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CROSS RATIO COORDINATES FOR THE DEFORMATION SPACES OF A MARKED MÖBIUS GROUP

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ABSTRACT. We introduce a new kind of coordinate systems for the deformation space of a finitely generated free Möbius group by using cross ratio functions induced by the fixed points of Möbius transformations. As an application, we give a new complete distance on the Schottky space by using such functions, which is not greater than the Teichmüller distance.

1. INTRODUCTION

Let G be a free group generated by n Möbius transformations, which we equip with an ordered set $(g_1, \cdots, g_n)$ of generators. Here we assume that $n \geq 2$. The *rough deformation space* $\mathrm{RDef}(G)$ of such a group G is the space of all homomorphisms of G into the space $\mathrm{PSL}_2(\mathbb{C})$ of all Möbius transformations. The natural identification between $\mathrm{RDef}(G)$ and $\mathrm{PSL}_2(\mathbb{C})^n$ is obtained by setting

$$\sigma \mapsto (\sigma(g_1), \cdots, \sigma(g_m)).$$

We say that σ and σ' in $\mathrm{RDef}(G)$ are *Möbius conjugate* if there is an element $M \in \mathrm{PSL}_2(\mathbb{C})$ such that

$$\sigma'(g) = M\sigma(g)M^{-1}$$

for every $g \in G$. The space of all Möbius conjugacy classes of elements in $\mathrm{RDef}(G)$ is called the *deformation space* of G and denoted by $\mathrm{Def}(G)$. For every $\sigma \in \mathrm{RDef}(G)$, $[\sigma]$ denotes the point in $\mathrm{Def}(G)$ containing σ. The natural projection of $\mathrm{RDef}(G)$ to $\mathrm{Def}(G)$ is denoted by π. Here, there is another quotient space $\mathrm{RDef}(G)//\mathrm{PSL}_2(\mathbb{C})$ in the sense of the invariant theory, which is called the *deformation variety* $X(G)$ for G in $\mathrm{PSL}_2(\mathbb{C})$. It is known that $X(G)$ is an affine algebraic set.

Remark 1.1. Standard parameters on $X(G)$ are trace functions. For a $\sigma \in \mathrm{RDef}(G)$, the *character* of σ is the map

$$\chi_\sigma : G \to \mathbb{C}, \quad g \in G \to \mathrm{tr}^2(\sigma(g)).$$

Since there is a natural bijection between $X(G)$ and the sets of such characters, $X(G)$ is also called the *character variety* for G. But trace functions do not give a system of coordinates for $X(G)$ in general. See, for instance, [4].

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It is a fundamental problem to give useful systems of coordinates, or parameters, on such deformation spaces as above. To discuss this issue, in the sequel we consider only such σ that $\sigma(G)$ *is non-abelian and* $\sigma(g_k)$ *is not the identical transformation for every* k. The subset $\mathrm{EL}(G)$ of $X(G)$ consisting of all conjugacy classes $[\sigma]$ of σ satisfying these conditions is generic and can be considered as a subset of $\mathrm{Def}(G)$. Here we say that a subset Y of an algebraic variety X is *generic* if $X-Y$ is contained in a finite union of proper algebraic sub-varieties.

Definition 1.2. We call $\mathrm{EL}(G)$ the *essential locus*.

It is well known that $\mathrm{EL}(G)$ has a standard complex manifold structure. Actually, the *rough essential locus* $\mathrm{REL}(G) = \pi^{-1}(\mathrm{EL}(G)) \subset \mathrm{RDef}(G)$ is the total space of a principal complex-analytic bundle over $\mathrm{EL}(G)$ with π as the projection. (See, for instance, [12].)

Recall that the deformation space $\mathrm{Def}(G)$ contains several important sub-loci other than the essential locus $\mathrm{EL}(G)$.

Definition 1.3. The sub-locus $F(G)$ consisting of all conjugacy classes of isomorphisms is called the *faithfulness locus*. The sub-locus $D(G)$ consisting of all $[\sigma]$ with discrete $\sigma(G)$ is called the *discreteness locus*.

The *Schottky space* $S(G)$, by definition, consists of all $[\sigma]$ such that $\sigma(G)$ are Schottky groups with n generators.

These sub-loci have been considered by many authors from various points of view, and many kinds of useful systems of coordinates have been considered. Also, it is clear that $F(D) \subset \mathrm{EL}(G)$. Further, we have the following strict inclusion relations. (Cf. Proposition 3.1 below.)

$$S(G) \subsetneq F(G) \cap D(G) \subsetneq \mathrm{EL}(G).$$

Remark 1.4. For various kinds of systems of classical coordinates for these deformation spaces, see, for instance, [1], [7], and [8]. Also cf. [15], and [5].

