CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 19, Pages 240–257 (October 28, 2015)
http://dx.doi.org/10.1090/ecgd/283

CONFORMAL GEOMETRIC INEQUALITIES ON THE KLEIN BOTTLE

CHADY EL MIR AND ZEINA YASSINE

Abstract. We prove three optimal conformal geometric inequalities of C. Blatter type on every Riemannian Klein bottle. These inequalities provide conformal lower bounds on the area and involve lengths of homotopy classes of curves that are natural candidates to realize the systole.

1. Introduction, preliminaries and results

The most interesting Riemannian metrics on a given compact differentiable manifold are those that extremize some Riemannian invariant. An interesting problem, for example, is to study metrics which maximize the ratio $sys(M,g)^n/vol(M,g)$ over the set of all Riemannian metrics g on a given n-dimensional differentiable manifold M, where $sys(M,g)$ denotes the systole of (M,g), i.e., the least length of a non-contractible closed curve. Concerning this problem, called the isosystolic problem, it is known that the extremal metric for the 2-dimensional torus is the flat hexagonal metric (unique up to a homothety). It is due to C. Loewner in 1949 (unpublished). His student, P. M. Pu, showed in 1952 that the extremal metric on the projective plane is the spherical metric (cf. [17]). Nevertheless, as in many works related to the isosystolic problem, various constraints can be put on the set of Riemannian metrics under consideration. For example, we may restrict ourselves to the set of constant or non-negative curvature metrics (cf. [1], [14], [8]) or to the set of metrics in a fixed conformal class (cf. [7], [4], [5], [6]).

In 1961, C. Blatter proved (cf. [7]) optimal conformal lower bounds on the area of the Mobius band with boundary (denoted by M) in terms of the product of the least lengths of two classes of curves. The first class consists of the family $\mathcal{F}$ of arcs joining two points on the boundary with non-trivial intersection with the soul of the Mobius band. The second class consists of the family $\mathcal{G}$ of loops in the homotopy class of a generator of $\pi_1(M)$. Let ℓ_v be the least length of an arc in $\mathcal{F}$ and ℓ_σ be the least length of a loop in $\mathcal{G}$. Then, Blatter (in his notation, $\ell_v = l^*$ and $\ell_\sigma = \ell_1$) obtained the following optimal *conformal* lower bound on the area of M :

$$\ell_\sigma(g)\ell_v(g) \leq C_\beta \cdot area(g) \tag{1.1}$$

where C_β is a positive constant that depends only on the conformal type β of g. Note that the optimal constant C_β is not bounded from above over the set of conformal types β.

Received by the editors April 17, 2014 and, in revised form, November 8, 2015, August 16, 2015, September 4, 2015.

2010 *Mathematics Subject Classification.* Primary 53C20, 53C22, 53C23.

Key words and phrases. Klein bottle, conformal metric, systole, isosystolic inequality.

Also, in the same article, Blatter proved the following conformal lower bound on the area of M :

$$sys(g)\ell_v(g) \leq C'_\beta \cdot area(g) \tag{1.2}$$

where C'_β is a positive constant that depends only on the conformal type β of g. Contrarily to the previous case, the optimal constant C'_β is bounded from above over the set of conformal types β. More precisely, $\sup_\beta C'_\beta = 2$. Hence, the optimal inequality

$$sys(g)\ell_v(g) \leq 2\ area(g)$$

holds for every Riemannian metric g on M.

Variants of C. Blatter's problem were studied by L. Keen in [16] and J. Hebda in [12] on the 2-dimensional torus. Keen obtained a lower bound on the area in terms of the product of the least lengths of two loops generating the fundamental group. On the other hand, Hebda obtained a lower bound on the square of the area in terms of the three primitive length spectrum. The equality in both cases is attained by the flat hexagonal metric.

In this article, we will prove three types of inequalities on the Klein bottle (denoted by K) in the same spirit as Blatter's inequalities. The classes of curves we consider are (free) homotopy classes of loops that are candidates to realize the systole. In particular, the role of the family $\mathcal{F}$ will be taken by a free homotopy class of loops representing the vertical translation in $\pi_1(K)$ (see Section 1.1 below). The main tool in our proof is the method of extremal length which was used by C. Bavard in 1988 to prove conformal *isosystolic* inequalities on the Klein bottle (cf. [4]). It can be applied to the Mobius band with boundary and provides an alternative proof of Blatter's inequalities (1.1) and (1.2).

1.1. **The Klein bottle.** A flat Klein bottle is the quotient of $\mathbb{C}$ by the group generated by the maps $\sigma : z \mapsto \bar{z} + \pi$ and $t_v : z \mapsto z + 2i\beta$. The induced flat metric on the Klein bottle will be denoted by g_β. By the uniformization theorem, every Riemannian metric g on the Klein bottle is conformally equivalent to a unique flat metric g_β for some $\beta \in (0, +\infty)$. The parameter β represents the conformal type of the metric g. We call $t_h := \sigma^2$ a horizontal translation and identify the fundamental group $\pi_1(K)$ with the deck group $\langle \sigma, t_v \rangle$. Note that any Klein bottle K is obtained by gluing two Mobius bands M_1 and M_2 along their boundaries. With the previous notation, fundamental domains of K, M_1 and M_2, are $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\beta, \beta]$, $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [\frac{-\beta}{2}, \frac{\beta}{2}]$ and $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\beta, \frac{-\beta}{2}] \cup [\frac{\beta}{2}, \beta]$ respectively.

By C. Bavard's theorem, cf. [3], every Riemannian Klein bottle K satisfies the isosystolic inequality

$$sys(K)^2 \leq \frac{\pi}{2\sqrt{2}} \cdot area(K)$$

where the equality is attained by a spherical metric outside a singular line (see [3], [19] and [8, p. 100] for a detailed description of the extremal Klein bottle). For details and (many) open problems in *systolic geometry* see the book of Katz [15] and the paper of M. Gromov [11].

Definition 1.1. In this paper, we will consider the following distinct families of homotopy classes of closed curves in K. They are the natural candidates for the systole of a Riemannian Klein bottle. These families are :

(1) the family $\mathcal{F}_\sigma$ of loops in the homotopy class of a glide reflection in $\pi_1(K)$ i.e. $\mathcal{F}_\sigma = \{\gamma : S^1 \to K \mid \gamma$ is in the homotopy class of a word in $\langle \sigma, t_v\rangle$ having odd number of letters $\sigma\}$;
(2) the family $\mathcal{F}_v$ of loops in the homotopy class of an element in the subgroup $\langle t_v\rangle$, i.e., $\mathcal{F}_v = \{\gamma : S^1 \to K \mid [\gamma] \in \langle t_v\rangle\}$;
(3) the family $\mathcal{F}_h$ of loops in the homotopy class of an element in the subgroup $\langle t_h\rangle$, i.e., $\mathcal{F}_h = \{\gamma : S^1 \to K \mid [\gamma] \in \langle t_h\rangle\}$.

Given a Riemannian metric on K, the least length of a closed curve in the first (resp. second, resp. third) family will be denoted by ℓ_σ (resp. ℓ_v, resp. ℓ_h). Also, denote by L_σ the least length of a loop freely homotopic to a non-trivial element in the subgroup $\langle\sigma\rangle$ generated by σ.

Note that an extremal metric g_{ext} for the isosystolic inequality on the Klein bottle K satisfies $\ell_\sigma(g_{ext}) = L_\sigma(g_{ext}) = \ell_v(g_{ext})$. Its conformal type β is equal to $2\ln(\tan(\frac{3\pi}{8}))$; cf. [3].

Finally, denote by a_0 the unique real $x \in (0, \frac{\pi}{2})$ such that

$$\tan(x) = 2x \ (a_0 \approx 1.1655).$$

1.2. **Two families of Riemannian metrics on the Klein bottle.** We will explain first how to write an arbitrary Riemannian metric on $\mathbb{C}$ in what we see as *the spherical coordinates*. This operation, useful in our later calculations, is commonly used in the field (cf. [3] and [4]). We denote by $C^{0,\infty}(\mathbb{R})$ the set of continuous and piecewise smooth functions on $\mathbb{R}$.

