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ON POINCARÉ EXTENSIONS OF RATIONAL MAPS

CARLOS CABRERA, PETER MAKIENKO, AND GUILLERMO SIENRA

Abstract. There is a classical extension of Möbius automorphisms of the Riemann sphere into isometries of the hyperbolic space $\mathbb{H}^3$ which is called the Poincaré extension. In this paper, we construct extensions of rational maps on the Riemann sphere over endomorphisms of $\mathbb{H}^3$ exploiting the fact that any holomorphic covering between Riemann surfaces is Möbius for a suitable choice of coordinates. We show that these extensions define conformally natural homomorphisms on suitable subsemigroups of the semigroup of Blaschke maps. We extend the complex multiplication to a product in $\mathbb{H}^3$ that allows us to construct an extension of any given rational map which is right equivariant with respect to the action of $PSL(2,\mathbb{C})$.

1. Introduction

In the literature there are some constructions of extensions of rational dynamics from $\mathbb{C}$ to $\mathbb{H}^3$; see for example [9], [12] and [14]. The constructions in [12] and [14] are based on Choquet's barycentric construction introduced and studied by A. Douady and C. Earle in [5]. Other important contributions on the barycentric constructions appear in [1] and [4].

As mentioned in the abstract, the basic idea of this paper is the following fact: "*Any holomorphic covering between Riemann surfaces is a Möbius map on suitable coordinates.*" Then this covering can be extended to suitable Möbius manifolds. Let us discuss this idea in detail.

First, let us remind the reader that a Möbius n-orbifold is an n-orbifold endowed with an atlas such that the transition maps are Möbius transformations.

Given a discrete subgroup Γ of Möbius transformations of the n-sphere S^n acting properly discontinuous and freely on a domain $\Omega \subset S^n$, the quotient manifold Ω/Γ admits a Möbius structure. In the case when $n = 3$, any manifold modeled on one of the following spaces $\mathbb{R}^3, S^3$, the unit ball B^3 in $\mathbb{R}^3$, $S^2 \times \mathbb{R}$ or $B^2 \times \mathbb{R}$ admits a Möbius structure; see [16].

Let S_1 and S_2 be two Möbius 2-orbifolds and let $R : S_1 \to S_2$ be a finite degree covering which is Möbius on the respective Möbius structures. Assume that there exist two Kleinian groups Γ_1 and Γ_2 and two components W_1 and W_2 of the discontinuity sets $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$ respectively, such that

$$S_i = W_i/Stab_{W_i}(\Gamma_i)$$

for $i = 1, 2$. Now assume that there exists a Möbius map $\alpha(R) : W_1 \to W_2$ making

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the following diagram commutative so that $\alpha(R)$ induces a homomorphism from Γ_1 to Γ_2:

(1)
$$\begin{array}{ccc} W_1 & \xrightarrow{\alpha(R)} & W_2 \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ S_1 & \xrightarrow{R} & S_2. \end{array}$$

If

$$M_i = (\bar{\mathbb{H}}^3 \cup W_i)/\Gamma_i,$$

then $\alpha(R)$ induces a unique Möbius morphism

$$\tilde{R} : M_1 \to M_2$$

which is an extension of $R : S_1 \to S_2$. We call the map $\tilde{R} : M_1 \to M_2$ a *Poincaré extension* of R. The map $\tilde{R}$ depends on the uniformizing groups Γ_1 and Γ_2. Hence, in general, for a given covering map R there are many possibilities to construct a Poincaré extension. Note that the degree $deg(\tilde{R})$ is equal to the index $[\Gamma_2 : \alpha(R) \circ \Gamma_1 \circ \alpha(R)^{-1}]$. Hence $deg(R) \leq deg(\tilde{R})$ with equality when

$$Stab_{W_i}(\Gamma_i) = \Gamma_i.$$

Given a Riemann surface S with a fixed Möbius structure, in [6, sect.8] R. Kulkarni and U. Pinkal constructed a Möbius 3-manifold M such that the surface S is canonically contained in the boundary of M. If the structure of S is uniformizable by a nontrivial Kleinian group then, this construction is given by the classical Poincaré extension of the uniformizing group and produces a complete hyperbolic manifold M. The construction is based on the following idea: Let D be a round disk on S with respect to the Möbius structure; that is, there exists a coordinate under which D is a round disk in the plane. Using this coordinate, we attach a round half-ball in $\mathbb{H}^3$ to D. Then the 3-manifold M is the union of all the half open balls over all round disks in S.

On the Riemann sphere $\bar{\mathbb{C}}$ there is a unique complete Möbius structure σ_0, this is the standard Möbius structure on $\bar{\mathbb{C}}$. The construction of Kulkarni and Pinkal is clearer when S is a planar surface with the standard Möbius structure.

When $S = \mathbb{C}^*$ with the standard Möbius structure, Kulkarni-Pinkal construction gives a canonical noncomplete Möbius 3-manifold which is Möbius equivalent to the 3-dimensional ball with the vertical diameter removed and endowed with the standard conformal structure on B^3. Now, consider a complete Möbius structure on $\mathbb{C}^*$. In this case, Kulkarni-Pinkal construction gives a complete hyperbolic 3-manifold with the same underlying space as before. More generally, if $S = \bar{\mathbb{C}} \setminus F$, where F is a closed set, then the Kulkarni-Pinkal extension M is homeomorphic to $\mathbb{H}^3 \setminus convhull(F)$, where $convhull(F)$ is the hyperbolic convex hull in $\mathbb{H}^3$ of all points in F. The standard Möbius structure on M is the extension of the standard Möbius structure on S.

The construction of Kulkarni and Pinkal motivates the idea of a model manifold for the Poincaré extension of a rational map. However, if R is a branched self-covering of $\bar{\mathbb{C}}$ and $deg(R) > 1$, then R is not a Möbius covering with respect to the standard structure on any domain in $\bar{\mathbb{C}}$. We will restrict our attention to the case when R is a rational map and S_1 and S_2 are two Riemannian orbifolds with underlying spaces contained in $\bar{\mathbb{C}}$, and such that $R : S_1 \to S_2$ is a holomorphic covering. Let σ_2 be a uniformizable Möbius structure on S_2 and suppose that the pullback $\sigma_1 := R_*(\sigma_2)$ is also a uniformizable Möbius structure on S_1. If Γ_1 and Γ_2 are the uniformizing groups. Let $\tilde{R}$ be a Poincaré extension of R, such that $\Omega(\Gamma_i)$ are connected. Let $\phi_i : \partial M_i \to S_i$ be some identification maps and assume that there are homeomorphic extensions $\Phi_i : M_i \to \bar{\mathbb{H}}^3$ for each ϕ_i. Then the map $\Phi_2 \circ \tilde{R} \circ \Phi_1^{-1}$ is called *geometric extension* if and only if it satisfies the following conditions.

(1) The sets $\Phi_i(M_i \cup \partial M_i)$ are of the form $\bar{\mathbb{H}}^3 \setminus \{\bigcup \gamma_j\}$ where each γ_j is either a quasi-geodesic or a family of finitely many quasigeodesic rays with common starting point. There are no more than countably many curves γ_j. Here by quasigeodesic we mean the image of a hyperbolic geodesic by a quasiconformal automorphism.
(2) There exists a continuous extension, on all $\mathbb{H}^3$, which maps complementary quasi-geodesics to complementary quasigeodesics.

Hence, a geometric extension is an endomorphism of $\mathbb{H}^3$ such that its restriction to $\Phi_1(M_1)$ is a Poincaré extension.

Remark. In a Poincaré extension construction the domain and the range are, in principle, different, and hence we loose dynamics. This explains why we introduce the notion of geometric extension. So, we have an endomorphism of $\bar{\mathbb{H}}^3$ whose restriction to the boundary is the given rational map R.

A geometric extension is unique up to the bi-action of a group G of automorphisms of $\bar{\mathbb{H}}^3$ which acts as the identity on $\partial\mathbb{H}^3$. However, different geometric extensions may have different dynamical behavior.

Let $Rat_d(\mathbb{C})$ denote the set of rational maps R of degree d. Let $A \subset Rat_d(\mathbb{C})$. Assume that there exist a map

$$Ext : A \to End(\bar{\mathbb{H}}^3)$$

such that $Ext(R)$ is an extension of R for every R in A. Then for every pair of maps h, g in the Möbius group Mob we define

$$\widetilde{Ext}(g \circ R \circ h) = \hat{g} \circ Ext(R) \circ \hat{h}$$

where $\hat{h}$ and $\hat{g}$ are the classical Poincaré extensions of the maps h and g in the hyperbolic space, respectively.

If $\widetilde{Ext}$ is a well-defined map from the Möbius bi-orbit of A to $End(\bar{\mathbb{H}}^3)$, then we call Ext a *conformally natural extension* of A.

In particular, when the bi-action of $PSL(2,\mathbb{C})$ on A has no fixed points, any map $Ext : A \to End(\mathbb{H}^3)$ defines a map $\widetilde{Ext}$ on the Möbius bi-orbit which is a conformal natural extension. If the action has fixed points then, in order to obtain a conformally natural extension, the map Ext has to be consistent with the Möbius action. The situation is tricky even in the case when A consists of a single point R.

Let $\hat{R}$ be a geometric extension of a rational map R, then any rational map on the Möbius bi-orbit of R has a geometric extension. Namely, if h and g are elements in $PSL(2,\mathbb{C})$, then $Q = g \circ R \circ h$ has a geometric extension with the same uniformizing groups Γ_1 and Γ_2, the projections $p_1 = \pi_1 \circ \hat{h}$ and $p_2 = \pi_2 \circ \hat{g}^{-1}$ and the associated manifolds are $N_1 = \hat{h}^{-1}(M_1)$ and $N_2 = \hat{g}(M_2)$. We define an extension of Q by the formula $\hat{Q} = \hat{g} \circ \hat{R} \circ \hat{h}$.