In this paper, we equip $\mathrm{EL}(G)$ with a natural marking induced by order of fixed points of generators, which gives a finite branched holomorphic cover $\widehat{\mathrm{EL}}(G)$ of $\mathrm{EL}(G)$, and we introduce a new kind of the global coordinate systems for generic sets of $\widehat{\mathrm{EL}}(G)$ by considering cross ratios of the fixed points considered as functions on $\widehat{\mathrm{EL}}(G)$. More precisely, see Theorem 2.4.

Also as an application of such cross ratio functions, we give a new distance on the Schottky space $S(G)$, which is complete and not greater than the Teichmüller distance (Theorem 3.7).

2. Coordinates induced by cross ratios

Schottky groups are marked by ordered pairs of disjoint non-nested closed topological discs. But such geometric markings are not so easy to generalize to general Möbius groups, and here we use different non-geometric marking as follows.

Definition 2.1. For every point $\sigma \in \mathrm{REL}(G)$, we say that σ is *marked* if every generator $\sigma(g_j)$ is decorated with a point $\mathbf{p}_j(\sigma) = (p_{2j-1}, p_{2j}) \in \widehat{\mathbb{C}}^2$ such that $\{p_{2j-1}, p_{2j}\}$ is the set $\mathrm{Fix}\,\sigma(g_j)$ of all fixed points of $\sigma(g_j)$. We write $\sigma \in \mathrm{REL}(G)$ with the marking $\mathbf{p} = \mathbf{p}(\sigma) = (\mathbf{p}_1(\sigma), \cdots, \mathbf{p}_n(\sigma)) \in \widehat{\mathbb{C}}^{2n}$ as $\widehat{\sigma} = (\sigma, \mathbf{p})$. The set of all ordered pairs $\widehat{\sigma} = (\sigma, \mathbf{p})$ with $\sigma \in \mathrm{REL}(G)$ and possible marking $\mathbf{p} = \mathbf{p}(\sigma)$ is called the *rough marked essential locus*, and denoted by $\widehat{\mathrm{REL}}(G)$.

We say that $(\sigma_1, \mathbf{p}_1)$ and $(\sigma_2, \mathbf{p}_2)$ in $\widehat{\mathrm{REL}}(G)$ are *Möbius equivalent* if there is an $M \in \mathrm{PSL}_2(\mathbb{C})$ such that $\mathbf{p}_2 = M\mathbf{p}_1$ and $\sigma_2(g) = M\sigma_1(g)M^{-1}$ for every $g \in G$. Here, $M(p_1, \cdots, p_{2n}) = (M(p_1), \cdots, M(p_{2n}))$.

Let $\widehat{\mathrm{EL}}(G)$ be the set of all Möbius equivalence classes $[\widehat{\sigma}]$ of $\widehat{\sigma} \in \widehat{\mathrm{REL}}(G)$. We call $\widehat{\mathrm{EL}}(G)$ the *marked essential locus.*

The following fact is clear from the definition.

Proposition 2.2. *The marked essential locus* $\widehat{\mathrm{EL}}(G)$ *has a natural complex structure such that the projection*

$$\tau : \widehat{\mathrm{EL}}(G) \to \mathrm{EL}(G)$$

obtained by forgetting the marking gives a 2^n*-sheeted holomorphic branched cover.*

Now, we introduce a system of global coordinates on a generic set of $\widehat{\mathrm{EL}}(G)$ by using cross ratios. Write the marking of $\widehat{\sigma}$ in $\widehat{\mathrm{REL}}(G)$ as

$$\mathbf{p} = (p_1, \cdots, p_{2n}).$$

Then for every $j = 1, \cdots, n$ and $k = 0, 1$, we can consider the *marked orbit* $MO(p_{2j-k})$ of p_{2j-k} which consists of the value p_{2j-k} marked with the identity and the values $\sigma(g)(p_{2j-k})$ marked with elements g of G, where g moves over all elements of G whose reduced word representations $g = g_{i_s}^{\epsilon_n} \cdots g_{i_1}^{\epsilon_1}$ ($s > 0$, $\epsilon_\nu = \pm 1$) satisfy the condition that $g_{i_1} \neq g_j$. Also, we fix an order of elements in $MO(p_m)$ for every $m = 1, \cdots, 2n$, and consider $MO(p_m)$ as an infinite-dimensional vector. Set

$$\mathrm{ES}(\widehat{\sigma}) = (MO(p_1), \cdots, MO(p_{2n})) \in (\widehat{\mathbb{C}}^\infty)^{2n},$$

and define a continuous map

$$\Sigma_{FP} : \widehat{\mathrm{REL}}(G) \to (\widehat{\mathbb{C}}^\infty)^{2n}$$

by sending $\widehat{\sigma}$ to $\mathrm{ES}(\widehat{\sigma})$. We call the image $\Sigma_{FP}(\widehat{\sigma}) = \mathrm{ES}(\widehat{\sigma})$ the *rough essential fixed point spectrum* of $\widehat{\sigma}$.

