CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 20, Pages 25–42 (March 18, 2016)
http://dx.doi.org/10.1090/ecgd/289

PARAMETRIZATIONS OF TEICHMÜLLER SPACES BY TRACE FUNCTIONS AND ACTION OF MAPPING CLASS GROUPS

GOU NAKAMURA AND TOSHIHIRO NAKANISHI

Dedicated to the memory of Professor Mika Seppälä

Abstract. We give a set of trace functions which give a global parametrization of the Teichmüller space $\mathcal{T}(g,n)(L_1,\dots,L_n)$ of hyperbolic surfaces of genus g with n geodesic boundary components of lengths $L_1,\dots,$ L_n such that the action of the mapping class group on the Teichmüller space can be represented by rational transformations in the parameters.

1. Introduction

Let $\bar{S} = \bar{S}_{g,n}$ be a surface of type (g,n), a smooth compact and oriented surface of genus g with n boundary curves $C_1,\dots, C_n$, and S the interior of $\bar{S}$. Throughout this paper we assume that $2g-2+n > 0$. If $n \geq 1$, let $L = (L_1,\dots,L_n)$ be a tuple of non-negative numbers. The Teichmüller space $\mathcal{T}(g,n)(L_1,\dots,L_n)$ is the space of equivalence classes of marked complete hyperbolic metrics of curvature -1 on S such that C_j is homotopic to a closed geodesic curve of length L_j in S for $j = 1,\dots,n$ (If $L_j = 0$, then C_j corresponds to a puncture). We mean by $\mathcal{T}(g,0)(L)$ the Teichmüller space of closed hyperbolic surfaces of genus g whatever L is. It is known that $\mathcal{T}(g,n)(L)$ is a real analytic manifold homeomorphic to $\mathbb{R}^{6g-6+2n}$ (see, for example, [22, Section 34]).

The free homotopy class of a homotopically non-trivial closed curve c in S defines a real analytic function ℓ_c on $\mathcal{T}(g,n)(L)$, called the *geodesic length function* of c, which assigns to each point in $\mathcal{T}(g,n)(L)$ the length of the geodesic curve freely homotopic to c on a marked hyperbolic surface representing the point. It is known that there are finitely many closed curves $c_1,\dots, c_N$ on S such that each point of $\mathcal{T}(g,n)(L)$ is recovered by the values of their geodesic length functions (see Section 3.2.) In other words,

$$(\ell_{c_1},\dots,\ell_{c_N}) : \mathcal{T}(g,n)(L) \to \mathbb{R}^N_{>0} \quad (\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\})$$

is a (real analytic) embedding (see, for example, [4], [11], [16], [17] and [18]). Then the problem of finding the minimal number $N(g,n)$ of geodesic length functions needed to parametrize $\mathcal{T}(g,n)(L)$ globally naturally arises. This problem was solved (Schmutz [15], Okumura [12]) and

$$N(g,n) = 6g-5+2n = \dim \mathcal{T}(g,n)(L) + 1. \tag{1.1}$$

The fact that $N(g,n) > \dim \mathcal{T}(g,n)(L)$ follows from convexity of geodesic length functions (Wolpert [21]).

Received by the editors July 2, 2015 and, in revised form, January 11, 2016.
2010 *Mathematics Subject Classification.* Primary 32G15; Secondary 30F35.
The first author was partially supported by the JSPS KAKENHI Grant No. 25400147.
The second author was partially supported by the JSPS KAKENHI Grant No. 22540191.

Let $G = G(g, n)$ be the fundamental group of S. Then G has a generator system

$$\mathscr{S} = (A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n) \tag{1.2}$$

satisfying

$$(\prod_{j=1}^{g} A_j B_j A_j^{-1} B_j^{-1}) C_1 \cdots C_n = 1, \tag{1.3}$$

where $(A_1, B_1), \ldots, (A_g, B_g)$ correspond to handles and $C_1, \ldots, C_n$ to the boundary curves of S. By the uniformization theorem and the lifting theorem ([5], [2], [19]), we may think of $\mathcal{T}(g, n)(L_1, \ldots, L_n)$ as the space of conjugacy classes of faithful and discrete (or *Fuchsian*) representations ρ of G into $SL(2, \mathbb{R})$ by fixing the signs of $\rho(X)$ for $X \in \mathscr{S}$ so that $\rho((\prod_{j=1}^{g} A_j B_j A_j^{-1} B_j^{-1}) C_1 \cdots C_n) = 1$ (the identity matrix).

Each $g \in G$ defines a *trace function*, $\chi_g([\rho]) = \mathrm{tr}\rho(g)$ on $\mathcal{T}(g, n)(L)$. Due to (2.1) below we can replace geodesic length functions by trace functions. We rephrase the solution of the problem mentioned above.

Theorem 1.1 ([12], [15]). *There exist $N = N(g, n) = 6g + 2n - 5$ elements $g_1, \ldots, g_N$ in G such that*

$$(\chi_{g_1}, \ldots, \chi_{g_N}) : \mathcal{T}(g, n)(L_1, \ldots, L_n) \to \mathbb{R}^N$$

is a real analytic embedding.

By a suitable choice of $g_1, \ldots, g_N$, the image of $\mathcal{T}(g, n)(L)$ under the embedding $(\chi_{g_1}, \ldots, \chi_{g_N})$ is contained in the zero locus of an RC function (compass and ruler constructible function) (Feng Luo [6]).

