

MAPPINGS OF FINITE DISTORTION BETWEEN METRIC MEASURE SPACES

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ABSTRACT. We establish the basic analytic properties of mappings of finite distortion between proper Ahlfors regular metric measure spaces that support a $(1, 1)$ -Poincaré inequality. As applications, we prove that under certain integrability assumption for the distortion function, the branch set of a mapping of finite distortion between generalized n -manifolds of type A has zero Hausdorff n -measure.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a domain. A mapping $f : \Omega \rightarrow \mathbb{R}^n$ is said to be a *mapping of finite distortion* if $f \in W_{loc}^{1,1}(\Omega, \mathbb{R}^n)$, $J_f \in L_{loc}^1(\Omega)$ and if there is a measurable function $K : \Omega \rightarrow [1, \infty]$ such that

$$|Df(x)|^n \leq K(x)J_f(x) \quad a.e. \text{ in } \Omega,$$

where $|Df(x)|$ and $J_f(x)$ are the operator norm and the Jacobian determinant of the matrix $Df(x)$, respectively. Associated with such a mapping the (outer) distortion $K(\cdot, f)$ is defined as

$$K(x, f) = \frac{|Df(x)|^n}{J_f(x)}, \quad \text{if } J_f(x) > 0,$$

and $K(x, f) = 1$ otherwise. When $K(\cdot, f)$ is (essentially) bounded, we recover the well-known class of *quasiregular mappings* or *mappings with bounded distortion*. The study of quasiregular mappings was initiated with Reshetnyak [33], and the basic theory was comprehensively laid and significantly advanced in a sequence of papers from the Finnish school of Martio, Rickman and Väisälä in the late 1960s [29–31, 34]. The study of mappings of finite distortion was (partially) motivated by nonlinear elasticity theory of Antman, Ball and Ciarlet [21] and the whole theory is well developed now; see [18, 20–23, 27].

In recent years, there has been growing interest to study geometric function theory beyond the Euclidean spaces, especially in the setting of nonsmooth metric measure spaces; see [10, 12, 14, 16, 25, 32, 42–44]. The purpose of this paper is to study mappings of finite distortion between nonsmooth metric measure spaces.

It was perhaps Heinonen and Koskela [12] who first systematically developed the basic theory of quasiconformal maps, that is, quasiregular homeomorphisms, between general metric spaces with controlled geometry. This theory also initiated

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a new way of looking at weakly differentiable maps between nonsmooth spaces. It was somehow surprising that such a rich theory exists merely assuming that the underlying measure is doubling and an abstract Poincaré inequality holds. Later, Cheeger systematically investigated the structure of doubling metric measure spaces with the abstract Poincaré inequality in [4] and proved that actually a strong measurable differentiable structure exists for such spaces. In [14], the Sobolev class of Banach space valued mappings was studied and several characterizations of quasiconformal mappings between Ahlfors regular Loewner spaces were established. Following the general Sobolev space theory [14, 15], Marshall Williams recently studied the equivalence of different definitions for quasiregular mappings between metric measure spaces of locally bounded geometry; see [44].

Of particular interest for us is the theory of a subclass of quasiregular maps, called *mappings of bounded length distortion* (BLD), between non-smooth spaces, developed by Heinonen and Rickman [16]. The motivation there was to use BLD-maps to study problems in geometric topology. Heinonen and Sullivan [17] applied the theory to study a widely open problem: characterize metric spaces that are locally bi-Lipschitz homeomorphic to some Euclidean spaces \mathbb{R}^n . The non-smooth setting, in both studies, is the so-called *generalized manifolds of type A*; see Section 5 and Section 6 below for precise definitions. Several interesting questions arose from [16], for example, it was implicitly asked *whether a general theory of quasiregular maps between generalized manifolds of type A exists*. Very recently, Onninen and Rajala [32] partially solved the problem. To be more precise, they developed a basic theory of quasiregular maps from Euclidean domains to generalized manifolds with restricted topology and locally controlled geometry (which are more general than generalized manifolds of type A). More recently, Kirsilä [25] has established the basic properties of mappings of finite distortion from generalized manifolds with controlled geometry to Euclidean spaces.

These are the motivations of the current paper and the paper is then naturally divided into two parts. In the first part, we develop analytic properties of mappings of finite distortion between proper Ahlfors n -regular metric measure spaces that support a $(1, 1)$ -Poincaré inequality. In fact, most results hold under weaker assumption, i.e. the space supports a $(1, n)$ -Poincaré inequality. We mainly investigate the following properties of mappings of finite distortion (possibly with some assumptions for the distortion function): Condition N , area formula, condition N^{-1} , Cheeger's differentiability, and \mathcal{H}^n -a.e. positiveness of the Jacobian. In the second part, we consider mappings of finite distortion between generalized manifolds of type A. We have shown that under suitable integrability condition for the distortion function, the branch set of a mapping of finite distortion f between generalized manifolds of type A has zero Hausdorff n -measure.

We have benefited much from the paper [16] considering the BLD mappings between generalized manifold of type A, and from the forthcoming monograph [15] considering the general Newtonian-Sobolev space theory on metric measure spaces.

2. SOBOLEV SPACES AND POINCARÉ INEQUALITIES ON METRIC MEASURE SPACES

In this section, we will briefly introduce the Sobolev spaces on metric measure spaces based on an upper gradient approach. For a detailed description of this approach, see the monograph [15].

2.1. **Metric measure spaces.**

Definition 2.1. A *metric measure space* is defined to be a triple (X, d, μ) , where (X, d) is a separable metric space and μ is a nontrivial locally finite Borel regular measure on X .

Definition 2.2. A Borel regular measure μ on a metric space (X, d) is called a *doubling measure* if every ball in X has positive and finite measure and there exists a constant $C_\mu \geq 1$ such that

$$(2.1) \quad \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r))$$

for each $x \in X$ and $r > 0$. We call the triple (X, d, μ) a doubling metric measure space if μ is a doubling measure on X . We call (X, d, μ) an Ahlfors n -regular space, $1 \leq n < \infty$, if there exists a constant $C \geq 1$ such that

$$(2.2) \quad C^{-1}r^n \leq \mu(B(x, r)) \leq Cr^n$$

for all balls $B(x, r) \subset X$ of radius $r < \text{diam } X$.

It is well known that if (X, d, μ) is an Ahlfors n -regular space, then

$$(2.3) \quad \mu(E) \approx \mathcal{H}^n(E)$$

for all Borel sets E in X ; see [9, Chapter 8]. In this regard, it will be convenient to use the Hausdorff n -measure for an Ahlfors n -regular metric spaces.

2.2. Modulus of a curve family. Let (X, d) be a metric space. A curve in X is a continuous map $\gamma : I \rightarrow X$, where $I \subset \mathbb{R}$ is an interval. We call γ compact, open, or half-open, depending on the type of the interval I .

Given a compact curve $\gamma : [a, b] \rightarrow X$, its length is the supremum of the sums

$$(2.4) \quad \sum_{i=1}^k d(\gamma(t_i), \gamma(t_{i-1})),$$

where the numbers t_i run over all finite sequences of points of the form

$$a = t_0 < t_1 < \dots < t_k = b.$$

If γ is not compact, its length is defined to be the supremum of the lengths of the compact subcurves of γ . Thus, every curve has a well-defined length in the extended nonnegative reals, and we denote it by $\text{length}(\gamma)$ or simply $l(\gamma)$.

A curve is said to be rectifiable if its length $l(\gamma)$ is finite, and locally rectifiable if each of its compact subcurves is rectifiable. For any rectifiable curve γ there are its associated length function $s_\gamma : I \rightarrow [0, l(\gamma)]$ and a unique 1-Lipschitz map $\gamma_s : [0, l(\gamma)] \rightarrow X$ such that $\gamma = \gamma_s \circ s_\gamma$. The curve γ_s is the arc length parametrization of γ .

When γ is rectifiable, and parametrized by arclength on the interval $[a, b]$, the integral of a Borel function $\rho : X \rightarrow [0, \infty]$ along γ is

$$\int_\gamma \rho ds = \int_0^{l(\gamma)} \rho(\gamma_s(t)) dt.$$

Similarly, the line integral of a Borel function $\rho : X \rightarrow [0, \infty]$ over a locally rectifiable curve γ is defined to be the supremum of the integral of ρ over all compact subcurves of γ .

Let $X = (X, d, \mu)$ be a metric measure space as defined in (2.1). Let Γ be a family of curves in X . A Borel function $\rho: X \rightarrow [0, \infty]$ is *admissible* for Γ if for every locally rectifiable curve $\gamma \in \Gamma$,

$$(2.5) \quad \int_{\gamma} \rho ds \geq 1.$$

The *p-modulus* of Γ , $p \geq 1$, is defined as

$$\text{Mod}_p(\Gamma) = \inf \left\{ \int_X \rho^p d\mu : \rho \text{ is admissible for } \Gamma \right\}.$$

Observe that

$$(2.6) \quad \text{Mod}_p(\emptyset) = 0,$$

$$(2.7) \quad \text{Mod}_p(\Gamma_1) \leq \text{Mod}_p(\Gamma_2), \text{ if } \Gamma_1 \subset \Gamma_2,$$

and

$$(2.8) \quad \text{Mod}_p \left(\bigcup_{i=1}^{\infty} \Gamma_i \right) \leq \sum_{i=1}^{\infty} \text{Mod}_p(\Gamma_i).$$

Moreover, if Γ_0 and Γ are two curve families such that each curve $\gamma \in \Gamma$ has a subcurve $\gamma_0 \in \Gamma_0$, then

$$(2.9) \quad \text{Mod}_p(\Gamma) \leq \text{Mod}_p(\Gamma_0).$$

We say in such a situation that Γ majorizes Γ_0 .

2.3. Sobolev spaces based on upper gradients. Let $X = (X, d, \mu)$ be a metric measure space and $Z = (Z, d_Z)$ a metric space.

Definition 2.3. A Borel function $g: X \rightarrow [0, \infty]$ is called an *upper gradient* for a map $f: X \rightarrow Z$ if for every rectifiable curve $\gamma: [a, b] \rightarrow X$, we have the inequality

$$(2.10) \quad \int_{\gamma} g ds \geq d_Z(f(\gamma(b)), f(\gamma(a))).$$

If inequality (2.10) merely holds for *p*-almost every compact curve, then g is called a *p-weak upper gradient* for f . When the exponent p is clear, we omit it.

The concept of upper gradient was introduced in [12]. It was initially called “very weak gradient”, but the befitting term “upper gradient” was soon suggested. Functions with *p*-integrable *p*-weak upper gradients were subsequently studied in [26], while the theory of Sobolev spaces based on upper gradient was systematically developed in [37] and [4].