Recall that the components $\sigma(g)(p_{2j-1})$ and $\sigma(g)(p_{2j})$ of $\mathrm{ES}(\widehat{\sigma})$ are the fixed points of the element $\sigma(gg_jg^{-1})$ for every j and $g \in G$ as above.

Definition 2.3. The cross ratio

$$\chi(q_1, q_2, q_3, q_4) = \frac{q_1 - q_2}{q_1 - q_3}\frac{q_4 - q_3}{q_4 - q_2} \quad (\text{if } q_j \in \mathbb{C})$$

of given components q_k of $\Sigma_{FP}(\widehat{\sigma})$ is well defined for the conjugacy class $[\widehat{\sigma}]$ and, furthermore, can be considered as a holomorphic function $\widehat{\chi}$ on a generic set $\Omega_{\widehat{\chi}}$ of $\widehat{\mathrm{EL}}(G)$ where no subsets consisting of three corresponding components are coincident to each other. We call such a holomorphic function

$$\widehat{\chi} : \Omega_{\widehat{\chi}} \to \widehat{\mathbb{C}}$$

a *FPCR* (*fixed point cross ratio*) *function.*

We can show that the set of all FPCR functions *separates points* of $\widehat{\mathrm{EL}}(G)$ in the sense that, for any pair of points in $\widehat{\mathrm{EL}}(G)$, there is a FPCR function defined at both points which takes distinct values at these points. Furthermore, we can introduce a system of global coordinates for a generic subset of $\widehat{\mathrm{EL}}(G)$.

Theorem 2.4. *There is an open cover* $\{\Omega_j\}$ *of* $\widehat{\mathrm{EL}}(G)$ *by a finite number of generic subsets such that, for every* Ω_j*, there are* $(3n-3)$ *FPCR functions* $\{\widehat{\chi}_1, \cdots, \widehat{\chi}_{3n-3}\}$ *defined on* Ω_j *which gives a holomorphic injection*

$$\Lambda_{\Omega_j} = (\widehat{\chi}_1, \cdots, \widehat{\chi}_{3n-3}) : \Omega_j \to \widehat{\mathbb{C}}^{3n-3}.$$

Proof. First note that, for every $[\widehat{\sigma}] \in \widehat{\mathrm{EL}}(G)$, there is a pair $\{i, j\}$ with $i \neq j$ such that $\mathrm{Fix}(\sigma(g_i)) \neq \mathrm{Fix}(\sigma(g_j))$. Set

$$\Omega(\{i,j\}) = \{[\widehat{\sigma}] \in \widehat{\mathrm{EL}}(G) \mid \mathrm{Fix}\,\sigma(g_i) \neq \mathrm{Fix}\,\sigma(g_j)\}$$

and

$$\Omega(i,j,k) = \{[\widehat{\sigma}] \in \widehat{\mathrm{EL}}(G) \mid p_{2j-k} \notin \mathrm{Fix}\,\sigma(g_i)\} \quad \text{if } k = 0, 1.$$

Then, clearly,

$$\bigcup_{k=0}^{1} (\Omega(i,j,k) \cup \Omega(j,i,k)) \supset \Omega(\{i,j\}),$$

and we have a finite open cover $\{\Omega(i,j,k) \mid i \neq j, k = 0, 1\}$ of $\widehat{\mathrm{EL}}(G)$.

Fix i, j, k with $i \neq j$, $k = 0, 1$. Set

$$A_{i,j,k} = \{p_{2i-1}, p_{2j-k}, \sigma(g_i)(p_{2j-k})\}.$$

Fix $(r_1, \cdots, r_n) \in (A_{i,j,k})^n$ arbitrarily and abbreviate it as $\mathbf{r}$. Set

$$\Omega(i,j,k)_{\mathbf{r}} = \{[\widehat{\sigma}] \in \Omega(i,j,k) \mid r_\ell \neq \sigma(g_\ell)(r_\ell) \text{ for every } \ell\}.$$

