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COMPACT KLEIN SURFACES OF GENUS 5 WITH A UNIQUE EXTREMAL DISC

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Abstract. A compact (orientable or non-orientable) surface of genus g is said to be extremal if it contains an extremal disc, that is, a disc of the largest radius determined only by g. The present paper concerns non-orientable extremal surfaces of genus 5. We represent the surfaces as side-pairing patterns of a hyperbolic regular 24-gon, that is, a generic fundamental region of an NEC group uniformizing each of the surfaces. We also describe the group of automorphisms of the surfaces with a unique extremal disc.

1. Introduction

The unit disc $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ is the universal covering for a compact hyperbolic surface S of genus g, where g denotes the number of handles ($g \geq 2$) if S is orientable, or the number of cross caps ($g \geq 3$) if S is non-orientable. The hyperbolic metric on S is the one induced by the hyperbolic metric $ds = 2|dz|/(1-|z|^2)$ on $\mathbb{D}$. C. Bavard showed in [B] that the radius r of a disc embedded in S satisfies the inequality

$$\cosh r \leq \frac{1}{2\sin\frac{\pi}{6-6\chi_g}}, \tag{1.1}$$

where χ_g denotes the Euler characteristic

$$\chi_g = \begin{cases} 2-2g & (S: \text{orientable}), \\ 2-g & (S: \text{non-orientable}). \end{cases}$$

For each case we denote by R_g the radius satisfying the equality in (1.1). A compact surface S of genus g is called an extremal surface if S contains an extremal disc, a disc of radius R_g. Our concern here is non-orientable surfaces (Klein surfaces). A non-orientable extremal surface of genus $g \geq 3$ is uniformized by an NEC group whose fundamental region is a hyperbolic regular $(6g-6)$-gon. The polygon, denoted by P_g, has angles of $2\pi/3$, so that three vertices project to one point on the surface. Hence the sides of P_g project to a trivalent graph with $2g-2$ vertices and $3g-3$ edges on the surface. The trivalent graphs are used to obtain side-pairing patterns of P_g. For $g = 3$ and 4, all non-orientable extremal surfaces of genus g are presented in [GN] and [N1] by showing the side-pairing patterns of P_g; their groups of automorphisms are also studied there. Furthermore it is revealed that non-orientable extremal surfaces of genus $g \geq 7$ admit a unique extremal disc (see [GN]). In the present paper we consider the case of $g = 5$. Using the trivalent

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graphs with 8 vertices and 12 edges, we give all the side-pairing patterns of a hyperbolic regular 24-gon that make non-orientable surfaces of genus 5. Furthermore we study the groups of automorphisms of non-orientable extremal surfaces of genus 5 with a *unique* extremal disc. The surfaces with more than one extremal disc are studied in [N2].

2. Side-pairing patterns

Let P be a hyperbolic regular 24-gon in $\mathbb{D}$. By identifying pairs of sides of P properly, we can construct a non-orientable extremal surface S of genus 5. We shall obtain all such side-pairing patterns of P. Since the sides of P project to a trivalent graph G with 8 vertices and 12 edges on S, a walk on the sides of P once in a certain direction corresponds to a walk on every edge of G twice, which is called a closed walk on G. Note that we can walk on edges of G twice in the same direction. Conversely, a closed walk on an arbitrary trivalent graph with 8 vertices and 12 edges corresponds to a side-pairing pattern of P. Then the surfaces obtained from the side-pairing patterns are necessarily non-orientable because an orientable extremal surface of genus g corresponds to a hyperbolic regular $(12g-6)$-gon (see [B]), so that 24 is never attained for any genus g. Consequently, we consider all the closed walks on each of the trivalent graphs G.

We shall describe how to obtain all the trivalent graphs G. We connect 8 vertices $1, 2, \ldots, 8$ as follows: First, there are three ways, (A), (B) and (C), to connect the vertex 1 to the others (Figure 1). Next we connect the vertex 2 to the others. For (A), there are two ways: 2 is connected to 3 by two edges or connected to 3 and 4 by one edge respectively. For (B), there are two ways: 2 is connected to 3 or 4 by one edge. For (C), there are three ways: 2 is connected to 3 and 4, 3 and 5, or 5 and 6. We only repeat this process for the rest of the vertices $3, \ldots, 8$, where loops are not needed in (B); loops and double edges are not needed in (C).

