CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 17, Pages 119–132 (November 1, 2013)
S 1088-4173(2013)00260-9

HYPERBOLIC GEOMETRIC VERSIONS OF SCHWARZ'S LEMMA

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Abstract. Let f be a holomorphic self-map of the unit disk $\mathbb{D}$. We prove monotonicity theorems which involve the hyperbolic area, the hyperbolic capacity, and the hyperbolic diameter of the images under f of hyperbolic disks in $\mathbb{D}$. These theorems lead to distortion and modulus growth theorems that generalize the classical lemma of Schwarz and to geometric estimates for the density of the hyperbolic metric.

1. Introduction

Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map of the unit disk $\mathbb{D}$ with $f(0) = 0$. Set $\mathbb{D}_r = \{z \in \mathbb{D} : |z| < r\}$ and $R(r) = \sup\{|f(z)| : z \in \mathbb{D}_r\}$. The Schwarz lemma can be viewed as a monotonicity result [8]: The function

$$r \mapsto \frac{R(r)}{r}, \qquad 0 < r < 1, \tag{1.1}$$

is increasing and

$$\lim_{r\to 0+} \frac{R(r)}{r} = |f'(0)|. \tag{1.2}$$

Another result of this kind involves the area of the image of $\mathbb{D}_r$ ([2], [8]): The function

$$r \mapsto \frac{\operatorname{Area} f(\mathbb{D}_r)}{\pi r^2}, \qquad 0 < r < 1, \tag{1.3}$$

is increasing and

$$\lim_{r\to 0+} \frac{\operatorname{Area} f(\mathbb{D}_r)}{\pi r^2} = |f'(0)|. \tag{1.4}$$

A great number of analogous monotonicity results have been proved. These results involve various geometric quantities such as area [2, 5, 8, 11, 20], diameter [5, 8, 9], length [8, 9, 11], width [6], inner radius [7], n-th diameter [8], and integral means [21].

The next result involves the hyperbolic areas of the images of hyperbolic disks under holomorphic self-maps of the unit disk. The hyperbolic area $A_h\Omega$ of a Borel set $\Omega \subset \mathbb{D}$ is given by

$$A_h\Omega = \int_\Omega \frac{1}{(1-|w|^2)^2}\, A(dw),$$

Received by the editors June 20, 2013 and, in revised form, September 14, 2013.
2010 *Mathematics Subject Classification.* Primary 30C80, 30C85, 30F45, 30H05.
Key words and phrases. Holomorphic function, Schwarz lemma, hyperbolic metric, hyperbolic area, hyperbolic capacity, hyperbolic diameter, condenser, symmetrization.

where A is the (Euclidean) area-measure. Hyperbolic area is invariant under conformal automorphisms of $\mathbb{D}$. The conformal automorphisms of $\mathbb{D}$ are the Möbius transformations of the form $z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z}$, $\theta \in \mathbb{R}, a \in \mathbb{D}$. We will use the notation

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z},\ z \in \mathbb{D} \quad \text{and} \quad q(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|,\ z,w \in \mathbb{D}.$$

Following Pólya and Szegö [16, p. 4], we define the hyperbolic-area-radius $R_h\Omega$ of Ω as the radius of the disk Δ centered at the origin with $A_h\Delta = A_h\Omega$. For $a \in \mathbb{D}$ and $0 < r < 1$, we will use the notation

$$\Delta(a,r) = \{z \in \mathbb{D} : |\phi_a(z)| < r\} = \{z \in \mathbb{D} : q(a,z) < r\}.$$

Observe that if $a = 0$, then $\Delta(a,r) = \mathbb{D}_r$, the disk of radius r centered at the origin; therefore, $R_h\Delta(0,r) = r$. It is easy to compute that

$$A_h\Delta(a,r) = A_h\Delta(0,r) = \frac{\pi r^2}{1-r^2}. \tag{1.5}$$

Theorem 1. *Let $f : \mathbb{D} \to \mathbb{D}$ be a nonconstant holomorphic function and let $a \in \mathbb{D}$.*
(a) *The function $\Pi : (0,1) \to (0,1)$ with*

$$\Pi(r) = \frac{R_h f(\Delta(a,r))}{r}, \quad 0 < r < 1,$$

is increasing.
(b) *The function Π is strictly increasing unless $f = \phi_b \circ F \circ \phi_a$, where $F(z) = \lambda z$ and λ, and b are constants in $\mathbb{D}$. In this case, Π is constant equal to $|\lambda|$.*
(c)

$$\lim_{r\to 0+} \Pi(r) = \frac{1-|a|^2}{1-|f(a)|^2}\,|f'(a)|.$$

(d) *If $A_h f(\mathbb{D}) < \infty$, then*

$$|f'(a)| \le \frac{1-|f(a)|^2}{1-|a|^2}\left(\frac{A_h f(\mathbb{D})}{\pi + A_h f(\mathbb{D})}\right)^{1/2}.$$

Equality holds if and only if $f = \phi_b \circ F \circ \phi_a$, where $F(z) = \lambda z$ and λ, and b are constants in $\mathbb{D}$.

Yamashita [22] proved parts (a) and (d) of this theorem by a different method. Part (d) is a distortion theorem for holomorphic functions with an image of finite hyperbolic area. The quantity

$$f^h(a) := \frac{1-|a|^2}{1-|f(a)|^2}\,f'(a)$$

that appears in (c) and (d) is the *hyperbolic derivative* of f at a; see [3] for the justification of this terminology and applications of hyperbolic derivatives.

The proof of Theorem 1 is in section 3. It uses an isoperimetric-type inequality for hyperbolic areas; this inequality was proved by Gehring [14]. A strong form of this result is proved in section 2.