Then the family

$$\{\Omega(i,j,k)_{\mathbf{r}} \mid i \neq j,\ k = 0, 1;\ \mathbf{r} \in (A_{i,j,k})^n\}$$

gives an open cover of $\widehat{\mathrm{EL}}(G)$. Indeed, every $[\widehat{\sigma}]$ in $\widehat{\mathrm{EL}}(G)$ belongs to some $\Omega(i,j,k)$, and then $A_{i,j,k}$ consists of 3 distinct points. Hence, for every $\ell \neq i$, there is a point $r_\ell \in A_{i,j,k}$ such that $r_\ell \neq \sigma(g_\ell)(r_\ell)$. Set $r_i = p_{2j-k}$ and take this $\mathbf{r} = (r_1, \cdots, r_n) \in (A_{i,j,k})^n$. Then $[\widehat{\sigma}]$ belongs to $\Omega(i,j,k)_{\mathbf{r}}$.

Next, fix $\Omega = \Omega(i,j,k)_{\mathbf{r}}$ arbitrarily, and write the corresponding $A_{i,j,k}$ as $\{s_1, s_2, s_3\}$. Consider $(3n-3)$ FPCR functions

$$\{\chi(s_1, s_2, s_3, x) \mid x \in \{p_1, \cdots, p_{2n}, q_1, \cdots, q_n\} - \{s_1, s_2, s_3\}\},$$

where we set $q_\ell = \sigma(g_\ell)(r_\ell)$ for every ℓ, and write each $\chi(s_1, s_2, s_3, x)$ as χ_x. Then, since s_1, s_2, s_3 are distinct, $\chi_{p_{2\ell-1}}, \chi_{p_{2\ell}}, \chi_{r_\ell}$, and χ_{q_ℓ} determine the equivalence class of the fixed points and the multiplier of $\sigma(g_\ell)$ for every ℓ and $\widehat{\sigma} \in \Omega$. (For a more explicit description, see Remark 2.6 below.) Therefore, a point $[\widehat{\sigma}] \in \Omega$ is determined uniquely from the vector

$$\Lambda_\Omega = (\widehat{\chi}_1, \cdots, \widehat{\chi}_{3n-3})$$

at $[\widehat{\sigma}]$, which implies that Λ_Ω defined as above gives a holomorphic injection. Thus we have proved the assertion. □

Definition 2.5. Such a map Λ_{Ω_j}, or the set $\{\widehat{\chi}_k\}_{k=1}^{3n-3}$, as in the proof of Theorem 2.4, is called a *global coordinate system* of Ω_j.

Remark 2.6. If $\sigma(g_\ell)$ is not parabolic, then we can take as a representative of $[\sigma(g_\ell)]$ the Möbius transformation g_ℓ^σ determined by

$$\frac{g_\ell^\sigma(z) - \chi_{p_{2\ell-1}}}{g_\ell^\sigma(z) - \chi_{p_{2\ell}}} = \frac{(\chi_{r_\ell} - \chi_{p_{2\ell}})(\chi_{q_\ell} - \chi_{p_{2\ell-1}})}{(\chi_{r_\ell} - \chi_{p_{2\ell-1}})(\chi_{q_\ell} - \chi_{p_{2\ell}})} \frac{z - \chi_{p_{2\ell-1}}}{z - \chi_{p_{2\ell}}}. \tag{2.1}$$

If $\sigma(g_\ell)$ is parabolic, we can take as a representative of $[\sigma(g_\ell)]$ the Möbius transformation g_ℓ^σ determined by

$$\frac{1}{g_\ell^\sigma(z)-\chi_{p_{2\ell}}}=\frac{1}{z-\chi_{p_{2\ell}}}+\frac{\chi_{r_\ell}-\chi_{q_\ell}}{(\chi_{r_\ell}-\chi_{p_{2\ell}})(\chi_{q_\ell}-\chi_{p_{2\ell}})}.$$

From these equations, we can see that non-parabolic g_ℓ^σ tend to parabolic ones continuously. Indeed, the equation (2.1) implies that

$$\begin{aligned}\frac{\chi_{p_{2\ell-1}}-\chi_{p_{2\ell}}}{g_\ell^\sigma(z)-\chi_{p_{2\ell}}}&=\frac{\chi_{p_{2\ell-1}}-\chi_{p_{2\ell}}}{z-\chi_{p_{2\ell}}}\\&\quad+\frac{(\chi_{r_\ell}-\chi_{q_\ell})(\chi_{p_{2\ell-1}}-\chi_{p_{2\ell}})}{(\chi_{r_\ell}-\chi_{p_{2\ell}})(\chi_{q_\ell}-\chi_{p_{2\ell}})}+O(|\chi_{p_{2\ell-1}}-\chi_{p_{2\ell}}|^2)\end{aligned}$$

as $\chi_{p_{2\ell-1}}$ tend to $\chi_{p_{2\ell}}$.