By [15, Lemma 5.2.3], f has a *p*-weak upper gradient in $L_{loc}^p(X)$ if and only if it has an actual upper gradient in $L_{loc}^p(X)$.

A *p*-weak upper gradient g of f is *minimal* if for every *p*-weak upper gradient \tilde{g} of f , $\tilde{g} \geq g$ μ -almost everywhere. If f has an upper gradient in $L_{loc}^p(X)$, then f has a unique (up to sets of μ -measure zero) minimal *p*-weak upper gradient by the following result from [15, Theorem 5.3.23]. In this situation, we denote the minimal upper gradient by g_f .

Proposition 2.4. *The collection of all p-integrable p-weak upper gradients of a map $u: X \rightarrow Z$ is a closed convex lattice inside $L^p(X)$ and, if nonempty, contains a unique element of smallest L^p -norm. In particular, if a map has a p-integrable p-weak upper gradient, then it has a minimal p-weak upper gradient.*

In view of the above result, the minimal p -weak upper gradient ρ_u should be thought of as a substitute for $|\nabla u|$, or the length of a gradient, for functions defined in metric measure spaces.

Let $\tilde{N}^{1,p}(X, Z)$ denote the collection of all maps $u \in L^p(X, Z)$ that have an upper gradient in $L^p(X)$. We equip it with seminorm

$$(2.11) \quad \|u\|_{\tilde{N}^{1,p}(X,Z)} = \|u\|_{L^p(X,Z)} + \|g_u\|_{L^p(X)},$$

where g_u is the minimal p -weak upper gradient of u guaranteed by Proposition 2.4.

We obtain a normed space $N^{1,p}(X, Z)$ by passing to equivalence classes of functions in $\tilde{N}^{1,p}(X, Z)$, where $u_1 \sim u_2$ if and only if $\|u_1 - u_2\|_{\tilde{N}^{1,p}(X,Z)} = 0$. Thus

$$(2.12) \quad N^{1,p}(X, Z) := \tilde{N}^{1,p}(X, Z) / \{u \in \tilde{N}^{1,p}(X, Z) : \|u\|_{\tilde{N}^{1,p}(X,Z)} = 0\}.$$

Let $\tilde{N}_{loc}^{1,p}(X, Z)$ be the vector space of functions $u : X \rightarrow Z$ with the property that every point $x \in X$ has a neighborhood U_x in X such that $u \in \tilde{N}^{1,p}(U_x, Z)$. Two functions u_1 and u_2 in $\tilde{N}_{loc}^{1,p}(X, Z)$ are said to be equivalent if every point $x \in X$ has a neighborhood U_x in X such that the restrictions $u_1|_{U_x}$ and $u_2|_{U_x}$ determine the same element in $\tilde{N}^{1,p}(U_x, Z)$. The local Sobolev space $N_{loc}^{1,p}(X, Z)$ is the vector space of equivalent classes of functions in $\tilde{N}_{loc}^{1,p}(X, Z)$ under the preceding equivalence relation.

With the notion of upper gradient, we are now ready to define the p -capacity of a general condenser in a metric measure space (X, d, μ) .

Suppose E and F are closed subsets of an open set Ω in X . The triple (E, F, Ω) is called a *condenser* and its p -capacity for $1 \leq p < \infty$ is defined as

$$(2.13) \quad \text{Cap}_p(E, F, \Omega) = \inf \int_{\Omega} g^p d\mu,$$

where the infimum is taken over all p -weak upper gradients of all functions u in Ω such that $u|_E \geq 1$ and $u|_F \leq 0$. Such a function u is called *admissible* for the condenser (E, F, Ω) .

Note that in the definition of p -capacity, we do not require any regularity of admissible functions. However, the existence of L^p -weak upper gradients in practice imposes restrictions. In particular, for a proper doubling metric measure space supporting a Poincaré inequality (in the sense of the next section), one can use locally Lipschitz continuous function as admissible functions for capacity; see [13, Proposition 2.4] or [12, Proposition 2.17]. Moreover, the following equation holds:

$$\text{Cap}_p(E, F, \Omega) = \text{Mod}_p(E, F, \Omega).$$

2.4. Poincaré inequalities and consequences.

Definition 2.5. We say that a metric measure space (X, d, μ) admits a $(1, p)$ -Poincaré inequality if there exist constants $C \geq 1$ and $\tau \geq 1$ such that

$$(2.14) \quad \int_B |u - u_B| d\mu \leq C \text{diam}(B) \left(\int_{\tau B} g^p d\mu \right)^{1/p}$$

for all open balls B in X , for every function $u : X \rightarrow \mathbb{R}$ that is integrable on balls and for every upper gradient g of u in X .

For a complete doubling metric measure space $X = (X, d, \mu)$ supporting a $(1, p)$ -Poincaré inequality, there are many important consequences. For example, one has

the important Sobolev embedding as in the Euclidean case. We need the following fact from [15, Theorem 7.3.2]; see also [8].

Lemma 2.6. *A complete and doubling metric measure space that supports a Poincaré inequality is quasiconvex.*

Recall that a metric space $Z = (Z, d_Z)$ is said to be C -*quasiconvex* or simply *quasiconvex*, $C \geq 1$, if each pair of points can be joined by a rectifiable curve in Z such that

$$(2.15) \quad l(\gamma) \leq Cd_Z(x, y).$$

Note that for a doubling metric space, completeness is equivalent to properness (cf. [2, Proposition 3.1]).

Recall that a metric space X is said to be θ -*linearly locally connected* (θ -LLC) if there exists $\theta \geq 1$ such that for $x \in X$ and $0 < r \leq \text{diam}(X)$:

- (i) every two points $a, b \in B(x, r)$ can be joined in $B(x, \theta r)$, and
- (ii) every two points $a, b \in X \setminus \overline{B}(x, r)$ can be joined in $X \setminus \overline{B}(x, \theta^{-1}r)$.

Here, by joining a and b in B we mean that there exists a path $\gamma : [0, 1] \rightarrow B$ with $\gamma(0) = a, \gamma(1) = b$.

Remark 2.7. Note that in the above definition, instead of a continuum, we choose to use a curve to join two points. This is in a prior stronger assumption. However, by [19, Theorem 3-17] a connected, locally connected, complete metric space is arcwise connected. Hence for such spaces, joining by continuum and joining by curve (or arc) are equivalent.

Based on Remark 2.7, throughout the paper, we assume that all the metric spaces in question are connected and locally connected.

The following result was proved in [12, Corollary 5.8], where the quasiconvexity condition is provided by Lemma 2.6.

Proposition 2.8. *Let (X, d, μ) be a complete Ahlfors n -regular metric measure space that supports a $(1, n)$ -Poincaré inequality. Then X is θ -linearly locally connected with θ depending only on the data associated with X .*

Definition 2.9. Let (X, d, μ) be a metric measure space. We say that X has n -*Loewner property* if there is a function $\phi : (0, \infty) \rightarrow (0, \infty)$ such that

$$\text{Mod}_n(\Gamma(E, F, X)) \geq \phi(\zeta(E, F))$$

for every nondegenerate compact connected sets $E, F \subset X$, where

$$\zeta(E, F) = \frac{\text{dist}(E, F)}{\min\{\text{diam } E, \text{diam } F\}}.$$

By [12, Theorem 3.6], if X is Ahlfors n -regular, and n -Loewner, then

$$(2.16) \quad \text{Mod}_n(\Gamma(E, F, X)) \geq C \left(\log \zeta(E, F) \right)^{1-n}$$

when $\zeta(E, F)$ is large enough with C depends only on the data of X . By [12, Corollary 5.13], a complete (or equivalently proper) Ahlfors n -regular metric measure space that supports a $(1, n)$ -Poincaré inequality is n -Loewner.

3. MAPPINGS OF FINITE DISTORTION

In this section, we define mappings of finite distortion between certain metric measure spaces. The metric measure spaces (X, d_X, μ) and (Y, d_Y, ν) in this paper, unless specialized, are always assumed to be proper, Ahlfors n -regular, and support a $(1, 1)$ -Poincaré inequality. We will always use the symbol Ω to denote a domain in X and in most situations drop the subscripts in the metrics d_X and d_Y when the context is clear. Without loss of generality, we only consider the case $\mu = \nu = \mathcal{H}^n$ in most situations.

Recall that a mapping $f : X \rightarrow Y$ between topological spaces is *discrete* if each fiber is a discrete set in X , i.e. for all $y \in Y$, $f^{-1}(y)$ is a discrete set in X , and is *open* if it maps an open set in X onto an open set in Y . For a mapping $f : X \rightarrow Y$ and a subset $A \subset X$, we will use the notation

$$(3.1) \quad N(y, f, A) = \#\{f^{-1}(y) \cap A\}$$

for the *multiplicity function* of f . We also write $N(f, A) := \sup_{y \in Y} N(y, f, A)$. A mapping $f : X \rightarrow Y$ is said to have *locally bounded multiplicity* if every point $x \in X$ has a neighborhood $U \subset X$ such that $N(f, U) < \infty$, and has *bounded multiplicity* if $N(f, X) < \infty$.

Let $f : X \rightarrow Y$ be a continuous, discrete and open mapping of locally bounded multiplicity between two locally compact metric measure spaces. We define a locally finite Borel regular outer measure (so-called pull-back measure) as

$$(3.2) \quad f^* \mathcal{H}^n(A) = \int_Y N(y, f, A) d\mathcal{H}^n(y).$$

The metric “Jacobian” above is given by

$$(3.3) \quad J_f(x) = \frac{d(f^* \mathcal{H}^n)(x)}{d\mathcal{H}^n(x)}.$$

Definition 3.1. A continuous mapping $f : \Omega \rightarrow Y$ is called a *mapping of finite distortion* if f is either constant or

- f is discrete, open, and of locally bounded multiplicity;
- $f \in N_{loc}^{1,n}(\Omega, Y)$;
- there is a measurable function $K = K(x) \geq 1$, finite almost everywhere, such that f satisfies the distortion inequality

$$(3.4) \quad g_f(x)^n \leq K(x) J_f(x) \text{ a.e. } \Omega.$$

If $K(\cdot) \in L^\infty(\Omega)$, then we call f *K-quasiregular* or simply *quasiregular*.

Next we introduce an useful concept called pseudomonotonicity.

Definition 3.2. A mapping $f : \Omega \rightarrow Y$ is said to be *T-pseudomonotone* or simply *pseudomonotone* if there exists a constant $T \geq 1$ such that

$$(3.5) \quad \text{diam } f(B(x, r)) \leq T \text{ diam } f(S(x, r))$$

for every $B(x, r) \subset\subset \Omega$. If T happens to be 1, then f is called *monotone*.