The mapping class group $\mathcal{MC}(g, n)$ of S acts on $\mathcal{T}(g, n)(L)$. So it is possible to describe each mapping class of $\mathcal{MC}(g, n)$ by using the parameters in $\mathbb{R}^N$. Our main theorem is

Theorem 1.2. *There exist $N = N(g, n)$ elements $g_1, \ldots, g_N$ in G such that*

$$\chi = (\chi_{g_1}, \ldots, \chi_{g_N}) : \mathcal{T}(g, n)(L_1, \ldots, L_n) \to \mathbb{R}^N$$

is a real analytic embedding and that each $\varphi \in \mathcal{MC}(g, n)$ acts on $\chi(\mathcal{T}(g, n)(L))$ as a rational transformation in the parameters of $\mathbb{R}^N$ and $c_j = 2\cosh(L_j/2)$, $j = 1, \ldots, n$.

In this note, a rational function in $x = (x_1, \ldots, x_m)$ means a function written in the form $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are *integer polynomials*. We denote the set of rational functions in $(x_1, \ldots, x_m)$ by $\mathscr{R}(x_1, \ldots, x_m)$. The theorem means that, if $x = (x_1, \ldots, x_N) \in \chi(\mathcal{T}(g, n)(L))$, then $\varphi(x) = (\varphi_1(x), \ldots, \varphi_N(x)) \in \mathscr{R}(x_1, \ldots, x_N, c_1, \ldots, c_n)^N$.

For $(g, n) = (0, 4)$ and $(1, 1)$ this is a classical theorem. For example, $\mathcal{T}(1, 1)(0)$ is identified with $\{(x, y, z) : x > 2, y > 2, z > 0, x^2 + y^2 + z^2 = xyz\}$ and $\mathcal{MC}(1, 1)$ with the group of Markov transformations, which are generated by

$$\varphi_*(x, y, z) = (x, z, y^{-1}(x^2 + z^2)) = (x, z, xz - y),$$
$$\psi_*(x, y, z) = (z, y, x^{-1}(y^2 + z^2)) = (z, y, yz - x).$$

(See [13, Example in p. 335]. It is known that the orbit of $(3, 3, 3)$ under $\mathcal{MC}(1, 1)$ provides all positive integer solutions of $x^2 + y^2 + z^2 = xyz$.) Our theorem may be viewed as a generalization of this result.

2. Preliminaries

2.1. **The Teichmüller space.** Our basic references for notation and terminology about Fuchsian groups and Teichmüller spaces are [1], [20] and [22]. The group $PSL(2,\mathbb{R})$ acts on the Riemann sphere $\bar{\mathbb{C}}$ by Möbius transformations. If its action is restricted to $\mathbb{H} = \{z = x + iy : y > 0\}$, a model of a hyperbolic plane, then it is the group of orientation-preserving isometries. A *Fuchsian group* Γ is a discrete subgroup of $PSL(2,\mathbb{R})$. If Γ is torsion free, then the space $\mathbb{H}/\Gamma$ is a hyperbolic surface. A *hyperbolic* element A of Γ has two fixed points p_A and q_A on the boundary $\partial\mathbb{H}$ in the Riemann sphere. The *axis* L_A of A is the hyperbolic line between p_A and q_A. The cyclic group $\langle A\rangle$ generated by A acts on L_A, and $L_A/\langle A\rangle$ is a closed geodesic curve c_A on $\mathbb{H}/\Gamma$. The trace of A and the length $\ell(A) = \ell_{c_A}$ of c_A (counting multiplicity) is related by

$$|\mathrm{tr}A| = 2\cosh\frac{\ell(A)}{2}. \tag{2.1}$$

Let $G = G(g,n)$ be the fundamental group of $S_{g,n}$. We fix a tuple $(c_1,\dots,c_n)$ of positive numbers ≥ 2. We consider faithful representations (injective homomorphisms) ρ from G into $SL(2,\mathbb{R})$ satisfying

$$(*)\begin{cases} \rho(G) \text{ is a Fuchsian group and } \mathbb{H}/\rho(G) \text{ a surface of type } (g,n), \\ c_j = -\mathrm{tr}\rho(C_j),\ j = 1,\dots,n, \text{ and if } c_j = 2, \text{ then } \rho(C_j) \text{ is parabolic.} \\ \mathrm{tr}\rho(A_j) > 0 \text{ and } \mathrm{tr}\rho(B_j) > 0 \text{ for } j = 1,\dots,g. \end{cases}$$

We remark that for all faithful Fuchsian representations $\tilde\rho$ of G into $PSL(2,\mathbb{R})$ with $c_j = |\mathrm{tr}\rho(C_j)|$, there exists a lift $\rho : G \to SL(2,\mathbb{R})$ of $\tilde\rho$ satisfying the above conditions (see the proofs of Theorems 3.5 and 3.6 in [19]). Two representations ρ_1 and ρ_2 satisfying $(*)$ are equivalent if there exists a $P \in SL(2,\mathbb{R})$ such that $\rho_2(g) = P^{-1}\rho_1(g)P$ for all $g \in G$.

Definition 2.1. The *Teichmüller space* $\mathscr{T}(g,n)(c_1,\dots,c_n)$ is the space of equivalence classes of faithful Fuchsian representations which satisfy $(*)$.

By the uniformization theorem, this space is identified with $\mathcal{T}(g,n)(L_1,\dots,L_n)$ in the introduction, if $c_j = 2\cosh(L_j/2)$. For $[\rho] \in \mathscr{T}(g,n)(c_1,\dots,c_n)$, ρ is determined by the images of generators in (1.2). So we can identify $[\rho]$ with the simultaneous conjugacy class of the tuple $(\rho(A_1),\rho(B_1),\dots,\rho(C_n))$.

