation set P_f' of P_f is a finite set.

Definition 1. Suppose $f, g : S^2 \to S^2$ are two branched coverings of degree $d \geq 2$. They are said to be combinatorially equivalent if there exists a pair of homeomorphisms $\phi,\ \varphi : S^2 \to S^2$ such that

(a) ϕ is isotopic to φ rel P_f and
(b) $\phi \circ f = g \circ \varphi$.

Note that in (a) of Definition 1, the statement that ϕ is isotopic to φ rel P_f means that there is a continuous map $H(x,t) : S^2 \times [0,1] \to S^2$ such that

(1) for each $t \in [0,1]$, $H_t(x) = H(x,t) : S^2 \to S^2$ is a homeomorphism;
(2) $H_0 = \phi$ and $H_1 = \varphi$;
(3) for any point $y \in P_f$ and any $t \in [0,1]$, $H_t(y) = \phi(y) = \varphi(y)$.

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Suppose $f : S^2 \to S^2$ is critically finite. Then there is an orbifold structure on S^2 associated to f as follows. Define the signature $\nu_f : S^2 \to \mathbb{Z}_+ \cup \{\infty\}$ as that $\nu_f(x)$ is the least common multiple of local degrees $\deg_y f^n$ over $y \in f^{-n}(x)$ for all $n \geq 1$. The *orbifold* associated to f is $\Omega_f = (S^2, \nu_f)$. The Euler characteristic of Ω_f, by definition, is

$$\chi(\Omega_f) = 2 - \Sigma_{x \in S^2}\Big(1 - \frac{1}{\nu_f(x)}\Big).$$

It is known that $\chi(\Omega_f) \leq 0$ (see Proposition 9.1(i) in [DH]). Moreover, the orbifold Ω_f is called *hyperbolic* if $\chi(\Omega_f) < 0$ and *parabolic* if $\chi(\Omega_f) = 0$.

Theorem A (see [Th, DH]). *Suppose f is a critically finite branched covering with a hyperbolic orbifold Ω_f. Then f is combinatorially equivalent to a rational map R if and only if f has no Thurston obstructions. Moreover, the rational map R is unique up to conjugations by automorphisms of the Riemann sphere.*

The reader can refer to §2 for the definition of a Thurston obstruction. If a critically finite branched covering f is not equivalent to a rational map, then there must exist Thurston obstructions. The canonical Thurston obstruction is the most interesting one among all Thurston obstructions. The reader can refer to §2 for the term non-peripheral and §4 for the definition of $l(\gamma, x)$.

Theorem B (see [Pi]). *Suppose f is a critically finite branched covering with a hyperbolic orbifold Ω_f, and let Γ_c denote the set of all homotopy class of non-peripheral curves γ in $S^2 \setminus P_f$ such that $l(\gamma, x_n) \to 0$ as $n \to \infty$. Then*

(1) *Γ_c is empty, and f is combinatorially equivalent to a rational map;*
(2) *otherwise, Γ_c is a Thurston obstruction and hence is a canonically defined Thurston obstruction to the existence of a rational map.*

The non-existence of Thurston's obstruction condition is essentially true for any rational map.

Theorem C (see [Mc]). *Suppose $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map. Let Γ be a multicurve on $\hat{\mathbb{C}} - P_R$. It can be a Thurston obstruction only in the following cases:*

(1) *R is critically finite with $\#P_R = 4$ and the orbifold $\Omega_R = (S^2, (2, 2, 2, 2))$ is parabolic. Moreover, R is a double covered by an integral torus endomorphism (it is a special case of a Lattés map).*
(2) *P_R is an infinite set and Γ includes the essential curves in a finite system of annuli permuted by R. These annuli lie in Siegel disks or Herman rings for R and each annulus is a connected component of $\widehat{\mathbb{C}} - P_R$.*

The reader can refer to [Mi] for a definition of a Lattés map and for definitions of a Siegel disk and a Herman ring.

For a geometrically finite branched covering f, the situation is much more complicated. It was first studied in a manuscript [CJS], which was divided into two parts [CJS1] and [CJS2]. The first part was eventually completed and published in [CJ] as follows.

Theorem D (see [CJ]). *There is a geometrically finite branched covering such that it has no Thurston obstruction and it is not combinatorially equivalent to any rational map.*

Due to this theorem, a semi-rational branched covering and a sub-hyperbolic semi-rational branched covering are introduced in [CJ] among the space of all geometrically finite branched coverings.

Definition 2. Suppose $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a geometrically finite branched covering of degree $d \geq 2$. We say f is *semi-rational* if

(1) f is holomorphic in a neighborhood of P'_f;
(2) each cycle $\langle p_0, \cdots, p_{k-1}\rangle$ of period $k \geq 1$ in P'_f is either attractive, that is, $0 < |(f^k)'(p_0)| < 1$, or super-attractive, that is, $(f^k)'(p_0) = 0$, or parabolic, that is, $|(f^k)'(p_0)| = 1$ and $\big((f^k)'(p_0)\big)^q = 1$ for some integer $q \geq 1$; and
(3) each attracting petal associated with a parabolic cycle in P'_f contains a point in the post-critical set P_f.

Furthermore, if all cycles in P'_f are either attractive or super-attractive, we call f a *sub-hyperbolic semi-rational* branched covering.

Clearly, every geometrically finite rational map is a semi-rational branched covering. Furthermore, we have the following theorem.

Theorem E (see [CJ])**.** *A semi-rational branched covering f is always combinatorially equivalent to a sub-hyperbolic semi-rational branched covering g.*

Thus, to study the combinatorial classification in the space of all semi-rational geometrically finite branched coverings, it is enough to study all sub-hyperbolic semi-rational branched coverings. Therefore, the CLH (combinatorially and locally holomorphically) equivalence was introduced in [CJ] in the space of all sub-hyperbolic semi-rational branched coverings.

Definition 3. Suppose f and g are two sub-hyperbolic semi-rational branched coverings. We say that they are CLH-equivalent if there exists a pair of homeomorphisms $\phi,\ \varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that:

(1) ϕ is isotopic to φ rel P_f;
(2) $\phi \circ f = g \circ \varphi$, and
(3) $\phi|U_f = \varphi|U_f$ is holomorphic on some open set $U_f \supset P'_f$.

We have then completed the second part of the study.

Theorem F (see [CJS2][1], [JZ])**.** *Suppose f is a sub-hyperbolic semi-rational branched covering. Then f is CLH-equivalent to a rational map R if and only if f has no Thurston obstructions. In this case, the rational map R is unique up to conjugations by automorphisms of the Riemann sphere.*

Thus, the study of canonical Thurston obstructions for sub-hyperbolic semi-rational branched coverings becomes our final goal, to have a complete understanding of combinatorial structures for geometrically finite branched coverings. In this paper we will complete our final goal.

For the critically finite case, the Teichmüller space of the Riemann sphere minus several points was considered in [DH]. They proved a crucial technical result that a sequence $\{x_n\}$ in the Teichmüller space converges if and only if its projection is a precompact set in the moduli space. The Mumford compactness theorem applies. But neither of them applies for the geometrically finite case. Therefore, we turn

[1] This paper was rewritten by Cui and Tan recently [CT].

to study the bounded geometry property (refer to [Ji]), which makes our approach different from the method applied in [DH]. Roughly speaking, our main result in this paper is that if a sub-hyperbolic semi-rational branched covering f is not CLH-equivalent to any rational map, then there exists a canonical Thurston obstruction. To have a more precise statement of our main result, let us first give an idea of a proof of Theorem F by using bounded geometry as follows.

Suppose f is a sub-hyperbolic semi-rational branched covering. Let $P'_f = \{a_i\}$ be the set of accumulation points of P_f. Then every a_i is periodic. There exists a collection of a finite number of open disks

$$\Lambda = \{D_i\} \tag{1}$$

centered at $\{a_i\}$ and a collection of a finite number of annuli $\{A_i\}$ (we call them the shielding rings) such that:

(i) $\overline{A_i} \cap P_f = \emptyset$;
(ii) $A_i \cap D_i = \emptyset$, but one component of ∂A_i is the boundary of D_i;
(iii) $(\overline{D_i \cup A_i}) \cap (\overline{D_j \cup A_j}) = \emptyset$ for $i \neq j$;
(iv) f is holomorphic on $\overline{D_i} \cup A_i$, and
(v) every $f(\overline{D_i} \cup A_i)$ is contained in D_{i+1} for $1 \leq i \leq k-1$ and $f(\overline{D_k} \cup A_k)$ is contained in D_1 where k is the period of a_i.