We will also prove an analogous theorem for hyperbolic capacity. This quantity is defined through a minimizing problem for an energy integral [19]: Let K be a compact set in the unit disk. Set

$$V_K = \inf_\mu \int_K\int_K \log\frac{1}{q(z,w)}\,\mu(dz)\,\mu(dw),$$

where the infimum is taken over all probability Borel measures μ on K. The hyperbolic capacity of K is

$$\operatorname{caph} K = e^{-V_K}.$$

It can be defined by various other (equivalent) ways; we refer to [19], [17], and [14] for its definitions and basic properties. The definition which is suitable for our purposes is the following:

$$\operatorname{caph} K = \exp(-2\pi/\operatorname{cap}(\mathbb{D}, K)), \tag{1.6}$$

where $\operatorname{cap}(\mathbb{D}, K)$ is the capacity of the condenser $(\mathbb{D}, K)$; see subsection 2.1. Note that $\operatorname{caph} \overline{\mathbb{D}_r} = r$, $0 < r < 1$. It follows from the invariance of hyperbolic capacity under conformal automorphisms of the unit disk that $\operatorname{caph} \overline{\Delta(a, r)} = r$, $0 < r < 1$, $a \in \mathbb{D}$.

Theorem 2. *Let $f : \mathbb{D} \to \mathbb{D}$ be a nonconstant holomorphic function and let $a \in \mathbb{D}$.*
(a) *The function $P : (0, 1) \to (0, 1)$ with*

$$P(r) = \frac{\operatorname{caph} f(\overline{\Delta(a, r)})}{r}, \quad 0 < r < 1,$$

is increasing.
(b) *The function P is not strictly increasing if and only if f is of the form $f = \phi_b \circ F \circ \phi_a$, where $F(z) = \lambda z$, $\lambda, b \in \mathbb{D}$. In this case, P is constant equal to $|\lambda|$.*
(c)

$$\lim_{r \to 0+} P(r) = \frac{1 - |a|^2}{1 - |f(a)|^2} |f'(a)|. \tag{1.7}$$

(d) *If $\overline{f(\mathbb{D})} \subset \mathbb{D}$, then*

$$|f'(a)| \le \frac{1 - |f(a)|^2}{1 - |a|^2} \operatorname{caph} \overline{f(\mathbb{D})} \tag{1.8}$$

with equality if and only if f is of the form $f = \phi_b \circ F \circ \phi_a$, where $F(z) = \lambda z$, $\lambda, b \in \mathbb{D}$.
(e) *If $\overline{f(\mathbb{D})} \subset \mathbb{D}$, then for every $z \in \mathbb{D}$,*

$$q(f(z), f(a)) \le \mu^{-1} \left(\mu(q(z, a)) + \log \operatorname{caph} \overline{f(\mathbb{D})} \right). \tag{1.9}$$

Equality holds for some $z \in \mathbb{D} \setminus \{a\}$ if and only if f maps $\mathbb{D}$ conformally onto the interior of a hyperbolic ellipse so that the points 0 and z are mapped to its foci.

Hyperbolic ellipses are defined by using hyperbolic distances instead of Euclidean ones. More precisely, the hyperbolic ellipse with foci $a, b \in \mathbb{D}$ is the set

$$\{z \in \mathbb{D} : d(z, a) + d(z, b) = t\},$$

where t is a constant greater than $d(a, b)$ and d is the hyperbolic distance in $\mathbb{D}$. We refer to [4] for a comprehensive introduction to the theory of the hyperbolic metric. The function μ appearing in part (e) of the theorem is a well-studied special function related to the Grötzsch ring capacity and has various applications in geometric function theory and quasiconformal mappings. It is defined by

$$\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}'(r)}{\mathcal{K}(r)}, \quad 0 < r < 1,$$

where $\mathcal{K}, \mathcal{K}'$ are the complete elliptic integrals of the first kind:

$$\mathcal{K}(r) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}, \qquad \mathcal{K}'(r) = \mathcal{K}(\sqrt{1-r^2}), \quad 0 < r < 1.$$

We refer to [1] for elliptic integrals and the function μ.

Since $\operatorname{caph} \overline{f(\mathbb{D})} < 1$, the inequality in part (e) and the monotonicity of the function μ yield the inequality $q(f(z), f(a)) \le q(z, a)$, which is the Schwarz-Pick Lemma. Under the additional assumptions that $a = 0$ and $f(0) = 0$, the inequality in (e) becomes a modulus growth bound:

$$|f(z)| \le \mu^{-1}(\mu(|z|) + \log \operatorname{caph} \overline{f(\mathbb{D})}), \tag{1.10}$$

which is stronger than that given by the Schwarz lemma.

The next result involves the pseudo-hyperbolic diameter. For $A \subset \mathbb{D}$, we set

$$D_q A = \sup\{q(z_1, z_2) : z_1, z_2 \in A\}.$$

Note that D_q is invariant under conformal automorphisms of the unit disk and that for $a \in \mathbb{D}$,

$$D_q \Delta(a, r) = \frac{2r}{1+r^2}, \quad 0 < r < 1.$$

If $D_q A > 0$, we define

$$R_q A = \frac{1 - \sqrt{1 - D_q A}}{D_q A};$$

this is the radius of the disk centered at the origin and having a pseudo-hyperbolic diameter equal to that of A.