2.2. **Trace identities for $SL(2,\mathbb{R})$.** We list basic trace identities for matrices in $SL(2,\mathbb{R})$ (see, for example, [7, §3.4]):

- (I1) $\mathrm{tr}A_1A_2\cdots A_n = \mathrm{tr}A_{i_1}A_{i_2}\cdots A_{i_n}$, where $(i_1,\dots,i_n)$ is a cyclic permutation of $(1,\dots,n)$.
- (I2) $\mathrm{tr}A = \mathrm{tr}A^{-1}$, hence, if $AB = 1$(the identity matrix), then $\mathrm{tr}A = \mathrm{tr}B$.
- (I3) $\mathrm{tr}AB + \mathrm{tr}AB^{-1} = \mathrm{tr}A\mathrm{tr}B$, hence $\mathrm{tr}AB^{-1}C = \mathrm{tr}B\mathrm{tr}AC - \mathrm{tr}ABC$.
- (I4) $\mathrm{tr}ABC = \mathrm{tr}A\mathrm{tr}BC + \mathrm{tr}B\mathrm{tr}CA + \mathrm{tr}C\mathrm{tr}AB - \mathrm{tr}A\mathrm{tr}B\mathrm{tr}C - \mathrm{tr}ACB$.
- (I5) For the commutator $[A,B] = ABA^{-1}B^{-1}$,
  $$\mathrm{tr}[A,B] = (\mathrm{tr}A)^2 + (\mathrm{tr}B)^2 + (\mathrm{tr}AB)^2 - \mathrm{tr}A\mathrm{tr}B\mathrm{tr}AB - 2.$$

(I6) $2\mathrm{tr}ABCD = (\mathrm{tr}A)(\mathrm{tr}BCD) + (\mathrm{tr}B)(\mathrm{tr}ACD) + (\mathrm{tr}C)(\mathrm{tr}ABD)$
$+(\mathrm{tr}D)(\mathrm{tr}ABC) + (\mathrm{tr}AB)(\mathrm{tr}CD) - (\mathrm{tr}AC)(\mathrm{tr}BD)$
$+(\mathrm{tr}AD)(\mathrm{tr}BC) - (\mathrm{tr}A)(\mathrm{tr}B)(\mathrm{tr}CD) - (\mathrm{tr}B)(\mathrm{tr}C)(\mathrm{tr}AD)$
$-(\mathrm{tr}C)(\mathrm{tr}D)(\mathrm{tr}AB) - (\mathrm{tr}A)(\mathrm{tr}D)(\mathrm{tr}BC)$
$+(\mathrm{tr}A)(\mathrm{tr}B)(\mathrm{tr}C)(\mathrm{tr}D)$.

(I7) $\mathrm{tr}A^{-1}BAC = \mathrm{tr}A\mathrm{tr}BAC + \mathrm{tr}B\mathrm{tr}C - \mathrm{tr}AB\mathrm{tr}AC - \mathrm{tr}BC$.

The identity (I7) is not listed in [7, §3.4], but it is a consequence of (I1)–(I4):

$$\begin{aligned}\mathrm{tr}A^{-1}(BA)C &= \mathrm{tr}A^{-1}\mathrm{tr}BAC + \mathrm{tr}BA\mathrm{tr}A^{-1}C + \mathrm{tr}C\mathrm{tr}A^{-1}BA \\ &\quad -\mathrm{tr}A^{-1}\mathrm{tr}BA\mathrm{tr}C - \mathrm{tr}(BA)A^{-1}C \\ &= \mathrm{tr}A\mathrm{tr}BAC + \mathrm{tr}AB(\mathrm{tr}A\mathrm{tr}C - \mathrm{tr}AC) + \mathrm{tr}B\mathrm{tr}C \\ &\quad -\mathrm{tr}A\mathrm{tr}AB\mathrm{tr}C - \mathrm{tr}BC.\end{aligned}$$

Definition 2.2. A tuple $\mathscr{S} = (A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n)$ of matrices in $SL(2,\mathbb{R})$ is said to be of *type* (g,n) if $\mathscr{S}$ represents a point of $\mathscr{T}_{g,n}(c_1,\ldots,c_n)$ with $c_j = -\mathrm{tr}C_j$, $j = 1,\ldots,n$.

Let (A,B,C,D) be of type $(0,4)$. So it satisfies $ABCD = 1$. Let $a = -\mathrm{tr}A$, $b = -\mathrm{tr}B$, $c = -\mathrm{tr}C$, $d = -\mathrm{tr}D$, $x = -\mathrm{tr}BC$, $y = -\mathrm{tr}CA$ and $z = -\mathrm{tr}AB$ and

$$\Sigma_{04} = \{x,y,z,a,b,c,d\}. \tag{2.2}$$

Then (see [6], [9])

$$\begin{aligned}&F_{04}(x,y,z,a,b,c,d)\\ &\quad = x^2+y^2+z^2-xyz+(ad+bc)x+(bd+ca)y+(cd+ab)z\\ &\qquad +a^2+b^2+c^2+d^2+abcd-4\\ &\quad = 0.\end{aligned} \tag{2.3}$$

For an ordered tuple $\mathscr{S} = (A_1,\ldots,A_n)$ of matrices in $SL(2,\mathbb{R})$ we define

$$T(\mathscr{S}) = \left\{ \begin{array}{l|l} A_i, & 1 \le i \le n, \\ A_{j_1}A_{j_2}, & 1 \le j_1 < j_2 \le n, \\ A_{k_1}A_{k_2}A_{k_3} & 1 \le k_1 < k_2 < k_3 \le n \end{array} \right\}. \tag{2.4}$$

If $G(\mathscr{S})$ denotes the group generated by $A_1,\ldots, A_n$, then

Lemma 2.3 (see [7, Lemma 3.5.3])**.** *If $g \in G(\mathscr{S})$, then its trace* $\mathrm{tr}g$ *is a rational polynomial in* $\{\mathrm{tr}g : g \in T(\mathscr{S})\}$.