Denote $D = \bigcup_i D_i$ and

$$P_1 = P_f \setminus D. \tag{2}$$

Since a_i are accumulation points of P_f, it follows that $\sharp P_1$ is finite. Without loss of generality, we assume that 0, 1, and ∞ belong to P_1. Define

$$Q = P_1 \cup \overline{D} \ \text{ and } \ X = \partial Q = P_1 \cup \partial D. \tag{3}$$

We associate with f the Teichmüller space $T_f = T(\widehat{\mathbb{C}} \setminus Q, X)$ which is the Teichmüller space of the Riemann surface $\widehat{\mathbb{C}} \setminus Q$ whose boundary is X. Note that T_f is also the Teichmüller space $T_0(Q)$ which is the space of all Q-equivalent classes of all Beltrami coefficients μ on $\widehat{\mathbb{C}}$ such that $\mu|Q = 0$. (Two Beltrami coefficients μ and ν are Q-equivalent if the normalized quasiconformal maps w^μ and w^ν are isotopic rel Q.) The space T_f is a complex manifold. The Teichmüller metric and the Kobayashi metric on T_f are also equal (refer to, for example, [EM, GJW, JMW]).

The map f induces a holomorphic map σ_f from T_f into itself and σ_f weakly contracts the Teichmüller metric. An equivalent statement of Theorem F is that σ_f has a unique fixed point if and only if f has no Thurston obstruction.

Every point x in T_f determines a complex structure on $\widehat{\mathbb{C}} \setminus Q$ up to homotopy. Then $(\widehat{\mathbb{C}} \setminus Q, x)$ is a Riemann surface R_x. We embed R_x into the Riemann sphere $\widehat{\mathbb{C}}$ by a quasiconformal map $\phi_x : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixing 0, 1, ∞. Then $\widehat{\mathbb{C}} \setminus \phi_x(Q)$ is a representative of R_x. The reader can refer to §4.

Let $d(\cdot, \cdot)$ mean the spherical distance on $\widehat{\mathbb{C}}$. We define a subspace $\mathcal{T}_{f,b}$ of T_f for each $b > 0$ as follows.

Definition 4. Let $b > 0$ be a constant. Let $\mathcal{T}_{f,b}$ be the subspace of $x = [\mu] \in \mathcal{T}_f$ satisfying the following conditions:

(1) for all $z_i \neq z_{i'} \in P_1$, $d(\phi_\mu(z_i), \phi_\mu(z_{i'})) \geq b$;
(2) for all $z_j \in P_1$ and all $D_i \in \Lambda$, $d(\phi_\mu(z_j), \phi_\mu(D_i)) \geq b$;
(3) for all $D_i \neq D_{i'} \in \Lambda$, $d(\phi_\mu(D_i), \phi_\mu(D_{i'})) \geq b$;

(4) every $D_i \in \Lambda$, $\phi_\mu(D_i)$ contains a round disk of radius b centered at $\phi_\mu(c_i)$.

We call $\mathcal{T}_{f,b}$ the subspace having the bounded geometry property determined by b.

Take an arbitrary $x_0 \in T_f$ and let $x_n = \sigma_f^n(x)$. If f has no Thurston obstructions, we can prove that $\{x_n\}_{n=0}^\infty \subset \mathcal{T}_{f,b}$ for some $b > 0$. This implies that the sequence $\{x_n\}_{n=0}^\infty$ converges in T_f. Thus σ_f has a unique fixed point, and f is CLH-equivalent to a unique sub-hyperbolic rational map.

For a non-peripheral curve γ in $\widehat{\mathbb{C}} \setminus Q$, let $l(\gamma, x)$ denote the hyperbolic length of the unique geodesic in R_x which is homotopic to γ in $\widehat{\mathbb{C}} \setminus Q$. If $\{x_n\} \subset \mathcal{T}_{f,b}$ for some $b > 0$, then there is a $\delta > 0$ such that $l(\gamma, x_n) \geq \delta$ for any non-peripheral curve γ in $\widehat{\mathbb{C}} \setminus Q$ and any $n \geq 0$. Therefore, if f is not CLH-equivalent to a sub-hyperbolic rational map, then there is a sequence of non-peripheral curves γ_n in $\widehat{\mathbb{C}} \setminus Q$ such that $l(\gamma_n, x_n) \to 0$ as $n \to \infty$.

Question. Suppose f is not CLH-equivalent to a rational map. Does there exist a non-peripheral curve γ, such that for any $x_0 \in T_f$ and $x_n = \sigma_f^n(x_0)$, $n > 0$, $l(\gamma, x_n) \to 0$ as $n \to \infty$?

We give an affirmative answer to this question. The positive answer to this question shows how x_n tends to the boundary of T_f. More precisely, we will prove a stronger result as follows.

Theorem 1 (Main Theorem). *Suppose f is a sub-hyperbolic semi-rational branched covering. Let Γ_c denote the set of all homotopy classes of non-peripheral curves γ in $\widehat{\mathbb{C}} \setminus Q$ such that $l(\gamma, x_n) \to 0$ as $n \to \infty$ for any initial $x_0 \in T_f = T_0(Q)$. Then we have that either*

(a) *$\Gamma_c = \emptyset$, then f is CLH-equivalent to a sub-hyperbolic rational map, or*
(b) *$\Gamma_c \neq \emptyset$ is a Thurston obstruction for f and f is not CLH-equivalent to a rational map. In this case, we call Γ_c the canonical Thurston obstruction for f.*

The paper is organized as follows. In §2, we define Thurston obstructions for sub-hyperbolic semi-rational branched coverings. In §3, we review non-negative matrices and study some properties for irreducible non-negative matrices. In §4, we study the Teichmüller space associated with a sub-hyperbolic semi-rational branched covering and short geodesics. For any Thurston obstruction Γ, we can decompose it into Γ_0 and Γ_∞ (see Definition 7). We estimate the upper bound for Γ_∞ in §5 and the lower bound for Γ_0 in §6. Finally, we prove Theorem 1 in §7.

2. Thurston obstructions

Suppose f is a sub-hyperbolic semi-rational branched covering. Let Q be the set as we defined in (3). Then

$$f : \widehat{\mathbb{C}} \setminus f^{-1}(Q) \longrightarrow \widehat{\mathbb{C}} \setminus Q$$

is a covering map of finite degree. If γ is a simple closed curve in $\widehat{\mathbb{C}} \setminus Q$, then all the components of $f^{-1}(\gamma)$ are simple closed curves in $\widehat{\mathbb{C}} \setminus f^{-1}(Q)$, which is a subset of $\widehat{\mathbb{C}} \setminus Q$. Thus all the components of $f^{-1}(\gamma)$ are simple closed curves in $\widehat{\mathbb{C}} \setminus Q$.

A simple closed curve γ is said to be *non-peripheral* if each component of $\widehat{\mathbb{C}} \setminus \gamma$ contains at least two points of Q. A *multi-curve*

$$\Gamma = \{\gamma_1, \cdots, \gamma_n\} \tag{4}$$

is a set of finitely many pairwise disjoint, non-homotopic, and non-peripheral curves in $\widehat{\mathbb{C}} \setminus Q$. For each multi-curve Γ in (4), let

$$\mathbb{R}^\Gamma = \langle \gamma_1, \cdots, \gamma_n \rangle$$

be the real vector space of dimension n with a basis Γ. We define a linear transformation

$$f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$$

as follows: For each $\gamma_j \in \Gamma$, let $\gamma_{i,j,\alpha}$ denote the components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\widehat{\mathbb{C}} \setminus Q$ and $d_{i,j,\alpha}$ be the degree of $f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j$. Define

$$f_\Gamma(\gamma_j) = \Sigma_i \Big(\Sigma_\alpha \frac{1}{d_{i,j,\alpha}} \Big) \gamma_i.$$

Let A_Γ be the corresponding matrix, that is

$$f_\Gamma \mathbf{v} = A_\Gamma \mathbf{v}, \quad \mathbf{v} \in \mathbb{R}^\Gamma.$$

Since the matrix A_Γ is non-negative, by the Perron-Frobenius Theorem, there exists a maximal non-negative eigenvalue $\lambda(A_\Gamma)$ which is the spectral radius of A_Γ.

A multi-curve Γ is said to be *f-stable* if for any $\gamma \in \Gamma$, every non-peripheral component of $f^{-1}(\gamma)$ is homotopic to an element of Γ rel Q.

Definition 5. A stable multi-curve Γ is called a *Thurston obstruction* for f if $\lambda(A_\Gamma) \geq 1$.

Remark 1. The definition of a Thurston obstruction for the critically finite case is similar by replacing Q by P_f.

3. Non-negative matrices

Since a Thurston obstruction is determined by a non-negative matrix, we give a brief review of some results in the matrix theory about non-negative matrices. We use [Ga] as a reference.

A non-negative $n \times n$ matrix A is called *irreducible*, if no permutation of the indices places the matrix in a block lower-triangular form. More precisely, there is no permutation matrix P, which is a matrix consisting of 0 and 1 such that each row or each column contains one and only one 1, such that

$$PAP^{-1} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square matrices. An equivalent definition of irreducibility is that for any $1 \leq i,\ j \leq n$, there exists a $0 \leq q = q(i,j) \leq n$ such that the ij-th entry of A^q is positive.