Theorem 3. *Let $f : \mathbb{D} \to \mathbb{D}$ be a nonconstant holomorphic function and let $a \in \mathbb{D}$.*
(a) *The function $Q : (0,1) \to (0,1)$ with*

$$Q(r) = \frac{R_q f(\Delta(a, r))}{r}, \quad 0 < r < 1,$$

is increasing.
(b) *The function Q is not strictly increasing if and only if f is of the form $f = \phi_b \circ F \circ \phi_a$, where $F(z) = \lambda z$, $\lambda, b \in \mathbb{D}$. In this case, Q is constant equal to $|\lambda|$.*
(c)

$$\lim_{r \to 0+} Q(r) = \frac{1 - |a|^2}{1 - |f(a)|^2} |f'(a)|. \tag{1.11}$$

(d)

$$|f'(a)| \le \frac{1 - |f(a)|^2}{1 - |a|^2} R_q f(\mathbb{D}) \tag{1.12}$$

with equality if and only if f is of the form $f = \phi_b \circ F \circ \phi_a$, where $F(z) = \lambda z$, $\lambda, b \in \mathbb{D}$.
(e) *For every $z \in \mathbb{D}$,*

$$q(f(z), f(a)) \le \nu\left(R_q f(\mathbb{D})\, \nu^{-1}(q(z, a))\right), \tag{1.13}$$

where $\nu(t) = 2t/(1+t^2)$, $t \in (0,1)$. Equality holds for some $z \in \mathbb{D} \setminus \{a\}$ if and only if f is of the form $f = \psi \circ F \circ \phi$, where ψ is a conformal automorphism of $\mathbb{D}$, ϕ is the conformal automorphism of $\mathbb{D}$ mapping a, z to a pair of points $\pm r$, $r > 0$, and $F(z) = \lambda z$, $\lambda \in \mathbb{D}$.

Since $R_q f(\mathbb{D}) \leq 1$, parts (d) and (e) generalize the Schwarz-Pick lemma. Part (e) also implies the following distortion estimates:

1. If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, $z \in \mathbb{D}$, and $R = R_q f(\mathbb{D})$, then

$$q(f(z), f(-z)) \leq \frac{R|z|}{1 + R^2|z|^2}. \tag{1.14}$$

2. If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, $R = R_q f(\mathbb{D})$, and the points $z_1, z_2 \in \mathbb{D}$ are mapped to the diametrically opposite points $w, -w$, then

$$q(z_1, z_2) \geq \frac{2R|w|}{|w|^2 + R^2}. \tag{1.15}$$

Open Problem. Find a modulus growth bound for holomorphic self-maps of $\mathbb{D}$ with $f(0) = 0$ and $A_h f(\mathbb{D}) < \infty$.

Estimates for the hyperbolic density. Suppose that Ω is a domain with $\Omega \subset \mathbb{D}$. By the uniformization theorem, for $w \in \Omega$, there exists a unique universal covering map $F_{w,\Omega} : \mathbb{D} \to \Omega$ of Ω with $F_{w,\Omega}(0) = w$ and $F'_{w,\Omega}(0) > 0$. The density $\sigma(w, \Omega)$ of the hyperbolic metric for Ω is defined by

$$\sigma(w, \Omega) = \frac{1}{F'_{w,\Omega}(0)}, \quad w \in \Omega.$$

Note that by the domain monotonicity of the hyperbolic density,

$$\sigma(w, \Omega) \geq \sigma(w, \mathbb{D}) = \frac{1}{1 - |w|^2}, \quad w \in \Omega.$$

If we apply part (d) of each of Theorems 1, 2, and 3 for $f = F_{w,\Omega}$, we obtain the following results.

Corollary 1. *Let $\Omega \subset \mathbb{D}$ be a domain of finite hyperbolic area $A_h\Omega$. Then for $w \in \Omega$,*

$$\sigma(w, \Omega) \geq \left(\frac{\pi + A_h\Omega}{A_h\Omega}\right)^{1/2} \sigma(w, \mathbb{D}). \tag{1.16}$$

Equality holds if and only if Ω is a disk of the form $\Delta(w, r)$, $0 < r < 1$.

Corollary 2. *Let $\Omega \subset \mathbb{D}$ be a domain with $\overline{\Omega} \subset \mathbb{D}$. Then for $w \in \Omega$,*

$$\sigma(w, \Omega) \geq \frac{\sigma(w, \mathbb{D})}{\operatorname{caph} \overline{\Omega}}. \tag{1.17}$$

Equality holds if and only if Ω is a disk of the form $\Delta(w, r)$, $0 < r < 1$.

Corollary 3. *Let $\Omega \subset \mathbb{D}$ be a domain. Then for $w \in \Omega$,*

$$\sigma(w, \Omega) \geq \frac{D_q\Omega}{1 - \sqrt{1 - D_q\Omega}} \sigma(w, \mathbb{D}). \tag{1.18}$$

Equality holds if and only if Ω is a disk of the form $\Delta(w, r)$, $0 < r < 1$.

2. Preparation for the proofs

2.1. **Symmetrization and condensers.** The definition and the basic properties of Steiner and circular symmetrizations can be found in [15], [10], [16]. Suppose that E is an open or closed set in $\mathbb{D}$. We denote by E^* the circular symmetrization of E with respect to the positive semi-axis. We will need the following facts that follow easily from the definitions:

1. $A_h E = A_h E^*$.
2. Suppose that E is circularly symmetric, namely that $E = E^*$. If $z_o = r_o e^{i\theta_o} \in E$, $\theta_o \in (-\pi, \pi]$, $0 < r_o < 1$, then the whole arc $\{r_o e^{i\theta} : |\theta| \le \theta_o\}$ lies in E.
3. If E is not simply connected, then E^* need not be simply connected. Every complementary component of E^* intersects the negative semi-axis.

We will also need some properties of condensers. A condenser is a pair (D, K), where D is a domain in $\mathbb{D}$ and K is a compact subset of D. We denote by $\operatorname{cap}(D, K)$ the capacity of the condenser (D, K). Note that (D^*, K^*) is again a condenser. It is well-known (see [15], [10], [16]) that

$$\operatorname{cap}(D^*, K^*) \le \operatorname{cap}(D, K). \tag{2.1}$$

Moreover, if the condenser (D, K) is regular (that is, the domain $D \setminus K$ is regular for the Dirichlet problem) and $\operatorname{cap}(D, K) > 0$, then equality holds in (2.1) precisely when D and K coincide with a rotation around the origin of D^* and K^*, respectively.