We shall give an example to show how a closed walk on G induces a side-pairing pattern of P. Figure 2 shows a trivalent graph, a closed walk on it, and a side-pairing pattern, where a line (resp. a dotted line) connecting two sides of the regular 24-gon denotes a pair with the opposite (resp. same) direction.

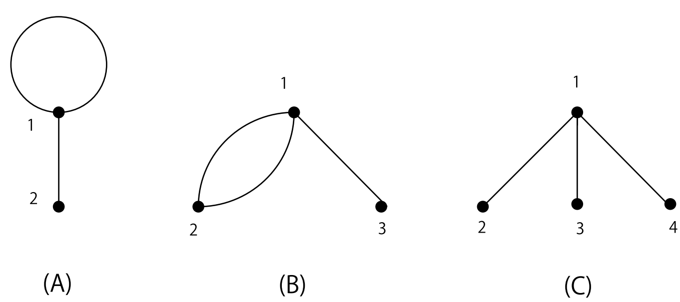

Figure 1. Edges connecting the vertex 1

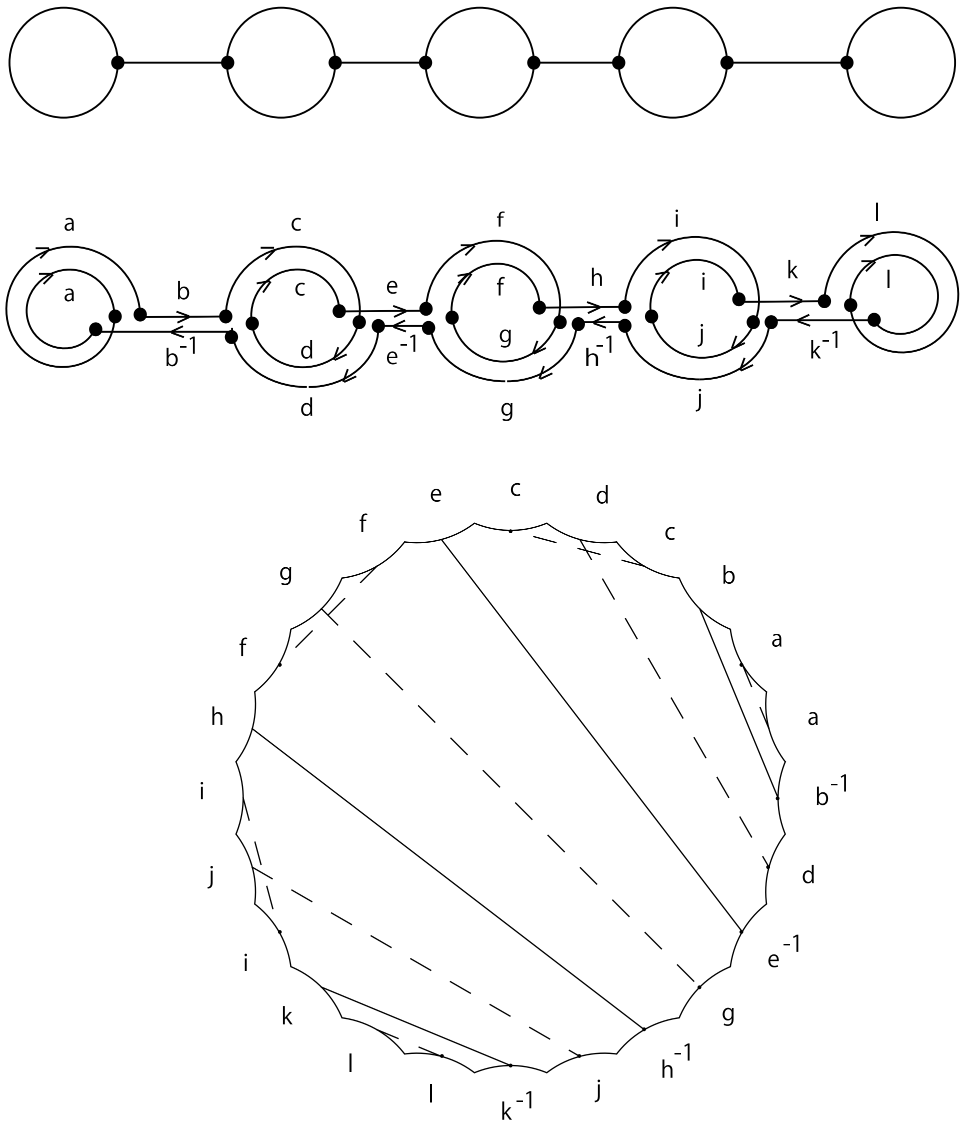

FIGURE 2. A graph, a closed walk and a corresponding side-pairing pattern

We obtained the following result via the use of a computer.