Finally, we show the following proposition.

Proposition 2.7. *For every two Ω_1, Ω_2 defined as above, $\Lambda_{\Omega_1}\circ\Lambda_{\Omega_2}^{-1}$ is a rational function on $\Lambda_{\Omega_2}(\Omega_2\bigcap\Omega_1)$. In particular, the coordinate changes between Ω_j are bi-rational.*

To prove Proposition 2.7, let $\Omega_h=\Omega(i_h,j_h,k_h)_{\mathbf{r}_h}$ for each $h=1,2$. Write the associated A_{i_h,j_h,k_h}, $\mathbf{r}_h$, and $\mathbf{q}_h$ as $A_h=\{s_{h,1},s_{h,2},s_{h,3}\}$, $(r_{h,\ell})$, and $(q_{h,\ell})$, respectively, for each h. Also set

$$\chi_x^{(h)}=\chi(s_{h,1},s_{h,2},s_{h,3},x)$$

for every $x\in\{p_1,\cdots,p_{2n},q_{h,1},\cdots,q_{h,n}\}-\{s_{h,1},s_{h,2},s_{h,3}\}$.

Now, if $(r_{1,\ell})=(r_{2,\ell})$, then we have the following lemma.

Lemma 2.8. *If $(r_{1,\ell})=(r_{2,\ell})$, then $(q_{1,\ell})=(q_{2,\ell})$ and we have*

$$\chi_x^{(2)}=\frac{\chi_{s_{2,1}}^{(1)}-\chi_{s_{2,2}}^{(1)}}{\chi_{s_{2,1}}^{(1)}-\chi_{s_{2,3}}^{(1)}}\cdot\frac{\chi_x{}^{(1)}-\chi_{s_{2,3}}^{(1)}}{\chi_x^{(1)}-\chi_{s_{2,2}}^{(1)}}$$

for every $x\in\{p_1,\cdots,p_{2n},q_{2,1},\cdots,q_{2,n}\}-\{s_{2,1},s_{2,2},s_{2,3}\}$.

Proof. Recall that $\chi(s_{1,1},s_{1,2},s_{1,3},z)$ is a Möbius transformation of z. Hence, we have that

$$\chi_x^{(2)}=\chi(s_{2,1},s_{2,2},s_{2,3},x)=\chi(\chi_{s_{2,1}}^{(1)},\chi_{s_{2,2}}^{(1)},\chi_{s_{2,3}}^{(1)},\chi_x^{(1)}),$$

which is the desired equation. □

Next, for general cases, we need the following lemma.

Lemma 2.9. *Suppose that $(i_1,j_1,k_1)=(i_2,j_2,k_2)$ and that there is only one ℓ_0 such that $r_{1,\ell_0}\neq r_{2,\ell_0}$. Then we have*

$$\chi_{q_{2,\ell_0}}^{(2)}=\frac{(\chi_{p_{2\ell_0-1}}^{(1)}+\chi_{p_{2\ell_0}}^{(1)})\chi_{q_{1,\ell_0}}^{(1)}-\chi_{p_{2\ell_0-1}}^{(1)}\chi_{p_{2\ell_0}}^{(1)}}{\chi_{q_{1,\ell_0}}^{(1)}}. \tag{2.2}$$

Proof. First consider only such $[\widehat{\sigma}]\in\Omega_1\bigcap\Omega_2$ that $\sigma(g_{\ell_0})$ are non-parabolic. Also, $(i_1,j_1,k_1)=(i_2,j_2,k_2)$ is denoted by (i,j,k). Then $3n-4$ coordinates $\chi_x^{(1)}$ and $\chi_x^{(2)}$ are the same, which we write as χ_x, except for $\chi_1=\chi_{q_{1,\ell_0}}^{(1)}$ and $\chi_2=\chi_{q_{2,\ell_0}}^{(2)}$. Since $\sigma(g_{\ell_0})$ are non-parabolic, $p_{2\ell_0-1},p_{2\ell_0},q_{1,\ell_0}$, and q_{2,ℓ_0} are distinct, and we have

$$\chi(\chi_{p_{2\ell_0-1}},\chi_{p_{2\ell_0}},\chi_{r_{1,\ell_0}},\chi_{r_{2,\ell_0}})=\chi(\chi_{p_{2\ell_0-1}},\chi_{p_{2\ell_0}},\chi_1,\chi_2).$$

Here $\chi_{r_{1,\ell_0}}$ and $\chi_{r_{2,\ell_0}}$ are constants in $\{0,1,\infty\}$, since $r_{h,\ell_0} \in A_1 = A_2$. We may assume without loss of generality that $\chi_{r_{1,\ell_0}} = 0$ and $\chi_{r_{2,\ell_0}} = \infty$. Then, we obtain the equation

$$\frac{1}{\chi_{p_{2\ell_0-1}}} = \frac{\chi_2 - \chi_1}{(\chi_{p_{2\ell_0-1}} - \chi_1)(\chi_2 - \chi_{p_{2\ell_0}})},$$

which, in turn, implies the desired equality on a generic subset of $\Omega_1 \bigcap \Omega_2$.