Another property of condensers is the following: If f is a nonconstant holomorphic self-map of $\mathbb{D}$ and (D, K) is a condenser of positive capacity, then $(f(D), f(K))$ is a condenser and

$$\operatorname{cap}(f(D), f(K)) \le \operatorname{cap}(D, K). \tag{2.2}$$

Equality holds in (2.2) if and only if f is univalent; see [18], [12] and the references therein.

It is well-known that for $0 < r < s < 1$,

$$\operatorname{cap}(\mathbb{D}_s, \overline{\mathbb{D}_r}) = 2\pi \left(\log \frac{s}{r}\right)^{-1}.$$

It follows from the conformal invariance of capacity that for every $a \in \mathbb{D}$,

$$\operatorname{cap}(\Delta(a, s), \overline{\Delta(a, r)}) = 2\pi \left(\log \frac{s}{r}\right)^{-1}. \tag{2.3}$$

2.2. **An isoperimetric-type inequality.**

Theorem 4. *Let (D, K) be a condenser with $A_h K > 0$. Let D^o be the disk centered at the origin with $A_h D^o = A_h D$. Let K^o be the closed disk centered at the origin with $A_h K^o = A_h K$. Then*

$$\operatorname{cap}(D^o, K^o) \le \operatorname{cap}(D, K). \tag{2.4}$$

Assume, in addition, that (D, K) is regular and that $A_h D < \infty$. Then equality holds in (2.4) precisely when there exists a conformal automorphism ϕ of $\mathbb{D}$ such that $D = \phi(D^o)$ and $K = \phi(K^o)$.

Proof. The inequality (2.4) is proved in [14]; see also [13]. Suppose that (D, K) is a regular condenser with $A_h D < \infty$ and that

$$\operatorname{cap}(D^o, K^o) = \operatorname{cap}(D, K). \tag{2.5}$$

The hyperbolic areas of D and K are invariant under conformal automorphisms of $\mathbb{D}$ and under circular symmetrization. It follows that the capacity of (D, K) is invariant under circular symmetrization.

Since K is a compact subset of $\mathbb{D}$, we may find a conformal automorphism ψ_1 of $\mathbb{D}$ with

$$\psi_1(K) \subset \{z \in \mathbb{D} : \operatorname{Re} z > 0\}.$$

Because of (2.5), $\psi_1(K)$ is up to rotation circularly symmetric. We may assume that it is circularly symmetric with respect to the positive semi-axis. Therefore, if $\psi_1(K)$ were not simply connected, it would have a hole intersecting the negative semi-axis; a contradiction. Hence $\psi_1(K)$ is simply connected and the same is true for K.

Let $a = \min(\psi_1(K) \cap \mathbb{R})$ and $b = \max(\psi_1(K) \cap \mathbb{R})$. Note that $0 < a < b < 1$. Let ψ_2 be the conformal automorphism of $\mathbb{D}$ with $\psi_2(a) = -r$ and $\psi_2(b) = r$ for a suitable $r \in (0, 1)$. Set $\psi = \psi_2 \circ \psi_1$. The set $\psi(K)$ is a simply connected compact set in $\mathbb{D}$ with

$$\min(\psi(K) \cap \mathbb{R}) = -r \quad \text{and} \quad \max(\psi(K) \cap \mathbb{R}) = r.$$

We may assume that $\psi(K)$ is circularly symmetric. Since $-r \in \psi(K)$, the whole circle $\{|z| = r\}$ lies in $\psi(K)$; it follows that $\{|z| \le r\} \subset \psi(K)$. Also, if $\zeta \in \psi(K)$, then $|\zeta| \in \psi(K)$. Therefore $\psi(K) = \{|z| \le r\}$, and this implies that

$$A_h K = A_h \psi(K) = A_h\{|z| \le r\}.$$

We conclude that $K^o = \{|z| \le r\}$.

The capacity of the condenser $(\psi(D), K^o)$ is invariant under circular symmetrization with respect to any semi-axis emanating from the origin. Hence $\psi(D)$ is a disk, too. Obviously, $\psi(D) = D^o$. So we have proved that $(D, K) = (\phi(D), \phi(K))$ with $\phi = \psi^{-1}$. The converse is trivial.

3. Proof of Theorem 1

(a) Let f be a nonconstant holomorphic self-map of $\mathbb{D}$ and let $a \in \mathbb{D}$. Let $0 < r < s < 1$. By (2.3), (2.2), and Theorem 4,

$$\begin{aligned} 2\pi \left(\log \frac{s}{r}\right)^{-1} &= \operatorname{cap}\left(\Delta(a, s), \overline{\Delta(a, r)}\right) \\ &\ge \operatorname{cap}\left(f(\Delta(a, s)), f(\overline{\Delta(a, r)})\right) \\ &\ge \operatorname{cap}\left(f(\Delta(a, s))^o, f(\overline{\Delta(a, r)})^o\right) \\ &= 2\pi \left(\log \frac{R_h f(\Delta(a, s))}{R_h f(\Delta(a, r))}\right)^{-1}. \end{aligned} \tag{3.1}$$

Hence the function Π is increasing.

(b) Suppose that $\Pi(r) = \Pi(s)$ for $0 < r < s < 1$. Then both inequalities in (3.1) become equalities. Hence (see subsection 2.1) f is univalent in $\Delta(a, s)$ and also we have equality in Theorem 4. Therefore, there exists a conformal automorphism ϕ of $\mathbb{D}$ such that $f(\Delta(a, s)) = \phi(\Delta(0, s^*))$ and $f(\Delta(a, r)) = \phi(\Delta(0, r^*))$, where $s^* = R_h f(\Delta(a, s))$ and $r^* = R_h f(\Delta(a, r))$. Set $b = -\phi(0) = -f(a)$. The function $F := \phi_{-b} \circ f \circ \phi_{-a}$ maps $\mathbb{D}_s = \Delta(0, s)$ conformally onto $\Delta(0, s^*) = \mathbb{D}_{s^*}$. By Schwarz's lemma $F(z) = e^{i\theta} \frac{s^*}{s} z,\ z \in \mathbb{C}$. Set $\lambda = e^{i\theta} \frac{s^*}{s}$ and note that $\lambda \in \mathbb{D}$ because holomorphic maps decrease hyperbolic areas. We conclude that $f = \phi_b \circ F \circ \phi_a$ and $F(z) = \lambda z$.