For the n-dimensional vector space $\mathcal{V}$, we will use the norm

$$\|\mathbf{v}\| = \max_{1 \leq i \leq n} |v_i|, \quad \mathbf{v} = (v_1, \cdots, v_n) \in \mathcal{V} \tag{5}$$

in the rest of this paper. For any linear map $L : \mathcal{V} \to \mathcal{V}$, let A be the corresponding matrix for L; define

$$\|A\| = \sup_{\|\mathbf{v}\|=1} \|A\mathbf{v}\|.$$

The spectral radius $\lambda(A)$ of A can be calculated as

$$\lambda(A) = \lim_{n \to \infty} \sqrt[n]{||A^n||} \geq 0.$$

If A is a non-negative, the Perron-Frobenius Theorem implies that $\lambda(A)$ is an eigenvalue of A. Thus it is a maximal eigenvalue of A. If A is irreducible, $\lambda(A)$ is a simple, positive, maximal eigenvalue with a positive eigenvector $\mathbf{v} = (v_1, \cdots, v_n)$, i.e., $v_i > 0$ for all $1 \leq i \leq n$. However, there may exist another eigenvalue $\mu \neq \lambda(A)$ but $|\mu| = \lambda(A)$. For example, consider

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is an irreducible matrix. The spectral radius is 1 which is a simple, positive, maximal eigenvalue with an eigenvector $\mathbf{v}_1 = (1, 1)$. However, -1 is also an eigenvalue with an eigenvector $\mathbf{v} = (1, -1)$. But if A is positive, that is, every entry is a positive number, the Perron-Frobenius theorem states that $\lambda(A)$ is a unique, simple, positive, maximal eigenvalue with a positive eigenvector $\mathbf{v} = (v_1, \cdots, v_n)$, i.e., $v_i > 0$ for all $1 \leq i \leq n$. Here the term "unique" means that all other eigenvalues μ of A satisfy that

$$|\mu| < \lambda(A).$$

Definition 6. We say that a multi-curve Γ is *irreducible* if the corresponding matrix A_Γ of the linear map $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ is irreducible.

For any non-negative matrix A, we can rearrange the order of the basis such that

$$A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{pmatrix} \tag{6}$$

and all the blocks A_{jj} on the diagonal are either irreducible or 0 matrices. It is not hard to calculate that

$$\lambda(A) = \max_j \lambda(A_{jj}).$$

Now we consider $A = A_\Gamma$ as the corresponding matrix of the linear map $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ for an f-stable multi-curve $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$. We assume that A is in the form of (6). Then we can use Γ_j to denote the subset of curves in Γ corresponding to the j-th block in A. That is, $A_{jj} = A_{\Gamma_j}$. It is clear that

$$\Gamma = \bigcup_j \Gamma_j.$$

We call $\{\Gamma_j\}$ an irreducible decomposition of Γ. The reader should note that Γ_j may not be f-stable.

Denote

$$\Gamma_{Ob} = \bigcup_j \Gamma_j,$$

where the union runs over all j such that $\lambda(A_{jj}) \geq 1$. We have the following definition to relate every element in Γ to Γ_{Ob} if it is not empty.

Definition 7. Suppose Γ is an f-stable multi-curve. For every $\gamma \in \Gamma$, if there exists a $\gamma_{ob} \in \Gamma_{Ob}$ and an integer $k \geq 0$ such that γ is homotopic to a component $f^{-k}(\gamma_{ob})$, then we define the depth of γ with respect to Γ to be the least such integer k. Otherwise, we define the depth as ∞. The set of all elements in Γ with finite depth is denoted by Γ_0. The set of all elements with infinite depth is denoted by Γ_∞.

Then

$$\Gamma = \Gamma_0 \cup \Gamma_\infty.$$

It is clear that if Γ is a Thurston obstruction, then Γ_0 is non-empty. Moreover, we have the following lemma.

Lemma 1. *If Γ is a Thurston obstruction, then Γ_0 is also a Thurston obstruction. In particular, under a permutation of the basis, we can write*

$$A_\Gamma = \begin{pmatrix} A_{\Gamma_\infty} & 0 \\ \bigstar & A_{\Gamma_0} \end{pmatrix}, \tag{7}$$

where $\lambda(A_{\Gamma_\infty}) < 1$ and $\lambda(A_\Gamma) = \lambda(A_{\Gamma_0}) \geq 1$.

Proof. First, for every curve $\gamma \in \Gamma_0$, there exists an integer $k \geq 0$ and an element $\gamma_{ob} \in \Gamma_{Ob}$ such that γ is homotopic to a component of $f^{-k}(\gamma_{ob})$. It follows that any non-peripheral component $\widetilde{\gamma}$ of $f^{-1}(\gamma)$ is homotopic to a component of $f^{-(k+1)}(\gamma_{ob})$. Since Γ is f-stable, then there exists an element $\gamma_i \in \Gamma$ which is homotopic to $\widetilde{\gamma}$. Therefore, any non-peripheral component of $f^{-1}(\gamma)$ is homotopic to an element $\gamma_i \in \Gamma$ whose depth is at most $k+1$. This implies that $\gamma \in \Gamma_0$. Thus Γ_0 is f-stable.

Let us write $\Gamma_\infty = \{\gamma_1, \cdots, \gamma_s\}$. Then $\Gamma_0 = \{\gamma_{s+1}, \cdots, \gamma_n\}$. Since Γ_0 is f-stable, A_Γ must be of the form (7). Furthermore, since $\Gamma_{Ob} \subset \Gamma_0$, we have that

$$\lambda(A_{\Gamma_\infty}) < 1 \quad \text{and} \quad \lambda(A_{\Gamma_0}) = \lambda(A_\Gamma) \geq 1. \qquad \square$$

Now we study the associated matrix A for a sub-hyperbolic semi-rational branched covering f. For each disk D_i in Λ, we take a point b_i on the boundary ∂D_i. Set

$$E = P_1 \cup \bigcup_i \{a_i, b_i\}. \tag{8}$$

Let $p = \sharp E$. It is obvious that every multi-curve Γ in $\widehat{\mathbb{C}} \setminus Q$ is a multi-curve in $\widehat{\mathbb{C}} \setminus E$. It follows that there are only a finite number of possible matrices for all linear transformations f_Γ (refer to [DH, Lemma 1.2]).

(There are infinitely many possible multi-curves Γ.) Therefore, we have the following proposition.

Proposition 1. *There is a number $0 < \beta \leq 1$ depending only on the degree d of f and the cardinality p of E such that for any irreducible multi-curve Γ in $\widehat{\mathbb{C}} \setminus Q$ (not necessarily f-stable) with $\lambda(A_\Gamma) \geq 1$, let $\mathbf{v}$ be the unique positive eigenvector of A_Γ corresponding to $\lambda(A_\Gamma) \geq 1$ with $\|\mathbf{v}\| = 1$. Then, the smallest coordinate of $\mathbf{v}$ is bounded below by β.*

Proof. Since there are only finitely many possible matrices for all irreducible multi-curves, there are finitely many simple, positive, maximal eigenvalues. Thus there are finitely many positive eigenvectors $\mathbf{v}$ with $\|\mathbf{v}\| = 1$. This gives the proposition. $\square$

Proposition 2. *There exists a positive integer m such that for any non-empty f-stable multi-curve Γ, if it is a Thurston obstruction,*

$$||A_{\Gamma_\infty}^m|| < 1/2.$$

Proof. Since there are only finitely many matrices A_Γ corresponding to all Γ, there are only finitely many A_{Γ_∞}. For each A_{Γ_∞}, $\lambda(A_{\Gamma_\infty}) < 1$. Thus we have an integer $m > 0$ such that

$$||A^m_{\Gamma_\infty}|| < 1/2.$$

□

Every multi-curve Γ can contain at most $p-3$ curves, so we have the following proposition.

Proposition 3. *There is a positive integer M depending on p such that for any f-stable multi-curve Γ in $\widehat{\mathbb{C}} \setminus Q$, the depth of every $\gamma \in \Gamma_0$ is less than or equal to M.*

4. Teichmüller space and short geodesics.

Suppose f is a sub-hyperbolic semi-rational branched covering. Recall Q and P_1 defined in (2) and (3) and the assumption that $0,\ 1,\ \infty \in P_1$. Let $\mathcal{M}(\mathbb{C})$ be the unit ball of the space $L^\infty(\mathbb{C})$. That is, it is the set of all measurable functions μ on $\mathbb{C}$ such that essential supremum norm $\|\mu\|_\infty < 1$. Each element $\mu \in \mathcal{M}(\mathbb{C})$ is called a Beltrami coefficient since the measurable Riemann mapping theorem [AB] says that the Beltrami equation

$$\phi_{\overline{z}} = \mu \phi_z$$

has a unique quasiconformal self-map ϕ^μ of $\widehat{\mathbb{C}}$ fixing 0, 1, and ∞ as a solution, which depends on $\mu \in \mathcal{M}(\mathbb{C})$ holomorphically. The map ϕ^μ is called the normalized solution.