Theorem 2.1. *There exist* 71 *trivalent graphs with* 8 *vertices and* 12 *edges (Figures* 3 *and* 4*). There exist* 3627 *side-pairing patterns for the regular* 24*-gon to be a non-orientable extremal surface of genus* 5*. The surfaces obtained from these side-pairings are not isomorphic to each other.*

Table 1 shows the numbers of side-pairing patterns (denoted by s.p.) derived from each of the 71 trivalent graphs G.

Since there are so many side-pairing patterns of the regular 24-gon, we show all of them in [N3].

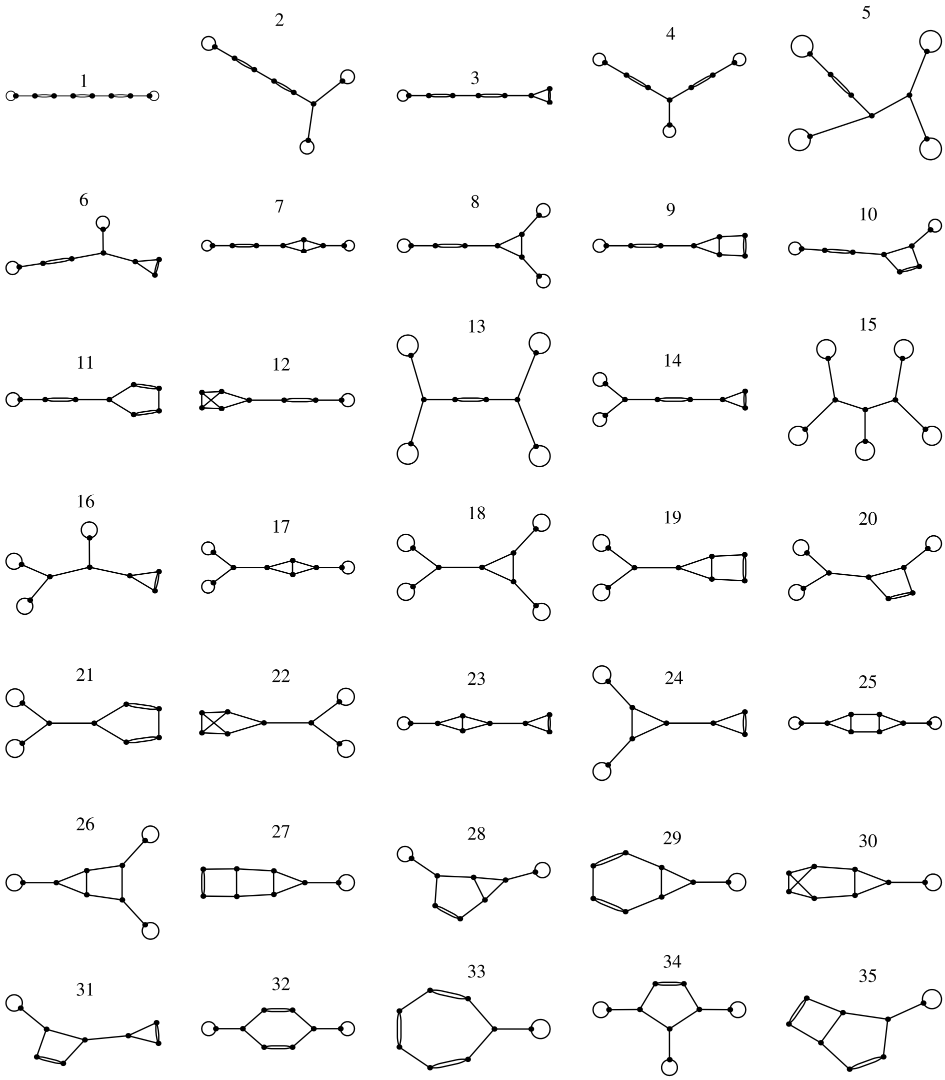

FIGURE 3. Trivalent graphs with 8 vertices and 12 edges

We shall consider the group of automorphisms of non-orientable surfaces of genus 5 which admit a unique extremal disc. The surfaces admitting more than one extremal disc are studied in [N2]. In the 3627 side-pairing patterns $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{3627}$, we showed that 17 of them correspond to the surfaces with more than one extremal disc [N2], which are those labeled as **684**, **1158**, **1353**, **1354**, **1356**, **1372**, **1471**, **1490**, **1514**, **1842**, **1985**, **1992**, **1994**, **2139**, **2240**, **3365**, **3379**. We therefore give all the groups of automorphisms of the other 3610 surfaces.