Let h^β be a Riemannian metric in $\mathbb{C} \simeq \mathbb{R}^2$, periodic with respect to the fundamental domain $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\beta, \beta]$ and satisfying

$$(h^\beta)_{(x,y)} = \varphi(y)(dx^2 + dy^2), \tag{1.3}$$

where $\varphi \in C^{0,\infty}(\mathbb{R})$ is a positive, even and 2β-periodic function. Then there exists a diffeomorphism $G : \mathbb{C} \to \mathbb{C}$ for which the Riemannian metric $h_b = (G^{-1})^* h^\beta$ satisfies

$$(h_b)_{(u,v)} = f^2(v)du^2 + dv^2, \tag{1.4}$$

where $f \in C^{0,\infty}(\mathbb{R})$ is a positive, even and $2b$-periodic function. To see this, let $\phi(y) = \int_0^y \sqrt{\varphi(t)}dt$ and $f = (\varphi \circ \phi^{-1})^{\frac{1}{2}}$ and define the map G by

$$G(x, y) = (x, \phi(y)). \tag{1.5}$$

The Riemannian metric h_b is then periodic with respect to the fundamental domain $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-2b, 2b]$. The quotient of $(\mathbb{C}, h_b)$ by the subgroup of isometries $\langle \sigma, t_{v'}\rangle$, where $\sigma : z \mapsto \bar{z} + \pi$ and $t_{v'} : z \mapsto z + 4ib$, is a Riemannian Klein bottle. Its conformal type is

$$\beta = \int_0^{2b} \frac{dt}{f(t)}.$$

Now, we will introduce two families of metrics on $\mathbb{C}$. They induce families of metrics on the Klein bottle that will be used later in the article as conformally extremal metrics for three types of inequalities on the Klein bottle. For every $b \in (0, +\infty)$ and $\omega \in (0, b]$, they are as follows:

(1) the metric S_b on $\mathbb{C}$ periodic with respect to the fundamental domain $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-2b, 2b]$ and defined by

$$(S_b)_{(u,v)} = f_b^2(v)du^2 + dv^2$$

where f_b is the unique one-variable function invariant by the translation $(u,v) \mapsto (u, v+2b)$ which agrees with *cosine* on $[-b,b]$. It is spherical outside the singular lines $v = nb$, where $n \in \mathbb{Z}$.

(2) The metric $SF_{b,\omega}$ on $\mathbb{C}$ periodic with respect to the fundamental domain $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-2b, 2b]$ and defined by

$$(SF_{b,\omega})_{(u,v)} = f^2_{b,\omega}(v)du^2 + dv^2$$

where $f_{b,\omega}$ is the unique one-variable function invariant by the translation $(u,v) \mapsto (u, v+2b)$ which agrees with *cosine* on $[-\omega, \omega]$ and is equal to the constant $\cos(\omega)$ on $[\omega, 2b-\omega]$ (cf. [4, Fig. 1, p. 351]). It is spherical on the band $\mathbb{R} \times [-\omega, \omega]$ and its images by the translations $(u,v) \mapsto (u, v+2nb)$, where $n \in \mathbb{Z}$, and flat elsewhere.

We will denote by (K, S_b) the Riemannian Klein bottle obtained by taking the quotient of $(\mathbb{C}, S_b)$ by the subgroup $\langle \sigma, t_{v'} \rangle$. Its conformal type is

$$\beta = 2\ln(\tan(\frac{\pi}{4} + \frac{b}{2})).$$

Similarly, $(K, SF_{b,\omega})$ will denote the Riemannian Klein bottle obtained by taking the quotient of $(\mathbb{C}, SF_{b,\omega})$ by the subgroup $\langle \sigma, t_{v'} \rangle$. Its conformal type is

$$\beta = 2\ln(\tan(\frac{\pi}{4} + \frac{\omega}{2})) + \frac{1}{\cos(\omega)}(2b - 2\omega).$$

1.3. **Geometric inequality of type $\ell_\sigma \ell_v$.** Our first result studies optimal conformal inequalities of the form $\ell_\sigma \ell_v \leq C_\beta \cdot area$, where C_β is a constant depending only on the conformal type β. We call such a relation a *"geometric inequality of type $\ell_\sigma \ell_v$"*. Actually, it extends the following result of C. Bavard to all the conformal classes of the Klein bottle.

Theorem 1.2 (C. Bavard [6]). *Let $a_0 \in (0, \frac{\pi}{2})$ be such that $\tan(a_0) = 2a_0$ and $0 < \beta \leq 2\ln\left(\tan\left(\frac{\pi}{4} + \frac{a_0}{2}\right)\right)$. Then, for every Riemannian metric g on the Klein bottle K of conformal type β, we have the following optimal inequality:*

$$\ell_\sigma(g)\ \ell_v(g) \leq \frac{\arcsin\left(\frac{e^\beta - 1}{e^\beta + 1}\right)}{\frac{e^\beta - 1}{e^\beta + 1}}\ area(g).$$

The equality is attained if and only if g is proportional to the spherical metric S_b for b satisfying $\beta = 2\ln(\tan(\frac{\pi}{4} + \frac{b}{2}))$.

Our first result completes C. Bavard's study by providing an optimal inequality for the remaining conformal classes.

Theorem 1.3. *Let $a_0 \in (0, \frac{\pi}{2})$ be such that $\tan(a_0) = 2a_0$ and*

$$\beta > 2\ln\left(\tan\left(\frac{\pi}{4} + \frac{a_0}{2}\right)\right).$$

Let $\omega_1 \in [a_0, \frac{\pi}{2})$ be defined by the equation

$$2\sin(\omega_1) = \left(\beta - 2\ln\left(\tan\left(\frac{\pi}{4} + \frac{\omega_1}{2}\right)\right)\right)\cos^2(\omega_1) + 4\omega_1\cos(\omega_1).$$

Then, for every Riemannian metric g on the Klein bottle K of conformal type β, we have the following optimal inequality:

$$\ell_\sigma(g)\ \ell_v(g) \leq \frac{1}{2\cos(\omega_1)}\ area(g).$$

Moreover, the equality is attained if and only if g is proportional to the spherical-flat metric SF_{b,ω_1} for $b = \tan(\omega_1) - \omega_1$.

Remark 1.4. Note that the bound $\beta \leq 2\ln\left(\tan\left(\frac{\pi}{4} + \frac{a_0}{2}\right)\right)$ found by C. Bavard in Theorem 1.2 is actually the critical value of the conformal type for the transition in the shape of the extremal metrics from *spherical* to *spherical-flat*.

Corollary 1.5. *There does not exist any (finite) universal constant C such that the inequality*

$$\ell_\sigma(g)\ell_v(g) \leq C \cdot area(g)$$

holds for every Riemannian metric g on the Klein bottle.

1.4. **Geometric inequality of type $L_\sigma \ell_v$.** The second part of the paper is devoted to establishing optimal conformal inequalities of the form $L_\sigma \ell_v \leq C_\beta \cdot area$. Note that in the case of the Mobius band with boundary M, L_σ is just the systole of M. We call such a relation a *"geometric inequality of type $L_\sigma \ell_v$"*.

Theorem 1.6. *Let $\beta > 0$. For every Riemannian metric g on the Klein bottle K of conformal type β, we have*

$$L_\sigma(g)\ell_v(g) \leq C_\beta \cdot area(g),$$

where

$$C_\beta = \begin{cases} \frac{e^\beta+1}{e^\beta-1}\arcsin\left(\frac{e^\beta-1}{e^\beta+1}\right) & \text{if} \quad 0 < \beta \leq 2\ln\left(2+\sqrt{3}\right), \\ \frac{2}{3}\cdot\frac{3\beta+4\pi-6\ln(2+\sqrt{3})}{4\sqrt{3}+\beta-2\ln(2+\sqrt{3})} & \text{if} \quad \beta > 2\ln\left(2+\sqrt{3}\right). \end{cases}$$

Moreover, the equality is attained if and only if g is proportional to the spherical metric S_b, for b satisfying $\beta = 2\ln(\tan(\frac{\pi}{4}+\frac{b}{2}))$, in the first case and to the spherical-flat metric $SF_{b,\frac{\pi}{3}}$, for b satisfying $\beta = 2\ln(2+\sqrt{3}) + 4(b - \frac{\pi}{3})$, in the second case.

Since the supremum of the conformal constant C_β over β is equal to 2, we have

Corollary 1.7. *For every Riemannian metric g on the Klein bottle K, we have*

$$L_\sigma(g)\ell_v(g) < 2\ area(g).$$

The inequality is optimal.