Definition 8. The Teichmüller space T_f is the equivalence class $[\mu]$ for $\mu \in \mathcal{M}(\mathbb{C})$ satisfying that $\mu|Q = 0$ a.e., where μ_1 and μ_2 are equivalent if and only if ϕ^{μ_1} is isotopic to ϕ^{μ_2} rel Q. Furthermore, we can define the Teichmüller distance between two points $x = [\mu]$ and $y = [\nu]$ in T_f as

$$\mathrm{d}_T(x,y) = \frac{1}{2}\min_{\widetilde{\mu}\in[\mu],\widetilde{\nu}\in[\nu]} \log K[\phi^{\widetilde{\mu}} \circ (\phi^{\widetilde{\nu}})^{-1}],$$

where $K[\phi]$ is the maximal dilation of the quasiconformal map ϕ.

From [Li] (or refer to [JZ]), we knew that T_f is the Teichmüller space $T(\widehat{\mathbb{C}} \setminus Q)$ of Riemann surface $\widehat{\mathbb{C}} \setminus Q$ with boundary ∂Q. It is a complex manifold and the projective map

$$\Phi : \mathcal{M}(\mathbb{C}) \to T_f$$

is a holomorphic split submersion.

Define the self-map σ_f of the Teichmüller T_f by

$$\sigma_f([\mu]) = [f^*(\mu)].$$

In formula,

$$(f^*\mu)(z) = \frac{\mu_f(z) + \mu(f(z))\theta(z)}{1 + \overline{\mu_f(z)}\mu(f(z))\theta(z)},$$

where $\theta(z) = \overline{f_z}/f_z$ and $\mu_f(z) = f_{\bar{z}}/f_z$, is the pull-back of μ by f. Since

$$\sigma_f = \Phi \circ f^* \circ \Phi^{-1},$$

where Φ^{-1} means a local holomorphic section of Φ. Thus

$$\sigma_f : T_f \to T_f$$

is a holomorphic map. Since the Teichmüller metric d_T coincides with the Kobayashi metric on the complex manifold T_f and σ_f is holomorphic, we have that

$$\mathrm{d}_T(\sigma_f(x), \sigma_f(y)) \leq \mathrm{d}_T(x, y), \quad \forall x, y \in T_f.$$

From [JZ], we also know that

$$\mathrm{d}_T(\sigma_f(x), \sigma_f(y)) < \mathrm{d}_T(x, y), \quad \forall x, y \in T_f. \tag{9}$$

We need more definitions and lemmas from [JZ] as follows.

Let Z be a subset of Q with $\sharp(Z) \geq 4$. Let $x = [\mu] \in T_f$ and let $\gamma \in \widehat{\mathbb{C}} \setminus Z$ be a simple closed and non-peripheral curve. We use $l_Z(\gamma, x)$ to denote the hyperbolic length of the unique simple closed geodesic which is homotopic to $\phi^\mu(\gamma)$ in the hyperbolic Riemann surface $\widehat{\mathbb{C}} \setminus \phi^\mu(Z)$. We say γ is a (μ, Z)-simple closed geodesic if $\phi^\mu(\gamma)$ is a simple closed geodesic in $\widehat{\mathbb{C}} \setminus \phi^\mu(Z)$.

Remark 2. From the definition of the Teichmüller space T_f, we know that the definition of $l_Z(\gamma, x)$ is independent of the choice of μ in x.

For $x_0 \in T_f$, let $x_n = \sigma_f^n(x_0)$, $n = 1, \cdots$, be a sequence in T_f. Recall our definition of E in (8).

Lemma 2. *If there is a real number $a > 0$ such that there is a point $x_0 \in T_f$ and every (x_n, E)-simple closed geodesic $\gamma \subset \widehat{\mathbb{C}} \setminus Q$ has hyperbolic length greater than or equal to a, then the sequence $\{x_n\}_{n=0}^\infty$ is convergent in T_f and the limiting point is the fixed point of σ_f in T_f.*

Remark 3. This lemma implies that if there exists an $x_0 \in T_f$ such that the length of the shortest geodesic on all the x_n has a uniform lower bound, then f has no Thurston obstructions.

Lemma 3. *There exists an $\eta > 0$ such that for any point $x = [\mu] \in T_f$ with $\mu(z) = 0$ on $\bigcup_i A_i$ and for any (x, E)-simple geodesic $\gamma \subset \widehat{\mathbb{C}} \setminus E$ with $l_E(\gamma, x) < \eta$, we have $\gamma \subset \widehat{\mathbb{C}} \setminus Q$. Moreover, for any $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$l_E(\gamma, x) > (1 - \epsilon) l_Q(\gamma, x)$$

whenever $l_E(\gamma, x) < \delta$.

Remark 4. The above lemma implies that for any $x = [\mu] \in T_f$ with $\mu(z) = 0$ for all $z \in \bigcup_i A_i$, sufficiently short geodesics in $\widehat{\mathbb{C}} \setminus \phi^\mu(E)$ are homotopic to the sufficiently short geodesics in $\widehat{\mathbb{C}} \setminus \phi^\mu(Q)$. More precisely, we can find a constant $\delta_0 > 0$ such that

$$\frac{1}{e} l_Q(\gamma, x) < l_E(\gamma, x) < l_Q(\gamma, x) \quad \text{whenever } l_E(\gamma, x) < \delta_0.$$

Suppose $x = [\mu] \in T_f$ and $Z \subset Q$. Define

$$w_Z(\gamma, x) = -\log l_Z(\gamma, x).$$

Consider the set

$$L_{Z,x} = \{w_Z(\gamma, x)\},$$

where γ ranges over all the non-peripheral simple closed curves in $\widehat{\mathbb{C}} \setminus Q$. Define

$$w_Z(x) = \sup\{w_Z(\gamma, x)\}$$

and

$$w_Z(\Gamma, x) = \max_{\gamma \in \Gamma} w_Z(\gamma, x).$$

The following lemma is a general result for hyperbolic Riemann surfaces (refer to [DH, JZ]). We just state it in our case.

Lemma 4. *Let $Z \subset Q$ be a finite subset with $\sharp Z \geq 4$ and let $\gamma \subset \widehat{\mathbb{C}} \setminus Q$ be a non-peripheral simple closed curve. Then the function*

$$x \mapsto w_Z(\gamma, x) : T_f \to \mathbb{R}$$

is Lipschitz with Lipschitz constant 2.

Let

$$A = \max\{-\log\log(2\sqrt{2}+3), -\log\delta_0\},$$

where δ_0 is the number in Remark 4. Note that $\log(2\sqrt{2}+3)$ is the magic number in the theory of hyperbolic Riemann surfaces such that for any hyperbolic Riemann surface S, any two simple closed geodesics γ and γ' in S are disjoint whenever the hyperbolic lengths of γ and γ' are less than $\log(2\sqrt{2}+3)$. This implies that for any point $x \in T_f$, there are at most $p-3$ curves γ with $l_E(\gamma, x) \leq \log(2\sqrt{2}+3)$.

For any $J > 0$, let (a, b) be the lowest interval in $\mathbb{R} \setminus L_{E,x}$ such that $a \geq A$ and $b - a = J$. For any $x = [\nu] \in T_f$, define

$$\Gamma_{J,x} = \{\gamma \mid \gamma \text{ is a simple closed geodesic on } R_x \text{ and } w_E(\gamma, x) \geq b\}.$$

Then $\Gamma_{J,x}$ is a multi-curve consisting of the geodesics which are sufficiently short on $\widehat{\mathbb{C}} \setminus \phi^\mu(E)$. This is equivalent saying that they are all the simple closed curves in $\widehat{\mathbb{C}} \setminus \phi^\mu(Q)$ which are homotopic to sufficiently short simply closed geodesics on $\widehat{\mathbb{C}} \setminus \phi^\mu(Q)$. There are at most $p-3$ elements in $\Gamma_{J,x}$ for any x and they are pairwise disjoint.

For any $x \in T_f$, let $D = d_T(x, \sigma_f(x))$.

Lemma 5. *If $J \geq \log d + 2D + 1$ and $\Gamma_{J,x} \neq \emptyset$, then $\Gamma_{J,x}$ is an f-stable multi-curve.*

See Lemma 7.3 in [JZ].

5. Upper bound for Γ_∞

We still keep the notation in the previous sections. Suppose $x_0 \in T_f$ and $x_n = \sigma_f^n(x_0)$ for all $n \geq 1$. Then we have a sequence $\{x_n\}_{n=0}^\infty$ in T_f.