However, one should be careful in the situation when, for a given R, there are other elements g' and h' in $PSL(2,\mathbb{C})$ such that $Q = g' \circ R \circ h'$. This situation happens when there are elements h_1 and h_2 in $PSL(2,\mathbb{C})$ such that

$$R \circ h_2 = h_1 \circ R. \qquad (*)$$

If there are no such h_1 and h_2, then any geometric extension of R is conformally natural. However, if such elements do exist but the Poincaré extensions of h_i are Möbius automorphisms of M_i with respect to their Möbius structures, then the extension $R \mapsto \hat{R}$ is conformally natural.

In this article, we will investigate the existence of extensions of R, defined in the hyperbolic space $\mathbb{H}^3$, satisfying as many as possible of the following desirable conditions.

(1) **Geometric.** As defined above.
(2) **Same degree.** The index $[\Gamma_2 : \Gamma_1]$ is equal to the degree of the map R.
(3) **Dynamical.** Let $\hat{R}$ be a Poincaré extension such that $M_1 = M_2$, then $\hat{R}$ is *dynamical*. In particular, these are extensions Ext such that $Ext(R^n) = Ext(R)^n$ for $n = 1, 2, \ldots$.
(4) **Semigroup homomorphisms.** A stronger version of the previous property is to find semigroups $\mathcal{S}$, of rational maps, for which there is an extension Ext defined in all $\mathcal{S}$ such that $Ext(R \circ Q) = Ext(R) \circ Ext(Q)$.
(5) **Equivariance under Möbius actions.** We look for conformally natural extensions of subsets A of $Rat_d(\mathbb{C})$.

Assume that R has a geometric extension and let Γ_2 be the group that uniformizes the Möbius structure on S_2. Then the discontinuity set $\Omega(\Gamma_2)$ consists of the orbit of a unique component C. The stabilizer of C uniformizes the surface S_2. In this case the group Γ_2 is either totally degenerated of Schottky type or of Web group type. When Γ_2 is a web group, the orbit of the component C is infinite. Thus, in general, it is possible that condition (2) may not be satisfied.

On the other hand, if the component C uniformizing the surface S_2 is invariant under Γ_2, then condition (2) is satisfied. For this reason, we restrict our discussion to this case. A group having an invariant component is called a function group. By Maskit's theorem any function group can be represented as a Klein-Maskit combination of the following groups:

- Totally degenerated groups.
- Groups of Schottky type.

According to this list of groups we call a geometric extension *totally degenerated*, or of *Schottky type*, whenever the uniformizing group has the corresponding property. Totally degenerated groups appear as geometric limits of quasifuchsian groups. In fact, totally degenerated groups belong to the boundary of a Bers slice.

This paper is organized as follows.

In Section 2 we will discuss Hurwitz spaces and quasifuchsian extensions. We will construct an extension, the radial extension, that satisfies (2), (3) and (4) in the previous list. We will give some conditions for which the radial extension is geometric.

Section 3 is devoted to Schottky type extensions.

In Section 4 we discuss the construction of a right equivariant extension, with respect to the action of $PSL(2,\mathbb{C})$, of all rational maps which is connected to a product structure defined on the hyperbolic space.

Finally, in Section 5, we discuss examples of extensions and surgeries of Maskit type.

2. Fuchsian structures, degenerated and radial extensions

Given a rational map R, let $CV(R)$ denote the critical values of R and take $S_2 = \bar{\mathbb{C}} \setminus CV(R)$ and $S_1 = R^{-1}(S_2)$, then $R : S_1 \to S_2$ is a covering. We assume that the set of critical values of R contains at least three points, say b_1, b_2 and b_3, so that S_2 is a hyperbolic Riemann surface. In order to get normalized maps we pick three points a_1, a_2 and a_3 in $\bar{\mathbb{C}}$ such that $R(a_i) = b_i$ for $i = 1, 2, 3$.

We say that two branched coverings R and Q, of the Riemann sphere onto itself, are *Hurwitz equivalent* if there are quasiconformal homeomorphisms ϕ and ψ, making the following diagram commutative:

$$\begin{array}{ccc} \bar{\mathbb{C}} & \xrightarrow{\psi} & \bar{\mathbb{C}} \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle Q} \\ \bar{\mathbb{C}} & \xrightarrow{\phi} & \bar{\mathbb{C}}. \end{array}$$

Given a rational map R, the Hurwitz space $H(R)$ is the set of all rational maps Q that are Hurwitz equivalent to R. The topology we are considering on $H(R)$ is the compact open topology.

Let $f : S_2 \to S_2'$ be a representative of a point in the Teichmüller space $T(S_2)$ with Beltrami coefficient μ and fixing the points b_i. Let $R_*(\mu)$ be the pullback of μ under R. Let h_f be the solution defined on the Riemann sphere of the Beltrami equation for the coefficient $R_*(\mu)$ and take $S_1' = h_f(S_1)$. Let us define the map $\tau : T(S_2) \to H(R)$ so that $\tau(f)$ is the rational map making the following diagram commutative:

$$\begin{array}{ccc} S_1 & \xrightarrow{h_f} & S_1' \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle \tau(f)} \\ S_2 & \xrightarrow{f} & S_2'. \end{array}$$

The map τ is well defined and continuous, since the solution of the Beltrami equation depends analytically on μ. We call the space $H_\tau(R) = \tau(T(S_2))$ the *reduced Hurwitz space* of R. The closure of the bi-orbit by the Möbius group of $H_\tau(R)$ is the whole of $H(R)$. The group $PSL(2, \mathbb{C})$ also acts by conjugation on $H(R)$. The space of orbits by conjugation fibers over the reduced Hurwitz space $H_\tau(R)$.

Let f be an element of the Mapping Class Group $MCG(S_2)$ such that $f(b_i) = b_i$. We say that f is *liftable* with respect to R if there exists a map $g : S_1 \to S_1$, such that $g(a_i) = a_i$ and makes the following diagram commutative:

$$\begin{array}{ccc} S_1 & \xrightarrow{g} & S_1 \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ S_2 & \xrightarrow{f} & S_2. \end{array}$$

In this case, we say that g is the lifted map of f with respect to R. Let $\mathcal{G}$ be the subgroup of the mapping class group $MCG(S_2)$ which consists of all liftable elements with respect to R. We identify $H_\tau(R)$ with the space $T_2(S_2)/\mathcal{G}$ as the following theorem suggests.

Theorem 1. *The space $H_\tau(R)$ is homeomorphic to $T(S_2)/\mathcal{G}$.*

Proof. The map τ induces a continuous map $\tilde{\tau}$ from $T(S_2)/\mathcal{G}$ onto $H_\tau(R)$. Now we will show that $\tau(\phi_1) = \tau(\phi_2)$ if and only if $\phi_2^{-1} \circ \phi_1 \in \mathcal{G}$.

Let $f = \phi_2^{-1} \circ \phi_1$, and assume that f belongs to $\mathcal{G}$ and let g be the lifted map of f with respect to R, so if h_{ϕ_1} and h_{ϕ_2} are the maps associated with $\tau(\phi_1)$ and $\tau(\phi_2)$ respectively, we have $h_{\phi_2} \circ g = h_{\phi_1}$ by the normalization of g; hence

$$\phi_2 \circ f \circ R = \phi_1 \circ R = \tau(\phi_1) \circ h_{\phi_1}.$$

On the other hand,

$$\phi_2 \circ f \circ R = \phi_2 \circ R \circ g = \tau(\phi_2) \circ h_{\phi_2} \circ g,$$

so we get

$$\tau(\phi_1) = \tau(\phi_2).$$

Reciprocally, if $\tau(\phi_1) = \tau(\phi_2)$, then $f = \phi_2^{-1} \circ \phi_1$ fixes the points b_i. Since $h_{\phi_2}^{-1} \circ h_{\phi_1}$ is a lift of f with respect to R, the map f belongs to $\mathcal{G}$.

Since $T(S_2)/\mathcal{G}$ is metrizable and of finite dimension, we can take a closed and bounded ball with respect to a metric D. Since $H_\tau(R)$ is a Hausdorff space then the restriction of $\tilde{\tau}$ on D is a homeomorphism. Hence $\tilde{\tau}$ itself is a homeomorphism. □

The following theorem gives a description of compact subsets of $H(R)$. For a quasiconformal map f, let $K_f(z)$ be the distortion of f at the point z.

Theorem 2. *If $\{f_i\}$ is a family of quasiconformal maps on the sphere $\bar{\mathbb{C}}$ fixing the points b_1, b_2 and b_3. Let*

$$A_n = \{z : K_{f_i}(z) \geq n \text{ for } i \text{ big enough}\}.$$

Assume that $A_\infty = \overline{\bigcap A_n}$ is a compact subset of S_2. So we have:

- *If A_∞ does not separate the critical values of R, then the family $\{[f_i]\}$ is bounded in $T(S_2)$.*
- *If there exist a domain D_0 contained in $S_2 \setminus A_\infty$ such that D_0 contains at least two of the points b_i and $W_0 = R^{-1}(D_0)$ is connected, then the respective classes $\{\tau([f_i])\}$ are bounded in $Rat_d(\mathbb{C})$.*

Proof. For the first item, let U be a neighborhood of A_∞ that is compactly contained in S_2, does not separate S_2 and such that every component of U is simply connected with analytic boundary.

The restrictions of f_i on ∂U are quasisymmetric maps with uniform bound of distortion. Using Douady-Earle extension, the maps $f_i|_{\partial U}$ extend to maps $\tilde{f}_i$, defined on the interior of U, with uniformly bounded distortion. Since f_i and $\tilde{f}_i$ have the same values on the boundary, the maps f_i and $\tilde{f}_i$ are homotopic, and define the same points in $T(S_2)$. Hence the family f_i have uniformly bounded Beltrami coefficients, so defines a bounded set in $T(S_2)$.