1.5. **Geometric inequality of type $\ell_\sigma\ell_v\ell_h$.** In the third part, we establish optimal conformal inequalities of the form $\ell_\sigma\ell_v\ell_h \leq C_\beta \cdot area^{\frac{3}{2}}$, which we call a *"geometric inequality of type $\ell_\sigma\ell_v\ell_h$"*. To our knowledge, this kind of inequality involving the product of three lengths has never been considered before. It is distinguished by the fact that the area to the power $\frac{3}{2}$ is bounded by the product of all the natural candidates for the systole:

Theorem 1.8. *Let $\beta > 0$ and let $\omega_2 \in (0, \frac{\pi}{2})$ be defined by the equation*

$$\beta = 2\ln(\tan(\frac{\pi}{4}+\frac{\omega_2}{2})) + \frac{2}{\cos(\omega_2)}\left(\tan(\omega_2) - \omega_2 + \sqrt{\tan^2(\omega_2) - \omega_2\tan(\omega_2) + \omega_2{}^2}\right).$$

Then, for every Riemannian metric g on the Klein bottle K of conformal type β, we have the following optimal inequality:

$$\ell_\sigma(g)\ell_v(g)\ell_h(g) \leq C_\beta \cdot area(g)^{\frac{3}{2}},$$

where

$$C_\beta = \frac{\sqrt{\pi}}{3\sqrt{3}} \cdot \frac{\left(b^4 - 4b\omega_2 + {\omega_2}^2 + {\omega_2}^4 - 2b^2(-2 + {\omega_2}^2)\right)^{\frac{1}{4}} (2b - \omega_2)}{(b - \omega_2)\sqrt{(b - \omega_2)b}},$$

with $b = \tan(\omega_2) + \sqrt{\tan^2(\omega_2) - \omega_2 \tan(\omega_2) + {\omega_2}^2}$.

Moreover, the equality is attained if and only if g is proportional to the spherical-flat metric SF_{b,ω_2}.

Now, if we replace b by $\tan(\omega_2) + \sqrt{\tan^2(\omega_2) - \omega_2 \tan(\omega_2) + {\omega_2}^2}$ in

$$C(\omega_2) = \frac{\left(b^4 - 4b\omega_2 + {\omega_2}^2 + {\omega_2}^4 - 2b^2(-2 + {\omega_2}^2)\right)^{\frac{1}{4}} (2b - \omega_2)}{(b - \omega_2)\sqrt{(b - \omega_2)b}},$$

we obtain a continuous increasing function $C :]\frac{2}{\pi}, +\infty[\to]\frac{2\sqrt{\pi}}{3\sqrt{3}}, +\infty[$ which tends to infinity as $\omega_2 \to 0$ (i.e. when $b \to 0$). Therefore, we derive

Corollary 1.9. *There does not exist any (finite) universal constant C such that the inequality*

$$\ell_\sigma(g)\ell_v(g)\ell_h(g) \leq C \cdot area(g)^{\frac{3}{2}}$$

holds for every Riemannian metric g on the Klein bottle K.

Remark 1.10. Note that unlike Corollaries 1.5 and 1.9, Corollary 1.7 provides a uniform upper bound on the Riemannian ratio $\frac{L_\sigma(g)\ell_v(g)}{area(g)}$.

2. The main tools for the proofs

The key tool in our proofs is the method of extremal length initiated by B. Fuglede in [9], J. A. Jenkins in [13] and M. Gromov in [11]. It was used later by C. Bavard in the setting of isosystolic geometry ([4] and [5]). This method characterizes a conformally extremal Riemannian manifold by means of its closed geodesics. See, e.g., L. V. Ahlfors' book [2] and Rodin's paper [18] and the references therein for more details and further applications of the method of extremal length.

2.1. **A maximality criterion.** Let (M, g) be a closed Riemannian manifold and Γ be a family of rectifiable curves on M. For every Radon measure μ on Γ, we associate a measure ${}^*\mu$ on M by setting, for $\varphi \in C^0(M, \mathbb{R})$,

$$\langle {}^*\mu, \varphi \rangle = \langle \mu, \overline{\varphi} \rangle$$

where $\overline{\varphi}(\gamma) = \int \varphi \circ \gamma(s) ds$ and ds is the arc-length of γ with respect to g.

Theorem 2.1 ([4],[5] and [13]). *Let M be a closed manifold. Let* S_i*, where* $i \in \{1, \cdots, p\}$*, be some families of rectifiable curves on M such that* $S_i \cap S_j = \emptyset$ *for all* $i \neq j$*. Denote by* $\ell_i(g)$ *the least length of a curve in* S_i *with respect to a Riemannian metric g on M. Suppose that* g_e *is a Riemannian metric on M for which there exists a positive measure* μ *on* $\Gamma = S_1 \cup \cdots \cup S_p$ *satisfying the following three conditions:*

1. *for each* $i \in \{1, \cdots, p\}$*, all the curves in* S_i *have the same length with respect to* g_e*,*
2. $m_1\ell_1(g_e) = m_2\ell_2(g_e) = \cdots = m_p\ell_p(g_e)$*,*
3. ${}^*\mu = dg_e$*,*

where m_i is the mass of the measure μ on S_i and dg_e is the volume measure of (M, g_e). Then, for every Riemannian metric g on M conformal to g_e, we have

$$\frac{\ell_1(g)\cdots\ell_p(g)}{vol^{\frac{p}{2}}(g)} \leq \frac{\ell_1(g_e)\cdots\ell_p(g_e)}{vol^{\frac{p}{2}}(g_e)}. \tag{2.1}$$

Furthermore, the equality holds if and only if g is homothetic to g_e.

Since our version of Theorem 2.1 is slightly more general than in the aforementioned references (as it holds for an arbitrary number of curve families S_i), we present a proof of it. It is straightforward by the following lemma.

Lemma 2.2. *Let S_i, where $i \in \{1, \cdots, p\}$, be families of rectifiable curves on a given closed manifold M such that $S_i \cap S_j = \emptyset$ for all $i \neq j$. Denote by $\ell_i(g)$ the least length of a curve in S_i with respect to a Riemannian metric g on M. Let g_e be a Riemannian metric on M such that there exists a positive measure μ on $\Gamma = S_1 \cup \cdots \cup S_p$ satisfying the three conditions of Theorem 2.1. Then,*

$$\ell_1(g)\cdots\ell_p(g) \leq \frac{vol^{\frac{p}{2}}(g_e)}{p^p m_1 \cdots m_p} vol^{\frac{p}{2}}(g), \tag{2.2}$$

where m_i is the mass of the measure μ on S_i. Furthermore, the equality holds if and only if g is homothetic to g_e.

Proof. Let g be a Riemannian metric conformal to g_e, that is, $g = \phi^2 g_e$. We have

$$\begin{aligned}
m_1\ell_1(g) + \cdots + m_p\ell_p(g) &\leq \int_{S_1} \bar{\phi}(\gamma)d\mu(\gamma) + \cdots + \int_{S_p} \bar{\phi}(\gamma)d\mu(\gamma) && (2.3)\\
&= \int_\Gamma \bar{\phi}(\gamma)d\mu(\gamma)\\
&= \int_M \phi(x)d(^*\mu)(x)\\
&= \int_M \phi(x)dg_e\\
&\leq \left(\int_M \phi^2(x)dg_e\right)^{\frac{1}{2}} \left(\int_M dg_e\right)^{\frac{1}{2}} && (2.4)\\
&= \sqrt{vol(g)vol(g_e)}.
\end{aligned}$$

When g is equal to g_e, equality in (2.2) is attained because g_e satisfies condition (1). Moreover, it is straightforward that the equality in (2.3) is attained if and only if g is equal to g_e by Cauchy-Schwartz. Using the arithmetic and geometric means inequality, we derive

$$p.\big(m_1\cdots m_p\ell_1(g)\cdots\ell_p(g)\big)^{\frac{1}{p}} \leq m_1\ell_1(g) + \cdots + m_p\ell_p(g)$$

with equality if and only if

$$m_1\ell_1(g) = m_2\ell_2(g) = \cdots = m_p\ell_p(g).$$

Finally, combining the two inequalities, we obtain that, under the required conditions, the following inequality holds:

$$p^p.\big(m_1\cdots m_p\big).\ell_1(g)\cdots\ell_p(g) \leq \big(vol(g)\ vol(g_b)\big)^{\frac{p}{2}},$$