For all $n > 0$ and all $z \in \bigcup_i A_i$, we have that $\mu_n(z) = 0$, where $[\mu_n] = x_n$, since $f(\bigcup_i A_i) \subset \bigcup_i D_i$ as we constructed $\{A_i\}$ as the shielding rings.

Recall the definition of $E = P_1 \cup \bigcup_i \{a_i, b_i\}$ in (8) and m in Proposition 2. Let

$$P_2 = E \cup f^m(E) \cup \bigcup_{1 \leq j \leq m} f^j(\Omega_f) \subset Q.$$

The following lemma is also from [JZ].

Lemma 6. *There exists an $\epsilon_0 > 0$, such that for any $x = [\mu] \in T_f$ with $\mu(z) = 0$ for all $z \in \cup_i A_i$, and for any (μ, P_2)-simple closed geodesic γ', if $l_{P_2}(\gamma', x) < \epsilon_0$, then there is a (μ, E)-simple closed geodesic γ such that γ' is homotopic to γ in $\widehat{\mathbb{C}} \setminus P_2$.*

The following lemma is also a general result in the theory of hyperbolic Riemann surfaces and the reader can find a proof in [DH].

Lemma 7. *Let X be a hyperbolic Riemann surface, $P \subset X$ is a finite subset, and $\sharp P < p$. Let $X' = X \setminus P$ and $L < \log(3+2\sqrt{2})$. Let γ be a simple closed geodesic on X, and let $\gamma_1', \cdots, \gamma_k'$ be all the geodesics on X' homotopic to γ in X whose hyperbolic length on X' is less than L. Set $l = l_X(\gamma)$ and $l_i' = l_{X'}(\gamma_i')$. Then:*

(1) *$k \leq p+1$;*
(2) *for all i, $l_i' \geq l$;*
(3) $\frac{1}{l} - \frac{1}{\pi} - \frac{(p+1)}{L} < \sum_{i=1}^{k} \frac{1}{l_i'} < \frac{1}{l} + \frac{(p+1)}{\pi}$.

The next proposition is essential for our proof.

Proposition 4. *Let m be the constant in Proposition 2. Let $x_0 \in T_f$ and $x_n = \sigma_f^n(x_0)$ for $n > 0$. There exists a constant $C(J) > 0$ depending on p, d, ϵ_0, $D = d_T(x_0, x_1)$ and $J \geq m(\log d + 2D + 1)$ such that if $w_E(x_0) > C(J)$, then $\Gamma = \Gamma_{J,x_0} \neq \emptyset$ is a stable multi-curve. Moreover, if $\Gamma_\infty \neq \emptyset$, then*

$$w_E(\Gamma_\infty, x_m) \leq w_E(\Gamma_\infty, x_0).$$

Proof. If $w_E(x_0) \geq A + (p-3)J$, then Γ_{J,x_0} is non-empty, since R_{x_0} has at most $(p-3)$ simple closed geodesics with hyperbolic length less than e^{-A} (they are not homotopic to each other). From Lemma 5, $\Gamma = \Gamma_{J,x_0}$ is also f-stable.

Suppose $\Gamma_\infty \neq \emptyset$ and A_Γ is in the form of (7). From Proposition 2, $||A_{\Gamma_\infty}^m|| < 1/2$.

For each $\gamma_j \in \Gamma_{J,x_0}$, let $\gamma_{i,j,\alpha}$ be any component of $f^{-m}(\gamma_j)$ homotopic to γ_i in $\widehat{\mathbb{C}} \setminus Q$. Then $\gamma_{i,j,\alpha}$ is also homotopic to γ_i in $\widehat{\mathbb{C}} \setminus E$. Let $g = \phi^\mu \circ f^m \circ (\phi^\nu)^{-1}$, where $[\mu] = x_0$ and $[\nu] = x_m$. Then g is a rational map and

$$g : \widehat{\mathbb{C}} \setminus \phi^\nu(f^{-m}(P_2)) \to \widehat{\mathbb{C}} \setminus \phi^\mu(P_2)$$

is a holomorphic covering map. Therefore

$$l_{f^{-m}(P_2)}(\gamma_{i,j,\alpha}, x_m) = d_{i,j,\alpha} l_{P_2}(\gamma_j, x_0),$$

where $d_{i,j,\alpha}$ is the degree of $f^m : \gamma_{i,j,\alpha} \to \gamma_j$. We get

$$\sum_\alpha \frac{1}{l_{f^{-m}(P_2)}(\gamma_{i,j,\alpha}, x_m)} = \Big(\sum_\alpha \frac{1}{d_{i,j,\alpha}}\Big) \frac{1}{l_{P_2}(\gamma_j, x_0)} = b_{ij} \frac{1}{l_{P_2}(\gamma_j, x_0)},$$

where b_{ij} is the ij-entry of A_Γ^m.

Since $E \subset P_2$, the inclusion

$$\iota : \widehat{\mathbb{C}} \setminus P_2 \hookrightarrow \widehat{\mathbb{C}} \setminus E$$

decreases the hyperbolic distances. So we have that $l_{P_2}(\gamma_j, x_0) > l_E(\gamma_j, x_0)$ for any γ_j. It follows that

$$\sum_\alpha \frac{1}{l_{f^{-m}(P_2)}(\gamma_{i,j,\alpha}, x_m)} < b_{ij} \frac{1}{l_E(\gamma_j, x_0)}.$$

From the definitions of P_2 and E, we know that $E \subset f^{-m}(P_2)$. Let $C = C(d, m, p) = \sharp(f^{-m}(P_2) \setminus E)$, where $p = \sharp E$.

We claim that for any $(\nu, f^{-m}(P_2))$-simple closed geodesic γ which is homotopic to γ_i in $\widehat{\mathbb{C}} \setminus E$, either γ is homotopic to some $\gamma_{i,j,\alpha}$ in $\widehat{\mathbb{C}} \setminus f^{-m}(P_2)$ or

$$l_{f^{-m}(P_2)}(\gamma, x_m) > \min\{e^{-(A+PJ)}, \epsilon_0\},$$

where ϵ_0 is the constant in Lemma 6.

We prove the claim. In fact, if γ is not homotopic in $\widehat{\mathbb{C}} \setminus f^{-m}(P_2)$ to some $\gamma_{i,j,\alpha}$, then $f^m(\gamma)$ is a (μ, P_2)-simple closed geodesic which is not homotopic to any γ_j in

$\widehat{\mathbb{C}} \setminus P_2$. Then there are two cases: either (1) $f^m(\gamma)$ is homotopic in $\widehat{\mathbb{C}} \setminus P_2$ to some (μ, E)-simple closed geodesic ξ which does not belong to Γ_{J,x_0}, then we have

$$l_{P_2}(f^m(\gamma), x_0) > l_E(f^m(\gamma), x_0) = l_E(\xi, x_0) > e^{-a} > e^{-(A+PJ)}$$

or (2) $f^m(\gamma)$ is not homotopic in $\widehat{\mathbb{C}} \setminus P_2$ to any (μ, E)-simple closed geodesic, then by Lemma 6, we have

$$l_{P_2}(f^m(\gamma), x_0) > \epsilon_0.$$

Thus we have

$$l_{f^{-m}(P_2)}(\gamma, x_m) \geq l_{P_2}(f^m(\gamma), x_0) > \min\{e^{-(A+PJ)}, \epsilon_0\}.$$

This proves the claim.

From the left hand of the inequality given by (3) in Lemma 7, for each $\gamma_i \in \Gamma$, we have

$$\frac{1}{l_E(\gamma_i, x_m)} - \frac{1}{\pi} - \frac{C+1}{\min\{e^{-(A+PJ)}, \epsilon_0\}} \leq \sum_{j,\alpha} \frac{1}{l_{f^{-m}(P_2)}(\gamma_{i,j,\alpha}, x_m)} \leq \sum_j b_{ij} \frac{1}{l_E(\gamma_j, x_0)}.$$

Suppose $\Gamma_\infty = \{\gamma_1, \cdots, \gamma_s\} \subset \Gamma$. Then for each $\gamma_i \in \Gamma_\infty$, from the form (7) of A_Γ,

$$\frac{1}{l_E(\gamma_i, x_m)} \leq \sum_{j=1}^{s} b_{ij} \frac{1}{l_E(\gamma_j, x_0)} + \frac{1}{\pi} + \frac{C+1}{\min\{e^{-(A+PJ)}, \epsilon_0\}}.$$

Let

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{l_E(\gamma_1, x_m)} \\ \vdots \\ \frac{1}{l_E(\gamma_s, x_m)} \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} \frac{1}{l_E(\gamma_1, x_0)} \\ \vdots \\ \frac{1}{l_E(\gamma_s, x_0)} \end{pmatrix}.$$

Since $\|A_\infty^m\| < 1/2$,

$$\|\mathbf{v}_1\| < \frac{1}{2}\|\mathbf{v}\| + \frac{1}{\pi} + \frac{C+1}{\min\{e^{-(A+PJ)}, \epsilon_0\}}.$$

Define

$$C(J) = \max\Big\{2\Big(\frac{1}{\pi} + \frac{C+1}{\min\{e^{-(A+PJ)}, \epsilon_0\}}\Big), A + (p-3)J\Big\}.$$

If $w_E(\Gamma_\infty, x_0) \geq C(J)$, then we have

$$w_E(\Gamma_\infty, x_m) < w_E(\Gamma_\infty, x_0). \qquad \square$$

Lemma 8. *Let $J \geq m(\log d + 2D + 1)$. Suppose $w_E(x_0) < C(J)$ and suppose $\Gamma = \Gamma_{J,x_k} \neq \emptyset$ for some $k \geq 0$. Let $E(J) = C(J) + 2mD$. If $\Gamma_\infty \neq \emptyset$, then for all n,*

$$w_E(\Gamma_\infty, x_n) < E(J).$$

Moreover, if $w_E(\gamma, x_k) \geq E(J)$, then $\gamma \in \Gamma_0$.