Theorem 2.2. *The groups of automorphisms of the non-orientable surfaces of genus* 5 *with a unique extremal disc are classified as follows:*

(1) D_3 : $\mathbf{803}, \mathbf{2765}, \mathbf{3431}, \mathbf{3509}$.
(2) $\mathbb{Z}_3$: $\mathbf{3436}, \mathbf{3486}$.

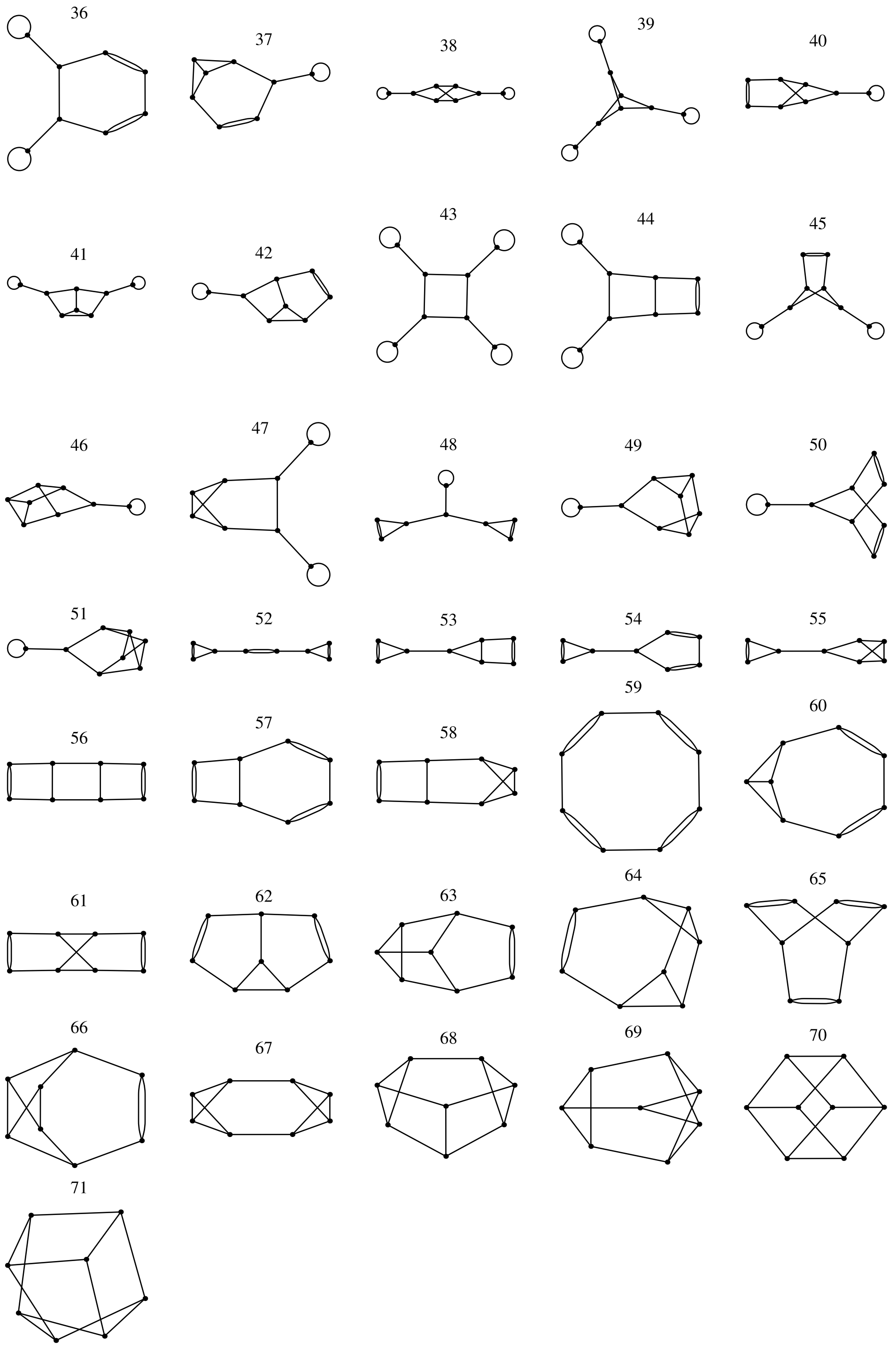

FIGURE 4. Trivalent graphs with 8 vertices and 12 edges

Table 1. The numbers of side-pairing patterns

G	s.p.	G	s.p.	G	s.p.	G	s.p.	G	s.p.	G	s.p.
1	3	13	1	25	26	37	88	49	110	61	40
2	2	14	4	26	18	38	16	50	24	62	144
3	8	15	1	27	84	39	4	51	56	63	328
4	3	16	4	28	48	40	46	52	10	64	162
5	2	17	8	29	34	41	60	53	52	65	15
6	8	18	3	30	90	42	176	54	26	66	88
7	16	19	14	31	32	43	2	55	58	67	87
8	4	20	8	32	8	44	30	56	76	68	388
9	24	21	7	33	18	45	13	57	62	69	137
10	16	22	15	34	10	46	208	58	154	70	40
11	12	23	32	35	80	47	32	59	8	71	118
12	28	24	10	36	14	48	10	60	64	Total	3627

(3) $\mathbb{Z}_2$:

1, 3, 14, 16, 127, 132, 146, 147, 154, 155, 156, 157, 171, 172, 184, 185, 225, 226, 232, 233, 242, 248, 258, 259, 272, 273, 274, 275, 347, 348, 349, 350, 352, 353, 354, 355, 438, 439, 440, 441, 528, 529, 530, 531, 568 569, 571, 572, 586, 588, 590, 591, 596, 597, 599, 600, 682, 686, 687, 785, 790, 798, 799, 802, 846, 847, 848, 849, 858, 860, 868, 879, 900, 905, 907, 909, 1086, 1087, 1088, 1089, 1095, 1098, 1099, 1100, 1103, 1104, 1105, 1106, 1107, 1109, 1129, 1130, 1339, 1341, 1346, 1351, 1358, 1375, 1378, 1380, 1473, 1475, 1476, 1479, 1480, 1482, 1484, 1486, 1487, 1488, 