with equality if and only if g is equal to g_e. □

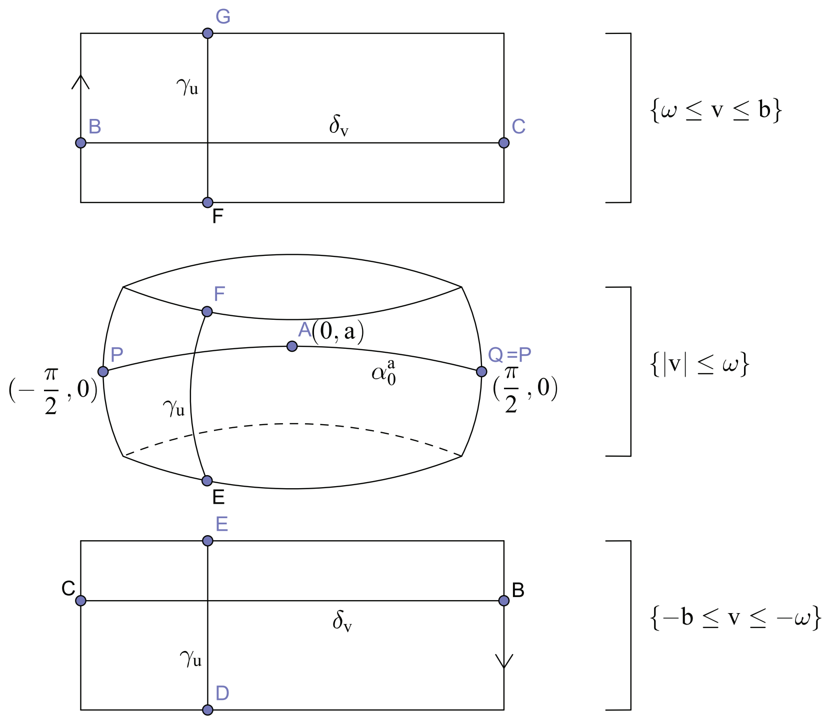

FIGURE 1. The curves α_θ^a, γ_u and δ_v in the Mobius band $\{(u,v) \mid |v| \leq b\}$ of $(K, SF_{b,\omega})$.

Thus, proving the conformal extremality of a metric requires finding disjoint families of rectifiable curves $S_1, \cdots, S_n$ on M and defining a measure on them that satisfies the three conditions of the previous theorem.

2.2. **Setting the curve families.** To prove our results, we will make use of three families of curves in $(K, SF_{b,\omega})$.

Definition 2.3. Let $(K, SF_{b,\omega})$ be the spherical-flat Klein bottle defined in Section 1.2. Define three families of loops as follows:

(1) For each $\theta \in \mathbb{R}/\pi\mathbb{Z}$ and each $a \in [-\omega, \omega] \cup [2b - \omega, 2b] \cup [-2b, -2b + \omega]$, the loop α_θ^a is the geodesic (image in $(K, SF_{b,\omega})$ of a great circle) going through the points $(\theta - \pi/2, 0)$, (θ, a). Let $S_1(\omega) = \{\alpha_\theta^a \mid |a| \leq \omega \text{ or } 2b - \omega \leq |a| \leq 2b, \theta \in \mathbb{R}/\pi\mathbb{Z}\}$.

(2) For each $u \in \mathbb{R}/\pi\mathbb{Z}$, γ_u is the vertical loop defined by $\gamma_u(t) = (u, t)$ with $|t| \leq 2b$. Let $S_2 = \{\gamma_u \mid u \in \mathbb{R}/\pi\mathbb{Z}\}$.

(3) For each $v \in \mathbb{R}$ satisfying $\omega \leq |v| \leq 2b - \omega$, δ_v is the horizontal loop defined by $\delta_v(t) = (t, |v|)$ with $|t| \leq \frac{\pi}{2}$. Let $S_3(\omega) = \{\delta_v \mid \omega \leq |v| \leq 2b - \omega\}$.

Remark 2.4. Note that for each $i \in \{1, 2, 3\}$, the curves in S_i have the same length with respect to the metric $SF_{b,\omega}$. Also note that a curve in S_1, resp. S_2, resp. S_3, belongs to the family $\mathcal{F}_\sigma$, resp. $\mathcal{F}_v$, resp. $\mathcal{F}_h$ (cf. Section 1.1).

We move now to the next step, i.e., verifying the three conditions of Theorem 2.1. Conditions (1) and (2) are easy to verify while condition (3) requires technical (but not simple) methods of calculation.

In the following we will consider a subset of the family of curves $S_1(\omega)$ in $(K, SF_{b,\omega})$ and equip it with a measure μ depending on a function h of one parameter (since $SF_{b,\omega}$ has an isometry group of dimension 1). Then, we will calculate the measure $^*\mu$ in terms of the volume measure of $SF_{b,\omega}$.

2.3. **The calculation of $^*\mu$ on a spherical region of K.** Let $b \in (0, +\infty)$ and $\omega \in (0, b]$. We consider on $(K, SF_{b,\omega})$ the family of curves $S'_1(\omega) \subset S_1(\omega)$ defined by

$$S'_1(\omega) = \{\alpha^a_\theta \subset (K, SF_{b,\omega}) \mid -\omega \leq a \leq \omega, \theta \in \mathbb{R}/\pi\mathbb{Z}\},$$

where α^a_θ is the great circle introduced in Definition 2.3.

Lemma 2.5. *Let μ be a measure on $S'_1(\omega)$ defined by*

$$d\mu(\alpha^a_\theta) = \begin{cases} h(a)\ da\ d\theta & \text{if} \quad a \geq 0, \\ h(-a)\ da\ d\theta & \text{if} \quad a < 0, \end{cases}$$

where $h : [0, \omega] \to \mathbb{R}$ is a continuous function. Then, we have

$$^*\mu = 2\chi_{\{|v|\leq\omega\}}\Big(\int_{|v|}^{\omega} \big(\cos^2(v) - \cos^2(a)\big)^{-\frac{1}{2}} h(a)\ da\Big) \cdot d(SF_{b,\omega})$$

where $d(SF_{b,\omega})$ is the volume measure of $SF_{b,\omega}$.

Proof. First, we compute the equation of α^a_θ. Suppose $\alpha^a_\theta(t) = (u(t), v(t))$. Then, from the classical geodesic equation $\frac{d^2x^i}{dt^2} + \sum_{j,k} \Gamma^i_{j,k} \frac{dx^k}{dt}\frac{dx^j}{dt} = 0$ (see for example [10, p. 81]), where $\Gamma^i_{j,k}$ are the Christoffel symbols, $x^1 = u$ and $x^2 = v$, we derive

$$u'' - 2\tan(v)u'v' = 0 \quad \text{and} \quad v'' + \sin(v)\cos(v)u'^2 = 0.$$

This shows that

$$\frac{du}{dv} = \frac{c}{\cos(v)\sqrt{\cos^2(v) - c^2}}, \tag{2.5}$$

where c is a constant. The solution of the differential equation (2.5) is $\sin(u - d) = c\tan(v)$, where d is a constant. Using the fact that α^a_θ goes through $(\theta - \pi/2, 0)$ and (θ, a), we obtain

$$\sin(u - \theta + \frac{\pi}{2}) = \frac{1}{\tan(a)}\tan(v). \tag{2.6}$$

Equation (2.6) shows that we can write $\alpha^a_\theta(u) = (u + \theta, v\,(u, a))$, with v verifying $v(u, -a) = -v(u, a)$.

Now, let $\phi \in C^0(K,\mathbb{R})$. Then, by the definition of ${}^*\mu$, we have

$$\int_K \phi(u,v)\ d({}^*\mu)(u,v) = \int_{S_1'(\omega)} \bar{\phi}(\alpha_\theta^a)\ d\mu(\alpha_\theta^a)$$

where

$$\begin{aligned}\bar{\phi}(\alpha_\theta^a) &= \int_{\alpha_\theta^a} \phi\left(\alpha_\theta^a(s)\right) ds \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi(u+\theta, v(u,a))\sqrt{\cos^2(v(u,a)) + \left(\frac{\partial v}{\partial u}(u,a)\right)^2}\, du.\end{aligned}$$

Then

$$\int_K \phi(u,v)\ d({}^*\mu)(u,v) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\omega}^{0} \bar{\phi}(\alpha_\theta^a)h(-a)\ da\ d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{\omega} \bar{\phi}(\alpha_\theta^a)h(a)\ da\ d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\omega}\int_{-\frac{\pi}{2}}^{0} \phi(u+\theta,-v(u,a))\sqrt{\cos^2(v(u,a)) + \left(\frac{\partial v}{\partial u}(u,a)\right)^2}\, h(a)\ du\ da\ d\theta$$

$$+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\omega}\int_{0}^{\frac{\pi}{2}} \phi(u+\theta,-v(u,a))\sqrt{\cos^2(v(u,a)) + \left(\frac{\partial v}{\partial u}(u,a)\right)^2}\, h(a)\ du\ da\ d\theta$$

$$+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\omega}\int_{-\frac{\pi}{2}}^{0} \phi(u+\theta,v(u,a))\sqrt{\cos^2(v(u,a)) + \left(\frac{\partial v}{\partial u}(u,a)\right)^2}\, h(a)\ du\ da\ d\theta$$

$$+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\omega}\int_{0}^{\frac{\pi}{2}} \phi(u+\theta,v(u,a))\sqrt{\cos^2(v(u,a)) + \left(\frac{\partial v}{\partial u}(u,a)\right)^2}\, h(a)\ du\ da\ d\theta.$$