Proof. We prove the first inequality by contradiction. Suppose there is an $n > 0$ such that $w_E(\Gamma_\infty, x_n) \geq C(J) + 2mD$. Suppose n_0 is the first integer having this property. Then we have $w_E(\Gamma_\infty, x_{n_0-m}) \geq C(J)$. Then by Proposition 4 and the fact that n_0 is the first integer such that $w_E(\Gamma_\infty, x_{n_0}) \geq C(J) + 2mD$, we have

$$w_E(\Gamma_\infty, x_{n_0}) \leq w_E(\Gamma_\infty, x_{n_0-m}) < C(J) + 2mD.$$

This is a contradiction.

If $w_E(\gamma, x_k) \geq E(J) > C(J) \geq A + (p-3)J$, then $\gamma \in \Gamma_{J,x_k} = \Gamma$ since there are at most $p-3$ simple closed curves in R_{x_k} such that $w_E(\gamma, x_k) > A$. But $\gamma \notin \Gamma_\infty$ because of the first conclusion and the assumption. Therefore, $\gamma \in \Gamma_0$. □

6. Lower bound for Γ_0

In order to get the lower bound for Γ_0, we need the following definition.

Definition 9. Let κ be a real number. A sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers is called κ-quasi-nondecreasing if for all $n_1 < n_2$ we have $a_{n_2} - a_{n_1} \geq \kappa$. A sequence is called quasi-nondecreasing if it is κ-quasi-nondecreasing for some κ.

It is easy to check that the following two properties are true.

Property 1. Suppose $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences. If $\{a_n\}_{n=0}^{\infty}$ is κ-quasi-nondecreasing and if $|a_n - b_n| < r$ for all n, then $\{b_n\}$ is $(\kappa - 2r)$-quasi-nondecreasing.

Property 2. Suppose $\{a_n\}$ is quasi-nondecreasing and unbounded. Then $a_n \to +\infty$ as $n \to +\infty$.

Recall that any $x = [\mu] \in T_f$ represents a complex structure on $\widehat{\mathbb{C}} \setminus Q$, which makes $\widehat{\mathbb{C}} \setminus Q$ a hyperbolic Riemann surface R_x. For any simple closed geodesic γ on R_x, let $A(\gamma, x)$ be the Riemann surface, conformally isomorphic to an annulus, obtained by taking the unit disk $\mathbb{D}$ modulo a $\mathbb{Z}$-subgroup of the fundamental group of R_x generated by γ. It is a covering space of R_x. The core curve of $A(\gamma, x)$ is a geodesic of length $l_Q(\gamma, x)$ and

$$\operatorname{mod}(A(\gamma, x)) = \frac{\pi}{l_Q(\gamma, x)}, \tag{10}$$

where $\operatorname{mod}(A)$ means the modulus of an annulus A.

If γ is a simple closed geodesic of hyperbolic length l on the Riemann surface R_x, then there is an embedding annulus $a(\gamma, x)$ of modulus $m(l)$ which is continuous and decreasing and satisfies

$$\frac{\pi}{l} - 1 < m(l) < \frac{\pi}{l}.$$

Thus for all $x \in T_f$, we have

$$\operatorname{mod}(A(\gamma, x)) - 1 < \operatorname{mod}(a(\gamma, x)) < \operatorname{mod}(A(\gamma, x)). \tag{11}$$

We need the following technical lemma.

Lemma 9. *If* $t \geq 1$, *then* $\log(t+1) - 1 < \log t$.

Proof. For $t \geq 1$,

$$\log(t+1) - \log t = \log(\frac{t+1}{t}) \leq \log 2 < 1.$$ □

If $w_Q(\gamma, x) \geq \log \frac{2}{\pi} = -0.451582705\cdots$, then we have $\operatorname{mod}(A(\gamma, x)) - 1 \geq 1$. By taking logarithms on all terms of inequality (11) and by applying Lemma 9 and equation (10), we have

$$\log \pi - 1 + w_Q(\gamma, x) < \log \operatorname{mod}(a(\gamma, x)) < \log \pi + w_Q(\gamma, x).$$

It follows that, if $w_Q(\gamma, x) \geq \log \frac{2}{\pi}$, then

$$|\log \operatorname{mod}(a(\gamma, x)) - w_Q(\gamma, x)| < \log \pi. \tag{12}$$

Given a multi-curve Γ, we denote vectors of moduli $(\mathrm{mod}(A(\gamma,x)))$ and $(\mathrm{mod}(a(\gamma,x)))$ by $\mathrm{mod}(A(\Gamma,x))$ and $\mathrm{mod}(a(\Gamma,x))$ respectively. Define

$$\underline{\mathrm{mod}}(A(\Gamma,x)) = \min_{\gamma\in\Gamma}\{\mathrm{mod}(A(\gamma,x))\}$$

and

$$\underline{\mathrm{mod}}(a(\Gamma,x)) = \min_{\gamma\in\Gamma}\{\mathrm{mod}(a(\gamma,x))\}.$$

Lemma 10. *Let β be the constant in Proposition 1. Let Γ be an irreducible multi-curve. Suppose the leading eigenvalue of the matrix A_Γ is greater than or equal to 1. Then for any $x_0 \in T_f$ and $x_n = \sigma_f^n(x_0)$, $n > 0$,*

(1) $\underline{mod}(A(\Gamma,x_n)) \geq \beta\underline{mod}(a(\Gamma,x_0))$ *and*
(2) $\underline{mod}(a(\Gamma,x_n)) \geq \beta\underline{mod}(a(\Gamma,x_0)) - 1$.

Proof. Since for any n, $f^n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a branched covering, we can similarly define the linear map $f_\Gamma^n : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$. Let B be the corresponding matrix for the linear map f_Γ^n with the basis Γ. It is easy to see that $B \geq A_\Gamma^n$.

Let $\mathbf{v}$ be the unique positive eigenvector of A_Γ with $\|\mathbf{v}\| = \mathbf{1}$. Let $\mathbf{1}$ denote the vector whose coordinates are all equal to 1. Then

$$\mathrm{mod}(a(\Gamma,x_0)) \geq \underline{\mathrm{mod}}(a(\Gamma,x_0)\mathbf{1} \geq \underline{\mathrm{mod}}(a(\Gamma,x_0))\mathbf{v}.$$

For any $n \geq 1$, let $\gamma_{i,j,\alpha}^n$ be the components of $f^{-n}(\gamma_j)$ homotopic to γ_i, and $a_{i,j,\alpha}^n$ be the components of $f^{-n}(a(\gamma_j,x_0))$ homotopic to γ_i. Then

$$\mathrm{mod}(a_{i,j,\alpha}^n) = \mathrm{mod}(a(\gamma_j,x_0))/d_{i,j,\alpha}^n,$$

where $d_{i,j,\alpha}^n = \deg f^n|_{\gamma_{i,j,\alpha}^n}$. Since $a_{i,j,\alpha}^n$ are disjoint annuli homotopic to the curve γ_i, we have

$$\sum_{\alpha,j}\mathrm{mod}(a_{i,j,\alpha}^n) \leq \mathrm{mod}(A(\gamma_i,x_n)).$$

(One can obtain this inequality by lifting them to the covering space $A(\gamma_i,x_n)$ of R_{x_n} and then by using Grötzsch's inequality.) Consequently we get

$$\begin{aligned}\mathrm{mod}(A(\Gamma,x_n)) &\geq \mathrm{mod}(a(\Gamma,x_n)) \geq B\mathrm{mod}(a(\Gamma,x_0))\\ &\geq A_\Gamma^n\mathrm{mod}(a(\Gamma,x_0)) \geq A_\Gamma^n\underline{\mathrm{mod}}(a(\Gamma,x_0))\mathbf{v}\\ &\geq \underline{\mathrm{mod}}(a(\Gamma,x_0))\mathbf{v} \geq \beta\underline{\mathrm{mod}}(a(\Gamma,x_0))\mathbf{1}.\end{aligned}$$

Hence for all $\gamma \in \Gamma$, we have $\mathrm{mod}(A(\gamma,x_n)) \geq \beta\underline{\mathrm{mod}}(a(\Gamma,x_0))$. The second conclusion follows the first one and inequality (11). $\square$

Lemma 11. *If $a,\ b > 0$, $\beta > 0$, and $e^a \geq \beta e^b - 1$, then $a - b \geq \log\beta - 1$.*

Proof. If $\beta\exp b - 1 \geq 1$, then by Lemma 9, we have

$$\log(\beta\exp b - 1) \geq \log(\beta e^b) - 1 \geq \log\beta + b - 1.$$

Hence by the assumption, we have $a - b \geq \log\beta - 1$.