Remark. The results in this subsection hold for $SL(2,\mathbb{C})$.

3. Parametrization of the Teichmüller spaces

3.1. **Teichmüller spaces of types $(1,1)$ and $(0,4)$.** For the results of this subsection we refer to [14], [10], [3] and [22, §33]. We often use

Lemma 3.1 ([22, Lemmas 33.4 and 33.17])**.** *If (A,B,C), $ABC = 1$, is of type $(0,3)$, then $(\mathrm{tr}A)(\mathrm{tr}B)(\mathrm{tr}AB) < 0$. If (A,B,C), $[A,B]C = 1$, is of type $(1,1)$, then $(\mathrm{tr}A)(\mathrm{tr}B)(\mathrm{tr}AB) > 0$ and $\mathrm{tr}[A,B] = x^2+y^2+z^2-xyz-2$ is negative.*

$\mathcal{T}(0,3)(a,b,c)$ is a one point set ([22, Proposition 33.6]). Next, let (A,B,C), $[A,B]C = 1$, be a tuple of type $(1,1)$ representing a point of $\mathcal{T}(1,1)(c)$. Since A and B have positive trace, $\mathrm{tr}AB > 0$ by Lemma 3.1. Let $x = \mathrm{tr}A$, $y = \mathrm{tr}B$ and $z = \mathrm{tr}AB$. Then (x,y,z) embeds $\mathcal{T}(1,1)(c)$ into $\mathbb{R}^3$ and its image is

$$\{(x,y,z) : x > 2, y > 2, z > 2, xyz - x^2 - y^2 - z^2 + 2 = c\}$$

[22, §33.D]. Next, let (A, B, C, D) be a tuple of type $(0, 4)$ representing a point of $\mathcal{T}(0, 4)(a, b, c, d)$. By condition $(*)$, $a = -\mathrm{tr}A$, $b = -\mathrm{tr}B$, $c = -\mathrm{tr}C$ and $d = -\mathrm{tr}D$ are positive. We define $x = -\mathrm{tr}BC$, $y = -\mathrm{tr}CA$, and $z = -\mathrm{tr}AB$. These values are all positive. For example, we know $\mathrm{tr}BC < 0$ by applying Lemma 3.1 to the tuple (B, C, DA) of type $(0, 3)$ with $\mathrm{tr}B < 0$, $\mathrm{tr}C < 0$. We see that (x, y, z) embeds $\mathcal{T}(0, 4)(a, b, c, d)$ into $\mathbb{R}^3$, and its image is

$$\{(x, y, z) : x > 2, y > 2, z > 2, F_{04}(x, y, z, a, b, c, d) = 0\},$$

where F_{04} is given in (2.3) ([10]).

3.2. **Geometric transformations on the tuples of type** (g, n)**.** We need the following lemma.

Lemma 3.2. *For tuples* $(F_1, \ldots, F_{2g+n})$ *of type* (g, n) *with* $n \geq 2$ *satisfying*

$$(\prod_{k=1}^{g} [F_{2k-1}, F_{2k}])F_{2g+1} \cdots F_{2g+n} = 1$$

and for $k = 2g + 1, \ldots, 2g + n$,

$$\sigma_k(F_1, \ldots, F_k, F_{k+1}, \ldots, F_{2g+n}) = (F_1, \ldots, F_k F_{k+1} F_k^{-1}, F_k, \ldots, F_{2g+n})$$
$$\sigma_k^{-1}(F_1, \ldots, F_k, F_{k+1}, \ldots, F_{2g+n}) = (F_1, \ldots, F_{k+1}, F_{k+1}^{-1} F_k F_{k+1}, \ldots, F_{2g+n})$$

are also of type (g, n).

In fact, σ_k arises from a Dehn twist of the loop on the surface $S_{g,n}$ bounding the $(k - 2g)$-th and $(k + 1 - 2g)$-th boundary curves.

3.3. **Parametrization of Teichmüller spaces.** It is known that there are trace functions $\chi_{g_1}, \ldots, \chi_{g_N}$ which embed the Teichmüller space $\mathcal{T}(g, n)(c_1, \ldots, c_n)$ into $\mathbb{R}^N$. For example, choose $g_{11}, \ldots, g_{m1} \in G(g, n)$ representing homotopy classes of simple closed curves $\gamma_1, \ldots, \gamma_m$, $m = 3g - 3 + n$, which give a pants-decomposition of S. For each $j = 1, \ldots, m$, choose a simple closed curve δ_j which meets γ_j once if γ_j is non-separating or twice if γ_j is separating. Let η_j be a curve obtained from δ_j by a Dehn twist along γ_j. Then, as the cases of types $(1, 1)$ and $(0, 4)$ indicate, the geodesic length functions on γ_j, η_j and δ_j determine the length and twist parameters along γ_j in the Fenchel-Nielsen coordinates. So, if g_{j2} and g_{j3} are the homotopy classes of δ_j and η_j, respectively, then the trace functions $\chi_{g_{jk}}$ $(j = 1, \ldots, m,\ k = 1, 2, 3)$ embed $\mathcal{T}(g, n)(L_1, \ldots, L_n)$ or $\mathscr{T}(g, n)(c_1, \ldots, c_n)$ with $c_j = 2\cosh(L_j/2)$ into $\mathbb{R}^{3m}$. Let $\mathscr{S} = (X_1, X_2, \ldots, X_{2g+n})$ represent a general point of $\mathscr{T}(g, n)(c_1, \ldots, c_n)$. Theorem 1.2 follows from Lemma 2.3 if we find $N = N(g, n)$ elements $g_1, \ldots, g_N$ in $G(\mathscr{S})$ such that the trace of each $g \in T(\mathscr{S})$ is a rational function in $\mathrm{tr}g_1, \ldots, \mathrm{tr}g_N$. The rest of paper is devoted to proving this.