Finally, since $\chi^{(2)}_{q_{2,\ell_0}}$ is holomorphic on Ω_2, the rational function on the right-hand side of (2.2) is continuous and hence has no indefinite singularity on the whole $\Omega_1 \bigcap \Omega_2$. Thus we conclude the assertion. □

Proof of Proposition 2.7. By using the formulas as in Lemmas 2.8 and 2.9, we can conclude the assertion by using the same argument as in the proof of Lemma 2.9. □

3. A new distance on the Schottky space

Recall that a Schottky group is, by definition, a purely loxodromic free Kleinian group having a proper subset of $\widehat{\mathbb{C}}$ as the limit set, which is also characterized as a geometrically finite purely loxodromic free Kleinian group. It is known that Schottky groups are quasi-conformally stable, and hence $S(G)$ is a connected open subset of $\mathrm{EL}(G)$. (See, for instance, [12].) Furthermore, the following fact is fundamental, and is a corollary of a famous theorem of Jørgensen [9].

Proposition 3.1. *The closure $\overline{S(G)}$ of the Schottky space $S(G)$ in $X(G)$ is a compact subset of $FD(G) = F(G) \cap D(G)$, which in turn is a proper subset of $\mathrm{EL}(G)$.*

Remark 3.2. The interior of $FD(G)$ is $S(G)$. Also the density theorem in [14] implies that $\overline{S(G)} = FD(G)$. (Cf. [6].)

As for the complement of $S(G)$, Minsky shows in [13] that the outer automorphism group $Out(G)$ acts properly discontinuously on a domain strictly larger than $S(G)$.

Now, we can equip every point $[\sigma]$ in $S(G)$ with the canonical marking $\mathbf{p} = (p_1, \cdots, p_{2n})$ by setting p_{2j-1}, and p_{2j} be the attracting fixed point and the repelling one of $\sigma(g_j)$, respectively, for every j. It is clear that, by this canonical marking, we have a holomorphic embedding

$$\rho : S(G) \to \widehat{\mathrm{EL}}(G).$$

Hence, we consider $S(G)$ as a subset of $\widehat{\mathrm{EL}}(G)$. We write points $[\sigma]$ of $S(G)$ with the canonical marking as $[\widehat{\sigma}]$.

Remark 3.3. The inverse image $\tau^{-1}(S(G))$ consists of 2^n components, each of which is bi-holomorphic to $S(G)$. While, the closure of $\tau^{-1}(S(G))$ is connected.

Here, we note the following fact.

Lemma 3.4. *Every generic set $\Omega = \Omega(i,j,k)_{\mathbf{r}}$ as in the proof of Theorem 2.4 contains $S(G)$ and the corresponding global coordinate system Λ_Ω of Ω gives a holomorphic injection of $S(G)$ into $(\mathbb{C} - \{0,1\})^{3n-3}$.*

Moreover, there is a point of $\overline{S(G)} - S(G)$ where Λ_Ω takes a value in $(\mathbb{C} - \{0,1\})^{3n-3}$.

Proof. As noted before, every component of $\Sigma_{FP}(\widehat{\sigma})$ is a fixed point of an element of $\sigma(G)$. On the other hand, if $[\widehat{\sigma}] \in S(G)$, then every non-identical element of $\sigma(G)$ has two distinct fixed points, and hence we can see that every two components of $\Sigma_{FP}(\widehat{\sigma})$ are distinct, which implies the first assertion.

Next, the existence of purely loxodromic free Kleinian groups on $\overline{S(G)} - S(G)$ implies the second assertion. □

The above proof also implies that every FPCR function is defined on the whole $S(G)$ and takes values in $\mathbb{C} - \{0,1\}$. Hence, we can define distances on $S(G)$ by using FPCR functions.