Lemma 3.3. *Suppose that $f : \Omega \rightarrow Y$ is continuous and open. Then for every $x \in \Omega$, there exists a radius $R = R(x) > 0$ such that $f|_{B(x,R)}$ is T -pseudomonotone, where $T \geq 1$ only depends on the data.*

Proof. By Proposition 2.8, we may assume that X and Y are C -linearly locally connected. Fix $x \in \Omega$. Since f is open, it is nonconstant. Thus, by the continuity of f , there exists $0 < R < 1$ and $p \in \Omega$ such that $B(x, C^2R) \subset\subset \Omega$, $5 \operatorname{diam} f(B(x, C^2R)) < \frac{\operatorname{diam}(Y)}{C}$ and

$$(3.6) \quad f(p) \in Y \setminus B(f(x), 5 \operatorname{diam} f(B(x, R))).$$

Now fix $B = B(y, r) \subset B(x, R)$ and a point $\omega \in \partial f(B)$. Recall that the openness of f implies that $\partial f(B) \subset f(\partial B)$, and thus

$$\operatorname{diam} f(\partial B) \geq \operatorname{diam} \partial f(B) =: A.$$

Then

$$\operatorname{diam} f(B) = \sup_{u, v \in f(B)} d(u, v) \leq \sup_{u, v \in f(B)} (d(u, \omega) + d(v, \omega)) = 2 \sup_{v \in f(B)} d(v, \omega).$$

To complete the proof, we show that, given $v \in f(B)$, $d(v, \omega) \leq CA$, where $C \geq 1$ does not depend on y or r .

Fix a constant $1 \leq M \leq 2 \operatorname{diam} f(B)/A$ such that

$$(3.7) \quad v \notin B(\omega, MA).$$

Since $\omega \in \partial f(B) \subset f(\partial B)$, there exists a point $\omega' \in \partial B$ such that $f(\omega') = \omega$. Thus, by (3.6) and the LLC-property of X , there exists a path

$$\gamma : [0, 1] \rightarrow (\Omega \setminus C^{-1}\overline{B}) \cup \{\omega'\}$$

such that $\gamma(0) = p$ and $\gamma(1) = \omega'$. We conclude that there exists a point $u \in f(|\gamma|)$ such that

$$(3.8) \quad u \in B(\omega, 3 \operatorname{diam} f(B)) \setminus (B(\omega, MA) \cup f(\overline{B})).$$

By (3.7), (3.8), and the LLC-property of Y , there exists a path

$$\alpha : [0, 1] \rightarrow Y \setminus B(\omega, C^{-1}MA)$$

such that $\alpha(0) = v \in f(B)$ and $\alpha(1) = u \notin \overline{f(B)}$. But then $M \leq C$, since otherwise $|\alpha| \cap \partial f(B) = \emptyset$, which contradicts the connectedness of α . We conclude that $v \in B(\omega, MA)$ for every $M \geq C$, i.e. $d(v, \omega) \leq 2C$. The proof is complete. \square

We end this section by pointing out the following modulus estimate, which is needed in the proof of [16, Lemma 6.19].

Lemma 3.4. *Let X be a proper, Ahlfors n -regular metric measure space supporting a $(1, n)$ -Poincaré inequality. Suppose that $0 < r < R$ and that $E, F \subset B(x, R) \subset X$ are disjoint compact sets such that*

$$E \cap S(x, s) \neq \emptyset, \quad F \cap S(x, s) \neq \emptyset$$

for every $s \in (r, R)$. Then there is a constant C_0 that depends on the data of X such that $R \geq C_0 r$ implies

$$(3.9) \quad \operatorname{Mod}_n(\Gamma) \geq C_0^{-1} \log \frac{R}{r},$$

where Γ is the family of curves in $B(x, R)$ joining E and F .

Proof. As pointed out earlier, by [13, Proposition 2.4] or [12, Proposition 2.17], one can use locally Lipschitz functions (in particular $N^{1,n}(B(x, R))$ -function) as admissible functions for the capacity. Let $u \in N^{1,n}(B(x, R))$ be an admissible function for Γ and g_u the minimal n -weak upper gradient of u . Then (u, g_u) satisfies the $(1, n)$ -Poincaré inequality. Moreover, by Proposition 2.8 and Lemma 2.6, we know that $B(x, R)$ is C -LLC. Hence [8, Corollary 7.3] implies that

$$(3.10) \quad |u(x) - u(y)| \leq Cr_0 \left(\int_{CB(x, r_0) \setminus C^{-1}B(x, r_0)} g_u^n d\mathcal{H}^n \right)^{1/n}$$

for all $x, y \in S(x, r_0)$ and $r \leq r_0 \leq R$. Thus, if we choose $x \in E$ and $y \in F$ on $S(x, r_0)$ in (3.10), then we obtain

$$(3.11) \quad \int_{CB(x, r_0) \setminus C^{-1}B(x, r_0)} g_u^n d\mathcal{H}^n \geq C_0.$$

Let $C_0 \geq C^2$. Then let A_i be the “annulus” defined by

$$A_i = \overline{B}(x, 2^i Cr) \setminus B(x, 2^{i-1} C^{-1} r), \quad i = 2, \dots, \lceil \log \frac{R}{Cr} \rceil.$$

Let Γ_i be the family of curves in A_i joining E and F . Then we obtain

$$\text{Mod}_n(\Gamma) \geq \sum_{i=2}^{\lceil \log \frac{R}{Cr} \rceil} \text{Mod}_n(\Gamma_i) \geq C_0 \left(\log \frac{R}{Cr} - 2 \right) \geq C' \log \frac{R}{r}$$

as desired. □

4. ANALYTIC PROPERTIES OF MAPPINGS OF FINITE DISTORTION

4.1. Condition N and area formula. In this subsection, we show that a mapping $f : \Omega \rightarrow Y$ of finite distortion satisfies *Condition N*, i.e. f maps sets of zero n -Hausdorff measure to sets of zero n -Hausdorff measure, and that the change of variable formula holds for f .

We begin with the following proposition, which shows the usefulness of pseudomonotonicity. It was proved in greater generality in [14, Theorem 7.2].

Proposition 4.1. *If $f : \Omega \rightarrow Y$ is a pseudomonotone map in $N_{loc}^{1,n}(\Omega, Y)$, then f satisfies Condition N.*

Since Condition N is a local property, we obtain the following corollary.

Corollary 4.2. *If $f : \Omega \rightarrow Y$ is mapping of finite distortion, then f satisfies Condition N.*

In order to prove (4.1) for mappings of finite distortion, we need the following properties of mappings in Newtonian spaces from [14, Proposition 4.6].

Lemma 4.3. *If $f : \Omega \rightarrow Y$ belongs to $N_{loc}^{1,p}(\Omega, Y)$ for some $1 \leq p < \infty$, then there exists a partition*

$$\Omega = \left(\bigcup_{k=1}^{\infty} A_k \right) \cup E, \quad E, A_1, A_2, \dots \text{ pairwise disjoint,}$$

such that $f|_{A_k}$ is k -Lipschitz for every $k \in \mathbb{N}$ and $\mathcal{H}^n(E) = 0$. In particular, if f satisfies Condition N and Ω is n -rectifiable with $n \in \mathbb{N}$, then $f(\Omega)$ is n -rectifiable.

Recall that a set $E \subset X$ is n -rectifiable if E has a countable union of Lipschitz images of the subset of \mathbb{R}^n plus a set of Hausdorff n -measure zero.

The following well-known change of variable formula (area formula) for Lipschitz mappings was proved in [28, Theorem 1].

Proposition 4.4. *Let $f : X \rightarrow Y$ be a Lipschitz mapping. Then the area formula*

$$(4.1) \quad \int_X u \circ f(x) J_f(x) d\mathcal{H}^n(x) = \int_Y u(y) N(y, f, X) d\mathcal{H}^n(y)$$

holds for each nonnegative measurable function u on Y .

Corollary 4.5. *Suppose that $f : \Omega \rightarrow Y$ is map in $N_{loc}^{1,p}(\Omega, Y)$ satisfying Condition N. Then (4.1) is valid. Moreover, if Ω is n -rectifiable with $n \in \mathbb{N}$, then $f(\Omega)$ is also n -rectifiable. In particular, the results hold for mappings of finite distortion.*

Proof. Note that (4.1) holds true for the restrictions $f|_{A_k}$ given in Lemma 4.3. Thus $\mathcal{H}^n(E) = \mathcal{H}^n(f(E)) = 0$, by Corollary 4.2 and Lemma 4.3. Therefore, (4.1) follows by applying the formula for each $f|_{A_k}$ and summing both sides over k . The second claim is clear from Corollary 4.2 and Lemma 4.3. \square

We end this subsection with the following simple fact.

Proposition 4.6. *Suppose that $f : \Omega \rightarrow Y$ is a mapping of finite distortion. Then*

$$\text{Lip} f(x) < \infty$$

for almost every $x \in \Omega$, where $\text{Lip} f$ is the pointwise (upper) Lipschitz constant function of f given by

$$(4.2) \quad \text{Lip} f(x) := \limsup_{r \rightarrow 0} \sup_{y \in B(x,r)} \frac{d(f(x), f(y))}{r}.$$

Proof. Note that the issue is local and the conclusion is bi-Lipschitz invariant, we may assume that Y is a subset of a Banach space. Then Lemma 3.3 and the proof of [14, Theorem 7.2] imply that there is a constant $\delta > 0$ such that for balls $B(x, r) \subset \Omega$ with $0 < r < \delta$ we have

$$(4.3) \quad \text{diam} f(B(x, r))^n \leq C \int_{B(x,r)} g_f^n(y) d\mathcal{H}^n(y),$$

where C is a constant depending only on the data of Ω and Y . The claim follows immediately from the Lebesgue differentiation theorem. \square

4.2. Differentiability of mappings of finite distortion. In this subsection, we consider differentiability result for mappings of finite distortion. We recall first the definition of strong measurable differentiable structure of a metric measure space.