Conversely, suppose that $f = \phi_b \circ F \circ \phi_{-a}$, where $b \in \mathbb{D}$ and $F(z) = \lambda z$. Then for every $r \in (0,1)$,

$$f(\Delta(a,r)) = \phi_b \circ F(\Delta(a,r)) = \phi_b(\Delta(0, r|\lambda|)).$$

Therefore, $R_h F(\Delta(a,r)) = r|\lambda|$ and $\Pi(r) = |\lambda|$.

(c) First we prove the following claim.

Claim 1:

$$\lim_{r\to 0} \frac{A_h f(\Delta(a,r))}{A_h \Delta(a,r)} = \lim_{r\to 0} \Pi(r)^2.$$

Proof of Claim 1: Set $r_* := R_h f(\Delta(a,r)),\ r \in (0,1)$. With this notation

$$A_h f(\Delta(a,r)) = A_h \Delta(0, r_*) = \frac{\pi r_*^2}{1 - r_*^2}$$

and

$$A_h \Delta(a,r) = \frac{\pi r^2}{1-r^2}.$$

Therefore,

$$\lim_{r\to 0} \frac{A_h f(\Delta(a,r))}{A_h \Delta(a,r)} = \lim_{r\to 0} \frac{r_*^2}{r^2} = \lim_{r\to 0} \Pi(r)^2$$

and Claim 1 is proved.

We make the additional assumption that $a = 0$. Suppose also that $f'(0) = 0$. By a nonunivalent change of variables and the Lebesgue differentiation theorem,

$$\begin{aligned}
\lim_{r\to 0} \frac{A_h f(\Delta(a,r))}{A_h \Delta(a,r)} &= \lim_{r\to 0} \frac{1-r^2}{\pi r^2} \int_{f(\Delta(0,r))} \frac{1}{(1-|w|^2)^2}\, A(dw) \\
&\le \lim_{r\to 0} \frac{1}{\pi r^2} \int_{f(\Delta(0,r))} \left(\sum_{f(z)=w} \frac{1}{(1-|f(z)|^2)^2} \right) A(dw) \\
&= \lim_{r\to 0} \frac{1}{\pi r^2} \int_{\Delta(0,r)} \frac{|f'(z)|^2}{(1-|f(z)|^2)^2}\, A(dz) \\
&= \frac{|f'(0)|^2}{(1-|f(0)|^2)^2} = 0.
\end{aligned}$$

By Claim 1, $\lim_{r\to 0} \Pi(r) = 0$.

We continue to assume that $a = 0$, but now we suppose that $f'(0) \neq 0$. In this case f is univalent in a neighborhood of the origin. So, as above (but with equality),

$$\begin{aligned}
\lim_{r\to 0} \Pi(r)^2 &= \lim_{r\to 0} \frac{A_h f(\Delta(a,r))}{A_h \Delta(a,r)} \\
&= \lim_{r\to 0} \frac{1}{\pi r^2} \int_{\Delta(0,r)} \frac{|f'(z)|^2}{(1-|f(z)|^2)^2}\, A(dz) \\
&= \frac{|f'(0)|^2}{(1-|f(0)|^2)^2}.
\end{aligned}$$

Finally, we remove the additional assumption that $a = 0$. By Claim 1, by the conformal invariance of the hyperbolic area, and by what we have already proved,

$$\begin{aligned}\lim_{r\to 0}\Pi(r)^2 &= \lim_{r\to 0}\frac{A_h f(\Delta(a,r))}{A_h\Delta(a,r)} \\ &= \lim_{r\to 0}\frac{A_h f\circ\phi_{-a}(\Delta(0,r))}{A_h\Delta(a,r)} \\ &= \frac{|(f\circ\phi_{-a})'(0)|^2}{(1-|(f\circ\phi_{-a})(0)|^2)^2} \\ &= \frac{|f'(a)|^2\,(1-|a|^2)^2}{(1-|f(a)|^2)^2}.\end{aligned}$$

(d) Because of (c) and (a),

$$\begin{aligned}\frac{|f'(a)|\,(1-|a|^2)}{1-|f(a)|^2} &= \lim_{r\to 0}\Pi(r) \le \lim_{r\to 1}\frac{R_h f(\Delta(a,r))}{r} \\ &\le R_h f(\mathbb{D}) = \left(\frac{A_h f(\mathbb{D})}{\pi + A_h f(\mathbb{D})}\right)^{1/2}.\end{aligned}$$

If we have equality, then Π is constant and we apply (b). The converse is easy.

4. Proof of Theorem 2

(a) Let $0 < r < s < 1$. Because of (1.6), the inequality $P(r) \le P(s)$ is equivalent to

$$\begin{aligned}&\operatorname{cap}(\mathbb{D},\overline{\Delta(a,s)})^{-1} - \operatorname{cap}(\mathbb{D}, f(\overline{\Delta(a,s)}))^{-1} \\ &\ge \operatorname{cap}(\mathbb{D},\overline{\Delta(a,r)})^{-1} - \operatorname{cap}(\mathbb{D}, f(\overline{\Delta(a,r)}))^{-1}.\end{aligned} \tag{4.1}$$

By Grötzsch's lemma [11, Lemma 1.2] and the fact that holomorphic functions reduce the capacity of condensers,

$$\begin{aligned}\operatorname{cap}(\mathbb{D},\overline{\Delta(a,s)})^{-1} - \operatorname{cap}(\mathbb{D},\overline{\Delta(a,r)})^{-1} &= \operatorname{cap}(\Delta(a,s),\overline{\Delta(a,r)})^{-1} \\ &\le \operatorname{cap}(f(\Delta(a,s)), f(\overline{\Delta(a,r)}))^{-1} \\ &\le \operatorname{cap}(\mathbb{D}, f(\overline{\Delta(a,r)}))^{-1} - \operatorname{cap}(\mathbb{D}, f(\overline{\Delta(a,s)}))^{-1},\end{aligned} \tag{4.2}$$

and (4.1) is proved.