Definition 3.5. Let Λ be a set of FPCR functions, and d_h is the hyperbolic distance on $\mathbb{C} - \{0,1\}$. Set

$$d_\Lambda([\widehat{\sigma}_1],[\widehat{\sigma}_2]) = \sup_{\widehat{\chi}\in\Lambda} d_h(\widehat{\chi}([\widehat{\sigma}_1]),\widehat{\chi}([\widehat{\sigma}_2]))$$

for every $[\widehat{\sigma}_1],[\widehat{\sigma}_2] \in S(G)$.

If Λ is the set of all FPCR functions, then we write Λ as Λ_{CR} and d_Λ as d_{CR}. We call d_{CR} the *FPCR-distance on* $S(G)$.

Note that d_Λ is a distance if Λ contains all FPCR functions in some global coordinate system. Also, recall that a classical theorem of Teichmüller (cf. [10]) implies that the FPCR-distance is not greater than the Teichmüller distance on $S(G)$. Furthermore, the proof of Lemma 3.4 implies the following fact.

Corollary 3.6. *For every set Λ of a finite number of FPCR functions such that d_Λ is a distance, d_Λ is not complete on $S(G)$.*

Here, we say that a distance d is *complete* on $S(G)$ if every closed ball

$$B = \{[\widehat{\sigma}] \in S(G) \mid d([\widehat{\sigma}_0],[\widehat{\sigma}]) \le R\}$$

with finite radius R and center $[\widehat{\sigma}_0]$ in $S(G)$ is compact.

Finally, we show the following theorem.

Theorem 3.7. *The FPCR-distance d_{CR} is complete on $S(G)$.*

Proof. Suppose that there were a non-compact closed ball B with a finite radius r in $S(G)$ with respect to the FPCR-distance. Then, there is a sequence $\{[\widehat{\sigma_n}]\}$ in B converging to some $[\widehat{\sigma_\infty}]$ in $\widehat{\mathrm{EL}}(G) - S(G)$. Since $G_\infty = \sigma_\infty(G)$ is a free Kleinian group by Proposition 3.1, either G_∞ contains a parabolic element g^*, or G_∞ is purely loxodromic and the limit set of G_∞ is $\widehat{\mathbb{C}}$.

In the former case, there are at least two distinct points in

$$\operatorname{Fix}\sigma_\infty(g_1)\bigcup\operatorname{Fix}\sigma_\infty(g_2)\bigcup g_1(\operatorname{Fix}\sigma_\infty(g_2))$$

which are not the single fixed point s^* of g^*. Fix such points and write them as $s_{\infty,1}, s_{\infty,2}$. Let g_n^*, $s_{n,1}$, and $s_{n,2}$ be the element in $G_n = \sigma_n(G)$ and the fixed points corresponding to g^*, $s_{\infty,1}$, and $s_{\infty,2}$, respectively, for every n. Here, letting $[\widehat{\sigma_0}]$ be the center of B, we set $G_0 = \sigma_0(G)$. Also, without loss of generality, we may assume that G_n converges to G_∞ algebraically. Then, since g_n^* converges to g^* in $\mathrm{PSL}_2(\mathbb{C})$ and both of the distinct fixed points $s_{n,1}^*$ and $s_{n,2}^*$ of g_n^* converge to s^*, the cross ratio $\chi_n = \chi(s_{n,1}, s_{n,2}, s_{n,1}^*, s_{n,2}^*) \in \mathbb{C} - \{0,1\}$ tends to

$$\chi_\infty = \chi(s_{\infty,1}, s_{\infty,2}, s^*, s^*) = 0$$

as $n \to \infty$. So we can find an N such that

$$d_h(\chi_0, \chi_N) > 2r.$$

On the other hand, since the orbit $G_0(s_0)$ of any component s_0 of $\Sigma_{FP}(\widehat{\sigma_0})$ (considered as a point) by G_0 is dense in the limit set $\Lambda(G_0)$ of G_0, we can find sequences of components $s_{0,j,k}$ of $\Sigma_{FP}(\widehat{\sigma_0})$ such that

$$\lim_{k\to\infty} s_{0,j,k} = s^*_{0,j}$$

for each $j = 1, 2$. Let $s_{n,j,k}$ be the component of $\Sigma_{FP}(\widehat{\sigma_n})$ corresponding to $s_{0,j,k}$ for every n, j, and k. Then, the cross ratio $\chi(s_{n,1}, s_{n,2}, s_{n,1,k}, s_{n,2,k})$ is the value of the same FPCR function, say $\widehat{\chi}^*_k$, at $[\widehat{\sigma}_n]$ for every k and n.