Definition 4.7. Let (X, d, μ) be a metric measure space, let $\mathcal{C} \subset \text{LIP}(X)$ be a vector space of functions, and let $\{(X_\alpha, \varphi_\alpha)\}$ be a countable collection of pairs $(X_\alpha, \varphi_\alpha)$, called coordinate patches, such that each $X_\alpha \subset X$ is measurable with positive measure, and such that each

$$\varphi_\alpha = (\varphi_\alpha^1, \dots, \varphi_\alpha^N) : X \rightarrow \mathbb{R}^{N(\alpha)}$$

is a function for some $N(\alpha) \in \mathbb{N} \cup \{0\}$, where $\varphi_\alpha^i \in \mathcal{C}$ for every $1 \leq i \leq N(\alpha)$. Then $\{(X_\alpha, \varphi_\alpha)\}$ is said to be a *strong measurable differentiable structure* for (X, d, μ) with respect to \mathcal{C} if the following conditions are true.

i) The sets X_α are pairwise disjoint and

$$\mu \left(X \setminus \bigcup_{\alpha} X_\alpha \right) = 0.$$

- ii) There exists a number $N \geq 0$ such that $N(\alpha) \leq N$ for every coordinate patch $(X_\alpha, \varphi_\alpha)$.
 iii) For every $f \in \mathcal{C}$ and coordinate patch $(X_\alpha, \varphi_\alpha)$, there is a unique (up to a set of measure zero) measurable function $df^\alpha : X_\alpha \rightarrow \mathbb{R}^{N(\alpha)}$ such that for μ -almost every $x \in X_\alpha$,

$$(4.4) \quad \lim_{y \rightarrow x, y \neq x} \frac{|f(y) - f(x) - df^\alpha(x) \cdot (\varphi_\alpha(y) - \varphi_\alpha(x))|}{d(y, x)} = 0.$$

The follow well-known result is proved by Cheeger [4].

Proposition 4.8. *Let (X, d, μ) be a complete doubling metric measure space supporting a $(1, p)$ -Poincaré inequality for some $p \geq 1$. Then (X, d, μ) admits a strong measurable differentiable structure for $\text{Lip}(X)$.*

We are now able to define the notion of Cheeger’s differentiability.

Definition 4.9. Let $\{(X_\alpha, \varphi_\alpha)\}$ be a strong measurable differentiability structure on a metric measure space (X, d, μ) and V be a Banach space. Given a measurable subset S of X , a measurable mapping $f : S \rightarrow V$ is *approximately differentiable* at $x \in X_\alpha \cap S$ with respect to $(X_\alpha, \varphi_\alpha)$ if there exists a unique (up to set of measure zero) collection of measurable mappings $\{\frac{\partial f}{\partial \varphi_n^\alpha} : X_\alpha \cap S \rightarrow V\}$ such that

$$(4.5) \quad \text{ap} \lim_{y \rightarrow x} \frac{|f(y) - f(x) - \sum_{n=1}^{N(\alpha)} (\varphi_n^\alpha(y) - \varphi_n^\alpha(x)) \frac{\partial f}{\partial \varphi_n^\alpha}(x)|}{d(y, x)} = 0,$$

i.e. for every $\varepsilon > 0$ the set

$$(4.6) \quad A_{x,\varepsilon} = \left\{ y \in X : \frac{|f(y) - f(x) - \sum_{n=1}^{N(\alpha)} (\varphi_n^\alpha(y) - \varphi_n^\alpha(x)) \frac{\partial f}{\partial \varphi_n^\alpha}(x)|}{d(y, x)} < \varepsilon \right\}$$

has x as a density point. If we change the approximate limit in (4.5) to limit, then f is said to be differentiable at x with respect to $(X_\alpha, \varphi_\alpha)$. Note that in the formulation of (4.5), we have used the symbol $|\cdot|$ to represent the Banach norm of V .

One can easily prove that the approximate differential or differential is well-defined, in the sense that if there exists a collection of measurable mappings $\{\frac{\partial f}{\partial \varphi_n^\alpha} : X_\alpha \rightarrow V\}$ satisfying (4.5) then it is unique for almost all points $x \in X_\alpha$; see [6, Lemma 1.10].

Definition 4.10. A Banach space V is said to have the *Radon-Nikodym property* if every Lipschitz map $f : \mathbb{R} \rightarrow V$ is differentiable Lebesgue almost everywhere.

Cheeger and Kleiner [5] proved that if X is a doubling metric measure space that supports a p -Poincaré inequality for some $1 \leq p < \infty$, and $\{(X_\alpha, \varphi_\alpha)\}$ is a strong measurable differentiable structure on X , then every Lipschitz mapping from X to a Banach space with the Radon-Nikodym property is differentiable almost everywhere in X with respect to the strong measurable structure.

We next show that the notion of approximately differentiability or differentiability does not depend on the choice of the strong measurable differentiable structure if the target V has the Radon-Nikodym property. The proof is similar to that used in [6, Lemma 1.13].

Lemma 4.11. *Let (X, d, μ) be a metric measure space that admits a strong measurable differentiable structure $\{(X_\alpha, \varphi_\alpha)\}$ and V has the Radon-Nikodym property. If $f : X \rightarrow V$ is approximately differentiable (or differentiable) μ -a.e. with respect to $\{(X_\alpha, \varphi_\alpha)\}$, then it is approximately differentiable (or differentiable) μ -a.e. with respect to any strong differentiable structure defined on X .*

Proof. Let $\{(X_\alpha, \varphi_\alpha)\}$ and $\{(X_\beta, \varphi_\beta)\}$ be two strong measurable differentiable structures defined on (X, d, μ) . We write $\{\frac{\partial f}{\partial \varphi_n^\alpha}\}$ for the approximate differential of f with respect to $\{(X_\alpha, \varphi_\alpha)\}$ at $x \in X_\alpha$.

For fixed x , the Banach-valued map $g_x(\cdot) = \sum_{n=1}^{N(\alpha)} \frac{\partial f}{\partial \varphi_n^\alpha}(x) \varphi_n^\alpha(\cdot)$ is a Lipschitz map from X to V , thus it is approximately differentiable μ -a.e. with respect to $\{(X_\beta, \varphi_\beta)\}$. Moreover, the set of points where g_x is approximately differentiable does not depend on the choice of x .

For almost every $x \in X_\beta$ we can choose α such that $x \in X_\alpha$ and x is a point of approximately differentiability of f with respect to $\{(X_\alpha, \varphi_\alpha)\}$. Thus for μ -a.e. $x \in X_\beta$ we have

$$\begin{aligned} |f(y) - f(x) - \sum_{n=1}^{N(\beta)} (\varphi_n^\beta(y) - \varphi_n^\beta(x)) \frac{\partial g_x}{\partial \varphi_n^\beta}(x)| \\ \leq |f(y) - f(x) - \sum_{n=1}^{N(\alpha)} (\varphi_n^\alpha(y) - \varphi_n^\alpha(x)) \frac{\partial f}{\partial \varphi_n^\alpha}(x)| \\ + |g_x(y) - g_x(x) - \sum_{n=1}^{N(\beta)} (\varphi_n^\beta(y) - \varphi_n^\beta(x)) \frac{\partial g_x}{\partial \varphi_n^\beta}(x)|. \end{aligned}$$

It follows easily from the above inequality that x is a point of approximately differentiability of f with respect to $\{(X_\beta, \varphi_\beta)\}$. Hence we conclude that f is approximately differentiable μ -a.e. on X with respect to $\{(X_\beta, \varphi_\beta)\}$. \square

Next we generalize the main results of [6] to Banach-valued mappings.

Theorem 4.12. *Let (X, d, μ) be a complete doubling metric measure space that supports a $(1, p)$ -Poincaré inequality for some $1 \leq p < \infty$, and V be a Banach space with Radon-Nikodym property. Suppose that $E \subset X$ is measurable and $f : E \rightarrow V$ is measurable. Then the following conditions are equivalent:*

- (1) f is approximately differentiable μ -a.e. in E with respect to the strong measurable differentiable structure.
- (2) For any $\varepsilon > 0$ there is a closed set $F \subset E$ such that $\mu(E \setminus F) < \varepsilon$ and $f|_F$ is locally Lipschitz.
- (3) f induces a Lusin decomposition of E , that is,

$$(4.7) \quad E = \bigcup_{j=1}^{\infty} E_j \cup Z,$$

where E_j are measurable sets, $f|_{E_j}$ are Lipschitz mappings and Z has measure zero.

Proof. The proof is similar to that used in [6, Theorem 2.1]. Let $\{(X_\alpha, \varphi_\alpha)\}$ be a strong measurable differentiable structure defined on (X, d, μ) . Let f be approximately differentiable μ -a.e. in E . Let $|Df|_\alpha(x) = \sum_{n=1}^{N(\alpha)} |\frac{\partial f}{\partial \varphi_\alpha^n}(x)|$. Then $|Df|_\alpha(\cdot)$ is a measurable function defined on X (in prior $|Df|_\alpha(\cdot)$ is only defined on X_α , but one can extend it to be zero in $X \setminus X_\alpha$). Finally, we let $|Df|(x) = \sum_\alpha |Df|_\alpha(x)$. Clearly, $|Df|$ is also a measurable function defined on X .

“(1) \implies (2)”

First, assume that $E \subset X$ is bounded and define

$$D = \{x \in E : f \text{ is approximately differentiable at } x\}.$$

We want to show that for any $\varepsilon > 0$ there exists a closed set $F = F_\varepsilon \subset D$, $\delta > 0$ and $L > 0$ such that $\mu(D \setminus F) < \varepsilon$ and

$$|f(x) - f(y)| \leq Ld(x, y)$$

for each $x, y \in F$ such that $d(x, y) < \delta$.

Since μ is a doubling measure, for any $r > 0$, $x, y \in X$ such that $d(x, y) \leq r/2$,

$$(4.8) \quad \mu(B(x, r) \cap B(y, r)) \geq \mu(B(x, r/2)) \geq 2a\mu(B(x, r)),$$

where $a = \frac{1}{2C_\mu}$ and C_μ is the doubling constant.

For any $\eta > 0$ define the following sequence of functions:

$$\psi_i^n(x) = \mu(B(x, 1/i) \setminus A_{x, \eta}) \quad x \in D, \quad i \in \mathbb{N},$$

where $A_{x, \eta}$ is given by formula (4.6). It is easy to see that for each $\eta > 0$ and $x \in D$ one has

$$\phi_i^n(x) = \frac{\psi_i^n(x)}{\mu(B(x, 1/i))} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Next, set $\eta = 1$. By Luzin’s and Egoroff’s theorem there exists a closed set $F \subset E$ such that

- i:** $\mu(E \setminus F) = \mu(D \setminus F) < \varepsilon$,
- ii:** $|Df||_F$ is continuous, moreover, since X is proper, $|Df||_F$ is bounded in F , i.e. $|Df||_F \leq C$,
- iii:** $\phi_i^1 \rightarrow 0$ uniformly on F .