4. Proof of the main theorem: Cases of types $(0, 5)$ and $(1, 2)$

4.1. **Case of type** $(0, 5)$**.** Let $\mathscr{S} = (A, B, C, D, E)$ with $ABCDE = 1$ be of type $(0, 5)$ and $a = -\mathrm{tr}A$, $b = -\mathrm{tr}B$, $c = -\mathrm{tr}C$, $d = -\mathrm{tr}D$ and $e = -\mathrm{tr}E$. The group $G(\mathscr{S})$ is generated by A, B, C and D. We define

$$\Sigma_{05} = \{x, y, z, u, v, a, b, c, d, e\}, \tag{4.1}$$

where

$$x = -\mathrm{tr}BC, \quad y = -\mathrm{tr}CA, \quad z = -\mathrm{tr}AB,$$
$$u = -\mathrm{tr}BCD, \quad v = -\mathrm{tr}AD.$$

Proposition 4.1. *Traces of all elements in $G(\mathscr{S})$ are rational functions in Σ_{05}. Moreover, Σ_{05} satisfies the polynomial identity $F_{05}(\Sigma_{05}) = 0$ given in (4.5) below.*

Proof. We shall show that traces of all elements in $T(\mathscr{S})$ in (2.4) are rational functions in Σ. The traces of elements in $T(\mathscr{S})$ not appearing in (4.1) are

$$t = -\mathrm{tr}ABC,\ x_1 = -\mathrm{tr}BD,\ x_2 = -\mathrm{tr}CD,\ x_3 = -\mathrm{tr}ABD,\ x_4 = -\mathrm{tr}ACD.$$

We have $\mathrm{tr}E = \mathrm{tr}(ABCD)^{-1} = \mathrm{tr}ABCD$, and by using (I7) and other trace identities we get

$$(4.2)\quad \begin{array}{llcl} \mathrm{tr}AE & = \mathrm{tr}BCD & = & -u, \\ \mathrm{tr}BE & = \mathrm{tr}B^{-1}E^{-1} = \mathrm{tr}B^{-1}AB(CD) & = & be + ax_2 - zu + x_4, \\ \mathrm{tr}CE & = \mathrm{tr}C^{-1}E^{-1} = \mathrm{tr}C^{-1}(AB)CD & = & ce + zd - tx_2 + x_3, \\ \mathrm{tr}DE & = \mathrm{tr}ABC & = & -t, \\ \mathrm{tr}ABE & = \mathrm{tr}CD & = & -x_2, \\ \mathrm{tr}ACE & = \mathrm{tr}C^{-1}A^{-1}E^{-1} = \mathrm{tr}C^{-1}BCD & = & cu + bd - xx_2 + x_1, \\ \mathrm{tr}ADE & = \mathrm{tr}BC & = & -x, \\ \mathrm{tr}BCE & = \mathrm{tr}(BC)^{-1}E^{-1} = \mathrm{tr}(BC)^{-1}A(BC)D & = & xe + ad - tu + v, \\ \mathrm{tr}BDE & = \mathrm{tr}D^{-1}B^{-1}E^{-1} = \mathrm{tr}B^{-1}ABC & = & bt + ac - xz + y, \\ \mathrm{tr}CDE & = \mathrm{tr}AB & = & -z. \end{array}$$

We consider two tuples of type $(0, 4)$, (A, B, C, DE) with

$$\{-\mathrm{tr}BC, -\mathrm{tr}CA, -\mathrm{tr}AB, -\mathrm{tr}A, -\mathrm{tr}B, -\mathrm{tr}C, -\mathrm{tr}DE\} = \{x, y, z, a, b, c, t\}$$

and (A, BC, D, E) with

$$\{-\mathrm{tr}BCD, -\mathrm{tr}DA, -\mathrm{tr}ABC, -\mathrm{tr}A, -\mathrm{tr}BC, -\mathrm{tr}D, -\mathrm{tr}E\} = \{u, v, t, a, x, d, e\}.$$

From (2.3) we have two equations:

$$F_{04}(x, y, z, a, b, c, t) = 0,\ F_{04}(u, v, t, a, x, d, e) = 0.$$

Solving $F_{04}(x, y, z, a, b, c, t) - F_{04}(u, v, t, a, x, d, e) = 0$ in t, we see that

$$(4.3)\qquad t = \frac{P(a, b, c, d, e, x, y, z, u, v)}{Q(a, b, c, d, e, y, z, u, v)}$$

where

$$\begin{aligned} P(a, b, c, d, e, x, y, z, u, v) \ =\ & adex + aeu + adv + xyz + dux + evx - bcx - acy \\ & -abz + d^2 + e^2 + u^2 + v^2 - b^2 - c^2 - y^2 - z^2 \end{aligned}$$

and

$$(4.4)\qquad Q(a, b, c, d, e, y, z, u, v) = abc - de + uv + by + cz.$$

By substituting this with t in $F_{04}(x, y, z, a, b, c, t) = 0$, we obtain a polynomial identity:

$$(4.5)\qquad F_{05}(\Sigma_{05}) = F_{05}(x, y, z, u, v, a, b, c, d, e) = 0.$$

The polynomial $F_{05}(\Sigma_{05})$ is quadratic in x and quartic and monic in b, c, d, e, y, z, u and v. Next we consider x_1, x_2, x_3 and x_4. By (I6) and (4.2) we have