Since $Rat_d(\mathbb{C})$ can be identified with an open and dense subset of the projective space, it is enough to prove that all the limit maps of $\{\tau(f_i)\}$ are rational maps of the same degree. Let g_i be quasiconformal automorphisms of $\bar{\mathbb{C}}$ fixing the points a_i and such that $f_i \circ R = \tau(f_i) \circ g_i$. Under the conditions of the second item, the accumulation functions of the families $\{f_i|_D\}$ and $\{g_i|_{W_0}\}$ are nonconstant quasiconformal functions. Let R_∞ be a rational map which is an accumulation point of the maps $\tau(f_i)$, then there exist two nonconstant quasiconformal functions f_∞ and g_∞ and domains $O = h_\infty(D)$ and $X = g_\infty^{-1}(W_0)$ on which $f_\infty \circ R = R_\infty \circ g_\infty$. Then $deg(R_{\infty|O}) = deg(R_{|W_0}) = deg(R)$, as we wanted to prove. $\square$

2.1. **Bers slices.** Let $\Delta = \{z : |z| < 1\}$ and $\Delta^* = \{z : |z| > 1\}$. We denote the boundary of Δ by $\mathbb{S}^1$. Now let us consider again the rational map R and surfaces S_i. Let Γ_i be Fuchsian groups that uniformizes the surfaces S_i. By the Monodromy Theorem there exist a Möbius map α such that $\tilde{\Gamma} = \alpha\Gamma_1\alpha^{-1}$ is a subgroup of Γ_2. If $p : \Delta \to S_1 \cong \Delta/\tilde{\Gamma}$ is the orbit projection. Then defining $\pi_1 = p \circ \alpha^{-1}$ gives the following diagram:

$$\begin{array}{ccc} \Delta & \xrightarrow{Id} & \Delta \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ S_1 \cong \Delta/\Gamma_1 & \xrightarrow{R} & S_2 \cong \Delta/\Gamma_2. \end{array}$$

Moreover, $[\Gamma_2 : \Gamma_1] = deg(R)$ since R is a covering. We call the pair (Γ_1, Γ_2) a uniformization of R. Simultaneously we have uniformization of the surfaces $S_i^* = \Delta^*/\Gamma_i$ and the map $Q : S_1^* \to S_2^*$ given by $Q(z) := \overline{R(\bar{z})}$. Now these groups are acting on the complement of the unit disk Δ^*.

Let $D(\Gamma_2)$ be the space of groups Γ such that there exist a quasiconformal map f such that $\Gamma = f \circ \Gamma_2 \circ f^{-1}$ and such that the Beltrami differential μ_f is equal to 0 in Δ^*. Now put

$$Def(\Gamma_2) = D(\Gamma_2)/PSL(2, \mathbb{C}).$$

Analogously, we define $Def^*(\Gamma_2)$ as the space of deformations of Γ_2 on Δ^*. By Bers Theorem, both spaces $Def(\Gamma_2)$ and $Def^*(\Gamma_2)$ have compact closure on the space of classes of faithful and discrete representations

$$\Gamma_2 \hookrightarrow PSL(2, \mathbb{C}).$$

These closures of $Def(\Gamma_2)$ and $Def^*(\Gamma_2)$ are called Bers slices of the Teichmüller space. In these cases, the Bers slices consist of function groups with a simply connected invariant component. Geometrically finite groups contained in the Bers slice are either quasifuchsian or cusps. Definitions and properties can be found in the papers by Bers [3], by Maskit [10] and by McMullen [11]. If a group G in the boundary of the Bers slice has a connected region of discontinuity, then G is totally degenerated. By theorems of Bers, Maskit and McMullen (see [3], [10] and [11]), totally degenerated groups and cusps are both dense on the boundary of the Bers slice.

Any group G in the Bers slice defines a 3-hyperbolic manifold $M(G)$ with boundary. Given a uniformization (Γ_1, Γ_2) of R, let us consider $M_1 := M(\Gamma_1)$ and $M_2 := M(\Gamma_2)$ the associated 3-hyperbolic manifolds with boundary. Then the inclusion Γ_1 in Γ_2 defines a Möbius map

$$F : M_1 \to M_2.$$

The restriction of F to the boundary components of M_1 define maps which are rational in coordinates. Let $\Sigma_i \subset \partial M_i$ be the respective invariant components of Γ_i. Then the map $F : \Sigma_1 \to \Sigma_2$ belongs either to the Hurwitz space $H(R)$ or to $H(Q)$.

If a group G in the Bers slice $Def(\Gamma_2)$ is geometrically finite, then G is a cusp or quasifuchsian. In those situations there is a boundary component S in $M(G)$ conformally equivalent to Δ^*/Γ_2. In the case that there are cusps on the group, we regard the set of all components as a connected surface with nodes $\tilde{S}$.

Question. Assume that G_i converges, in the Bers slice, to a totally degenerated group G. Is it true that the accumulation set of the associated rational maps may contain constants maps?

Now we are ready to prove the main result of this section. Let us begin with the following definition. Let G be a totally degenerated group. Then we call the group G *acceptable* for the rational map R if and only if the following conditions hold:

- There are two uniformizable Möbius orbifolds S_i supported on the Riemann sphere, such that $R : S_1 \to S_2$ is a holomorphic covering.
- If Γ_2 is a Fuchsian group uniformizing S_2, then G belongs to $Def^*(\Gamma_2)$.
- The manifold $M(G)$ is homeomorphic to $\partial M(G) \times \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of nonnegative real numbers.

Let $\pi : \mathbb{H}^3 \cup \Omega(G) \to M(G)$ be the orbit projection. Under the homeomorphism of the last item, let us define, for every t in $\mathbb{R}_+$ the set $\Omega(G)_t = \pi^{-1}(\partial M \times t)$. Hence, the space $\mathbb{H}^3 \cup \Omega(G)$ is foliated by the sets $\Omega(G)_t$ and there exists a continuous family of homeomorphisms $f_t : \Omega(G) \to \Omega(G)_t$ which commutes with G and $f_0 = Id$.

Let $\mathbb{B}^3$ denote the unit ball model for the hyperbolic space. Given a rational map R, we define the radial extension $\tilde{R}$ as follows. For every $\lambda \in [0, 1]$ and $(x, y, z) \in \mathbb{R}^3$, let $H_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda z)$. Then we have

$$\bar{\mathbb{B}}^3 = \bigcup_{\lambda \in [0,1]} H_\lambda(\partial \mathbb{B}^3).$$

Now define $\hat{R}(0, 0, 0) = (0, 0, 0)$ and for $v \in \bar{\mathbb{B}}^3$, different from 0, define $\hat{R}(v) = H_{\|v\|} \circ R \circ H^{-1}_{\|v\|}$ where $\|v\|$ denotes the euclidean distance to v from the origin.

Theorem 3. *If there exist an acceptable group for R then the radial extension of R is geometric.*

Proof. Let G_2 be an acceptable group for R, then there exists $\phi : \Omega(G_2) \to \Delta$ such that it induces an isomorphism $\phi_* : G_2 \to \Gamma_2$, then G_2 has a finite index subgroup $G_1 = \phi_*^{-1}(\Gamma_1)$, such that the map $\alpha(R) : M(G_1) \to M(G_2)$ is Möbius. Moreover, the manifold $M(G_1)$ is a manifold homeomorphic to $\partial M(G_1) \times \mathbb{R}_+$. We have to show that the radial extension is equivalent to $\alpha(R)$, so it is geometric. Again, each $M(G_i)$ is homeomorphic to $S_i \times \mathbb{R}_+$, and the horizontal foliation in $M(G_1)$ is the pullback by $F : M(G_1) \to M(G_2)$ of the horizontal foliation in $M(G_2)$. Hence, there exist a covering $\phi : S_1 \times \mathbb{R}_+ \to S_2 \times \mathbb{R}_+$ and homeomorphisms h_1, h_2 such that the following diagram commutes:

$$\begin{array}{ccc} M(G_1) & \xrightarrow{F} & M(G_2) \\ \downarrow h_1 & & \downarrow h_2 \\ S_1 \times \mathbb{R}_+ & \xrightarrow{\phi} & S_2 \times \mathbb{R}_+. \end{array}$$

Hence, $\phi(x) = F(x) = R(x)$ for x in S_1. There are two families of homeomorphisms ψ_t and χ_t such that

$$\begin{array}{ccc} S_1 \times \{t\} & \xrightarrow{\phi} & S_2 \times \{t\} \\ \uparrow \psi_t & & \chi_t \uparrow \\ S_1 \times \{0\} & \xrightarrow{\phi} & S_2 \times \{0\}, \end{array}$$

so that ϕ preserves the parameter t. Now consider a homeomorphism $k : [0, \infty] \to [0, 1]$ such that $k(0) = 1$ and $k(\infty) = 0$. For $i = 0, 1$, let us identify the sets $S_i \times 0$ with the corresponding S_i on the Riemann sphere. Hence, we define two homeomorphisms ψ and χ such that $\psi(x, t) = H_{k(t)}(\psi_t^{-1}(x))$ and $\chi(x, t) = H_{k(t)}(\chi_t^{-1}(x))$, where $H_t(x, y, z) = (tx, ty, tz)$. Since ψ and χ are the identity on the boundary, these homeomorphisms uniformize the extension of ϕ over S_1 and S_2. □

We have not found a reference that shows that the manifold of a totally degenerated group in the Bers slice of a finitely generated group is always a product. However, Michael Kapovich kindly gave us arguments to show that this happens. The arguments are based upon a work of Waldhausen and the solution of the Tame Conjecture.

3. Schottky type extensions of rational maps

In this section, we prove that any map in the Hurwitz space of a Blaschke map has an extension of Schottky type that satisfies the properties (1), (2) and (3) in the introduction. We also construct an extension of Blaschke maps that satisfies almost all five conditions: this extension does not satisfy condition (4). However, if we take the group of Möbius transformations preserving the unit circle instead of $PSL(2, \mathbb{C})$, then the modified condition (4) holds.