Choose i_0 such that

$$\phi_i^1(x) < \frac{a}{C_\mu}, \quad x \in F, \quad i \geq i_0.$$

Fix $x, y \in F$ such that $d(x, y) < \frac{1}{2i_0}$. Points x and y may belong to two different charts, we write $x \in X_\alpha$ and $y \in X_\beta$. Choose $i \geq i_0$ such that

$$\frac{1}{2(i+1)} < d(x, y) \leq \frac{1}{2i}.$$

For such i we have that $i \geq i_0$ and

$$\psi_i^1(y) < \frac{a}{C_\mu} \mu(B(y, 1/i)) \leq \frac{a}{C_\mu} \mu(B(x, 2/i)) \leq a\mu(B(x, 1/i)).$$

Hence

$$(4.9) \quad \psi_i^1(x) < a\mu(B(x, 1/i)) \text{ and } \psi_i^1(y) < a\mu(B(x, 1/i)).$$

Combining (4.8) and (4.9), we deduce that there exists a point $z \in B(x, 1/i) \cap B(y, 1/i)$ which does not belong to either $B(x, 1/i) \setminus A_{x,1}$ or $B(y, 1/i) \setminus A_{y,1}$. For such a point z , we have $d(x, z) < 1/i$, $d(y, z) < 1/i$,

$$(4.10) \quad \frac{|f(z) - f(x) - \sum_{n=1}^{N(\alpha)} (\varphi_n^\alpha(z) - \varphi_n^\alpha(x)) \frac{\partial f}{\partial \varphi_n^\alpha}(x)|}{d(z, x)} < 1$$

and

$$(4.11) \quad \frac{|f(z) - f(y) - \sum_{n=1}^{N(\beta)} (\varphi_n^\beta(z) - \varphi_n^\beta(y)) \frac{\partial f}{\partial \varphi_n^\beta}(y)|}{d(z, y)} < 1.$$

By combining (4.10), (4.11), $d(y, z) < 5d(x, y)$ and $d(x, z) < 5d(x, y)$ we obtain

$$\begin{aligned} |f(y) - f(x)| &\leq |f(z) - f(x) - \sum_{n=1}^{N(\alpha)} (\varphi_n^\alpha(z) - \varphi_n^\alpha(x)) \frac{\partial f}{\partial \varphi_n^\alpha}(x)| \\ &\quad + |f(z) - f(y) - \sum_{n=1}^{N(\beta)} (\varphi_n^\beta(z) - \varphi_n^\beta(y)) \frac{\partial f}{\partial \varphi_n^\beta}(y)| \\ &\quad + \left| \sum_{n=1}^{N(\alpha)} (\varphi_n^\alpha(z) - \varphi_n^\alpha(x)) \frac{\partial f}{\partial \varphi_n^\alpha}(x) \right| + \left| \sum_{n=1}^{N(\beta)} (\varphi_n^\beta(z) - \varphi_n^\beta(y)) \frac{\partial f}{\partial \varphi_n^\beta}(y) \right| \\ &\leq 10(1 + C)d(x, y), \end{aligned}$$

which shows that $f|_F$ is locally Lipschitz and finishes the proof of the implication for the case $E \subset X$ bounded.

The corresponding arguments in [6] allow us to proceed to the general case and thus the claim holds.

“(2) \implies (3)”

The same proof as in [6].

“(3) \implies (1)”

If decomposition (4.7) holds, then the restriction $f|_{E_i}$ is Lipschitz for every i . By [15, Theorem 3.1.25], we can extend $f|_{E_i}$ to a Lipschitz map \tilde{f}_i defined on the whole space X . Then by the results in [5], \tilde{f}_i is (approximately) differentiable μ -a.e. with respect to the strong measurable differentiable structure on X . Since every point of a measurable set E_i is its point of density, f is also approximately differentiable μ -a.e. on E_i with respect to the strong measurable differentiable structure on X . \square

Remark 4.13. It is easily seen from the proof that if $E \subset X$ is bounded, then condition (2) in Theorem 4.12 can be replaced by:

(2') For any $\varepsilon > 0$ there exists a closed set $F \subset E$ and a Lipschitz map $g : X \rightarrow V$ such that $\mu(E \setminus F) < \varepsilon$ and $f|_F = g$.

The following Stepanov-type theorem is essentially proved in [6, Theorem 2.3] (it is there stated for \mathbb{R} -valued functions, but the proof works for V -valued mappings as well).

Theorem 4.14. *Let (X, d, μ) be a complete doubling metric measure space and V be a Banach space. A measurable map $f : E \rightarrow V$ defined on a measurable subset*

$E \subset X$ satisfies the condition

$$\operatorname{ap} \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{d(y, x)} < \infty$$

for a.e. $x \in E$ if and only if for any $\varepsilon > 0$ there is a closed set $G \subset E$ such that $\mu(E \setminus G) < \varepsilon$ and f is locally Lipschitz in G .

The following lemma is a variant of Lemma 3.1.5 in [7].

Lemma 4.15. *Let (X, d, μ) be a complete doubling metric measure space that supports a $(1, p)$ -Poincaré inequality and V be a Banach space. Let $C \subset B(a, r_0) \subset X$, and $z \in C$ implies that $B(z, \eta) \subset B(a, r_0)$ and $f|_{B(z, \eta)}$ is M -Lipschitz. If $a \in C$, $X \setminus C$ has density zero at a , and f is approximately differentiable at a with respect to the strong measurable differentiable structure, then f is differentiable with respect to the strong measurable differentiable structure.*

Proof. Let $\{(X_\alpha, \varphi_\alpha)\}$ be the strong differentiable structure on X and $a \in X_\alpha$. Let $\{\frac{\partial f}{\partial \varphi_n^\alpha}(a)\}$ be the “approximate differential”. We write $L_n = \frac{\partial f}{\partial \varphi_n^\alpha}(a)$. Fix $0 < \varepsilon < 1$ and $0 < \delta < \eta$ such that

$$W = C \cap \left\{ z : |f(z) - f(a) - \sum_{n=1}^{N(\alpha)} (\varphi_n^\alpha(z) - \varphi_n^\alpha(a))L_n| \leq \varepsilon d(z, a) \right\}$$

and $\mu(B(a, r) \setminus W) < \varepsilon \mu(B(a, r))$ whenever $0 < r < \delta$.

For $x \in B(a, (1 - \varepsilon')\delta)$, we set $r = \frac{d(x, a)}{1 - \varepsilon'} < \delta$ and observe that

$$B(x, \varepsilon' r) \subset B(a, r), \quad B(x, \varepsilon' r) \cap W \neq \emptyset$$

if we choose ε' such that $\mu(B(x, \varepsilon' r)) \geq \varepsilon \mu(B(a, r))$. By the doubling condition of μ , we may choose $\varepsilon' = \varepsilon^c$, where c is a positive constant depending only on C_μ . Choose $z \in B(x, \varepsilon' r) \cap W$ and it follows that

$$\begin{aligned} & |f(x) - f(a) - \sum_{n=1}^{N(\alpha)} (\varphi_n^\alpha(x) - \varphi_n^\alpha(a))L_n| \\ & \leq |f(z) - f(a) - \sum_{n=1}^{N(\alpha)} (\varphi_n^\alpha(z) - \varphi_n^\alpha(a))L_n| \\ & \quad + |f(x) - f(z)| + \left| \sum_{n=1}^{N(\alpha)} (\varphi_n^\alpha(z) - \varphi_n^\alpha(x))L_n \right| \\ & \leq \varepsilon d(z, a) + M d(x, z) + \sum_{n=1}^{N(\alpha)} |L_n| \operatorname{Lip} \varphi_n^\alpha \cdot d(z, x) \\ & = d(x, a) o(\varepsilon). \quad \square \end{aligned}$$

As a corollary of Theorem 4.14 and Lemma 4.15, we formulate the following Stepanov-type result; see also [41, Theorem 4.1].

Theorem 4.16. *Let (X, d, μ) be a complete doubling metric measure space supporting a $(1, p)$ -Poincaré inequality for some $p \in [1, \infty)$, and let V be a Banach*

space with the Radon-Nikodym property. Then any measurable map $f : X \rightarrow V$ is differentiable at almost every point of the set

$$\left\{ x : \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{d(y, x)} < \infty \right\}$$

with respect to the strong measurable differentiable structure.

4.3. A general fact about the Jacobian for discrete and open mappings.

Let $f : \Omega \rightarrow Y$ be a discrete, open mapping of locally bounded multiplicity such that the area formula (4.1) holds.

We point out that the Jacobian $J_f(x)$ can be alternately described by

$$(4.12) \quad J_f(z) = \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(f(B(z, r)))}{\mathcal{H}^n(B(z, r))}$$

for a.e. $z \in \Omega$.

To see this, let $\Omega_n = \{x \in \Omega : N(f(x), f, \Omega) = n\}$ and $\nu_n = \mathcal{H}^n \llcorner_{f(\Omega_n)}$. Then it is clear that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Let $z \in \Omega$ be a Lebesgue point of f . Without loss of generality, we may assume $z \in \Omega_n$. Then

$$(4.13) \quad \begin{aligned} \lim_{r \rightarrow 0} \frac{f^* \nu_n(B(z, r))}{\mathcal{H}^n(B(z, r))} &= \lim_{r \rightarrow 0} \frac{\nu_n(f(B(z, r)))}{\mathcal{H}^n(B(z, r))} \\ &\leq \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(f(\Omega_n) \cap f(B(z, r)))}{\mathcal{H}^n(B(z, r))} \leq \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(f(B(z, r)))}{\mathcal{H}^n(B(z, r))}. \end{aligned}$$

On the other hand,

$$\mathcal{H}^n(f(B(z, r))) \leq f^* \mathcal{H}^n(B(z, r)) = \int_Y N(y, f, B(z, r)) d\mathcal{H}^n(y)$$

for each $r > 0$. Thus

$$(4.14) \quad \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(f(B(z, r)))}{\mathcal{H}^n(B(z, r))} \leq \lim_{r \rightarrow 0} \frac{f^* \mathcal{H}^n(B(z, r))}{\mathcal{H}^n(B(z, r))} = \lim_{r \rightarrow 0} \frac{f^* \nu_n(B(z, r))}{\mathcal{H}^n(B(z, r))}.$$

Combining (4.13) and (4.14) we obtain that

$$(4.15) \quad J_f(z) = \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(f(B(z, r)))}{\mathcal{H}^n(B(z, r))}$$

for all Lebesgue points $z \in \Omega$.

Remark 4.17. The different descriptions of the Jacobian were first pointed out by Williams [44]. The first description implies that the validity of area formula (in some sense) does not depend on the differentiable structure of the space. Meanwhile, the second description reveals that the validity of the analytic description (4.12) is independent of the size of the branch set B_f .

4.4. Condition N^{-1} and positiveness of the Jacobian. In this subsection, we show that $J_f(x)$ is positive for \mathcal{H}^n -a.e. $x \in \Omega$. The idea of the proof comes from [27]. In the following proposition, $n' = \frac{n}{n-1}$ is the conjugate exponent of n .