$$\begin{aligned}
-2b &= 2\mathrm{tr}ACDE \\
&= \mathrm{tr}A\mathrm{tr}CDE + \mathrm{tr}C\mathrm{tr}ADE + \mathrm{tr}D\mathrm{tr}ACE + \mathrm{tr}E\mathrm{tr}ACD + \mathrm{tr}AC\mathrm{tr}DE \\
&\quad -\mathrm{tr}AD\mathrm{tr}CE + \mathrm{tr}AE\mathrm{tr}CD - \mathrm{tr}A\mathrm{tr}C\mathrm{tr}DE - \mathrm{tr}C\mathrm{tr}D\mathrm{tr}AE - \mathrm{tr}D\mathrm{tr}E\mathrm{tr}AC \\
&\quad -\mathrm{tr}E\mathrm{tr}A\mathrm{tr}CD + \mathrm{tr}A\mathrm{tr}C\mathrm{tr}D\mathrm{tr}E \\
&= az + cx + d(xx_2 - x_1 - cu - bd) + ex_4 + yt + v(ce + zd - tx_2 + x_3) \\
&\quad +ux_2 + act + cdu + dey + aex_2 + acde \\
&= -dx_1 + (dx - vt + u + ae)x_2 + vx_3 + ex_4 \\
&\quad +az + cx - bd^2 + yt + vce + vzd + act + dey + acde.
\end{aligned}$$

In a similar way we apply (I6) and (4.2) to $-c = \mathrm{tr}ABDE$, $-d = \mathrm{tr}ABCE$ and $-e = \mathrm{tr}ABCD$ to see that x_1, x_2, x_3 and x_4 satisfy

$$\begin{aligned}
&\begin{pmatrix} -d & u + dx - tv + ae & v & e \\ u + ae & d + av & e & v \\ -b & c + bx + tz + ay + abt & -z - ab & y \\ -y & z + ab & c & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \\
&= \begin{pmatrix} -2b - az - cx + bd^2 - yt - cev + dvz - act - dey - acde \\ -2c - axz + a^2c + ay - bx - tz - bev + zuv - bdu - dez - abde \\ -2d - atu + a^2d + av + b^2d - et + dz^2 - bey + uyz - ux + abdz \\ -2e - au - dt - vx - bcv - cdz - adx - abcd \end{pmatrix}.
\end{aligned} \tag{4.6}$$

So far our calculations are in a formal manner. If (A, B, C, D, E) represents a point of the Teichmüller space $\mathscr{T}(0, 5)(a, b, c, d, e)$, the parameters in (4.1) and $t = -\mathrm{tr}ABC$ are greater than 2, owing to the condition $(*)$ and that each of them corresponds to a boundary curve of a subsurface of type $(0, 3)$ ([22, Lemma 33.4]). Since the quadratic equation

$$F_{04}(u, v, t, a, x, d, e) = t^2 - (uv - de - ax)t + c_0 = 0,$$

with

$$c_0 = u^2 + v^2 + (ae + xd)u + (xe + ad)v + a^2 + x^2 + d^2 + e^2 + axde - 4$$

positive, has a solution $t > 2$, we must have $uv - de > ax > 0$. Therefore (4.3) holds, since its denominator (4.4) is positive over $\mathscr{T}(0, 5)(a, b, c, d, e)$. As mentioned in Section 3.3, (x, y, z, u, v, t) gives a global parameter system of $\mathscr{T}(0, 5)(a, b, c, d, e)$ and now we can omit t. The traces x_1, x_2, x_3 and x_4 are real analytic functions on the Teichmüller space. Since t in (4.3) is linear in x, the determinant of the matrix on the left-hand side of (4.6) is also a linear polynomial in x. By also using the quadratic equation (4.5) in x, we see that the determinant vanishes only in a lower dimensional subvariety in the four dimenisonal (y, z, u, v)-space. (The authors guess that the determinant never vanishes on the Teichmüller space.) So by solving (4.6) and by the continuity of the traces, we can express x_1, x_2, x_3 and x_4 as rational functions in $\{x, y, z, u, v, a, b, c, d, e\}$. □

4.2. **Case of type** $(1, 2)$**.** Let $\mathscr{S} = (A, B, C, D)$ be of type $(1, 2)$. The matrices satisfy $[A, B]CD = 1$, $c = -\mathrm{tr}C$ and $d = -\mathrm{tr}D$.

4.2.1. We define $\Sigma_{12}^{(1)} = \{a, b, z, x, y, c, d\}$, where

$$\begin{array}{lll} a = \mathrm{tr}A, & b = \mathrm{tr}B, & d = -\mathrm{tr}D = -\mathrm{tr}[A,B]C, \\ z = \mathrm{tr}AB, & x = \mathrm{tr}BA^{-1}B^{-1}C, & y = \mathrm{tr}AC. \end{array} \tag{4.7}$$

Note that a equals $\mathrm{tr}A$ but not $-\mathrm{tr}A$. This is the reason for the choice of x and y.

Proposition 4.2. *Let A, B and C be matrices in $SL(2,\mathbb{R})$ and suppose that $\mathscr{S} = (A, B, C, D)$, $D = C^{-1}[B, A]$ is of type $(1, 2)$. Then*

$$x_1 = \mathrm{tr}BC \quad \textit{and} \quad x_2 = \mathrm{tr}ABC$$

are rational functions in $\Sigma_{12}^{(1)}$, and hence so are the traces of all elements in $G(\mathscr{S})$. The traces in $\Sigma_{12}^{(1)}$ satisfy the polynomial identity $F_{12}(\Sigma_{12}^{(1)}) = 0$ given in (4.8) below.