A *Blaschke map* $B : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a rational map that leaves the unit disk Δ invariant. If d is the degree of B, then there exist $\theta \in [0, 2\pi]$ and d points $\{a_1, \dots, a_d\}$ in Δ such that

$$B(z) = e^{i\theta} \left(\frac{z - a_1}{1 - \bar{a}_1 z} \right) \dots \left(\frac{z - a_d}{1 - \bar{a}_d z} \right).$$

Let us denote by $B_1 = B|_\Delta$ and $B_2 = B|_{\Delta^*}$, then $B_2(z) = \frac{1}{\bar{z}} \circ B_1(z) \circ \frac{1}{\bar{z}}$.

Let $CV(B)$ be the set of critical values of B, then define $S_2 = \bar{\mathbb{C}} \setminus CV(B)$ and $S_1 = B^{-1}(S_2)$. Thus $B : S_1 \to S_2$ is a holomorphic covering and the surfaces S_i are symmetric with respect to $\mathbb{S}^1$.

The class of Blaschke maps allows us to build a specific topological construction based upon Schottky coverings of Riemann surfaces. What is special about Blaschke maps is that every Blaschke map commutes with the involution $\tau(z) = \frac{1}{\bar{z}}$. Recall that Fuchsian groups of second type are Schottky type Fuchsian groups.

Theorem 4. *Given a Blaschke map B, such that $B : S_1 \to S_2$ is a covering and the surfaces S_i are symmetric surfaces with respect to $\mathbb{S}^1$. There are two Fuchsian groups of second type, Γ_1 and Γ_2 such that $\Omega(\Gamma_i)/\Gamma_i = S_i$, where $\Omega(\Gamma_i)$ is the* *discontinuity set of* Γ_i *for* $i = 1, 2$. *Furthermore, there exist a Möbius map* $\alpha : \Omega(\Gamma_1) \to \Omega(\Gamma_2)$ *making the following diagram commutative:*

$$\begin{array}{ccc} \Omega(\Gamma_1) & \xrightarrow{\alpha} & \Omega(\Gamma_2) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ S_1 & \xrightarrow{B} & S_2. \end{array}$$

Also $[\Gamma_2 : \alpha\Gamma_1\alpha^{-1}] = deg(B)$.

Proof. For $i = 1, 2$ let $\Delta_i = \bar{\Delta} \cap S_i$ and $\Delta_i^* = \bar{\Delta}^* \cap S_i$.

The Simultaneous Uniformization Theorem of Bers [2], ensures that there exist a Fuchsian group Γ_2 acting in $\bar{\mathbb{C}}$, with $\Delta/\Gamma_2 = \Delta_2$ and $\Delta^*/\Gamma_2 = \Delta_2^*$. The limit set $\Lambda(\Gamma_2)$ is contained in $\mathbb{S}^1$.

Similarly, there is a Fuchsian group Γ_1 such that $\Delta^*/\Gamma_1 = \Delta_1^*$ and $\Delta/\Gamma_1 = \Delta_1$ and $\Lambda(\Gamma_1) \subset \mathbb{S}^1$.

The map B_i lifts to Möbius maps $\alpha_1 : \Delta \to \Delta$ and $\alpha_2 : \Delta^* \to \Delta^*$. Moreover, since Ω_2 is a Riemann surface anti-conformally equivalent to Ω_1, we can choose α_2 such that $\alpha_2(z) = \frac{1}{\bar{z}} \circ \alpha_1 \circ \frac{1}{\bar{z}}$ and these maps agree at $\mathbb{S}^1 \setminus \Lambda(G)$. Being that $\Lambda(G)$ is a Cantor set, then the map $\alpha : \bar{\mathbb{C}} \setminus \Lambda(G) \to \bar{\mathbb{C}} \setminus \Lambda(G)$ defined as $\alpha|\Delta_i = \alpha_i$ extends to a Möbius map α defined on the Riemann sphere. □

By Theorem 4, a Blaschke map admits a Poincaré extension which follows from the diagram below:

$$\begin{array}{ccc} \mathbb{B}^3 & \xrightarrow{\alpha} & \mathbb{B}^3 \\ \downarrow & & \downarrow \\ \mathbb{B}^3/\Gamma_1 & \xrightarrow{\hat{B}} & \mathbb{B}^3/\Gamma_2. \end{array}$$

Let us observe that Γ_1 and Γ_2 are Schottky type groups with parabolic generators. Hence $\mathbb{B}^3/\Gamma_1$ and $\mathbb{B}^3/\Gamma_2$ are homeomorphic to the complement in $\mathbb{B}^3$ of a finite number of geodesics connecting symmetric perforations of the surfaces S_i.

The arguments in Theorem 4 work in a more general situation. The key facts are Bers Simultaneous Uniformization Theorem and the symmetry of respective orbifolds. So we have the following.

Corollary 5. *Let* W_1 *and* W_2 *be any two given connected symmetric orbifolds supported on the Riemann sphere. If* $B : W_1 \to W_2$ *is a covering symmetric with respect to* $\mathbb{S}^1$, *then the conclusion of Theorem* 4 *holds for* B.

Let $G\mathcal{O}(P(B))$ be the grand orbit of the postcritical set $P(B)$ and take $S = \bar{\mathbb{C}} \setminus \overline{G\mathcal{O}(P(B))}$. When S is a symmetric surface such that $D = S \cap \Delta$ and $D^* = S \cap \Delta^*$ are connected, then $B : S \to S$ is a holomorphic self-covering, and we have the following corollary.

Corollary 6. *There exist a Fuchsian group* Γ *and* α *in* $PSL(2,\mathbb{R})$ *such that* $\alpha_*(\Gamma) = \alpha\Gamma\alpha^{-1}$ *is a subgroup of* Γ. *Moreover, we have that* $\Omega(\Gamma) = \Delta \cup \Delta^*$, $\Delta^*/\Gamma = D^*$, *and* $\Delta/\Gamma = D$. *Finally, for every* n *the following diagram is commutative:*

$$\begin{array}{ccc} \Omega(\Gamma) & \xrightarrow{\alpha^n} & \Omega(\Gamma) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\ S & \xrightarrow{B^n} & S. \end{array}$$

Proof. The proof is essentially the same as in Theorem 4 using the symmetry of the surfaces plus the fact that B defines a self-covering of the surface S. □

As noted after Theorem 4, we have that the Poincaré extension of B is an endomorphism of $\mathbb{B}^3/\Gamma$, so this Poincaré extension is dynamical.

Remark. By the classification of periodic Fatou components, D and D^* are disconnected if and only if the postcritical set is infinite and there exist fixed critical points in D and D^*, respectively. In other words, D and D^* are connected for an open and everywhere dense set in the space of Blaschke maps of any given degree. In particular, D and D^* are connected for every Blaschke map of degree 2.

Again, the argument in the corollary above can be generalized as in the following corollary:

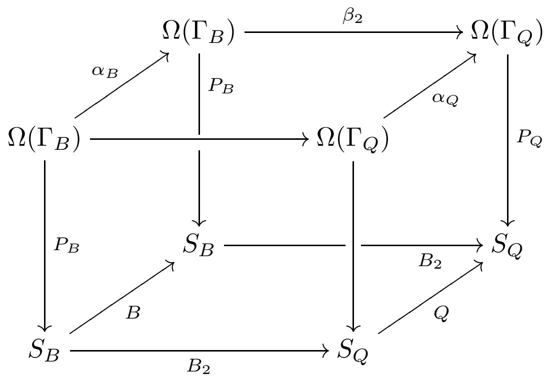
Corollary 7. *Given a Blaschke map* B, *let* A *be a completely invariant symmetric closed subset of* $\bar{\mathbb{C}}$ *which contains all critical points of* B. *Let* $W = \bar{\mathbb{C}} \setminus A$ *and assume that* $U = W \cap \Delta$ *and* $U^* = W \cap \Delta^*$ *are connected. Then* $\{B^n : U^* \to U^*\}$ *and* $\{B^n : U \to U\}$ *are semigroups of coverings and the Poincaré extensions of these semigroups are dynamical.*

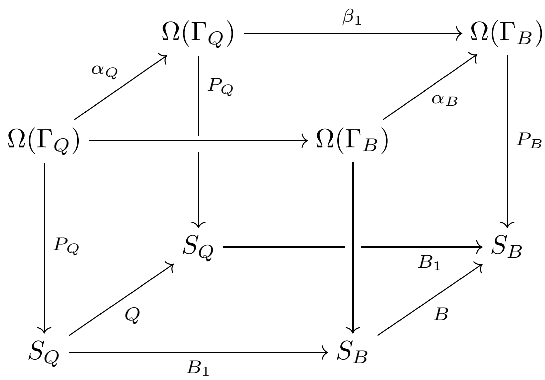
Now let us consider the case of decomposable Blaschke maps. Assume that $B = B_1 \circ B_2$ where B_1 and B_2 are Blaschke maps. Then, if $Q = B_2 \circ B_1$, there are semiconjugacies:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{B=B_1\circ B_2} & \mathbb{C} \\ {\scriptstyle B_2}\downarrow & & \downarrow{\scriptstyle B_2} \\ \mathbb{C} & \xrightarrow{Q=B_2\circ B_1} & \mathbb{C} \\ {\scriptstyle B_1}\downarrow & & \downarrow{\scriptstyle B_1} \\ \mathbb{C} & \xrightarrow{B} & \mathbb{C}. \end{array}$$

Corollary 8. *Let* $\hat{B}$ *be the dynamical extension of* $B = B_1 \circ B_2$ *constructed in Corollary* 6, *then* $Q = B_2 \circ B_1$ *has a dynamical extension such that there exist two Fuchsian groups* $\Gamma(B)$ *and* $\Gamma(Q)$ *and elements* α_B, α_Q, β_1, *and* β_2 *in* $PSL(2,\mathbb{R})$

making the following diagrams commutative:





Proof. Note that $B_2 : S_B \to S_Q$ and $B_1 : S_Q \to S_B$ define coverings, hence there are Möbius maps $\beta_1 : \Omega(\Gamma_Q) \to \Omega(\Gamma_B)$ and $\beta_2 : \Omega(\Gamma_B) \to \Omega(\Gamma_Q)$ which make the diagrams commutative. □

As a consequence of Corollary 8 we have the following conclusion.