Proposition 4.18. *Let $f : \Omega \rightarrow Y$ be a nonconstant mapping of finite distortion and $K(\cdot) \in L_{loc}^{n'-1}(\Omega)$. Then $J_f(x) > 0$ for \mathcal{H}^n -a.e. $x \in \Omega$. Moreover, f satisfies Condition N^{-1} , i.e. for any set $E \subset \Omega$ we have*

$$\mathcal{H}^n(f(E)) = 0 \Rightarrow \mathcal{H}^n(E) = 0.$$

We need the following lemma to prove Proposition 4.18.

Lemma 4.19. *There is a constant $\tau = \tau(n)$ with the following property: For each atomless probability Borel measure μ on Y there is a point $y \in Y$ and a radius $R > 0$ such that*

$$\mu(B(y, 2R)) \geq \tau \quad \text{and} \quad \mu(Y \setminus B(y, 3R)) \geq \tau.$$

Proof. Since Y is Ahlfors n -regular, there is a constant L depending only on n such that any ball with radius $3r$ can be covered by L balls with radius r ; see [35, Lemma B.4]. Set

$$\tau = \frac{1}{L + 1}.$$

Let

$$\rho = \inf\{r > 0 : \text{there exists a point } y \text{ in } Y \text{ such that } \mu(B(y, r)) \geq \tau\}.$$

Since μ does not have atoms and $\mu(Y) < \infty$, it easily follows that $\rho > 0$. We find a point $y \in Y$ and a radius $R > 0$ such that

$$\mu(B(y, 2R)) \geq \tau \quad \text{and} \quad R < \rho < 2R.$$

Consider a collection B_1, \dots, B_L of balls with radii R such that they cover $B(y, 3R)$. Since $R < \rho$, we have

$$\mu(B_i) \leq \tau, \quad i = 1, \dots, L$$

and thus

$$\mu(B(y, 3R)) \leq \sum_{i=1}^L \mu(B_i) \leq L\tau,$$

which implies, since $\mu(Y) = 1$,

$$\mu(Y \setminus B(y, 3R)) \geq 1 - L\tau = \tau. \quad \square$$

Proof of Proposition 4.18. Since the issue is local, we may assume that f has bounded multiplicity in Ω , i.e.

$$N(f, \Omega) \leq N,$$

for some positive constants N . We divide the proof into four steps.

Step 1. We first prove an auxiliary estimate. Let $E \subset \Omega$ be measurable. Consider a Lipschitz function u with compact support in Y . A direct computation gives

$$\begin{aligned} (4.16) \quad \int_E g_{u \circ f} d\mathcal{H}^n &\leq \int_E g_u \circ f \cdot g_f d\mathcal{H}^n \leq \int_E g_u \circ f J_f(x)^{1/n} K^{1/n} d\mathcal{H}^n(x) \\ &\leq \left(\int_E K^{n'-1} d\mathcal{H}^n(x) \right)^{1/n'} \left(\int_E (g_u \circ f)^n J_f(x) d\mathcal{H}^n(x) \right)^{1/n} \\ &\leq \left(\int_E K^{n'-1} d\mathcal{H}^n(x) \right)^{1/n'} \left(N \int_Y g_u^n d\mathcal{H}^n \right)^{1/n}. \end{aligned}$$

Step 2. We claim that

$$(4.17) \quad y_0 \in Y \Rightarrow \mathcal{H}^n(f^{-1}(\{y_0\})) = 0.$$

For this, consider an arbitrary ball $B \subset\subset \Omega$ and $y_0 \in Y$. Suppose that f differs from y_0 on a set of positive measures in B . Then there is $R > 0$ such that

$$(4.18) \quad \kappa := \mathcal{H}^n(B \setminus f^{-1}(B(y_0, R))) > 0.$$

Since singletons have zero n -capacity, given $\varepsilon > 0$ there is a Lipschitz function u on Y such that

$$(4.19) \quad \text{spt } u \in B(y_0, R), \quad u(y_0) = 1 \quad \text{and} \quad \int_Y g_u^n(x) d\mathcal{H}^n(x) < \varepsilon^n.$$

Then

$$(4.20) \quad \min \left\{ \kappa, \mathcal{H}^n(B \cap f^{-1}(y_0)) \right\} \leq C \text{diam}(B) \int_B g_{u \circ f} d\mathcal{H}^n.$$

For this we used the well-known trick

$$(4.21) \quad \frac{1}{2} \min \left\{ \mathcal{H}^n(B \cap \{v \leq 0\}), \mathcal{H}^n(B \cap \{v \geq 1\}) \right\} \\ \leq \inf_{c \in \mathbb{R}} \int_B |v - c| d\mathcal{H}^n \leq C \text{diam}(B) \int_B g_v d\mathcal{H}^n,$$

based on the Poincaré inequality. By (4.16), (4.19) and (4.20) we have

$$\min \left\{ \kappa, \mathcal{H}^n(B \cap f^{-1}(y_0)) \right\} \\ \leq C \text{diam}(B) \left(\int_Y g_u^n d\mathcal{H}^n \right)^{1/n} \left(\int_B K^{n'-1} d\mathcal{H}^n \right)^{1/n'} \\ \leq C \text{diam}(B) \varepsilon \left(\int_B K^{n'-1} d\mathcal{H}^n \right)^{1/n'}.$$

Letting $\varepsilon \rightarrow 0$ and using (4.18) we obtain that $\mathcal{H}^n(B \cap f^{-1}(y_0)) = 0$ whenever f differs from y_0 on a set of positive measure in B . Hence (4.17) follows by taking the connectedness of Ω and the assumption that f is not constant into account.

Step 3. Now we prove that $J_f(x) > 0$ for \mathcal{H}^n -a.e. $x \in \Omega$. Denote by Z the zero set of the Jacobian. Fix a ball $B(x_0, r) \subset \Omega$. Consider the Borel measure defined by

$$\mu(A) = \frac{\mathcal{H}^n(B(x_0, r) \cap f^{-1}(A))}{\mathcal{H}^n(B(x_0, r))}, \quad A \subset Y.$$

By (4.17), μ does not have atoms. Using Lemma 4.19 we find a point $y \in Y$ and a radius $R > 0$ such that

$$(4.22) \quad \mu(B(y, 2R)) \geq \tau \quad \text{and} \quad \mu(Y \setminus B(y, 3R)) \geq \tau.$$

Choose a locally Lipschitz function u on Y such that

$$(4.23) \quad \text{spt } u \in B(y, 3R), \quad u = 1 \text{ in } B(y, 2R) \quad \text{and} \quad \int_Y g_u^n(x) d\mathcal{H}^n(x) < C(n).$$

Set

$$v = u \circ f.$$

Then by (4.22) and (4.23),

$$(4.24) \quad \frac{\mathcal{H}^n(B(x_0, r) \cap \{v = 1\})}{\mathcal{H}^n(B(x_0, r))} \\ \geq \frac{\mathcal{H}^n(B(x_0, r) \cap f^{-1}(B(y, 2R)))}{\mathcal{H}^n(B(x_0, r))} = \mu(B(y, 2R)) \geq \tau,$$

and

$$(4.25) \quad \frac{\mathcal{H}^n(B(x_0, r) \cap \{v = 0\})}{\mathcal{H}^n(B(x_0, r))} \geq \frac{\mathcal{H}^n(B(x_0, r) \setminus f^{-1}(B(y, 3R)))}{\mathcal{H}^n(B(x_0, r))} = \mu(Y \setminus B(y, 3R)) \geq \tau.$$

By the Poincaré inequality (4.21) we have

$$(4.26) \quad 1 \leq \frac{C}{\mathcal{H}^n(B(x_0, r))} \inf_{c \in \mathbb{R}} \int_{B(x_0, r)} |v - c| d\mathcal{H}^n \leq \frac{C \operatorname{diam}(B(x_0, r))}{\mathcal{H}^n(B(x_0, r))} \int_{B(x_0, r)} g_v d\mathcal{H}^n = \frac{C \operatorname{diam}(B(x_0, r))}{\mathcal{H}^n(B(x_0, r))} \int_{B(x_0, r)} g_{u \circ f} d\mathcal{H}^n.$$

Since f is a mapping of finite distortion, we have $g_f = 0$ and thus $g_{u \circ f} = 0$, \mathcal{H}^n -a.e. on Z . Hence by (4.26), (4.16) and (4.23),

$$1 \leq \frac{C \operatorname{diam}(B(x_0, r))}{\mathcal{H}^n(B(x_0, r))} \int_{B(x_0, r) \setminus Z} g_{u \circ f} d\mathcal{H}^n \leq C \left(\operatorname{diam}(B(x_0, r))^{-n} \int_{B(x_0, r) \setminus Z} K^{n'-1} d\mathcal{H}^n \right)^{1/n'}.$$

If x_0 is a Lebesgue point for $g := K^{n'-1} \chi_{\Omega \setminus Z}$, where $\chi_{\Omega \setminus Z}$ is the characteristic function of $\Omega \setminus Z$, it follows that $g(x_0) > 0$. This means that the set Z does not contain any Lebesgue points for g , and hence it must be of measure zero. Thus $J_f > 0$ \mathcal{H}^n -a.e. in Ω .

Step 4. We prove that f satisfies Condition N^{-1} , i.e. $\mathcal{H}^n(f(E)) = 0$ implies $\mathcal{H}^n(E) = 0$. Given $E \subset \Omega$ with $\mathcal{H}^n(f(E)) = 0$, we find a Borel measurable set $A \subset Y$ of measure zero which contains $f(E)$. Then E is contained in the measurable set $E' = f^{-1}(A)$. Let h be the characteristic function of A . By the area formula we have

$$\int_{E'} J_f(x) d\mathcal{H}^n(x) = \int_{\Omega} h(f(x)) J_f(x) d\mathcal{H}^n(x) \leq N \int_Y h(y) d\mathcal{H}^n(y) = 0.$$

Since $J_f > 0$ \mathcal{H}^n -a.e. in Ω , it follows that $\mathcal{H}^n(E) = 0$. \square

Remark 4.20. In the proof of Proposition 4.18, we have implicitly used the following fact: Let (f, g_f) be a function- minimal p -weak upper gradient pair and u a Lipschitz function. Then $(u \circ f, g_u \circ f \cdot g_f)$ is a function p -weak upper gradient pair, which can be found in [15].

5. GENERALIZED MANIFOLD

From now on, all spaces are assumed to be proper (and hence locally compact), Hausdorff, connected and locally connected. $H_c^*(X)$ denotes the Alexander-Spanier cohomology groups of a space X with compact supports and coefficients in \mathbb{Z} . From now on, $n \in \mathbb{N}$ will be a positive integer.