Next, we apply for the first two integrals in the previous expression the change of variables $y \mapsto -v(u,a)$, whose Jacobian is equal to $-\frac{\partial v}{\partial u}(u,a)$, and we write $k_1(y,a) = u$. For the last two integrals, we make use of the change of variables $y \mapsto v(u,a)$, whose Jacobian is equal to $\frac{\partial v}{\partial u}(u,a)$, and we write $k_2(y,a) = u$. In either case, let $z(y,a) = \frac{\partial v}{\partial u}(u,a)$. We derive

$$\begin{aligned}&\int_K \phi(u,v)\ d({}^*\mu)(u,v) \\ &\quad = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\omega}\int_{-a}^{0} \phi(k_1(y,a)+\theta,y)\sqrt{\cos^2(y)+z^2(y,a)}\frac{h(a)}{|z(y,a)|}\ dy\ da\ d\theta \\ &\quad\quad + 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\omega}\int_{0}^{a} \phi(k_2(y,a)+\theta,y)\sqrt{\cos^2(y)+z^2(y,a)}\frac{h(a)}{|z(y,a)|}\ dy\ da\ d\theta.\end{aligned}$$

Next, in the previous expression, we make the change of variables $x \mapsto k_1(y,a)+\theta$ for the first integral and $x \mapsto k_2(y,a)+\theta$ for the second integral. The Jacobian for both changes of variables is equal to 1. We obtain

$$\int_K \phi(u,v)\ d({}^*\mu)(u,v) = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\omega}\int_{-a}^{0} \phi(x,y)\sqrt{\cos^2(y)+z^2(y,a)}\frac{h(a)}{|z(y,a)|}\ dy\ da\ dx$$

$$+2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\omega}\int_{0}^{a} \phi(x,y)\sqrt{\cos^2(y)+z^2(y,a)}\frac{h(a)}{|z(y,a)|}\ dy\ da\ dx$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\omega}^{0}\int_{-y}^{\omega}\phi(x,y)\sqrt{\cos^2(y)+z^2(y,a)}\frac{h(a)}{|z(y,a)|}\ da\ dy\ dx$$

$$+2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{\omega}\int_{y}^{\omega}\phi(x,y)\sqrt{\cos^2(y)+z^2(y,a)}\frac{h(a)}{|z(y,a)|}\ da\ dy\ dx$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\omega}^{\omega}\phi(x,y)\int_{|y|}^{\omega}\sqrt{\left(\left(\frac{\cos(y)}{z(y,a)}\right)^2+1\right)}h(a)\ da\ dy\ dx.$$

Hence, we derive

$$^*\mu = 2\frac{\chi_{\{|v|\le\omega\}}}{\cos(v)}\int_{|v|}^{\omega}\sqrt{\left(\left(\frac{\cos(v)}{z(v,a)}\right)^2+1\right)}h(a)da\cdot d(SF_{b,\omega})$$

where $d(SF_{b,\omega})$ is the volume element of $(K, SF_{b,\omega})$.

From equation (2.5), we obtain

$$z^2(v,a) = \frac{\cos^2(v)}{\cos^2(a)}\left(\cos^2(v)-\cos^2(a)\right).$$

This shows that

$$^*\mu = 2\chi_{\{|v|\le\omega\}}\left(\int_{|v|}^{\omega}\left(\cos^2(v)-\cos^2(a)\right)^{-\frac{1}{2}}h(a)da\right)\cdot d(SF_{b,\omega}).$$

□

3. Proof of the geometric inequality of type $\ell_\sigma\ell_v$

In the following, we prove the geometric inequality of type $\ell_\sigma\ell_v$ for

$$\beta > 2\ln\left(\tan\left(\frac{\pi}{4}+\frac{a_0}{2}\right)\right).$$

As in the case $\beta \le 2\ln\left(\tan\left(\frac{\pi}{4}+\frac{a_0}{2}\right)\right)$, we will consider both curve families $S_1(\omega_1)$ and S_2 (cf. [6, Corollary 2]).

Proof of Theorem 1.3. Let $\beta > 2\ln\left(\tan\left(\frac{\pi}{4}+\frac{a_0}{2}\right)\right)$. There exists a unique $\omega_1 \in [a_0, \frac{\pi}{2})$ such that

$$2\sin(\omega_1) = \left(\beta - 2\ln\left(\tan\left(\frac{\pi}{4}+\frac{\omega_1}{2}\right)\right)\right)\cos^2(\omega_1) + 4\omega_1\cos(\omega_1). \tag{3.1}$$

We endow K with the metric SF_{b,ω_1} defined in Section 1.2, where

$$b = \tan(\omega_1) - \omega_1. \tag{3.2}$$

Consider the two families of curves $S_1(\omega_1)$ and S_2 (cf. Section 2.2). Each curve in $S_1(\omega_1)$ (resp. S_2) has length equal to π (resp. $4b$) with respect to SF_{b,ω_1}. Hence, condition (1) of Theorem 2.1 is satisfied. Now, let

$$\begin{array}{rccc} h: & [-\omega_1,\omega_1] & \longrightarrow & \mathbb{R} \\ & a & \mapsto & \frac{\sin(|a|)}{\pi\cos(a)}\sqrt{\cos^2(a)-\cos^2(\omega_1)} \end{array}.$$

We define on the family $S_1(\omega_1)$ the measure

$$\mu_1 = \widetilde{h}(a)\ da \otimes d\theta$$

where

$$\widetilde{h}(a) = \begin{cases} h(a) & \text{if} \quad a \in [-\omega_1, \omega_1], \\ h(a-2b) & \text{if} \quad a \in [2b-\omega_1, 2b], \\ h(a+2b) & \text{if} \quad a \in [-2b, -2b+\omega_1]. \end{cases}$$

On the family S_2, we define the measure

$$\mu_2 = \cos(\omega_1) du.$$

The mass m_1 of the measure μ_1 on $S_1(\omega_1)$ is equal to $4\sin(\omega_1) - 4\omega_1 \cos(\omega_1)$ and the mass m_2 of the measure μ_2 on S_2 is equal to $\pi \cos(\omega_1)$. Now, since $\ell_\sigma(SF_{b,\omega_1}) = \pi$ and $\ell_v(SF_{b,\omega_1}) = 4b$, condition (2) of Theorem 2.1 is satisfied. Next, it can be easily verified that

$$^*\mu_2 = \frac{\cos(\omega_1)}{f_{b,\omega_1}(v)} \cdot d(SF_{b,\omega_1})$$

where f_{b,ω_1} is the unique one-variable function invariant by the translation $(u, v) \mapsto (u, v+2b)$ which agrees with *cosine* on $[-\omega_1, \omega_1]$ and is equal to the constant $\cos(\omega_1)$ on $[\omega_1, 2b - \omega_1]$ (see Section 1.2). Moreover, by Lemma 2.5, we have

$$\chi_{\{|v| \leq \omega_1\}} \, {}^*\mu_1 = 2\chi_{\{|v| \leq \omega_1\}} \Big(\int_{|v|}^{\omega_1} \big(\cos^2(v) - \cos^2(a)\big)^{-\frac{1}{2}} \widetilde{h}(a) \, da \Big) \cdot d(SF_{b,\omega_1}).$$

Then, we derive

$$\chi_{\{|v| \leq \omega_1\}} \, {}^*\mu_1 = \chi_{\{|v| \leq \omega_1\}} \, \big(1 - \frac{\cos(\omega_1)}{f_{b,\omega_1}(v)}\big) \cdot d(SF_{b,\omega_1}).$$

Finally, since f_{b,ω_1} and $\widetilde{h}$ are invariant by the translation $(u, v) \mapsto (u, v + 2b)$, we have

$$^*\mu_1 = \big(1 - \frac{\cos(\omega_1)}{f_{b,\omega_1}(v)}\big) \cdot d(SF_{b,\omega_1}).$$

Hence, we get $^*\mu_1 +^* \mu_2 = d(SF_{b,\omega_1})$. This is condition (3) of Theorem 2.1. Finally, from Theorem 2.1 and inequality (2.2), we derive

$$\ell_\sigma(g)\ell_v(g) \leq \frac{1}{2\cos(\omega_1)} area(g)$$

which holds for every Riemannian metric g conformal to SF_{b,ω_1}. The equality is attained if and only if g is homothetic to SF_{b,ω_1}, and the result follows. □

Recall that M denotes the Mobius band with boundary obtained by taking the quotient of $\mathbb{R} \times [-\beta, \beta]$ by the group generated by the map $\sigma : z \mapsto \bar{z} + \pi$. We denote by g_β the flat metric induced by such a quotient. The parameter β represents the conformal type of any metric conformal to g_β.