Proposition 9. *Let $B = B_1 \circ B_2$, let $\hat{B}$ be a dynamical extension, then there are Poincaré extensions $\hat{B}_1$ and $\hat{B}_2$ of B_1 and B_2, respectively, such that*

$$\hat{B} = \hat{B}_1 \circ \hat{B}_2.$$

Moreover, there exist $\hat{Q}$ a dynamical extension of $Q := B_2 \circ B_1$ such that

$$\hat{Q} = \hat{B}_2 \circ \hat{B}_1.$$

The following theorem summarizes the results above and shows that the corresponding extensions are geometric.

Theorem 10. *Let B be a Blaschke map, then:*

(i) *The extension constructed in Theorem 4 is geometric.*
(ii) *The dynamical extension constructed in Corollary 6 is geometric.*
(iii) *The extensions in Corollary 8 are all geometric.*

Each of the extensions in items (i)–(iii) is conformally natural with respect to the group of Möbius transformations that leaves the unit circle invariant.

Proof. We will show item (i), the proof of the other items apply similar arguments to the extensions constructed in Corollary 6 and Proposition 9. Again, the important feature is that the corresponding surfaces are symmetric with respect to the unit circle. According to Theorem 4 there are manifolds M_1 and M_2, Möbius projections p_1 and p_2, and a Poincaré extension $\hat{B}$ of B such that the following diagram is commutative such that $p_i|_{\Omega(\Gamma_i)} = \pi_i$:

$$\begin{array}{ccc} \mathbb{H}^3 \cup \Omega(\Gamma_1) & \xrightarrow{Id} & \mathbb{H}^3 \cup \Omega(\Gamma_2) \\ \downarrow p_1 & & \downarrow p_2 \\ M_1 & \xrightarrow{\hat{B}} & M_2 . \end{array}$$

Now, we construct universal coverings q_i which maps $\mathbb{H}^3 \cup \Omega(\Gamma_i)$ in $\bar{\mathbb{B}}^3$ and $q_i(x) = q_i(y)$ if and only if there exist a γ_i in Γ_i with $\gamma_i(x) = y$, so that

$$q_i|_{\Omega(\Gamma_i)} = p_i|_{\Omega(\Gamma_i)} = \pi_i.$$

By Theorem 4, the group Γ_2 acts on the unit disk which belongs to the boundary of $\bar{\mathbb{B}}^3 \cap \mathbb{R}^2$ so that $\pi_2(\Delta) = S_2 \cap \Delta$ also belongs to the boundary of $\bar{\mathbb{B}}^3 \cap \mathbb{R}^2$. Let τ_ϕ be the Möbius rotation, in $\mathbb{R}^3$, with respect to $\partial\Delta$ of angle ϕ. Then

$$\bar{\mathbb{B}}^3 = \bigcup_{0 \le \phi \le \pi} \tau_\phi(\Delta)$$

and $\tau_\pi : \Delta \to \Delta^*$ is the map $z \mapsto 1/\bar{z}$ in the holomorphic coordinate of Δ. Define $q_i(z, \phi) = (\tau_\phi \circ \pi_i(z))$ such that τ_ϕ commutes with $Aut(\Delta) \simeq PSL(2, \mathbb{R})$. Furthermore, τ_ϕ commutes with any Möbius map that leaves the unit circle invariant (for instance, $z \mapsto 1/z$).

Then $M_i = q_i(\mathbb{H}^3 \cup \Omega(\Gamma))$ are subsets of $\bar{\mathbb{B}}^3$ and the respective Poincaré extension is conformally natural with respect to the group of all Möbius maps that leave the unit circle invariant. □

Corollary 11. *If in Proposition 9 we have that $B_1 = B_2$, then $\hat{B}_1 = \hat{B}_2$. Using induction, if $B = B_1^n$, then for every dynamical extension $\hat{B}$ of B, there exists a dynamical extension $\hat{B}_1$ of B_1 such that $\hat{B} = \hat{B}_1^n$.*

In the case (i) of Theorem 10, let $Q_i : \mathbb{H}^3 \to \bar{\mathbb{B}}^3$ be other extensions of the projections π_i, then there are continuous maps $h_i : M_i \to \bar{\mathbb{B}}^3$ such that $Q_i = h_i \circ q_i$. Where q_i are the extensions constructed on the proof of Theorem 10.

Let us put $K = Q_2 \circ Q_1^{-1}$ where the composition is defined. If $\hat{B}$ is the geometric extension of B from Theorem 10, then $K \circ h_2 = h_1 \circ \hat{B}$.

In the case (ii) of Theorem 10, assume Q_i is another extension of π_i. Again, put $K = Q \circ \alpha_B \circ Q^{-1}$. If K is a map, then K is semiconjugated to $\hat{B}$.

Theorem 12. *Let E be a semigroup of Blaschke maps, then the extension constructed in Theorem 10 defines in E a geometric homomorphic conformally natural extension preserving degree if, and only if, E does not intersect the bi-orbit of $f(z) = z^n$ with respect to $Aut(\Delta)$.*

Proof. If the extension is not conformally natural with respect to $PSL(2, \mathbb{C})$ then there are two elements B_1 and B_2 in S with two maps g_1 and g_2 in $PSL(2, \mathbb{C})$ such that $B_1 \circ g_1 = g_2 \circ B_2$. Then by Theorem 10, the maps g_1 and g_2 cannot leave the unit circle invariant. Then there are two circles $C_1 = g_1(\mathbb{S}^1)$ and $C_2 = g_2(\mathbb{S}^1)$ with

$B_1^{-1}(C_2) = C_1$. Let us show that the circles C_i do not intersect $\mathbb{S}^1$. Assume that there is x in $C_1 \cap \mathbb{S}^1$, then C_2 intersects $\mathbb{S}^1$ in all the preimages of x with respect to B_1. But this is possible only if the preimage of x under B_1 is a single critical point, but a Blaschke map cannot have critical points on $\mathbb{S}^1$. Therefore, C_1 and C_2 cannot intersect $\mathbb{S}^1$. Then C_1 and $\mathbb{S}^1$ bound an annulus. By the reflection principle, we have that B_1 has in the unit disk a critical point of multiplicity $d-1$. Hence either B_1 or $1/B_1$ belongs to the bi-orbit of z^n with respect to $Aut(\Delta)$, but the extension in Theorem 4 is compatible with the map $z \mapsto 1/z$. Hence the conclusion of the theorem holds. Reciprocally, the extension of Theorem 4 is not conformally natural on the map z^n. □

Corollary 13. *Assume B is a Blaschke map of the form $B = g_1 \circ z^d \circ g_2$ such that $g_2(z) \neq e^{i\alpha} \circ g_1^{-1}(z)$, then the extensions constructed on Theorem 10 of $\langle B^n \rangle$ are conformally natural whenever $n \geq 2$.*

Proof. If, for $n \geq 2$, B^n does not satisfy the conditions of Theorem 12, then B^n should be in the bi-orbit of z^{d^n}. Hence B^n has one critical point and one critical value, this implies that the critical point x of B is fixed. Hence $g_1 \circ z^d \circ g_2(x) = x$ then $g_1(0) = g_2^{-1}(0)$, which implies $g_2 \circ g_1(0) = 0$ and $g_2 = e^{i\alpha} \circ g_1^{-1}(z)$. □

3.1. **Geometric extensions in Hurwitz spaces.** Let us recall that a branched covering $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ of degree d is in general position, if the number of critical points is the same as the number of critical values and equal to $2d-2$. According to [17] a theorem of Luroth and Clebsch states that:

Lemma 14. *Any two branched coverings $f_i : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ of the same degree in general position are Hurwitz equivalent.*

The existence of a Schottky type extension is a property of the whole Hurwitz space as we show in the following.

Lemma 15. *Let B be a map with a Schottky type geometric extension and let R be a rational map in $H(B)$, then R also has an extension of Schottky type.*

Proof. Let Γ_1 and Γ_2 be the uniformizing groups for B. Let $\alpha : \mathbb{C} \to \mathbb{C}$ be the Möbius map extending B. By definition of $H(B)$, there are two quasiconformal maps f, g on the Riemann sphere such that $f \circ B = R \circ g$. Solving the Beltrami equation, we get quasiconformal extensions $\tilde{f} : \mathbb{C} \to \mathbb{C}$ and $\tilde{g} : \mathbb{C} \to \mathbb{C}$ of f and g, respectively, such that $\beta = \tilde{g}^{-1} \circ \alpha \circ \tilde{f}$ is a Möbius map extending R. Let us assume first that $\tilde{f}$ and $\tilde{g}$ have small distortion, then by a theorem in [5, Th. 5] each map, $\tilde{f}$ and $\tilde{g}$, admits a homeomorphic extension, say $\hat{f}$ and $\hat{g}$, to $\mathbb{B}^3$ compatible with the groups Γ_1 and Γ_2. Hence we obtain two Kleinian groups $\tilde{\Gamma}_1 = \hat{f} \circ \Gamma_1 \circ \hat{f}^{-1}$ and $\tilde{\Gamma}_2 = \hat{g} \circ \Gamma_2 \circ \hat{g}^{-1}$ with manifolds $M(\tilde{\Gamma}_1)$ and $M(\tilde{\Gamma}_2)$ that extend the map R. To see that this extension is geometric we have to embed each manifold $M(\tilde{\Gamma}_1)$ and $M(\tilde{\Gamma}_2)$ into $\mathbb{B}^3$, such that the image of these embeddings are the complement of a finite set of quasigeodesics.

To do so, we use the geometric extension of B. We know that $M(\Gamma_1)$ and $M(\Gamma_2)$ are already realized as submanifolds of $\mathbb{B}^3$, hence by conjugating $M(\Gamma_1)$ by $\hat{f}$ and $M(\Gamma_2)$ by $\hat{g}$, we obtain the desired embeddings of $M(\tilde{\Gamma}_1)$ and $M(\tilde{\Gamma}_2)$ into $\mathbb{B}^3$.