If $\beta\exp b - 1 < 1$, then $b < \log 2 - \log\beta$. Since $a > 0$,

$$a - b > 0 - b = -b > \log\beta - \log 2 > \log\beta - 1. \qquad\square$$

For any $x \in T_f$ and any multi-curve Γ, define

$$\underline{w}(\Gamma,x) = \min_{\gamma\in\Gamma}w_Q(\gamma,x).$$

Lemma 12. *Suppose Γ is an irreducible multi-curve and suppose the leading eigenvalue of the matrix A_Γ is greater than or equal to 1. For any $x_0 \in T_f$, if $\underline{w}(\Gamma, x_0) \geq \log(3/\beta) + \log \pi$, then the sequence $\{\underline{w}(\Gamma, x_n)\}_{n=0}^{\infty}$, where $x_n = \sigma_f^n(x_0)$, is $(\log \beta - 1 - 2\log \pi)$-quasi-nondecreasing.*

Proof. For $\underline{w}(\Gamma, x_0) \geq \log(3/\beta) + \log \pi > \log \frac{2}{\pi}$, by inequality (12), we have

$$\log \underline{\mathrm{mod}}(a(\Gamma, x_0)) \geq \log(\frac{3}{\beta}).$$

That is, $\underline{\mathrm{mod}}(a(\Gamma, x_0)) \geq 3/\beta$. So $\beta \underline{\mathrm{mod}}(a(\Gamma, x_0)) - 1 \geq 2$. By Lemma 10, we have that for all $n \geq 0$,

$$\underline{\mathrm{mod}}(a(\Gamma, x_n)) \geq 2. \tag{13}$$

Now consider the sequence $y_n = \log \underline{\mathrm{mod}}(a(\Gamma, x_n))$. Choose arbitrarily $n_2 > n_1 \geq 0$, and let $a = y_{n_2}$, $b = y_{n_1}$ and $n = n_2 - n_1$. By Lemma 10, we have $e^a \geq \beta e^b - 1$. Applying Lemma 11, we have $a - b \geq \log \beta - 1$, so the sequence $\{y_n\}$ is a $(\log \beta - 1)$-quasi-nondecreasing.

By inequalities (11) and (13), we have $\underline{\mathrm{mod}}(A(\Gamma, x)) \geq 2$. This implies that $\log \pi + \underline{w}(\Gamma, x_n) \geq \log 2$. That is, $\underline{w}(\Gamma, x_n) \geq \log(2/\pi)$. Since $\mathrm{mod}(a(\gamma, x_n))$ is continuous and decreasing with $l_Q(\gamma, x_n)$, we obtain

$$\underline{\mathrm{mod}}(a(\Gamma, x_n)) = \mathrm{mod}(a(\gamma, x_n)) \quad \text{and} \quad \underline{w}(\Gamma, x_n) = w_Q(\gamma, x_n)$$

at the same $\gamma \in \Gamma$. This further implies that

$$|y_n - \underline{w}(\Gamma, x_n)| < \log \pi.$$

From Property 1, we finally have that $\underline{w}(\Gamma, x_n)$ is $(\log \beta - 1 - 2\log \pi)$-quasi-nondecreasing. □

Lemma 13. *Let $k \geq 1$ be an integer. For any $x_0 \in T_f$, let $x_n = \sigma_f^n(x_0)$ for $n > 0$. Let $D = d_T(x_0, x_1)$. If γ_1, γ_2 are non-peripheral curves in $\widehat{\mathbb{C}} \setminus Q$ such that some component of $f^{-k}(\gamma_1)$ is homotopic to γ_2, then*

$$w_Q(\gamma_2, x_0) \geq w_Q(\gamma_1, x_0) - k(\log d + 2D).$$

Proof. Let $Y = f^{-k}(R_{x_0})$. Then $Y \subset R_{x_k}$ is a Riemann surface and $f^k : Y \to R_{x_0}$ is a holomorphic covering map of degree d^k. Then

$$l_Y(\gamma_2) \leq d^k l_Q(\gamma_1, x_0).$$

Since the inclusion map $\iota : Y \hookrightarrow R_{x_k}$ decreases the hyperbolic lengths,

$$l_Q(\gamma_2, x_k) \leq d^k l_Q(\gamma_1, x_0).$$

It follows that

$$w_Q(\gamma_2, x_k) > w_Q(\gamma_1, x_0) - k \log d.$$

Since σ_f decreases the Teichmüller distance d_T,

$$d_T(x_i, x_{i+1}) \leq d_T(x_0, x_1) = D.$$

The map $\gamma \mapsto w_Q(\gamma, x)$ for any $x \in T_f$ is a Lipschitz function with Lipschitz constant 2 (see Lemma 4), so we have that

$$w_Q(\gamma_2, x_0) \geq w_Q(\gamma_2, x_k) - 2kD \geq w_Q(\gamma_1, x_0) - k(2D + \log d). \quad \square$$

Lemma 14. *Suppose Γ is an irreducible multi-curve. Then for all $\gamma_i,\ \gamma_j \in \Gamma$ and all $x \in T_f$,*

$$|w_Q(\gamma_i, x) - w_Q(\gamma_j, x)| \le (p-3)(\log d + 2D).$$

Proof. Since Γ is irreducible, there is an integer $q \le \sharp\Gamma \le p-3$ such that γ_i is homotopic to a preimage of $f^{-q}(\gamma_j)$. By Lemma 13, we see that $w_Q(\gamma_i, x) \ge w_Q(\gamma_j, x) - (p-3)(\log d + 2D)$. By exchanging i and j, we complete the proof. □

Proposition 5. *Suppose Γ is an f-stable multi-curve satisfying $\Gamma = \Gamma_0$. Let $x_0 \in T_f$ and $x_n = \sigma_f^n(x_0)$, $n > 0$. Let $D = d_T(x_0, x_1)$. Suppose $min_\gamma w_Q(\gamma, x_0) \ge \log(3/\beta) + \log\pi$, where β is the number in Proposition 1. Write $\Gamma = \Gamma' \sqcup \Gamma''$, where $\Gamma' = \Gamma_{Ob}$ is the union of the irreducible component Γ_j of Γ for which $\lambda(A_{\Gamma_j}) \ge 1$. Then*

1. *for all $\gamma \in \Gamma'$, $\{w_Q(\gamma, x_n)\}_{n\ge 0}$ is κ-quasi-nondecreasing, where $\kappa = \log\beta - 1 - 2\log\pi - 2(p-3)(\log d + 2D)$;*
2. *for all $\gamma \in \Gamma''$ and all $n \ge 0$,*
$$w_Q(\gamma, x_n) \ge min_{\gamma'\in\Gamma'}\{w_Q(\gamma', x_n)\} - M(\log d + 2D),$$
where M is the constant in Proposition 3.
3. *Suppose $min_{\gamma\in\Gamma} w_Q(\gamma, x_0) \ge J_A - 1$, where*
$$J_A = max\{\log(3/\beta) + \log\pi, A\} + \kappa + M(\log d + 2D) + 1.$$
Then for all $\gamma \in \Gamma$ and for all $n \ge 0$, we have
$$w_Q(\gamma, x_n) \ge A.$$

Proof. Let Γ_j be an irreducible component of Γ for which $\lambda(\Gamma_j) \ge 1$. By the assumption that $\underline{w}(\Gamma, x_0) \ge \log(3/\beta) + \log\pi$, we have $\{\underline{w}(\Gamma_j, x_n)\}$ is $\log\beta - 1 - 2\log\pi$-quasi-nondecreasing.