Proof. Note that the traces of elements in $T(\mathscr{S})$ in (2.4) not appearing in (4.7) are x_1 and x_2. We have two linear equations in x_1 and x_2:

$$\begin{aligned} x &= \mathrm{tr}BA^{-1}B^{-1}C = \mathrm{tr}BA^{-1}B^{-1}\mathrm{tr}C - \mathrm{tr}B^{-1}CBA \ (\because \text{(I3)}) \\ &= \mathrm{tr}A\mathrm{tr}C - (\mathrm{tr}B\mathrm{tr}CBA + \mathrm{tr}C\mathrm{tr}A - \mathrm{tr}CB\mathrm{tr}BA - \mathrm{tr}CA) \ (\because \text{(I7)}) \\ &= -b(\mathrm{tr}C\mathrm{tr}BA + \mathrm{tr}B\mathrm{tr}CA + \mathrm{tr}A\mathrm{tr}CB - \mathrm{tr}A\mathrm{tr}B\mathrm{tr}C - \mathrm{tr}ABC) \\ &\quad + x_1 z + y \quad (\because \text{(I4)}) \\ &= bcz - b^2y - abx_1 - ab^2c + bx_2 + x_1z + y \\ &= -(ab - z)x_1 + bx_2 + bcz - b^2y - ab^2c + y, \end{aligned}$$

$$\begin{aligned} -d &= \mathrm{tr}ABA^{-1}B^{-1}C = \mathrm{tr}A^{-1}(B^{-1}C)AB \\ &= \mathrm{tr}A\mathrm{tr}AC + \mathrm{tr}B^{-1}C\mathrm{tr}B - \mathrm{tr}B^{-1}CA\mathrm{tr}AB - \mathrm{tr}C \quad (\because \text{(I7)}) \\ &= ay + (\mathrm{tr}B\mathrm{tr}C - \mathrm{tr}BC)\mathrm{tr}B - (\mathrm{tr}B\mathrm{tr}AC - \mathrm{tr}ABC)\mathrm{tr}AB + c \\ &= -bx_1 + zx_2 - byz + ay - b^2c + c. \end{aligned}$$

Since $\mathrm{tr}[A, B] < -2$ by Lemma 3.1,

$$\begin{vmatrix} ab - z & -b \\ b & -z \end{vmatrix} = \mathrm{tr}[A,B] - a^2 + 2 < 0.$$

Solving equations above, we may express x_1 and x_2 as rational functions in $\Sigma_{12}^{(1)}$. □

The tuple $(-A, -BA^{-1}B^{-1}, C, D)$ is of type $(0, 4)$. Its corresponding Σ_{04} in (2.2) is $\{x, y, abz - a^2 - b^2 - z^2 + 2, a, a, c, d\}$. So we obtain the identity

$$F_{04}(x, y, abz - a^2 - b^2 - z^2 + 2, a, a, c, d) = 0. \tag{4.8}$$

The polynomial in the left-hand side is monic and quadratic in c and d.

4.2.2. For our later use, we introduce the second system of traces [8, (3.2)],

$$\Sigma_{12}^{(2)} = \{z, u, v, w, k, c, d\}, \tag{4.9}$$

where

$$\begin{array}{l} z = \mathrm{tr}AB, \quad u = \mathrm{tr}CABA^{-1}, \quad v = \mathrm{tr}CAB^2, \quad w = \mathrm{tr}CAB, \\ k = -\mathrm{tr}[A,B] = -\mathrm{tr}CD, \end{array} \tag{4.10}$$

Proposition 4.3. *Traces of all elements in $G(\mathscr{S})$ are rational functions in $\Sigma_{12}^{(2)}$.*

A proof of this proposition is given in [8, Section 3]. If $S^2 = uvz - u^2 - v^2 - z^2$, then we have

$$\begin{aligned} \mathrm{tr}A &= \frac{w(uz-v)(c+d+zw)+u(2+cd+k+S^2+z^2-w^2)}{w(S^2+z^2)}, \\ \mathrm{tr}B &= \frac{uw(c+d+zw)+v(2+cd+k+S^2+z^2-w^2)}{w(S^2+z^2)}, \\ \mathrm{tr}AC &= bw - v, \\ \mathrm{tr}BC &= -bzw + zv + aw - bc - u. \end{aligned} \tag{4.11}$$

From [8, (3.10)], the parameters satisfy the polynomial identity

$$F_{12}(u,v,w,z,k,c,d) = k^2 + Pk + Q = 0, \tag{4.12}$$

where

$$P = 2\left(w^2 + \frac{c+d}{2}zw + S^2 + z^2 + cd + 2\right) - w^2(S^2+4),$$

$$Q = \left(w^2 + \frac{c+d}{2}zw + S^2 + z^2 + cd + 2\right)^2 - 2w^2(S^2+4) - \left(\frac{c-d}{2}\right)^2 w^2(z^2-4).$$

5. Cases of types $(0,n)$ and $(1,n)$

5.1. **type** $(0,n)$**.** Let $\mathscr{S} = (A_1, A_2, \dots, A_n)$ represent a point of the Teichmüller space $\mathscr{T}(0,n)(a_1,\dots,a_n)$ with $n \geq 4$. The matrices $A_1, \dots, A_n$ in $SL(2,\mathbb{R})$ satisfy $A_1A_2\cdots A_n = 1$ and $a_j = -\mathrm{tr}A_j$ for $j = 1,\dots,n$. The group $G(\mathscr{S})$ is generated by $A_1, A_2, \dots, A_{n-1}$. We define

$$\Sigma_{0n} = \{z, x_3, y_3, \dots, x_{n-1}, y_{n-1}, a_1, \dots, a_n\},$$

where

$$\begin{aligned} z &= -\mathrm{tr}A_1A_2, \\ x_j &= -\mathrm{tr}A_2\cdots A_j \\ y_j &= -\mathrm{tr}A_1A_j \end{aligned} \Bigg\} \quad j = 3,\dots,n-1.$$

Proposition 5.1. *Traces of all elements in $G(\mathscr{S})$ are rational functions in Σ_{0n}. The parameters in Σ_{0n} satisfy a polynomial identity*

$$F_{0n}(\Sigma_{0n}) = F_{0n}(z, x_3, y_3, \dots, x_{n-1}, y_{n-1}, a_1, a_2, \dots, a_n) = 0,$$

where, in each a_k, $k = 2,\dots,n$, x_{n-1} and y_{n-1}, $F_{0n}(\Sigma_{0n})$ is monic and has degree 2^{n-3}.