Since G_0 and G_n are quasiconformally conjugate, we can see that

$$\lim_{k\to\infty} s_{n,j,k} = s^*_{n,j}$$

and hence

$$\lim_{k\to\infty} \widehat{\chi}^*_k([\widehat{\sigma_n}]) = \chi_n$$

for every n. Hence, we can find a K such that

$$d_h(\widehat{\chi}^*_K([\widehat{\sigma_0}]), \widehat{\chi}^*_K([\widehat{\sigma_N}])) > r,$$

which implies that

$$d_{CR}([\widehat{\sigma}_0], [\widehat{\sigma}_N]) > r.$$

But this contradicts the assumption that $[\widehat{\sigma}_N] \in B$, which shows the assertion in this case.

Next, in the latter case, G_∞ is purely loxodromic and geometrically infinite. Here, we recall the following characterization of geometrically infinite ends by Bonahon [2]. □

Lemma 3.8 (Cf. [3, Proposition 6.2])**.** *Let N be a hyperbolic 3-manifold with finitely generated fundamental group, and let E be an end of the complement of ϵ-thin cusps of N (with sufficiently small $\epsilon > 0$). Then E is not geometrically finite if and only if there exists a sequence of closed geodesics exiting E.*

Proof of Theorem 3.7*, continued.* In our case, the above characterization implies that there is a sequence of disjoint closed geodesics γ_m in the hyperbolic manifold $N_\infty = \mathbb{H}^3/G_\infty$ such that, for every compact set A in N_∞, there are only a finite number of γ_m which intersect A. In other words, for every sequence $\{g^*_m\}$ of elements $g^*_m \in G_\infty$ whose axes project to γ_m, we have that

$$\chi_\infty(g^*_m) = \chi(s_{\infty,1}, s_{\infty,2}, q^*_{m,1}, q^*_{m,2}) \to 0$$

as $m \to \infty$, where $s_{\infty,1}, s_{\infty,2}$ and $q^*_{m,1}, q^*_{m,2}$ are the two fixed points of $g^\infty_1 = \sigma_\infty(g_1)$ and those of g^*_m, respectively, for every $m > 1$.

On the other hand, since G_0 is geometrically finite, i.e., the convex core of G_0 is compact in $\mathbb{H}^3/G_0$, we can find a sequence of elements $g^0_m \in G_0$ which corresponds to a sequence $\{g^{*,0}_m\}$ in G_∞ such that the axis of $g^{*,0}_m$ projects to γ_m for every m and satisfies that the hyperbolic distance between the axis of $g^0_1 = \sigma_0(g_1)$ and that of g^0_m is bounded for all m. Namely, there is a constant $\eta > 0$ such that

$$\chi_0(g^0_m) = \chi(s_{0,1}, s_{0,2}, q^0_{m,1}, q^0_{m,2}) \geq \eta$$

for every m, where $s_{0,1}, s_{0,2}$ and $q^0_{m,1}, q^0_{m,2}$ are the fixed points of g^0_1 and those of g^0_m. Fix such a sequence $\{g^0_m\}$ and hence also $\{g^{*,0}_m\}$. Then, since $\chi_\infty(g^{*,0}_m)$ tends to 0, we can find an M such that

$$d_h(\chi_0(g^0_M), \chi_\infty(g^{*,0}_M)) > 3r.$$

Now, we can define

$$\chi_n(g^n_M) = \chi(s_{n,1}, s_{n,2}, q^n_{M,1}, q^n_{M,2})$$

corresponding to $\chi_0(g^0_M)$ by using the fixed points $s_{n,1}, s_{n,2}$ and $q^n_{M,1}, q^n_{M,2}$ of g^n_1 and g^n_M in G_n which correspond to g^0_1 and g^0_M, respectively, for every n. Then, since G_n converges to G_∞ algebraically,

$$\lim_{n\to\infty} \chi_n(g^n_M) = \chi_\infty(g^{*,0}_M)$$

and hence we can find an N such that

$$d_h(\chi_0(g^0_M), \chi_N(g^N_M)) > 2r.$$

Finally, by using the same argument as in the former case, we can find a FPCR function, say $\widehat{\chi}$, such that

$$d_h(\widehat{\chi}([\widehat{\sigma_0}]), \widehat{\chi}([\widehat{\sigma_N}]) > r.$$

Thus, again we have a contradiction, which shows the assertion. □

Remark 3.9. On the Schottky space $S(G)$, every FPCR function is a rational function of $3n-3$ coordinates in a fixed global coordinate system, which can be shown similarly as in the proof of Proposition 2.7.

Also note that the proof of Theorem 3.7 shows that the set Λ of all FPCR functions obtained from a given single marked orbit $MO(p_m)$ gives a complete distance d_Λ ($\leq d_{CR}$).

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