Definition 5.1. A space X is called an n -dimensional, $n \geq 2$, *cohomology manifold* (over \mathbb{Z}), or a *cohomology n -manifold* if

- (a) the cohomological dimension $\dim_{\mathbb{Z}} X$ is at most n , and

- (b) the local cohomology groups of X are equivalent to \mathbb{Z} in degree n and to zero in degree $n - 1$.

Condition (a) means that $H_c^p(U) = 0$ for all open $U \subset X$ and $p \geq n + 1$. Condition (b) means that for each point $x \in X$, and for each open neighborhood U of x , there is another open neighborhood V of x contained in U such that

$$H_c^p(V) = \begin{cases} \mathbb{Z} & \text{if } p = n, \\ 0 & \text{if } p = n - 1, \end{cases}$$

and the standard homomorphism

$$(5.1) \quad H_c^n(W) \rightarrow H_c^n(V)$$

is a surjection whenever W is an open neighborhood of x contained in V . As for examples of cohomology n -manifolds, we point out all topological n -manifolds are cohomology n -manifolds. More examples can be found in [16].

Definition 5.2. A space X is called a *generalized n -manifold*, $n \geq 2$, if it is a finite-dimensional cohomology n -manifold.

If a generalized n -manifold X satisfies $H_c^n(X) \simeq \mathbb{Z}$, then X is said to be orientable and a choice of a generator g_X in $H_c^n(X)$ is called an orientation; X together with g_X is an oriented generalized n -manifold. If X is oriented, we can simultaneously choose an orientation g_U for all connected open subsets U of X via the isomorphisms

$$H_c^n(U) \rightarrow H_c^n(X).$$

Let X and Y be oriented generalized n -manifolds, $\Omega \subset X$ be an oriented domain and let $f : \Omega \rightarrow Y$ be continuous. For each domain $D \subset\subset \Omega$ and for each component V of $Y \setminus f(\partial D)$, the map

$$f|_{f^{-1}(V) \cap D} : f^{-1}(V) \cap D \rightarrow V$$

is proper. Hence we have a sequence of maps

$$(5.2) \quad H_c^n(V) \rightarrow H_c^n(f^{-1}(V) \cap D) \rightarrow H_c^n(D),$$

where the first map is induced by f and the second map is the standard homomorphism. The composition of these two maps sends the generator g_V to an integer multiple of the generator g_D ; this integer, denoted by $\mu(y, f, D)$, is called the *local degree of f at a point $y \in V$ with respect to D* . The local degree is an integer-valued locally constant function

$$y \mapsto \mu(y, f, D)$$

defined in $Y \setminus f(\partial D)$. If $V \cap f(D) = \emptyset$, then $\mu(y, f, D) = 0$ for all $y \in V$.

From now on, we make the convention that whenever we write $f : \Omega \rightarrow Y$ is a continuous mapping between two generalized n -manifolds, it means that X and Y are two generalized n -manifolds and $\Omega \subset X$ is an oriented domain.

Definition 5.3. A continuous map $f : \Omega \rightarrow Y$ between two oriented generalized n -manifolds is said to be *sense-preserving* if

$$\mu(y, f, D) > 0$$

whenever $D \subset\subset \Omega$ is a domain and $y \in f(D) \setminus f(\partial D)$.

The following properties of the local degree can be found in [16].

Proposition 5.4 (Basic properties of the local degree). (a) If $f, g : \Omega \rightarrow Y$ are homotopic through proper maps h_t , $0 \leq t \leq 1$, such that $y \in Y \setminus h_t(\partial D)$ for all t , then

$$\mu(y, f, D) = \mu(y, g, D).$$

(b) If $y \in Y \setminus h_t(\partial D)$ and if $f^{-1}(y) \subset D_1 \cup \dots \cup D_p$, where D_i are all disjoint domains and contained in D such that $y \in Y \setminus f(\partial D_i)$, then

$$\mu(y, f, D) = \sum_{i=1}^p \mu(y, f, D_i).$$

(c) If $f : D \rightarrow f(D)$ is a homeomorphism, then $\mu(y, f, D) = \pm 1$ for each $y \in f(D)$. In particular, if f is a local homeomorphism, there is for each $x \in X$ a connected neighborhood D such that $\mu(f(x), f, D) = \pm 1$. More generally, if f is discrete and open and $x \in X$, then there is a relatively compact neighborhood D of x such that $\{f^{-1}(f(x))\} \cap \overline{D} = \{x\}$; the number $\mu(f(x), f, D) =: i(x, f)$ is independent of D and called the local index of f at x .

(d) If f is open, discrete, and sense-preserving, then for each $x \in X$ there is a connected neighborhood D as above such that $f(\partial D) = \partial f(D)$; D is called a normal neighborhood of x , and

$$(5.3) \quad i(x, f) = \max_{y \in f(D)} \text{card}\{f^{-1}(y) \cap D\}.$$

If D is any domain such that $f(\partial D) = \partial f(D)$, then D is called a normal domain.

Definition 5.5. Let $f : \Omega \rightarrow Y$ be a continuous mapping between oriented generalized n -manifolds. We say that f is a branched cover if f is both discrete and open.

The branch set B_f of $f : \Omega \rightarrow Y$ is the closed set of points in Ω where f does not define a local homeomorphism. It was proved by Väisälä in [40] that for a branched cover $f : \Omega \rightarrow Y$ the branch set B_f has topological dimension at most $n - 2$. It follows that $\Omega \setminus B_f$ is connected if $f : \Omega \rightarrow Y$ is a branched cover; see [3, I, Cor 4.7].

For a branched cover $f : \Omega \rightarrow Y$ between two oriented generalized n -manifolds, it is either sense-preserving or sense-reversing. The latter means that $\mu(y, f, D) < 0$ for each domain $D \subset\subset \Omega$ and all $y \in f(D) \setminus f(\partial D)$; see [39, Sec. 5.2].

6. SPACES OF TYPE A

As in [16], we isolate a class of generalized manifolds whose Lipschitz analysis is similar to that on Riemannian manifolds.

Definition 6.1. Let X be a metric space, and let $n \geq 2$ be an integer. Consider the following properties that X may or may not possess:

- (A1) X is n -rectifiable and has locally finite \mathcal{H}^n -measure;
- (A2) X is Ahlfors n -regular;
- (A3) X is locally bi-Lipschitz embeddable in Euclidean space;
- (A4) X is locally linearly contractible.

We call X a space of type A_1, A_2, \dots, A_{12} , if it satisfies axioms (A1), (A2), \dots , $A(1)$ and (A2), \dots . If X satisfies all four axioms (A1)–(A4), then we simply call X of type A . Note that the dimension n is fixed in the terminology.

As already mentioned before, condition (A1) means that X is a countable union of Lipschitz images of subset of \mathbb{R}^n plus a set of Hausdorff n -measure zero.

Condition (A2) means that there exists a constant $C > 0$ such that the Hausdorff n -measure in X satisfies

$$(6.1) \quad \frac{1}{C}r^n \leq \mathcal{H}^n(B(x, r)) \leq Cr^n$$

for all $x \in X$ and $0 < r \leq \text{diam}(X)$.

In the definition of (A3), the receiving Euclidean space is allowed to depend on the local embedding.

The last condition (A4) means that there exists a constant $C > 0$ such that every ball $B(x, r)$ in X with $0 < r \leq \frac{\text{diam}(X)}{C}$ is contractible in $B(x, Cr)$.

We point out, for example, that every Riemannian manifold is a space of type A and every Lipschitz n -manifold admits metrics that make the manifold into a space of type A . In particular, by Sullivan's theorem [38], every topological n -manifold, $n \neq 4$, admit type A metrics. However, spaces of type A need not be manifolds; see [36].

Remark 6.2. For a homology n -manifold X , the assumption X is n -rectifiable in (A1) is indeed implied by condition (A2), (A3) and (A4). In fact, by the work of Semmes [35], X admits a (1,1)-Poincaré inequality. Then it follows from [4, Theorem 14.2] that X is n -rectifiable; see also [11, Proposition 2.1].

As pointed out by Heinonen and Rickmann [16], spaces of type A will allow for differential calculus akin to that in Euclidean spaces. This follows from the work of Semmes [35] which shows that (locally) a space of type A admits a (1,1)-Poincaré type inequality in the sense of Section 2. Properties (A1) and (A2) guarantee that the usual Lipschitz calculus on rectifiable sets is at our disposal.

Assume that X is an oriented generalized n -manifold, and assume that X is of type A123. Let U be an open connected neighborhood of a point in X that can be embedded bi-Lipschitz in some \mathbb{R}^N and that U has finite Hausdorff n -measure. Because of properties (A1) and (A2), the set U has a tangent n -plane $T_x U$ at a.e. point $x \in U$. The collection of these planes is called a measurable tangent bundle of U , and it is denoted by TU .

Each n -plane $T_x U$, whenever it exists, is an n -dimensional subspace of \mathbb{R}^N , and a measurable choice of orientations $\xi = (\xi_x)$ on each $T_x U$ is called an orientation of the tangent bundle TU . Such orientations always exist. Because X is an oriented generalized n -manifold, there is another orientation on U , provided U is connected; this is a generator g_U in the group $H_c^n(U) = \mathbb{Z}$ determined by the fixed orientation on X .

Fix a point $x \in U$ such that $T_x U$ exists. Then the projection

$$\pi_x : \mathbb{R}^N \rightarrow T_x U + x$$

satisfies $x \notin \pi_x(\partial D)$ whenever D is a sufficiently small open connected neighborhood of x in U . To see it, consider the set

$$(6.2) \quad \Lambda = \left\{ y \in \mathbb{R}^N : \text{dist}(y - x, T_x U) > \varepsilon |y - x| \right\}.$$

We claim for sufficiently small open connected neighborhood D of x , it must hold that $\Lambda \cap D = \emptyset$ for each $\varepsilon > 0$. If not, the n -regularity condition (A2) implies that U has positive n -density along the set Λ , contradicting the definition for approximate tangent planes; see [7, Theorem 3.2.19]. Moreover, one can easily show that for any

$\varepsilon > 0$,

$$(6.3) \quad \frac{|x - x_0|}{2} \leq \pi_{x_0}(x - x_0) \leq \varepsilon|x - x_0|$$

whenever $|x - x_0|$ is small enough; see [7, Section 3.2.16]. Thus, if V is the x -component of $(T_x U + x) \setminus \pi_x(\partial D)$, we have

$$(6.4) \quad H_c^n(T_x U) \leftarrow H_c^n(V) \xrightarrow{\pi_x^*} H_c^n(\pi_x^{-1}(V) \cap D) \rightarrow H_c^n(D) \rightarrow H_c^n(U),$$

where the unnamed arrows represent a canonical isomorphism. If the orientations ξ_U and g_U correspond to each other under the map in (6.4), we say that $T_x U$ and U are *coherently oriented at x* by ξ_x and g_U ; if a measurable coherent orientation $\xi = (\xi_x)$ is chosen at almost every point, we say that TU is *metrically oriented* by ξ and g_U .