We recover Blatter's result for the Mobius band ([7, Theorem 2]) by a simpler method. Indeed, consider in Bavard's proof of Theorem 1.2 and in our previous proof the restriction of S_b and SF_{b,ω_1} on the set $E_b = \{(u, v) \in \mathbb{C} \mid |v| \leq b\}$.

Corollary 3.1 ([7, Satz 2]). *Let* $\beta > 0$ *and* $a_0 \in (0, \frac{\pi}{2})$ *such that* $\tan(a_0) = 2a_0$. *Let* $\omega_1 \in [a_0, \frac{\pi}{2})$ *be defined by the equation*

$$\sin(\omega_1) = \Big(\beta - \ln\Big(\tan\Big(\frac{\pi}{4} + \frac{\omega_1}{2}\Big)\Big)\Big) \cos^2(\omega_1) + 2\omega_1 \cos(\omega_1).$$

Then, for every Riemannian metric g *on the Mobius band* M *of conformal type* β, *we have*

$$\ell_\sigma(g)\ell_v(g) \leq C_\beta \cdot area(g),$$

where

$$C_\beta = \begin{cases} \frac{e^{2\beta}+1}{e^{2\beta}-1} \arcsin\left(\frac{e^{2\beta}-1}{e^{2\beta}+1}\right) & \text{if } 0 < \beta \leq \ln\left(\frac{\pi}{4} + \frac{a_0}{2}\right), \\ \frac{1}{2\cos(\omega_1)} & \text{if } \beta > \ln\left(\frac{\pi}{4} + \frac{a_0}{2}\right). \end{cases}$$

Moreover, the equality is attained if and only if g is proportional to the spherical metric S_b restricted to $E_b = \{(u,v) \in \mathbb{C} \mid |v| \leq b\}$, for b satisfying

$$\beta = 2\ln(\tan(\frac{\pi}{4} + \frac{b}{2})),$$

in the first case, and to the spherical-flat metric SF_{b,ω_1} restricted to $E_b = \{(u,v) \in \mathbb{C} \mid |v| \leq b\}$, for $b = \tan(\omega_1) - \omega_1$, in the second case.

4. Proof of the geometric inequality of type $L_\sigma \ell_v$

In the following, we prove the geometric inequality of type $L_\sigma \ell_v$ on the Klein bottle. The curve families we consider this time are $S_1(\frac{\pi}{3})$, S_2 and $S_3(\frac{\pi}{3})$.

Proof of Theorem 1.6. The inequality in the first case can be deduced from Theorem 1.2 since when $0 < \beta \leq 2\ln\left(2+\sqrt{3}\right)$, i.e., when $b \leq \frac{\pi}{3}$, we have $L_\sigma(S_b) = \ell_\sigma(S_b)$. Now, since the conformal type of (K, S_b) is

$$\beta = 2\ln(\tan(\frac{\pi}{4} + \frac{b}{2})),$$

we deduce that for $0 < \beta \leq 2\ln\left(2+\sqrt{3}\right)$,

$$L_\sigma(g)\ell_v(g) \leq \frac{\arcsin\left(\frac{e^\beta-1}{e^\beta+1}\right)}{\frac{e^\beta-1}{e^\beta+1}} \, area(g).$$

Note that when b becomes greater than $\frac{\pi}{3}$, the horizontal geodesic loops closed by the horizontal translation t_h (corresponding to the singularity line of S_b) become shorter than the curves α_θ^a and therefore L_σ is attained by such lines.

To prove the inequality in the second case, we fix $\beta > 2\ln\left(2+\sqrt{3}\right)$ and let b such that $\beta = 2\ln(2+\sqrt{3}) + 4(b - \frac{\pi}{3})$ (we have $b \geq \frac{\pi}{3}$). Then we endow K with the metric $SF_{b,\frac{\pi}{3}}$. We consider the two families of curves $S = S_1(\frac{\pi}{3}) \cup S_3(\frac{\pi}{3})$ and S_2 (cf. Section 2.2). Each curve in S, (resp. S_2) has length equal to π, (resp. $4b$) with respect to the metric $SF_{b,\frac{\pi}{3}}$. Hence, condition (1) of Theorem 2.1 is satisfied. Now, let

$$\begin{array}{cccc} k: & [-\frac{\pi}{3}, \frac{\pi}{3}] & \longrightarrow & \mathbb{R} \\ & a & \mapsto & \frac{\tan(|a|)}{24\pi b} \cdot \frac{24b\cos^2(a) - 3\sqrt{3} - 3b + \pi}{\sqrt{\cos^2(a) - \frac{1}{4}}}. \end{array}$$

We define on the family S the measure

$$\mu_1 = \widetilde{k}(a)\ da \otimes d\theta + \frac{-3\sqrt{3} + 3b + \pi}{6b} dv,$$

where

$$\widetilde{k}(a) = \begin{cases} k(a) & \text{if} \quad a \in [-\frac{\pi}{3}, \frac{\pi}{3}], \\ k(a-2b) & \text{if} \quad a \in [2b - \frac{\pi}{3}, 2b], \\ k(a+2b) & \text{if} \quad a \in [-2b, -2b + \frac{\pi}{3}]. \end{cases}$$

On the family S_2, we define the measure

$$\mu_2 = \frac{3\sqrt{3} + 3b - \pi}{12b} du.$$

Note that these measures are positive since $b \geq \frac{\pi}{3}$. The mass m_1 of the measure μ_1 on S is equal to $\sqrt{3}+3b-\frac{\pi}{3}$, and the mass m_2 of the measure μ_2 on S_2 is equal to $\frac{3\sqrt{3}+3b\pi-\pi^2}{12b}$. Now, since $L_\sigma(SF_{b,\frac{\pi}{3}}) = \pi$ and $\ell_v(SF_{b,\frac{\pi}{3}}) = 4b$, condition (2) of Theorem 2.1 is satisfied. Next, it can be easily verified that

$$^*\mu_2 = \frac{3\sqrt{3}+3b-\pi}{12bf_{b,\frac{\pi}{3}}(v)} \cdot d(SF_{b,\frac{\pi}{3}})$$

where $f_{b,\frac{\pi}{3}}$ is the unique one-variable function invariant by the translation $(u,v) \mapsto (u, v+2b)$ which agrees with *cosine* on $[-\frac{\pi}{3}, \frac{\pi}{3}]$ and is equal to the constant $\frac{1}{2}$ on $[\frac{\pi}{3}, 2b-\frac{\pi}{3}]$. Moreover, by Lemma 2.5 we obtain

$$\chi_{\{|v|\leq\frac{\pi}{3}\}}\ ^*\mu_1 = 2\chi_{\{|v|\leq\frac{\pi}{3}\}}\Big(\int_{|v|}^{\frac{\pi}{3}} \left(\cos^2(v)-\cos^2(a)\right)^{-\frac{1}{2}} \widetilde{k}(a)\ da\Big) \cdot d(SF_{b,\frac{\pi}{3}}).$$

Then, we derive

$$\chi_{\{|v|\leq\frac{\pi}{3}\}}\ ^*\mu_1 = \chi_{\{|v|\leq\frac{\pi}{3}\}}\ \big(1-\frac{3\sqrt{3}+3b-\pi}{12bf_{b,\frac{\pi}{3}}(v)}\big) \cdot d(SF_{b,\frac{\pi}{3}}).$$

Finally, since $f_{b,\frac{\pi}{3}}$ and $\widetilde{k}$ are invariant by the translation $(u,v) \mapsto (u, v+2b)$, we have

$$^*\mu_1 = \begin{cases} \big(1-\frac{3\sqrt{3}+3b-\pi}{12bf_{b,\frac{\pi}{3}}(v)}\big)\cdot d(SF_{b,\frac{\pi}{3}}) & \text{if} \quad |v| \leq \frac{\pi}{3} \text{ or } 2b-\frac{\pi}{3} \leq |v| \leq 2b, \\ \frac{-3\sqrt{3}+3b+\pi}{6b} \cdot d(SF_{b,\frac{\pi}{3}}) & \text{if} \quad \frac{\pi}{3} \leq |v| \leq 2b-\frac{\pi}{3}. \end{cases}$$

Hence, we get $^*\mu_1 +^* \mu_2 = d(SF_{b,\frac{\pi}{3}})$. Therefore, condition (3) of Theorem 2.1 is satisfied. It follows from Theorem 2.1 and inequality (2.2) that for every Riemannian metric g conformal to $SF_{b,\frac{\pi}{3}}$, we have

$$L_\sigma(g)\ell_v(g) \leq \frac{2b}{\sqrt{3}+b-\frac{\pi}{3}} area(g)$$

with equality if and only if g is homothetic to $SF_{b,\frac{\pi}{3}}$. □

We can also derive C. Blatter's similar result on the Mobius band M (Theorem 3 in [7]) using a minor adaptation of the previous proof.