(b) Suppose that $P(r) = P(s)$ for some r, s; $0 < r < s < 1$. Then both inequalities in (4.2) become equalities. Hence [18] f is univalent in $\Delta(a,s)$ and [7, Theorem 2.2] $P(\rho) = P(s)$ for every $\rho \in (0,s]$. Let r_n be a decreasing sequence with $0 < r_n < s$ and $r_n \to 0$. For $n = 1, 2, \dots$, let ψ_n be the conformal mapping of the doubly-connected domain $\mathbb{D}\setminus f(\overline{\Delta(a,r_n)})$ onto the annulus $\mathbb{D}\setminus\overline{\mathbb{D}_{c_n}}$ with $\psi_n(1) = 1$, where c_n is a suitable positive constant (in fact, $c_n = \operatorname{caph} f(\overline{\Delta(a,r_n)})$).

Fix s_1 with $0 < r_n < s_1 < s$. The curve $\partial f(\Delta(a,s_1))$ is an equipotential curve of the condenser $(\mathbb{D}, f(\overline{\Delta(a,r)}))$; see [7]. Hence, it is mapped by ψ_n onto a circle $\{|z| = \hat{s_1}\}$. Note that the radius $\hat{s_1}$ is the same for all n because

$$\operatorname{cap}(\mathbb{D},\overline{\mathbb{D}_{\hat{s_1}}}) = \operatorname{cap}(\mathbb{D}, f(\overline{\Delta(a,s_1)})).$$

The Carathéodory kernel of the increasing sequence of domains $\mathbb{D}\setminus f(\overline{\Delta(a,r_n)})$ is the domain $\mathbb{D}\setminus\{a\}$. By the Carathéodory convergence theorem, the sequence ψ_n converges uniformly on compact subsets of $\mathbb{D}\setminus\{a\}$ to a function ψ that maps $\mathbb{D}\setminus\{a\}$ onto $\mathbb{D}\setminus\{0\}$ with $\psi(1) = 1$. It follows that $\psi = e^{i\theta}\phi_a$ for some $\theta\in\mathbb{R}$. Therefore,

$f(\Delta(a,s_1))$ is a disk. By the Schwarz lemma, $f=\phi_b\circ F\circ\phi_a$, where $F(z)=\lambda z$, $\lambda, b\in\mathbb{D}$.

Conversely, if f is of this form, then for every $r\in(0,1)$,

$$\begin{aligned}P(r) &= \frac{\operatorname{caph} f(\Delta(a,r))}{r}=\frac{\operatorname{caph} F(\Delta(0,r))}{r}\\ &= \frac{\operatorname{caph}\Delta(0,\lambda r))}{r}=\frac{\lambda r}{r}=\lambda.\end{aligned}\tag{4.3}$$

(c) Suppose first that $a=0$ and $f(0)=0$. Let $\operatorname{cap}\overline{f(\mathbb{D}_r)}$ be the logarithmic capacity of $\overline{f(\mathbb{D}_r)}$. By [8, p.136],

$$\lim_{r\to 0}\frac{\operatorname{cap}\overline{f(\mathbb{D}_r)}}{r}=|f'(0)|.\tag{4.4}$$

By the Schwarz lemma, $f(\mathbb{D}_r)\subset\mathbb{D}_r$. Hence (see [17, p.53])

$$\frac{\operatorname{cap}\overline{f(\mathbb{D}_r)}}{1+r^2}\le\operatorname{caph}\overline{f(\mathbb{D}_r)}\le\frac{\operatorname{cap}\overline{f(\mathbb{D}_r)}}{1-r^2}.\tag{4.5}$$

Therefore

$$\lim_{r\to 0}P(r)=\lim_{r\to 0}\frac{\operatorname{caph}\overline{f(\mathbb{D}_r)}}{r}=\lim_{r\to 0}\frac{\operatorname{caph}\overline{f(\mathbb{D}_r)}}{\operatorname{cap}\overline{f(\mathbb{D}_r)}}\frac{\operatorname{cap}\overline{f(\mathbb{D}_r)}}{r}=|f'(0)|.\tag{4.6}$$

To remove the additional assumptions, we consider the function

$$\phi_{f(a)}\circ f\circ\phi_{-a}$$

which sends 0 to 0; we omit the simple details.

(d) By (c) and (a),

$$\begin{aligned}|f'(a)| &= \frac{1-|f(a)|^2}{1-|a|^2}\lim_{r\to 0}P(r)\\ &\le \frac{1-|f(a)|^2}{1-|a|^2}\lim_{r\to 1}P(r)\\ &\le \frac{1-|f(a)|^2}{1-|a|^2}\operatorname{caph}\overline{f(\mathbb{D})}.\end{aligned}\tag{4.7}$$

We have equality if and only if the function P is constant, and so we can apply (b).