Proposition 6.3 ([17], Exam.3.10). *Every generalized n -manifold of type A can be (locally) metrically oriented.*

7. SIZE OF THE BRANCH SET

From now on, mappings of finite distortion are assumed to be sense-preserving. The fact that both B_f and $f(B_f)$ has measure zero is of great importance in the theory of mappings of finite distortion in Euclidean space. In this subsection, we show that, under suitable integrability condition for the distortion function for the map f , both B_f and $f(B_f)$ has zero n -Hausdorff measure for a mapping f of finite distortion between generalized n -manifolds of type A .

Before we get to the statement of the main result of this section, let us recall that a mapping f of an n -rectifiable subset Ω of \mathbb{R}^N into \mathbb{R}^M is *approximately differentiable* at $x_0 \in \Omega$ if x_0 has a neighborhood in Ω of finite \mathcal{H}^n -measure and if there exists $y_0 \in \mathbb{R}^M$ and a linear map $T : \mathbb{R}^N \rightarrow \mathbb{R}^M$ such that

$$\lim_{r \rightarrow 0} \mathcal{H}^n(\{x : |x - x_0| < r \text{ and } |f(x) - y_0 - T(x - x_0)| \geq \varepsilon|x - x_0|\})r^{-n} = 0$$

for each $\varepsilon > 0$. Equivalently, if there exist a neighborhood U of x_0 in \mathbb{R}^N and a mapping $g : U \rightarrow \mathbb{R}^M$, which is differentiable at a , such that

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}^n(\{x : f(x) \neq g(x)\} \cap B(x_0, r))}{r^n} = 0.$$

If a tangent n -plane $T_{x_0} \Omega$ exists at x_0 , the restriction of T to $T_{x_0} \Omega$ is the approximate differential of f at x_0 , and it is denoted by $\text{ap-}Df(x_0)$. The approximately differential $\text{ap-}Df(x_0)$ is unique; see [16, Section 6.5] or [24, Remark 2.13].

Theorem 7.1. *Let $f : \Omega \rightarrow Y$ be a mapping of finite distortion with $K(\cdot) \in L_{loc}^{n'-1}(\Omega)$. Then it holds that $\mathcal{H}^n(B_f) = \mathcal{H}^n(f(B_f)) = 0$.*

Proof. Since f satisfies condition N , it suffices to show that $\mathcal{H}^n(B_f) = 0$. Because the issue is local, and the conclusion is bi-Lipschitz invariant, we may assume that Ω and Y are metrically oriented subsets of some Euclidean space. In this framework, the claim follows from Proposition 7.2 and Proposition 7.3 below. \square

Proposition 7.2. *Let Ω and Y be metrically oriented subsets of some Euclidean spaces and $f : \Omega \rightarrow Y$ a mapping of finite distortion. Then for \mathcal{H}^n -a.e. $x_0 \in \Omega$,*

there exists a linear map $T : \mathbb{R}^N \rightarrow \mathbb{R}^M$ such that

$$(7.1) \quad \lim_{x \rightarrow x_0, x \in \Omega} \frac{|f(x) - f(x_0) - T(x - x_0)|}{|x - x_0|} = 0.$$

Proof. We first note that f is approximately differentiable \mathcal{H}^n -a.e. To see it, let $\Omega = \left(\bigcup_{k=1}^{\infty} A_k\right) \cup E$ be as in Lemma 4.3. Then $f|_{A_k}$ is k -Lipschitz. By [7, Theorem 3.2.19], $f|_{A_k}$ is approximately differentiable at \mathcal{H}^n -almost every point in E_k . Since every point in A_k is a point of \mathcal{H}^n density, we conclude that f is approximately differentiable \mathcal{H}^n -a.e. in Ω . Note also that if the approximately differentiable point is such that the tangent n -plane $T_{x_0}\Omega$ exists, then the restriction of T to $T_{x_0}\Omega$ is unique (by the Ahlfors n -regularity). On the other hand, Theorem 8.1 in [1] implies that (Ω, \mathbf{x}) is a strong measurable differentiable structure on $(\Omega, |\cdot|, \mathcal{H}^n)$, where $\mathbf{x} : \Omega \rightarrow \mathbb{R}^N$ is defined by $\mathbf{x}(x) = x$. Theorem 4.16 and Proposition 4.6 then allow us to complete the proof. \square

If (7.1) holds at $x_0 \in \Omega$ and the tangent n -plane $T_{x_0}\Omega$ exists, we denote by $Df(x_0)$ the restriction of T to $T_{x_0}\Omega$ and call it a *differentiable point* of f . In this sense, Proposition 7.2 implies that f is a differentiable at \mathcal{H}^n -a.e. points in Ω .

Next assume that Ω and Y are two metrically oriented generalized n -manifolds of type A lying in some Euclidean space. Denote the respective orientations by (ξ, g_Ω) and (η, g_Y) . Let OT_Ω be the set of all points in Ω such that $T_x\Omega$ exists, that $T_x\Omega$ and Ω are coherently oriented by ξ_x and g_Ω , and that Ω has Hausdorff n -measure density 1 at x ; see [7, Theorem 3.2.22]. The set OT_Ω has full measure in Ω . We denote OT_Y similarly.

If $f : \Omega \rightarrow Y$ is as in Theorem 7.1, it follows from Proposition 4.18 and Corollary 4.2 that the sets $f^{-1}(OT_Y) \cap OT_\Omega$ and $f(OT_\Omega) \cap OT_Y$ have full measure in Ω and $f(\Omega)$, respectively. We also denote by P_f the set of points in Ω where the Jacobian is positive. It follows from Proposition 4.18 that P_f have full measure in Ω .

Proposition 7.3. *Let $f : \Omega \rightarrow Y$ be as in Theorem 7.1, with an additional assumption that Ω and Y are metrically oriented subsets of some Euclidean space. Then f cannot be differentiable at any point in $B_f \cap f^{-1}(OT_Y) \cap OT_\Omega \cap P_f$.*

Proof. Let $x_0 \in f^{-1}(OT_Y) \cap OT_\Omega \cap P_f$ be a point where f is differentiable and $J_f(x_0) > 0$. Then the differential $Df(x_0)$ cannot be degenerate; that is, it has to map the tangent plane $T_{x_0}\Omega$ onto the tangent plane $T_{y_0}Y$, where $f(x_0) = y_0$; see [7, Theorem 3.2.22].

Moreover, we claim that for a sufficiently small neighborhood $D := B(x_0, r)$ of x_0 in Ω , the maps $\overline{Df}(x_0) \circ \pi_{x_0}$ and $\pi_{y_0} \circ f$ are homotopic as maps $D \rightarrow T_{y_0}Y + y_0$ and y_0 has a positive distance away from the image of ∂D under the homotopy, where π_* denotes the projection map, and $\overline{Df}(x_0)$ is the affine map between the planes $T_{x_0}\Omega + x_0$ and $T_{y_0}Y + y_0$ determined by the differential $Df(x_0)$. In fact, we may use the straight-line homotopy, i.e. $H_t(x) = (1-t)\overline{Df}(x_0) \circ \pi_{x_0}(x) + t\pi_{y_0} \circ f(x)$. Now we only need to show that $y_0 \notin H_t(\partial D)$ for all $t \in [0, 1]$. To this end, set

$$\lambda = \inf_{|x|=1} |\overline{Df}(x_0)x|.$$

Since $J_f(x_0) > 0$, $\lambda > 0$, it follows that

$$(7.2) \quad |\overline{Df}(x_0)\pi_{x_0}(x - x_0)| \geq \frac{\lambda}{2} |\pi_{x_0}(x - x_0)| \geq \frac{\lambda}{4} |x - x_0|,$$

when $|x - x_0|$ are small enough. Then for any $x \in \partial D$,

$$\begin{aligned} |H_t(x) - y_0| &= |(1-t)\overline{Df}(x_0) \circ \pi_{x_0}(x) + t\pi_{y_0} \circ f(x) - y_0| \\ &\geq |\overline{Df}(x_0) \circ \pi_{x_0}(x) - y_0| - t|\pi_{y_0} \circ f(x) - \overline{Df}(x_0) \circ \pi_{x_0}(x)| \\ &= |\overline{Df}(x_0) \circ \pi_{x_0}(x - x_0)| - t|\pi_{y_0} \circ f(x) - \overline{Df}(x_0) \circ \pi_{x_0}(x)| \\ &\geq \frac{\lambda}{4}|x - x_0| - t|\pi_{y_0} \circ f(x) - \overline{Df}(x_0) \circ \pi_{x_0}(x)|. \end{aligned}$$

We next estimate the second term,

$$\begin{aligned} |\pi_{y_0} \circ f(x) - \overline{Df}(x_0) \circ \pi_{x_0}(x)| &= |\pi_{y_0} \circ (f(x) - y_0) - \overline{Df}(x_0) \circ \pi_{x_0}(x - x_0)| \\ &\leq |\pi_{y_0}(f(x) - y_0) - (f(x) - y_0)| + |f(x) - f(x_0) - \overline{Df}(x_0)(x - x_0)| \\ &\quad + |\overline{Df}(x_0) \circ \pi_{x_0}(x - x_0) - \overline{Df}(x_0)(x - x_0)| := I_1 + I_2 + I_3. \end{aligned}$$

The differentiability of f at x_0 implies that

$$(7.3) \quad I_1 \leq |Df(x_0)|o(|x - x_0|)$$

and

$$(7.4) \quad I_2 \leq |x - x_0|o(|x - x_0|).$$

The property of π_* easily implies

$$(7.5) \quad I_3 \leq |Df(x_0)|o(|x - x_0|)$$

Combining the estimates (7.2) with (7.3), (7.4) and (7.5) gives

$$(7.6) \quad |H_t(x) - y_0| \geq \frac{\lambda}{5}|x - x_0|$$

when $|x - x_0|$ is sufficiently small. Thus $y_0 \notin H_t(\partial D)$ for all $t \in [0, 1]$ if D is sufficiently small.

By using this and the definition for coherent orientation, it is clear that f must have local index $i(x_0, f) = 1$ at x_0 . In particular, f is a local homeomorphism at x_0 , and the proposition follows. \square

Remark 7.4. i) The assumption that f is sense-preserving in Proposition 7.3 is crucial; see [16, Remark 6.7(a)].

ii) Theorem 7.1 is sharp in the following sense: for each $n \geq 3$ and $\varepsilon > 0$ there is a quasiregular map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that the Hausdorff dimension of B_f is at least $n - \varepsilon$; see [16, Remark 6.7(b)].

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