To complete the proof we note that $H(B)$ is connected, so for maps f and g with big distortion, we can use a path on $H(B)$ and extend the maps along the path with small distortion changes. □

Now let us show that there are Schottky type extensions for a large set of rational maps.

Theorem 16. *There is an open and everywhere dense subset in* $Rat_d(\mathbb{C})$ *which has a geometric extension of Schottky type of the same degree.*

Proof. By Lemma 14, the union of all the Hurwitz spaces of all Blaschke maps of fixed degree is open and everywhere dense in $Rat_d(\mathbb{C})$. Hence Theorem 4 and Lemma 15 imply this theorem. □

It follows that structurally stable rational maps have a Schottky type extension. We believe that any rational map has a geometric extension such that the respective manifold belongs to the closure of the Schottky space. Since Hurwitz space of any branched covering of finite degree of the sphere contains a rational map, we conjecture that the closure of the Schottky space of given degree d contains all realizable Hurwitz combinatorics.

3.2. **Extensions of exceptional maps.** A dynamical Poincaré extension, in general, requires a uniformizable geometric structure which is invariant under a rational map R. However, it is rare for a sphere with finitely many punctures to admit a self-covering, with the exception of z^n. Nevertheless, there are orbifolds W, supported on the Riemann sphere, and rational maps $R : W \to W$, so the map R is an orbifold self-covering. The class of maps R are called *exceptional*; the reader will find a more detailed discussion of these maps in [13]. In particular, the Euler characteristic $\chi(W_1)$ is zero. Hence W_1 is a parabolic orbifold, this only occurs when the map R is conjugate to either a Tchebychev type map, a Lattès map or $z \mapsto z^n$.

Theorem 17. *Let G be a semigroup of rational maps which are self-coverings of a parabolic orbifold W supported on the Riemann sphere. Then there exist a geometric extension satisfying the following conditions:*

- *For every $g \in G$, the extension $\hat{g}$ has the same degree as g.*
- *Each extension is geometric.*
- *The set of extensions $\hat{G}$ is a semigroup, and the extension map is a homomorphism from G to $\hat{G}$.*

Proof. The proof exploits the fact that the elements in G have known uniformizations. Consider the lattice

$$L_\tau := \langle z \mapsto z + 1, z \mapsto z + \tau : \Im\tau > 0\rangle$$

in the Lattès case, and the lattice

$$L_0 := \langle z \mapsto z + 1\rangle$$

in the case of z^n and Tchebychev. According to the classification of Lattès maps given by Milnor (see Theorem 1.1 in [13]), for any holomorphic semigroup of endomorphisms of a flat orbifold structure supported on $\bar{\mathbb{C}}$ there exist τ_0 and a Kleinian group Γ_n such that Γ_n contains the lattice L_{τ_0} as a subgroup of finite index n and W is equivalent to $\mathbb{C}/\Gamma_n$. Here n is necessarily either $2, 3, 4$ or 6 when $\tau_0 \neq 0$. In the case where $\tau_0 = 0$, then n is either 1 or 2.

In terms of the group Γ_n, G is a semigroup of affine endomorphisms of Γ_n. In each case, each element in G has a simultaneous Poincaré extension on the orbifold $\mathbb{H}^3/\Gamma_n$. This Poincaré extension satisfies the properties of the theorem. □

Now we discuss the geometric extension of semigroups of the exceptional rational maps mentioned in the theorem above. First, we discuss the case when $n = 2$. In this case, R is a holomorphic endomorphism of the orbifold of type $(\bar{\mathbb{C}}, 2, 2, 2, 2)$.

Let us consider the torus T in $\mathbb{C}^2$ given in coordinates (z_1, z_2) by $|z_1| \leq 1$ and $|z_2| = 1$. The core of T is the unit circle $C = \{(z_1, z_2) : z_1 = 0, |z_2| = 1\}$. The space $T \setminus C$ is uniformized by the Poincaré extension of a lattice L_τ in $\mathbb{H}^3$ parallel to the plane, for a suitable choice of τ.

Let I be the involution map

$$(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2).$$

The map I acts on the filled torus T as an involution. The quotient T/I gives an orbifold O supported in $\bar{\mathbb{B}}^3$ with two ramification lines. Let $\pi : T \to T/I$ be the orbit projection.

Let us consider the family of endomorphisms $\Psi(l, m, k)$ of T given by the formulae

$$(z_1, z_2) \mapsto (z_1^l z_2^k, z_2^m),$$

where l, m, k are integers. In other words, the family $\{\Psi(l, m, k)\}$, for all l, m, k, contains the family of all the Poincaré extensions of semigroups of integer multiplications on $T \setminus C$. Then this family commutes with the involution I and generates a semigroup J of endomorphisms of the orbifold O.

In contrast with the case $n = 2$, when $n > 2$ we obtain a finite group G_n which acts on the boundary of T as a rotation. The action of G_n does not have a continuous extension on the whole of T, it is discontinuous on the core. Hence we cannot construct a compact orbifold analogous to O. However, there exist an invariant foliation, which implies that, for the cases where n is either $3, 4$ or 6, there exist a geometric extension which is isometric to the radial extension. This construction is similar to the construction given in Theorem 3.

Corollary 18. *Let $\hat{R}$ be an element of J, then $\pi(C)$ is an interval which is invariant with respect to $\hat{R}$ and the restriction of $\hat{R}$ on $\pi(C)$ is topologically conjugate to a Tchebychev polynomial of degree k restricted on its Julia set. Moreover, there exist a continuous projection $h : \partial O \to \pi(C)$ so that $h \circ \hat{R} = \hat{R} \circ h$.*

Proof. Note that the projection $P : T \to C$ given by $(z_1, z_2) \to (0, z_2)$ commutes with the maps $\Psi(l, m, k)$ and the involution I. Moreover, the restriction of the map $\Psi(l, m, k)$ on C is the power map $z_2 \mapsto z_2^m$. The action of $\hat{R}$ restricted on $\pi(C)$ is topologically conjugate to the action of a Tchebychev polynomial on its Julia set. Since the projection P commutes with the action of I, then P descends to a projection $h : \partial O \to \pi(C)$ which satisfies the desired properties. □

One would expect that the dynamics of Tchebychev polynomials on its Julia set are obtained by pinching T onto C. The projection P defines a foliation on ∂T by circles. Hence, the projection h defines a foliation F on ∂O, all leaves in F are topological circles with the exception of two leaves homeomorphic to the interval. In other words, one would think that the foliation F shrinks to a model of the Tchebychev polynomial restricted to its Julia set. So there would be a deformation of the foliation in the boundary ∂O that produces a Tchebychev map. The corollary above suggests an argument to construct such a deformation. So, it is natural to ask: is it true that the closure of the space of quasiconformal deformations of flexible Lattès maps in $\bigcup_{d<deg(R)} Rat_d(\mathbb{C})$ contains Tchebychev maps? Is it true

that any point in the boundary of quasiconformal deformations of flexible Lattès map is rigid? Finally, is it true that any point in the boundary has degree strictly smaller than degree of the given Lattès map?

3.3. **Non-Galois affine extensions.** The main idea of this paper is to transform a rational map to a Möbius morphism. On the sections above, we discussed Galois coverings, which is the uniformizable situation. In this subsection, let us consider non-Galois coverings which also transform rational dynamics into Möbius dynamics.

Simple examples of non-Galois coverings are given by Poincaré functions associated to repelling cycles of R. These are functions f satisfying the functional equation $f(\lambda z) = R^n \circ f(z)$ for some n and some complex number λ.

Let us suppose that there exist an extension $\hat{f}$ of f in $\mathbb{H}^3$ and consider the multivalued map defined by

$$K_f(z) = \hat{f}(\lambda \hat{f}^{-1}(z)).$$

When K_f is an ordinary map, we have a dynamical extension. If f is a Galois covering, then we are in the parabolic situation described in the previous subsection. However, is not clear when K_f is a map. In this case, we call $\hat{K}_f$ a *non-Galois* extension of R with respect to f.

Let B be a Blaschke map, and R a quasiconformal deformation of B. Now we show that R has a non-Galois extension with respect to any Poincaré function f.

Theorem 19. *Let R be quasiconformally conjugate to a Blaschke map. For every Poincaré function of R there exist a non-Galois extension of R.*

Proof. Let us first consider a Blaschke map B. Any Poincaré function of B satisfies

$$f(\bar{z}) = \frac{1}{\overline{f(z)}},$$

hence f maps the lower half-plane $\mathbb{H}^2_-$ to the unit disk. Now we are in position to use a similar argument of the proof of Theorem 10 to define an extension $\hat{f} : \mathbb{H}^3 \to \mathbb{B}^3$ as follows: first identify $\mathbb{H}^3$ with the "open book" coordinates (z, ϕ) where $z \in \mathbb{H}^2_-$ and ϕ in the interval $(0, \pi)$ and put

$$\hat{f}(z, \phi) = \tau_\phi(f(z))$$

where τ_ϕ is the Möbius rotation of angle ϕ in $\mathbb{R}^3$ with respect to the unit circle. In this case, from the equation satisfied by a Poincaré function, we have that $K_f = \hat{B}^n$ for some iterate of $\hat{B}$, where $\hat{B}$ is the dynamical extension constructed in Theorem 10. Let $R = \phi \circ B \circ \phi^{-1}$, then any Poincaré function for R belongs to the Hurwitz space of a suitable Poincaré function of B. Now we can apply the arguments in Lemma 15 to finish the proof. □

Let us recall that, for every complex affine line L there is a process of hyperbolization $T : L \to H(L)$ which associates a hyperbolic manifold $H(L)$ to L. This hyperbolization process is used in the construction of 3-hyperbolic Lyubich-Minsky laminations in [8]. By this process there is an identification $H(L) \cong \mathbb{C} \times \mathbb{R}_+$. Given an affine line L, let us assume that we have fixed any such identification.