1506, 1508, 1509, 1510, 1511, 1512, 1513, 1566, 1567, 1569, 1570, 1571, 1575, 1578, 1580, 1611, 1612, 1613, 1614, 1625, 1626, 1627, 1628, 1646, 1647, 1653, 1654, 1694, 1695, 1709, 1710, 1734, 1737, 1738, 1739, 1740, 1741, 1742, 1743, 1744, 1745, 1758, 1764, 1779, 1781, 1782, 1783, 1784, 1785, 1786, 1787, 1789, 1790, 1791, 1792, 1797, 1798, 1799, 1800, 1828, 1829, 1834, 1839, 1843, 1848, 1853, 1883, 1884, 1885, 1886, 1952, 1953, 1960, 1962, 1963, 1965, 1967, 1972, 1984, 1995, 1997, 1999, 2004, 2009, 2010, 2012, 2013, 2015, 2016, 2068, 2070, 2071, 2072, 2074, 2077, 2078, 2080, 2090, 2093, 2098, 2099, 2100, 2101, 2102, 2106, 2115, 2117, 2119, 2120, 2125, 2131, 2144, 2199, 2205, 2237, 2245, 2249, 2253, 2256, 2260, 2261, 2263, 2264, 2480, 2483, 2487, 2488, 2504, 2507, 2518, 2519, 2540, 2543, 2547, 2548, 2564, 2567, 2578, 2579, 2635, 2637, 2639, 2640, 2643, 2644, 2646, 2648, 2650, 2651, 2652, 2654, 2707, 2709, 2720, 2728, 2733, 2735, 2746, 2754, 2756, 2758, 2760, 2761, 2762, 2763, 2764, 2793, 2794, 2796, 2797, 2831, 2833, 2836, 2841, 2847, 2849, 2852, 2857, 2863, 2879, 2897, 2898, 2901, 2911, 2914, 2919, 2920, 2925, 2927, 2933, 2940, 2944, 2971, 2980, 2981, 2992, 2996, 2997, 3009, 3010, 3030, 3031, 3050, 3066, 3123, 3139, 3144, 3171, 3176, 3197, 3201, 3204, 3207, 3212, 3218, 3228, 3259, 3263, 3266, 3276, 3279, 3280, 3289, 3294, 3299, 3300, 3304, 3313, 3328, 3329, 3331, 3332, 3360, 3382, 3387, 3388, 3389, 3390, 3404, 3428, 3449, 3450, 3451, 3452, 3454,