(e) Let $z\in\mathbb{D}$. Let $\widehat{az}$ denote the closed hyperbolic geodesic arc joining a and z. By the conformal invariance of condenser capacity and the explicit formula for the capacity of Grötzsch's condenser (see [1, p.167, p.87]),

$$\operatorname{cap}(\mathbb{D},\widehat{az})=\operatorname{cap}(\mathbb{D},\phi_a(\widehat{az}))=\operatorname{cap}(\mathbb{D},[0,q(a,z)])=\frac{2\pi}{\mu(q(a,z))}.\tag{4.8}$$

Since holomorphic functions reduce the condenser capacity,

$$\operatorname{cap}(\mathbb{D},\widehat{az})\ge\operatorname{cap}(f(\mathbb{D}),f(\widehat{az})).\tag{4.9}$$

Let ϕ be the conformal automorphism of $\mathbb{D}$ that maps $f(a)$ to $-\sqrt{r}$ and $f(z)$ to $\sqrt{r}$, where r is the unique number in $(0,1)$ such that

$$q(f(a),f(z))=q(-\sqrt{r},\sqrt{r})=\frac{2\sqrt{r}}{1+r}.\tag{4.10}$$

Let Ω_1 be the Steiner symmetrization of the domain $\phi \circ f(\mathbb{D})$ with respect to the real axis. Then

$$\operatorname{cap}(f(\mathbb{D}), f(\widehat{az})) = \operatorname{cap}(\phi \circ f(\mathbb{D}), \phi \circ f(\widehat{az})) \geq \operatorname{cap}(\Omega_1, [-\sqrt{r}, \sqrt{r}]) \tag{4.11}$$

because the segment $[-\sqrt{r}, \sqrt{r}]$ is a subset of the Steiner symmetrization of the set $\phi \circ f(\widehat{az})$. Note also that

$$\operatorname{caph} \overline{f(\mathbb{D})} \geq \operatorname{caph} \overline{\Omega_1}. \tag{4.12}$$

Let g be the conformal mapping of the doubly connected domain $\mathbb{D} \setminus [-\sqrt{r}, \sqrt{r}]$ onto the doubly connected domain $\mathbb{D} \setminus \overline{\mathbb{D}_t}$ with $g(1) = 1$. This mapping is described in detail in [1, §6.26]. The number $t \in (0,1)$ is equal to the hyperbolic capacity of $[-\sqrt{r}, \sqrt{r}]$ and is given by

$$t = e^{-\mu(r)/2}. \tag{4.13}$$

Set $\Omega_2 = \mathbb{D} \setminus g(\mathbb{D} \setminus \Omega_1)$. Note that Ω_2 is a simply connected domain containing $\overline{\mathbb{D}_t}$ and that

$$\operatorname{cap}(\Omega_1, [-\sqrt{r}, \sqrt{r}]) = \operatorname{cap}(\Omega_2, \overline{\mathbb{D}_t}) \tag{4.14}$$

and

$$\operatorname{caph} \overline{\Omega_1} = \operatorname{caph} \overline{\Omega_2}. \tag{4.15}$$

Let Ω_3 be the disk of radius $\operatorname{caph} \overline{\Omega_2}$ centered at the origin. Then

$$\operatorname{caph} \overline{\Omega_2} = \operatorname{caph} \overline{\Omega_3} \tag{4.16}$$

and also, by Grötzsch's lemma,

$$\begin{aligned} \operatorname{cap}(\Omega_2, \overline{\mathbb{D}_t}) &\geq \frac{1}{\operatorname{cap}(\mathbb{D}, \overline{\mathbb{D}_t})^{-1} - \operatorname{cap}(\mathbb{D}, \overline{\Omega_2})^{-1}} \\ &= \frac{1}{\operatorname{cap}(\mathbb{D}, \overline{\mathbb{D}_t})^{-1} - \operatorname{cap}(\mathbb{D}, \overline{\Omega_3})^{-1}} \\ &= \operatorname{cap}(\Omega_3, \overline{\mathbb{D}_t}) \\ &= 2\pi \left(\log \frac{\operatorname{caph} \overline{\Omega_3}}{t} \right)^{-1}. \end{aligned} \tag{4.17}$$

It follows from (4.8), (4.9), (4.11), (4.14), and (4.17) that

$$\frac{2\pi}{\mu(q(a,z))} \geq \frac{2\pi}{\log \operatorname{caph} \overline{\Omega_3} - \log t}. \tag{4.18}$$

Because of (4.13), (4.12), (4.15), and (4.16), the inequality (4.18) implies

$$\frac{2\pi}{\mu(q(a,z))} \geq \frac{2\pi}{\log \operatorname{caph} \overline{f(\mathbb{D})} + \mu(r)/2}. \tag{4.19}$$

Because of (4.10) and the identity [1, p.80] $\mu(r)/2 = \mu(\frac{2\sqrt{r}}{1+r})$, the inequality (4.19) becomes

$$\frac{2\pi}{\mu(q(a,z))} \geq \frac{2\pi}{\log \operatorname{caph} \overline{f(\mathbb{D})} + \mu(q(f(a), f(z))} \tag{4.20}$$

which is equivalent to (1.9).

Suppose that we have equality in (1.9) for some $z \in \mathbb{D} \setminus \{a\}$. Then we have equality in (4.9). Therefore ([12], [18]), f is a conformal mapping. We also have equality in (4.12) and in (4.17) (namely, in Grötzsch's lemma). Hence Ω_3 is a disk of radius equal to caph $\overline{f(\mathbb{D})}$. The function g^{-1} maps circles of radius greater than t onto hyperbolic ellipses (see [1, p. 124, p. 131] and the references therein). Hence $\phi \circ f(\mathbb{D})$ is the interior of a hyperbolic ellipse with foci $\pm\sqrt{r}$. Since ϕ preserves hyperbolic distances in $\mathbb{D}$, $f(\mathbb{D})$ is the interior of a hyperbolic ellipse with foci $f(a), f(z)$.

Conversely, suppose that f maps $\mathbb{D}$ conformally onto the interior of a hyperbolic ellipse with $f(a)$ being one of its foci. Let w be the other focus and set $z = f^{-1}(w)$. It easily follows that for this z, the inequality (1.9) becomes an equality.