Now let L_1 and L_2 be complex affine lines. Let $F : L_1 \to L_2$ be any map, then for any $\lambda > 0$ there is a family of extensions $\hat{F}_\lambda : H(L_1) \to H(L_2)$ given in coordinates by

$$\hat{F}_\lambda(x, t) = (F(x), \lambda t).$$

Note that if F is affine then there exists a unique λ_0 such that F_{λ_0} is the Poincaré extension of F in $H(L)$. Let $q_i : L_i \to \mathbb{C}$ be maps for $i = 1, 2$. Assume we have a polynomial P and an affine map $\gamma : L_1 \to L_2$ satisfying

$$P \circ q_1 = q_2 \circ \gamma.$$

Then for all λ, ω and ρ positive real numbers, we have

$$\begin{aligned}(q_2)_\omega \circ \gamma_\lambda(x,t) &= (q_2)_\omega(\gamma(x), \lambda t)\\ &= (q_2(\gamma(x)), \omega\lambda t) = (P \circ \pi_1(x), \omega\lambda t)\\ &= P_\rho(q_1(x), \frac{\omega\lambda}{\rho} t) = P_\rho \circ (q_1)_{\frac{\omega\lambda}{\rho}}(x,t).\end{aligned}$$

Now let λ_0 be the number such that γ_{λ_0} is the Poincaré extension of γ in $\mathbb{H}^3$ and put $\rho = \lambda_0$ in the formula above, then we have

$$(q_2)_\omega \circ \gamma_{\lambda_0} = P_{\lambda_0} \circ (q_1)_\omega.$$

Assume that $q_1 = q_2 = f$ where f is the Poincaré function of a fixed point with multiplier λ_0, then for every ω the map P_{λ_0} is a geometric dynamical extension with the same degree. In this extension, the orbit of every point in $\mathbb{H}^3$ converges to infinity. In other words, the Julia set of the extension belongs to $\bar{\mathbb{C}}$. To show that P_{λ_0} is geometric let M_1 be the complement in $\mathbb{H}^3$ of all vertical lines based on the P-preimages of the postcritical set, then M_2 is the complement in $\mathbb{H}^3$ of all vertical lines based on the postcritical set and $P_{\lambda_0} : M_1 \to M_2$. Then f_ω endows M_i with incomplete Möbius structures, making P_{λ_0} geometric.

Another situation is when $\rho = 1$, again let $q_1 = q_2 = f$ as above. Then P_1 is a Poincaré extension, with the manifolds M_i. However, the Möbius structures on M_i are different, on M_1 is given with f_ω and on M_2 is given by $f_{\lambda_0\omega}$. Note that $\rho = 1$ gives a homomorphic extension defined on the semigroup of polynomials.

For the reader familiar with the construction of Lyubich-Minksy [8], we note that natural extension of either P_{λ_0} or P_1 is equivalent to the 3-hyperbolic Lyubich-Minsky lamination.

4. Product extension

At least for us, it is very surprising that there is a product structure on $\mathbb{H}^3$ which, in a sense, is a "conformal natural" extension of the complex product on $\mathbb{C}$. To construct this product, first let us extend the exponential map $Exp(z) = e^z$. We consider the coordinates (z, t) in $\mathbb{H}^3$. Let $h_\alpha : \mathbb{H}^3 \to \mathbb{H}^3$ be the translation given by

$$(x, y, t) \mapsto (x, y - \alpha, t),$$

the map $H_\beta : \mathbb{H}^3 \to \mathbb{H}^3$ is the dilation given by

$$(x, y, t) \mapsto (\beta x, \beta y, \beta t),$$

and let $p : \triangle \to \mathbb{H}^3$ be the stereographic projection that maps the unit disk $\triangle$ into the unit semisphere in $\mathbb{H}^3$.

Write

$$\Phi(x, 0, t) = p \circ Exp(x + it)$$

and let V be the vertical semiplane over the imaginary line. Then Φ maps V onto the unit semisphere in $\mathbb{H}^3$. Finally, for $w = (x, y, t)$ let

$$\widehat{Exp}(w) = H_{e^{-2\pi y}} \circ \Phi \circ h_y^{-1}(w).$$

By construction $\widehat{Exp}$ maps $\mathbb{H}^3$ onto M, the complement of the t-axis in $\mathbb{H}^3$, and is a covering. When $t = 0$ the map $\widehat{Exp}$ coincides with the function Exp. Also, $\widehat{Exp}$ defines a complete Möbius structure δ on M. Any Möbius map that leaves M invariant is Möbius in δ.

Since Exp defines an homomorphism of the additive structure on $\mathbb{C}$ onto the multiplicative structure of $\mathbb{C}^*$. Then $\widehat{Exp}$ gives a multiplication "$*$" in M, which is the pushforward of the additive structure on $\mathbb{H}^3$. Indeed, let a and b elements in M, and let a_1 and b_1 be elements such that $a = \widehat{Exp}(a_1)$ and $b = \widehat{Exp}(b_1)$. Then write

$$a * b = \widehat{Exp}(a_1 + b_1).$$

This is well defined and M is closed under the $*$ multiplication.

Lemma 20. *The multiplicative structure in M extends to a multiplicative structure on $\mathbb{H}^3$.*

Proof. Let $\|.\|$ be the standard norm in the euclidean space $\mathbb{R}^3$. Then for every x and y in $\mathbb{H}^3$. We have

$$\|x * y\| = \|x\|\|y\|.$$

Now, let $x = (0, 0, t)$. For any $y \in \mathbb{H}^3$ define $y * \tau = (0, 0, t\|y\|)$. The restriction of this multiplication to points in the axis t in $\mathbb{H}^3$ coincides with the usual multiplication on $\mathbb{R}_+$. This definition continuously extends the multiplication $*$ to the whole of $\mathbb{H}^3$. □

The multiplication in $\mathbb{H}^3$ is commutative and associative and has the following properties:

- The multiplication restricted on the boundary is the usual complex multiplication in $\mathbb{C}$.
- Let λ be a nonzero complex number then, for any $x \in \mathbb{H}^3$, we have $\lambda * x = H_\lambda(x)$. Where $H_\lambda(x)$ is the Poincaré extension of the map $z \mapsto \lambda z$.
- The unique unit element is $(1, 0, 0)$. For any $x \neq 0, \infty$ in $\bar{\mathbb{H}}^3$ there exists y such that $x * y = y * x = (1, 0, 0)$ and $y = H(x)$ where H is the Poincaré extension of the map $z \mapsto 1/z$.

Now we can define an extension of rational maps. Let R be a rational map and consider a decomposition of R as a product of Möbius maps:

$$R(z) = \prod \gamma_i(z)$$

where $\gamma_i \in PSL(2, C)$. Hence for $x \in \bar{\mathbb{H}}^3$ we have an extension with respect to the maps γ_i,

$$\hat{R}(x) = \prod_* P(\gamma_i)(x),$$

where $P(\gamma_i)$ is the Poincaré extension of γ_i in $\bar{\mathbb{H}}^3$. Since the multiplication is commutative then the definition does not depend on the order of the factors γ_i. The following proposition follows from the definition of the product extension.

Proposition 21. *The product extension has the following properties:*

(1) *If σ_i is the geodesic that connects the pole with the zero of γ_i. Then $\hat{R}(\sigma_i)$ is the t-axis.*
(2) *The extension $R \mapsto \hat{R}$ is right equivariant with respect to the action of $PSL(2, \mathbb{C})$.*

(3) *Any rational map R has a finite number of decompositions in Möbius factors. Hence there are only finitely many product extensions for each rational map R.*

There is another extension from $\mathbb{C}$ to $\mathbb{H}^3$ which is induced by the product. This extension is given by a monomorphism Φ from the ring of formal series with the usual multiplication on the complex plane to the ring of formal series with the $*$ multiplication. The map Φ is continuous on the subring of polynomials. However, it is not clear whether it is still continuous on the subring of absolutely convergent series. Note that this extension is not conformally natural, and is not even right equivariant with respect to the action of $PSL(2,\mathbb{C})$. The map Φ is not a homomorphism with respect to composition. The biggest semigroup E, where Φ defines a homomorphism with respect to composition, is the generated by λz^n for any complex λ and n a natural number. Moreover, Φ on E is conformally natural, geometric and the same degree. In general is not clear when product or ring extensions are geometric. However, numerical calculations of the ring extension of z^2+c, with c real, suggests that this extension is geometric.

4.1. **Some examples of Poincaré extensions of quadratic polynomials.** Here we compute some Poincaré extensions. These computations are based on the following formula for the exponential map defined in the previous subsection.

$$\widehat{Exp}(x,y,t) = (\frac{2e^t cos(y)}{1+e^{2t}}e^x, \frac{2e^t sin(y)}{1+e^{2t}}e^x, \frac{e^{2t}-1}{1+e^{2t}}e^x).$$

We have the following facts:

- The map $\widehat{Exp}$ is a Poincaré extension of the map e^z.
- Let T be group generated by the translation $z \mapsto z+2\pi i$, then the action of T in $\mathbb{C}$ extends an action in $\mathbb{H}^3$ generated by the map $(z,t) \mapsto (z+2\pi i, t)$.
- The orbit space $\mathbb{H}^3/T$ is homeomorphic to $B_L := \mathbb{H}^3 \setminus L$ where L is the t-axis.
- There exist a complete hyperbolic Möbius structure on B_L so that $\widehat{Exp}$: $\mathbb{H}^3 \to B_L$ defines a Möbius universal covering map.
- The extension from Exp to $\widehat{Exp}$ is conformally natural.

Let H_2 be the Poincaré extension of the Möbius map $z \mapsto 2z$, hence the map $\hat{Q} = \widehat{Exp} \circ H_2 \circ \widehat{Exp}^{-1} : B_L \to B_L$ is a Poincaré extension of the map $Q(z) = z^2$.