Corollary 4.1 (see [7, Satz 3])**.** *Let $\beta > 0$. Then, for every Riemannian metric g on the Mobius band M of conformal type β, we have*

$$sys(g)\ell_v(g) \leq C_\beta \cdot area(g),$$

where

$$C_\beta = \begin{cases} \frac{e^{2\beta}+1}{e^{2\beta}-1}\arcsin\left(\frac{e^{2\beta}-1}{e^{2\beta}+1}\right) & \textit{if} \quad 0 < \beta \leq \ln\left(2+\sqrt{3}\right), \\ \frac{2}{3}.\frac{3\beta+2\pi-3\ln(2+\sqrt{3})}{2\sqrt{3}+\beta-\ln(2+\sqrt{3})} & \textit{if} \quad \beta > \ln\left(2+\sqrt{3}\right). \end{cases}$$

Moreover, the equality is attained if and only if g is proportional to the spherical metric S_b restricted to $E_b = \{(u,v) \in \mathbb{C} \mid |v| \leq b\}$, for b satisfying $\beta = 2\ln(\tan(\frac{\pi}{4}+\frac{b}{2}))$, in the first case, and to the spherical-flat metric $SF_{b,\frac{\pi}{3}}$ restricted to $E_b = \{(u,v) \in \mathbb{C} \mid |v| \leq b\}$, for $b = \frac{1}{4}\beta+\frac{\pi}{3}-\frac{1}{2}\ln(2+\sqrt{3})$, in the second case.

5. Proof of the geometric inequality of type $\ell_\sigma \ell_v \ell_h$

In the following, we prove the geometric inequality of type $\ell_\sigma \ell_v \ell_h$ on the Klein bottle. We will consider the curve families $S_1(\omega_2)$, S_2 and $S_3(\omega_2)$.

Proof of Theorem 1.8. Let $\beta > 0$. There exists a unique $\omega_2 \in (0, \frac{\pi}{2})$ such that

$$\beta = 2\ln(\tan(\frac{\pi}{4}+\frac{\omega_2}{2}))+\frac{2}{\cos(\omega_2)}\left(\tan(\omega_2)-\omega_2+\sqrt{\tan^2(\omega_2)-\omega_2\tan(\omega_2)+{\omega_2}^2}\right). \tag{5.1}$$

We equip K with the metric SF_{b,ω_2}, where

$$b = \tan(\omega_2)+\sqrt{\tan^2(\omega_2)-\omega_2\tan(\omega_2)+{\omega_2}^2}\,. \tag{5.2}$$

Then, we consider the families of curves $S_1(\omega_2)$, S_2 and $S_3(\omega_2)$ (cf. Section 2.2). Each curve in $S_1(\omega_2)$, (resp. S_2, resp. $S_3(\omega_2)$) has length equal to π, (resp. $4b$, resp. $2\pi\cos(\omega_2)$) with respect to the metric SF_{b,ω_2}. Hence, condition (1) of Theorem 2.1 is satisfied. Let

$$\begin{array}{cccc} l: & [-\omega_2,\omega_2] & \longrightarrow & \mathbb{R} \\ & a & \mapsto & \frac{\tan(|a|)}{\pi}.\frac{\cos^2(a)-\cos^2(\omega_2)\frac{b-\omega_2}{2b-\omega_2}}{(\cos^2(a)-\cos^2(\omega_2))^{\frac{1}{2}}}. \end{array}$$

We define on the family $S_1(\omega_2)$ the measure

$$\mu_1 = \widetilde{l}(a)\ da \otimes d\theta$$

where

$$\widetilde{l}(a) = \begin{cases} l(a) & \text{if} \quad a \in [-\omega_2,\omega_2], \\ l(a-2b) & \text{if} \quad a \in [2b-\omega_2, 2b], \\ l(a+2b) & \text{if} \quad a \in [-2b,-2b+\omega_2]. \end{cases}$$

On the family S_2, we define the measure

$$\mu_2 = \cos(\omega_2).\frac{b-\omega_2}{2b-\omega_2}du.$$

Finally, on $S_3(\omega_2)$, we define the measure

$$\mu_3 = \frac{b}{2b-\omega_2}dv.$$

The masses m_i of the measures μ_i are $m_1 = 4\sin(\omega_2) - \frac{4\omega_2(b-\omega_2)\cos(\omega_2)}{2b-\omega_2}$, $m_2 = \frac{\pi\cos(\omega_2)(b-\omega_2)}{2b-\omega_2}$ and $m_3 = \frac{2b(b-\omega_2)}{2b-\omega_2}$. Now, since $\ell_\sigma(SF_{b,\omega_2}) = \pi$, $\ell_v(SF_{b,\omega_2}) = 4b$, and $\ell_h(SF_{b,\omega_2}) = 2\pi\cos(\omega_2)$, condition (2) of Theorem 2.1 is satisfied. Next, it can be easily verified that

$$^*\mu_2 = \frac{\cos(\omega_2)}{f_{b,\omega_2}(v)}.\frac{b-\omega_2}{2b-\omega_2}\cdot d(SF_{b,\omega_2})$$

where f_{b,ω_2} is the unique one-variable function invariant by the translation $(u,v)\mapsto(u,v+2b)$ which agrees with *cosine* on $[-\omega_2,\omega_2]$ and is equal to the constant $\cos(\omega_2)$ on $[\omega_2, 2b-\omega_2]$. Moreover,

$$^*\mu_3 = \begin{cases} 0 & \text{if} \quad |v|\le\omega_2 \text{ or } 2b-\omega_2\le|v|\le 2b, \\ \frac{b}{2b-\omega_2}\cdot d(SF_{b,\omega_2}) & \text{if} \quad \omega_2\le|v|\le 2b-\omega_2. \end{cases}$$

Next, by Lemma 2.5, we have

$$\chi_{\{|v|\le\omega_2\}}\ {^*\mu_1} = 2\chi_{\{|v|\le\omega_2\}}\Big(\int_{|v|}^{\omega_2}\left(\cos^2(v)-\cos^2(a)\right)^{-\frac{1}{2}}\widetilde{l}(a)\ da\Big)\cdot d(SF_{b,\omega_2}).$$

Then, we derive

$$\chi_{\{|v|\leq\omega_2\}}\ {}^*\mu_1 = \chi_{\{|v|\leq\omega_2\}}\ (1 - \frac{\cos(\omega_2)(b-\omega_2)}{f_{b,\omega_2}(v)(2b-\omega_2)}) \cdot d(SF_{b,\omega_2}).$$

Finally, since f_{b,ω_2} and $\widetilde{l}$ are invariant by the translation $(u, v) \mapsto (u, v + 2b)$, we have

$${}^*\mu_1 = \begin{cases} (1 - \frac{\cos(\omega_2)(b-\omega_2)}{f_{b,\omega_2}(v)(2b-\omega_2)}) \cdot d(SF_{b,\omega_2}) & \text{if} \quad |v| \leq \omega_2 \text{ or } 2b - \omega_2 \leq |v| \leq 2b, \\ 0 & \text{if} \quad \omega_2 \leq |v| \leq 2b - \omega_2. \end{cases}$$

Thus, we obtain ${}^*\mu_1 +^* \mu_2 +^* \mu_3 = dSF_{b,\omega_2}$. Hence, condition (3) of Theorem 2.1 is satisfied. Hence,

$$area(g)^{\frac{3}{2}} \geq C'_\beta \cdot \ell_\sigma(g)\ell_v(g)\ell_h(g)$$

where $C'_\beta = \frac{3\sqrt{3}}{\sqrt{\pi}} \cdot \frac{(b-\omega_2)\sqrt{(b-\omega_2)b}}{(b^4-4b\omega_2+\omega_2{}^2+\omega_2{}^4-2b^2(-2+\omega_2{}^2))^{\frac{1}{4}}(2b-\omega_2)}$, with equality if and only if g is homothetic to SF_{b,ω_2}. □

6. Some remarks on the extremal metrics

In this paper, we present four families of conformally extremal metrics for different types of inequalities on the Klein bottle. They are as follows:

(1) The spherical metric S_b, where $\beta = 2\ln(\tan(\frac{\pi}{4} + \frac{b}{2}))$, which is conformally extremal for the inequality of type $\ell_\sigma\ell_v$ for $0 < \beta \leq 2\ln\big(\tan(\frac{\pi}{4} + \frac{a_0}{2})\big)$, and for the inequality of type $L_\sigma\ell_v$ for $0 < \beta \leq 2\ln(2 + \sqrt{3})$.