Since Γ_j is an irreducible multi-curve, by Lemma 13 and Property 1, we have for each $\gamma \in \Gamma_j$, the sequence $\{w_Q(\gamma, x_k)\}_{k=0}^{\infty}$ is a $\kappa = \log\beta - 1 - 2\log\pi - 2(p-3)(\log d + 2D)$-quasi-nondecreasing. This completes (1).

By Lemma 12 and (1), we have for all $\gamma \in \Gamma''$ and all $n \ge 0$,

$$w_Q(\gamma, x_n) \ge \min_{\gamma'\in\Gamma'}\{w_Q(\gamma', x_n)\} - M(\log d + 2D).$$

This is (2).

(3) follows from (1) and (2) immediately. □

Proposition 6. *Suppose Γ is an f-stable multi-curve satisfying $\Gamma = \Gamma_0$. Let $x_0 \in T_f$ and $x_n = \sigma_f^n(x)$, $n > 0$, and $D = d_T(x_0, x_1)$. Suppose $min_{\gamma\in\Gamma} w_E(\gamma, x_0) \ge J_A$. Write $\Gamma = \Gamma' \sqcup \Gamma''$, where $\Gamma' = \Gamma_{Ob}$ is the union of the irreducible component Γ_j of Γ for which $\lambda(A_{\Gamma_j}) \ge 1$. Then:*

1. *For all $\gamma \in \Gamma$, $w_E(\gamma, x_n) \ge A$ for any $n \ge 0$.*
2. *For all $\gamma \in \Gamma'$, $\{w_E(\gamma, x_n)\}_{n\ge 0}$ is $(\kappa - 2)$-quasi-nondecreasing.*
3. *For all $\gamma \in \Gamma''$ and all $n \ge 0$,*
$$w_E(\gamma, x_n) \ge min_{\gamma'\in\Gamma'}\{w_E(\gamma', x_n)\} - 2 - M(\log d + 2D).$$

Proof. From Lemma 3, we have, for any $x \in T_f$,

$$w_Q(\gamma, x) \le w_E(\gamma, x) \le w_Q(\gamma, x) + 1$$

if $w_E(\gamma, x) \geq A$. If $\min_\gamma w_E(\gamma, x_0) \geq J_A$, then $\min_\gamma w_Q(\gamma, x_0) \geq J_A - 1$; then, by Proposition 5, for any $n \geq 0$ and $\gamma \in \Gamma$, $w_Q(\gamma, x_n) \geq A$. Consequently, $w_E(\gamma, x_n) \geq A$. We get (1).

From (1), we have $|w_E(\gamma, x_n) - w_Q(\gamma, x_n)| < 1$ for all $n \geq 0$. Then by Property 1 and Proposition 5, we have (2) and (3). □

7. Proof of Theorem 1

Choose any $x_0 \in T_f$, we can find a $J \geq J_A$ such that $w_E(x_0) < C(J)$. Without loss of generality, we assume that $J = J_A$. Since $C(J)$ is an increasing function of J, we have $w_E(x_0) < C(J)$ for all $J \geq J_A$. Let $x_n = \sigma_f^n(x_0)$, $n > 0$, and $D = d_T(x_0, x_1)$.

Suppose that f is not equivalent to a rational map. By Lemma 2, the sequence $\{w_E(x_n)\}_{n\geq 0}$ is unbounded. Thus there exists γ_k and x_{n_k} with $w_E(\gamma_k, x_{n_k}) \to \infty$, as $k \to \infty$.

Fix $J > J_0 = J_A + |A|$. Then $w_E(\gamma_k, x_{n_k}) > E(J) = C(J) + 2mD$ for some k. So by Lemma 8, the set of the finite depth curves in $\Gamma_{J,x_{n_k}}$, denoted by $\Gamma_{J,x_{n_k},0}$, is nonempty.

Moreover, if for some n_0, $\gamma \in \Gamma_{J,n_0,0}$, then $w_E(\gamma, x_{n_0}) > a + J \geq J_A$, which implies $w_E(\gamma, x_n) \geq A$ for all $n \geq n_0$ by Proposition 6. This implies that $\Gamma_J = \bigcup_n \Gamma_{J,x_n,0}$ and $\mathcal{G} = \bigcup_{J\geq J_0} \Gamma_J$ are multi-curves, since $\gamma \in \Gamma_J$ satisfies $w_E(\gamma, x_n) \geq A$ for all n sufficiently large.

Since $w_E(\gamma_k, x_{n_k}) \to \infty$, as $k \to \infty$, given any fixed $J \geq J_0$, $w_E(\gamma_k, x_{n_k}) \geq E(J)$ for infinitely many k. Hence $\gamma_k \in \Gamma_J \subset \mathcal{G}$ infinitely often. Since $\mathcal{G}$ is finite, for some $\gamma \in \mathcal{G}$, we have $\gamma_k = \gamma$ for infinitely many k. Hence the set

$$\Gamma_u = \{\gamma \mid \{w_E(\gamma, x_n)\}_{n\geq 0} \text{ is unbounded}\}$$

is nonempty.

Proposition 7. $\Gamma_u = \bigcap_{J\geq J_0} \Gamma_J$.

Proof. The inclusion $\bigcap_{J>J_0} \Gamma_J \subset \Gamma_u$ is clear. To see the other inclusion, let $\gamma \in \Gamma_u$. Given J, there exists some n such that $w(\gamma, x_n) > E(J)$. By Lemma 8, $\gamma \in \Gamma_{J,x_n,0}$. Thus $\bigcap_{J>J_0} \Gamma_J \supset \Gamma_u$. This proves the proposition. □

Proposition 8. $\Gamma_u = \Gamma_{J_c}$ *for some* $J_c \geq J_A$.

Proof. We prove it by contradiction. Since $\Gamma_u = \bigcap_{J\geq J_0} \Gamma_J$, for all $J \geq J_0$, if $\Gamma_u \neq \Gamma_J$, then there exists a curve γ_J such that $\gamma_J \in \Gamma_J \subset \mathcal{G}$ but $\gamma_J \notin \Gamma_u$. Since $\mathcal{G}$ is finite, this implies that there is some $\gamma \in \mathcal{G}$ such that $\gamma = \gamma_J \in \Gamma_J$ for infinitely many J, while also $\gamma \notin \Gamma_u$. This is a contradiction, since $\gamma \in \Gamma_J$ for infinitely many J implies that the sequence $\{w_E(\gamma, x_n)\}$ is unbounded. The contradiction proves the proposition. □

Now consider $\Gamma_u = \Gamma_{J_c} = \bigcup_n \Gamma_{J_c,x_n,0}$.

For each k such that $\Gamma = \Gamma_{J_c,x_k,0}$ is nonempty, applying Proposition 6, we know that if $\gamma' \in \Gamma'$, then the sequence $\{w_E(\gamma', x_n)\}_{n\geq 0}$ is both unbounded and quasi-nondecreasing, so $w_E(\gamma', x_n) \to \infty$, as $n \to \infty$. (3) of Proposition 6 implies that $w_E(\gamma, x_n) \to \infty$, as $n \to \infty$, for all $\gamma \in \Gamma''$. Hence

$$\Gamma_u = \{\gamma \mid w_E(\gamma, x_n) \to \infty \text{ as } n \to \infty\}.$$

Proposition 9. $\Gamma_u = \Gamma_{J_c, x_{n_c}, 0}$ *for some* $n = n_c$.

Proof. Since $\Gamma_u = \bigcup_n \Gamma_{J_c, x_n, 0}$, the inclusion $\Gamma_{J_c, x_n, 0} \subset \Gamma_u \subset \mathcal{G}$ holds for all n.

Since there are finitely many elements in $\mathcal{G}$, there exists an n_c such that for all $\gamma \in \Gamma_u$,

$$w_E(\gamma, x_{n_c}) > E(J_c).$$

By Lemma 8, we have $\gamma \in \Gamma_{J_c, x_{n_c}, 0}$. Thus $\Gamma_u = \Gamma_{J_c, n_c, 0}$. □

From Proposition 9, Γ_u is a Thurston obstruction. Furthermore, Γ_u depends only on f and is independent of the initial point x_0, since for any γ, the map $x \mapsto w_E(\gamma, x)$ is a Lipschitz map with Lipschitz constant 2 (see Lemma 4) and since σ_f decreases the Teichmüller distance d_T.

Finally, since

$$w_Q(\gamma, x) \leq w_E(\gamma, x) \leq 1 + w_Q(\gamma, x),$$

if $w_E(\gamma, x) \geq A$ (refer to Remark 4), we have that

$$\begin{array}{rcl} \Gamma_c & = & \{\gamma \mid w_Q(\gamma, x_n) \to \infty \text{ as } n \to \infty\} \\ & = & \{\gamma \mid w_E(\gamma, x_n) \to \infty \text{ as } n \to \infty\} = \Gamma_u. \end{array}$$

Therefore, Γ_c is a Thurston obstruction. This completes the proof of Theorem 1.

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