5. Proof of Theorem 3

(a) Let $0 < r < s < 1$. Set

$$\tilde{r} = R_q f(\Delta(a,r)), \qquad \tilde{s} = R_q f(\Delta(a,s)).$$

Let $w_r, \tilde{w}_r$ be points of $f(\overline{\Delta(a,r)})$ such that

$$D_q f(\Delta(a,r)) = q(w_r, \tilde{w}_r).$$

Let ϕ be the conformal automorphism of $\mathbb{D}$ mapping $w_r, \tilde{w}_r$ to $-\tilde{r}, \tilde{r}$ respectively. Let $z_r, \tilde{z}_r$ be points on $\partial\Delta(a,r)$ with

$$\phi \circ f(z_r) = -\tilde{r}, \qquad \phi \circ f(\tilde{z}_r) = \tilde{r}.$$

Set $\Omega_s = \phi \circ f(\Delta(a,s))$. Let Ω_1 be the Steiner symmetrization of Ω_s with respect to the real axis, and let Ω_2 be the Steiner symmetrization with respect to the imaginary axis. Then Ω_2 is contained in the disk $\mathbb{D}_{\tilde{s}}$. By the Schwarz-Pick lemma and the symmetrization inequalities,

$$\begin{aligned}
\frac{2r/s}{1+r^2/s^2} &= q(-r/s, r/s) = q_{\Delta(0,s)}(-r,r) \geq q_{\Delta(a,s)}(z_r, \tilde{z}_r) \\
&\geq q_{\phi\circ f(\Delta(a,s))}(-\tilde{r}, \tilde{r}) \geq q_{\Delta(0,\tilde{s})}(-\tilde{r}, \tilde{r}) \\
&= q(-\tilde{r}/\tilde{s}, \tilde{r}/\tilde{s}) = \frac{2\tilde{r}/\tilde{s}}{1+\tilde{r}^2/\tilde{s}^2}.
\end{aligned} \tag{5.1}$$

This implies that $\tilde{r}/\tilde{s} \leq r/s$, which is equivalent to $Q(r) \leq Q(s)$.

(b) The proof is similar to that of Theorem 3(b) and so we omit it.

(c) Suppose that $a = 0$ and $f(0) = 0$; the general case is proved by composing with suitable automorphisms of $\mathbb{D}$.

Claim 2:

$$\lim_{r\to 0} Q(r) = \lim_{r\to 0} \frac{D_q f(\mathbb{D}_r)}{2r}.$$

Proof of Claim 2: By the Schwarz lemma, $f(\mathbb{D}_r) \subset \mathbb{D}_r$. Therefore, $\lim_{r\to 0} R_q f(\mathbb{D}_r) = 0$. It follows that

$$\begin{aligned}
\lim_{r\to 0} \frac{D_q f(\mathbb{D}_r)}{2r} &= \lim_{r\to 0} \frac{2R_q f(\mathbb{D}_r)}{1+R_q f(\mathbb{D}_r)^2}\,\frac{1}{2r} \\
&= \lim_{r\to 0} \frac{R_q f(\mathbb{D}_r)}{r} = \lim_{r\to 0} Q(r),
\end{aligned}$$

and Claim 2 is proved.

For $r \in (0,1)$, let $z_r, \tilde{z}_r \in \partial\Delta(0,r)$ be such that $D_q f(\Delta(0,r)) = q(f(z_r), f(\tilde{z}_r))$. Then by Claim 2,

$$\begin{aligned}\lim_{r\to 0} Q(r) &= \lim_{r\to 0} \frac{q(f(z_r), f(\tilde{z}_r))}{2r} \\ &\le \lim_{r\to 0} \left(\frac{q(f(z_r),0) + q(0, f(\tilde{z}_r))}{2r} \right) \\ &= \lim_{r\to 0} \frac{|f(z_r)|}{2r} + \lim_{r\to 0} \frac{|f(\tilde{z}_r)|}{2r} = |f'(0)|.\end{aligned} \tag{5.2}$$

If $f'(0) = 0$, then (5.2) implies that $\lim_{r\to 0} Q(r) = |f'(0)| = 0$. Suppose that $f'(0) \neq 0$. Then f is univalent in a neighborhood of 0. For r small enough, the Jordan curve $f(\partial\Delta(0,r))$ intersects the real axis at points $\tilde{a}_r < 0 < a_r$. Then by Claim 2,

$$\begin{aligned}\lim_{r\to 0} Q(r) &= \lim_{r\to 0} \frac{D_q(f(\Delta(0,r))}{2r} \ge \lim_{r\to 0} \frac{a_r - \tilde{a}_r}{2r} \\ &= \lim_{r\to 0} \frac{a_r}{2r} + \lim_{r\to 0} \frac{|\tilde{a}_r|}{2r} = |f'(0)|.\end{aligned} \tag{5.3}$$

By (5.2) and (5.3), $\lim_{r\to 0} Q(r) = |f'(0)|$.

(d) The proof is similar to that of Theorem 2(d) and so we omit it.

(e) Let $z \in \mathbb{D} \setminus \{a\}$. Take $r \in (0,1)$ with

$$q(z,a) = \frac{2r}{1+r^2}$$

and $\tilde{r} \in (0,1)$ with

$$q(f(z), f(a)) = \frac{2\tilde{r}}{1+\tilde{r}^2}.$$

Set $R = R_q f(\mathbb{D})$. Let ϕ be the conformal automorphism of $\mathbb{D}$ with $\phi(z) = -r$ and $\phi(a) = r$. Then, as in (a),

$$\begin{aligned}\frac{2r}{1+r^2} &= q(-r,r) = q(z,a) \ge q_{f(\mathbb{D})}(f(z), f(a)) \\ &\ge q_{\Delta(0,R)}(-\tilde{r}, \tilde{r}) = q(-\tilde{r}/R, \tilde{r}/R) = \frac{2\tilde{r}/R}{1+\tilde{r}^2/R^2}.\end{aligned} \tag{5.4}$$

It follows that $r \ge \tilde{r}/R$, which is equivalent to (1.13). The proof of the equality statement is similar to that of Theorem 2(e) and so we omit it.

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