(2) The spherical-flat metric SF_{b,ω_1}, where ω_1 and b verify the equations (3.1) and (3.2), which is conformally extremal for the inequality of type $\ell_\sigma\ell_v$ for $2\ln\big(\tan(\frac{\pi}{4} + \frac{a_0}{2})\big) < \beta < \infty$.

(3) The spherical-flat metric $SF_{b,\frac{\pi}{3}}$, where $\beta = 2\ln(2 + \sqrt{3}) + 4(b - \frac{\pi}{3})$, which is conformally extremal for the inequality of type $L_\sigma\ell_v$ for $2\ln(2 + \sqrt{3}) < \beta < \infty$.

(4) The spherical-flat metric SF_{b,ω_2}, where ω_2 and b verify the equations (5.1) and (5.2), which is conformally extremal for the inequality of type $\ell_\sigma\ell_v\ell_h$ for any β.

Remark 6.1. We emphasize the fact that the metrics SF_{b,ω_1} and SF_{b,ω_2} do not have the same conformal type since ω_1 verifies the equation (3.1) whereas ω_2 verifies the equation (5.1). Note that the equality $\omega_1 = \omega_2$ leads to

$$\sqrt{\tan^2(\omega_1) - \omega_1\tan(\omega_1) + \omega_1{}^2} + \omega_1 = 0$$

which is not possible.

Moreover, the β-extremal metric (extremal metric in the conformal class of β) for the inequality of type $\ell_\sigma\ell_v\ell_h$ agrees with the β-extremal metric for the inequality of type $L_\sigma\ell_v$ if and only if

$$\beta = \beta_0 = 2\ln(2 + \sqrt{3}) + 4(\sqrt{3} + (3 - \frac{\pi}{\sqrt{3}} + \frac{\pi^2}{9})^{\frac{1}{2}} - \frac{\pi}{3})$$

(i.e., $b = \sqrt{3} + (3 - \frac{\pi}{\sqrt{3}} + \frac{\pi^2}{9})^{\frac{1}{2}}$ and $\omega_2 = \frac{\pi}{3}$). On the other hand, the β-extremal metric for the inequality of type $\ell_\sigma\ell_v$ agrees with the β-extremal metric for the inequality of type $L_\sigma\ell_v$ if and only if $\beta \leq 2\ln(2 + \sqrt{3})$. Finally, the β-extremal metric for the inequality of $\ell_\sigma \ell_v$ and the β-extremal metric for the inequality of type $\ell_\sigma \ell_v \ell_h$ never agree.

The following table summarizes our study of the extremal metrics for the various inequality types relative to the conformal type β:

Ineq. \ β	$(0, 2\ln(2+\sqrt{3})]$	$\left(2\ln(2+\sqrt{3}), 2\ln\left(\tan(\frac{\pi}{4}+\frac{a_0}{2})\right)\right]$	$\left(2\ln\left(\tan(\frac{\pi}{4}+\frac{a_0}{2})\right), +\infty\right)$
$\ell_\sigma \ell_v \leq C_\beta area$	S_b	S_b	SF_{b,ω_1}
$L_\sigma \ell_v \leq C_\beta area$	S_b	$SF_{b,\frac{\pi}{3}}$	$SF_{b,\frac{\pi}{3}}$
$\ell_\sigma \ell_v \ell_h \leq C_\beta area^{\frac{3}{2}}$	SF_{b,ω_2}	SF_{b,ω_2}	SF_{b,ω_2}

Acknowledgements

The authors would like to thank Stéphane Sabourau for useful discussions and many interesting remarks and comments which helped to improve the paper. Many thanks to the referee for the professional work in revising an earlier version of the paper which led to a better presentation of some details that can enhance the understanding of the reader.

References

[1] Colin C. Adams and Alan W. Reid, *Systoles of hyperbolic* 3*-manifolds*, Math. Proc. Cambridge Philos. Soc. **128** (2000), no. 1, 103–110, DOI 10.1017/S0305004199003990. MR1724432 (2000h:53049)

[2] Lars V. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973. McGraw-Hill Series in Higher Mathematics. MR0357743 (50 #10211)

[3] C. Bavard, *Inégalité isosystolique pour la bouteille de Klein* (French), Math. Ann. **274** (1986), no. 3, 439–441, DOI 10.1007/BF01457227. MR842624 (87i:53059)

[4] C. Bavard, *Inégalités isosystoliques conformes pour la bouteille de Klein* (French, with English summary), Geom. Dedicata **27** (1988), no. 3, 349–355, DOI 10.1007/BF00181499. MR960206 (89k:53012)

[5] Christophe Bavard, *Inégalités isosystoliques conformes* (French), Comment. Math. Helv. **67** (1992), no. 1, 146–166, DOI 10.1007/BF02566493. MR1144618 (93f:53033)

[6] Christophe Bavard, *Une remarque sur la géométrie systolique de la bouteille de Klein* (French, with English summary), Arch. Math. (Basel) **87** (2006), no. 1, 72–74, DOI 10.1007/s00013-006-1665-2. MR2246408 (2007b:53083)

[7] Christian Blatter, *Zur Riemannschen Geometrie im Grossen auf dem Möbiusband* (German), Compositio Math. **15** (1961), 88–107 (1961). MR0140060 (25 #3484)

[8] Chady El Mir and Jacques Lafontaine, *Sur la géométrie systolique des variétés de Bieberbach* (French, with English summary), Geom. Dedicata **136** (2008), 95–110, DOI 10.1007/s10711-008-9276-7. MR2443345 (2009f:53057)

[9] Bent Fuglede, *Extremal length and functional completion*, Acta Math. **98** (1957), 171–219. MR0097720 (20 #4187)

[10] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine, *Riemannian geometry*, 3rd ed., Universitext, Springer-Verlag, Berlin, 2004. MR2088027 (2005e:53001)

[11] Mikhael Gromov, *Filling Riemannian manifolds*, J. Differential Geom. **18** (1983), no. 1, 1–147. MR697984 (85h:53029)

[12] James J. Hebda, *Two geometric inequalities for the torus*, Geom. Dedicata **38** (1991), no. 1, 101–106, DOI 10.1007/BF00147738. MR1099924 (92d:58210)

[13] James A. Jenkins, *Univalent functions and conformal mapping*, Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 18. Reihe: Moderne Funktionentheorie, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. MR0096806 (20 #3288)

[14] Mikhail G. Katz and Stéphane Sabourau, *An optimal systolic inequality for CAT(0) metrics in genus two*, Pacific J. Math. **227** (2006), no. 1, 95–107, DOI 10.2140/pjm.2006.227.95. MR2247874 (2007e:53043)

[15] Mikhail G. Katz, *Systolic geometry and topology*, Mathematical Surveys and Monographs, vol. 137, American Mathematical Society, Providence, RI, 2007. With an appendix by Jake P. Solomon. MR2292367 (2008h:53063)

[16] Linda Keen, *An extremal length on a torus*, J. Analyse Math. **19** (1967), 203–206. MR0216436 (35 #7269)

[17] P. M. Pu, *Some inequalities in certain nonorientable Riemannian manifolds*, Pacific J. Math. **2** (1952), 55–71. MR0048886 (14,87e)

[18] Burton Rodin, *Extremal length and geometric inequalities*, Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., La Jolla, Calif., 1966), Amer. Math. Soc., Providence, R.I., 1968, pp. 370–376. MR0237771 (38 #6052)

[19] Takashi Sakai, *A proof of the isosystolic inequality for the Klein bottle*, Proc. Amer. Math. Soc. **104** (1988), no. 2, 589–590, DOI 10.2307/2047017. MR962833 (89m:53068)

Laboratoire de Mathématiques et Applications (LaMA), Université Libanaise, Tripoli, Liban

E-mail address: chady.mir@gmail.com

Laboratoire D'analyse et Mathématiques Appliquées (UMR 8050), Université Paris-Est, UPEC, UPEMLV, CNRS, F-94010, Créteil, France

E-mail address: zeina.yassine@u-pec.fr