3476, 3485, 3489, 3490, 3496, 3499, 3500, 3503, 3504, 3505, 3508, 3521, 3528, 3532, 3537, 3543, 3545, 3548, 3557, 3560, 3564, 3570, 3573, 3574, 3576, 3577, 3578, 3584, 3585, 3587, 3588, 3590, 3592, 3594, 3595, 3598, 3605, 3610, 3614, 3616, 3617, 3621, 3622, 3623, 3624, 3625, 3627 *(402 surfaces).*

(4) $\{1\}$: *the others (3202 surfaces).*

Proof. Suppose that the regular 24-gon is located in such a way that the center is the origin 0 of $\mathbb{D}$. Hence 0 corresponds to the center of the unique extremal disc. Since an automorphism T of each surface fixes the center of the extremal disc, we can assume that the lift $\tilde{T}:\mathbb{D}\to\mathbb{D}$ fixes 0. Therefore $\tilde{T}$ is a rotation around 0 or a reflection in a line passing through 0. Considering the side-pairing patterns, we see that the side-pairing patterns in (1) admit the rotation by an angle $2\pi/3$ about 0 and a reflection, so that the surfaces derived from these side-pairing patterns have the group of automorphisms isomorphic to D_3, the dihedral group of order 6. For the side-pairing patterns in (2) or in (3), we see that they admit the rotation by an angle $2\pi/3$ about 0 or a reflection, which implies the cyclic group of order 3 or 2, respectively. The side-pairing patterns in (4) imply only an identity mapping. It is proved in a similar fashion that there exists no isomorphism between any two of the surfaces. ☐

Figure 5 shows the side-pairing patterns of the regular 24-gons corresponding to the surfaces admitting $\mathbb{Z}_3$ or D_3 as the group of automorphisms.

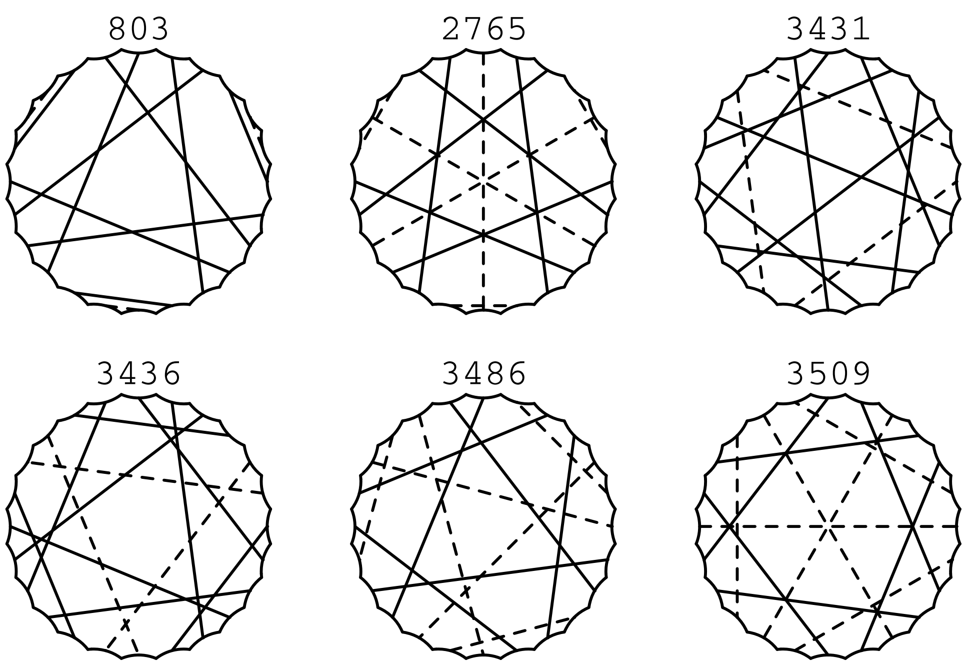

FIGURE 5. 6 side-pairing patterns

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