For a circle C in $\partial\mathbb{H}^3$, let us define the *dome* over C as the 2-sphere with equator C intersected with $\mathbb{H}^3$ and will be denoted by $Dome(C)$.

Using the equations above, we obtain the equation

$$\hat{Q}(\lambda(X,Y,T)) = \lambda^2(\frac{X^2-Y^2}{1+T^2}, \frac{2XY}{1+T^2}, \frac{2T}{1+T^2})$$

with $\lambda \in \mathbb{R}$ and $(X,Y,T) \in Dome(\mathbb{S}^1)$.

If $x = \lambda X$, $y = \lambda Y$ and $t = \lambda T$, then $||p||^2 = x^2+y^2+t^2 = \lambda^2$. We have that

$$\hat{Q}(x,y,t) = (||p||^2\frac{x^2-y^2}{||p||^2+t^2}, ||p||^2\frac{2xy}{||p||^2+t^2}, ||p||^2\frac{2t||p||}{||p||^2+t^2}).$$

In this case, by the formula above, we have that in fact, $\hat{Q}$ extends to the whole of $\mathbb{H}^3$ and $\hat{Q}(0,0,t) = (0,0,t^2)$. Moreover, for every w in $\bar{\mathbb{H}}^3$ we have $\hat{Q}(w) = w * w$ where $*$ is the product defined above. The map $\hat{Q}$ commutes with the reflection with respect to the dome.

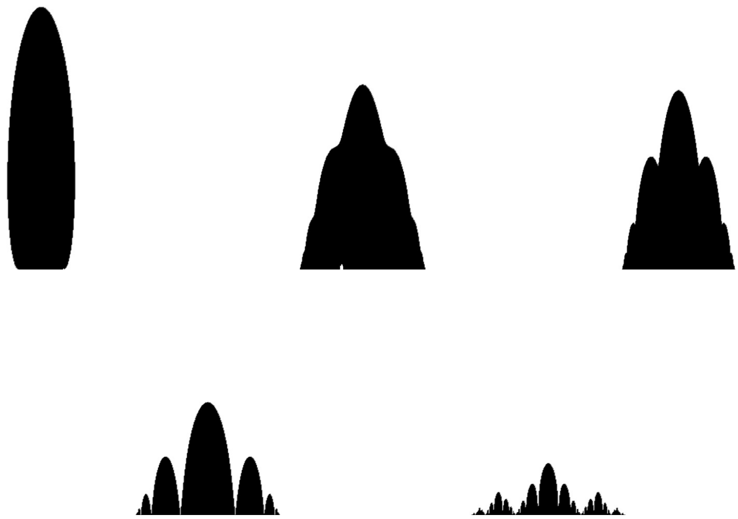


FIGURE 1. The sets $K(\hat{Q}_c) \cap V_0$ for $c = .25, -0.75, -.77, -1, -1.28$ from left to right and top to bottom.

We have the following invariant foliations for the action of $\hat{Q}$:

(1) **Semispheres centered at the origin**. Observe that the family of parallel planes of the form (x_0, y, t) is invariant under H_2. The map $\widehat{Exp}$ sends this foliation to a foliation of domes over circles centered at the origin. Hence this domes is a foliation invariant under $\hat{Q}$.

(2) **Cones centered at the origin**: Horizontal planes (horocyclic foliation of $\mathbb{B}^3$) of the form (x, y_0, z) are invariant under H_2, hence their image under $\widehat{Exp}$ is also invariant. These are cones centered at 0.

(3) **Onion like foliation**: Planes of the form (x, y, kx), are invariant under H_2. Its image is the onion-like foliation surrounding the vertical axis $(0, 0, t)$. This is a book decomposition, here the binding of the book is the unit circle in the boundary plane.

Let $T_c(x, y, t) = (x + Re(c), y + Im(c), t)$ be the Poincaré extension of the map $z \to z + c$ and L_c the vertical line over c in $\mathbb{H}^3$, and let $B_{L_c} = \mathbb{H}^3 \setminus L_c$. If $Q_c(z) = z^2 + c$, then we have an extension $\hat{Q}$ of Q_c depicted in the following commutative diagram:

$$\begin{array}{ccc} \mathbb{H}^3 & \xrightarrow{H_2} & \mathbb{H}^3 \\ \downarrow{\scriptstyle \widehat{Exp}} & & \downarrow{\scriptstyle T_c \circ \widehat{Exp}} \\ B_L & \xrightarrow{\hat{Q}_c} & B_{L_c}. \end{array}$$

The diagram implies that $\hat{Q}_c = T_c \circ \hat{Q}$. In this case, the line L is also the critical line but has complicated dynamics. Let us define $K(\hat{Q}_c)$:= **spatial filled Julia set** of $\hat{Q}_c$, the set of (x, y, t) such that $\hat{Q}_c^n(x, y, t)$ does not tend to ∞ as $n \to \infty$.

Using similar arguments as in the one-dimensional case, one can show that $K(\hat{Q}_c)$ is always bounded in $\mathbb{H}^3$. Also, for parameters c with $|c|$ large enough, the critical line converges to infinity. If V_0 denotes the semiplane $\{(x, y, t) \in \mathbb{H}^3 : y = 0\}$. The section $K(\hat{Q}_c) \cap V_0$ is a bi-dimensional set that we have illustrated in Figure 1 for different values of c.

5. Remarks and conclusions

As mentioned in the introduction, there are some constructions in the literature of extensions of rational maps into endomorphisms of $\mathbb{H}^3$. Most of these extensions are based on the barycentric construction suggested by Choquet's Theorem. Let us briefly describe the barycentric extension. Let $\bar{\mathbb{B}}^3$ be the closed unit ball in $\mathbb{R}^3$. Let $\mathcal{M}$ be the space of probability measures on $\partial\mathbb{B}^3$. Then, for every μ in $\mathcal{M}$ the barycenter of μ is the unique point x such that for every functional L on $\mathbb{R}^3$, the following equation holds:

$$L(x) = \int_{\partial\mathbb{B}^3} L(y) d\mu(y).$$

We define $Bar(\mu) = x$, by Choquet's theorem (See [15], page 48) the map Bar sends $\mathcal{M}$ onto $\bar{\mathbb{B}}^3$. The semigroup $Rat(\mathbb{C})$ acts in $\mathcal{M}$ by a pushforward, for every R in $Rat(\mathbb{C})$ we denote by $R\mu = R_*(\mu)$ the pushforward of μ by R. For every point $x \in \mathbb{B}^3$, let ν_x be the visual measure based on x we define the barycentric extension of R by

$$\hat{R}(x) = Bar(R\nu_x).$$

Then the barycentric extension is visual as is proved in [12]. If instead we use the conformal barycenter we obtain a conformally natural extension as is proved in [14]. It is very difficult to get any geometric information of these extensions. For instance, it is not clear that these extensions define a branched covering of the same degree from $\bar{\mathbb{H}}^3$ to $\bar{\mathbb{H}}^3$.

The following proposition was already mentioned in [7]. We include the proof for completeness.

Proposition 22. *Let R be a rational map, then all conformally natural extensions of R are homotopic, with a homotopy that consists of conformally natural extensions of R.*

Proof. Let $\hat{Q}$ and $\hat{S}$ be extensions of a rational map R, and let $x \in \mathbb{H}^3$. For λ in $[0, 1]$, let $E_\lambda(x)$ be the point along the geodesic from $\hat{Q}(x)$ to $\hat{S}(x)$, which is at distance $\lambda d(\hat{Q}(x), \hat{S}(x))$. Since for every $x \in \partial\mathbb{H}^3$, $\hat{Q}(x) = \hat{S}(x) = R(x)$. The map $E_\lambda(x)$ extends to $\partial\mathbb{H}^3$ as an extension of R. If the maps $\hat{Q}$ and $\hat{S}$ are either right equivariant with respect to the action of $PSL(2, \mathbb{C})$, conformally natural or Poincaré, the map $E_\lambda(x)$ also holds the same property. □

It follows that if for a given rational map R there are two right equivariants with respect to the action of $PSL(2, \mathbb{C})$ (or conformally natural) extensions, then there are uncountably many right equivariants with respect to the action of $PSL(2, \mathbb{C})$ (or conformally natural) extensions. This situation contrasts with the product extensions which are only finitely many.

The extension discussed in [9] is uniformly quasiregular dynamical and has the same degree as the starting map R; moreover, it can be shown that it is geometric. However, for most rational maps these extensions do not exist [9].

Another aspect of our geometric construction is about Maskit surgery on the respective Möbius manifolds. A rational map $R : S_1 \to S_2$ is modeled with two groups $\Gamma_1 < \Gamma_2$. Let us assume that $\Gamma_2 = \langle \gamma_1, \dots, \gamma_n \rangle$. For $1 < k < n$ $G = \langle \gamma_1, \dots, \gamma_k \rangle$ and $H = \langle \gamma_{k+1}, \dots, \gamma_n \rangle$, and consider the intersections $G_i = \Gamma_i \cap G$ and $H_i = \Gamma_i \cap H$. Then G_1 and H_1 are subgroups of finite index in G_2 and H_2 respectively. This construction defines two rational maps R_G and R_H associated to groups G_i and H_i respectively. It is possible to define analogously amalgamated products and HNN-extensions.

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Universidad Nacional Autonoma de Mexico, Unidad Cuernavaca del Instituto de Matematicas, Universidad s/n, Col Lomas de Chamilpa, 62210 Cuernavaca, Mexico
E-mail address: carloscabrerao@im.unam.mx

Universidad Nacional Autonoma de Mexico, Unidad Cuernavaca del Instituto de Matematicas, Universidad s/n, Col Lomas de Chamilpa, 62210 Cuernavaca, Mexico
E-mail address: makienko@matcuer.unam.mx

Universidad Nacional Autonoma de Mexico, Facultad de Ciencias, Av. Universidad 3000, 04510 Mexico, Mexico
E-mail address: gsl@dinamical.fciencias.unam.mx