Proof. For $n = 4$ the theorem is known and classical (see Section 3.1.) Our proof proceeds by induction on n. We assume that the proposition holds for n. Let $\mathscr{S} = (A_1,\dots,A_{n+1})$ represent a point of $\mathscr{T}(0,n+1)(a_1,\dots,a_{n+1})$. If $B_n = A_nA_{n+1}$ and $b = -\mathrm{tr}B_n$, then $\mathscr{S}_0 = \{A_1,\dots,A_{n-1},B_n\}$ is of type $(0,n)$ and traces of all elements in $G(\mathscr{S}_0)$ are rational functions in

$$\Sigma_{0n} = \{z, x_3, y_3, \dots, x_{n-1}, y_{n-1}, a_1, \dots, a_{n-1}, b\}. \tag{5.1}$$

We also have an identity

$$\begin{aligned} &F_{0n}(z, x_3, y_3, \dots, x_{n-1}, y_{n-1}, a_1, \dots, a_{n-1}, b) \\ &\qquad = b^m + \alpha_{m-1}b^{m-1} + \cdots + \alpha_1 b + \alpha_0 = 0, \end{aligned} \tag{5.2}$$

where $m = 2^{n-3}$ and α_j are polynomials in $\Sigma_{0n} \setminus \{b\}$. By our assumption α_0 includes the sum $a_2^m + \cdots + a_{n-1}^m + x_{n-1}^m + y_{n-1}^m$. Let $C = A_2 \cdots A_{n-1}$. Then the tuple (A_1, C, A_n, A_{n+1}) is of type $(0, 4)$ and its corresponding Σ_{04} in (2.2) is

$$\{-\mathrm{tr}CA_n, -\mathrm{tr}A_nA_1, -\mathrm{tr}A_1C, -\mathrm{tr}A_1, -\mathrm{tr}C, -\mathrm{tr}A_n, -\mathrm{tr}A_{n+1}\}$$
$$= \{x_n, y_n, b, a_1, x_{n-1}, a_n, a_{n+1}\}.$$

We have (2.3) of the form

$$F_{04}(x_n, y_n, b, a_1, x_{n-1}, a_n, a_{n+1}) = b^2 + Ab + B = 0, \tag{5.3}$$

where A and B are polynomials of $\{x_n, y_n, a_1, x_{n-1}, a_n, a_{n+1}\}$. From (2.3), for $x = a_1$, x_{n-1}, a_n, a_{n+1}, x_n and y_n, $\deg_x A = 1$, $\deg_x B = 2$ and B is monic, where $\deg_x$ means the degree in the variable x.

Dividing (5.2) by (5.3) results in the remainder $Pb + Q = 0$ with P and Q as in (5.6) below. From this and (5.3) we obtain a polynomial equation

$$F_{0\,n+1}(\Sigma_{0,n+1}) = BP^2 - APQ + Q^2 = 0. \tag{5.4}$$

In order to prove the last statement of Proposition 5.1 by induction, we must show

Lemma 5.1. *$F_{0\,n+1}(\Sigma_{0,n+1})$ is a monic polynomial of degree $2m = 2^{n-2}$ in variables $a_2, \ldots, a_n, a_{n+1}, x_n$ and y_n.*

Proof. From (5.2) and (5.3) we have

$$\sum_{j=0}^{m} \alpha_j b^j = (\sum_{j=0}^{m-2} x_j b^j)(b^2 + Ab + B) + Pb + Q,$$

with $\alpha_m = x_{m-2} = 1$ and

$$\left\{ \begin{array}{l} x_{m-3} + Ax_{m-2} = \alpha_{m-1}, \\ x_{j-2} + Ax_{j-1} + Bx_j = \alpha_j, \quad j = 2, \ldots, m-2, \\ Ax_0 + Bx_1 + P = \alpha_1, \\ Bx_0 + Q = \alpha_0. \end{array} \right. \tag{5.5}$$

Let

$$M = \begin{pmatrix} -A & 1 \\ -B & 0 \end{pmatrix}, \; L = \begin{pmatrix} 1 & 0 \\ 0 & -B \end{pmatrix}$$

and ${}^tM$ be the transpose of M. Then $L\,{}^tM = ML$ and

$$ML \begin{pmatrix} 1 \\ 0 \end{pmatrix} = M^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

From (5.5) we have

$$\begin{pmatrix} P \\ Q \end{pmatrix} = ML \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \alpha_0 \end{pmatrix}, \; \begin{pmatrix} x_{m-3} \\ x_{m-2} \end{pmatrix} = {}^tM \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \alpha_{m-1} \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} x_{j-2} \\ x_{j-1} \end{pmatrix} = {}^tM \begin{pmatrix} x_{j-1} \\ x_j \end{pmatrix} + \begin{pmatrix} \alpha_j \\ 0 \end{pmatrix}, \quad j = 2, \ldots, m-2.$$

Therefore,

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \sum_{j=0}^{m} \alpha_j \begin{pmatrix} -A & 1 \\ -B & 0 \end{pmatrix}^j \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.6}$$

with $\alpha_m = 1$. Let x stand for one of a_n, a_{n+1}, x_n and y_n and write $A = \alpha x + \beta$ and $B = x^2 + \gamma x + \delta$ with α, β, γ and δ independent of x. By induction on j we see the first and second entry of $M^j \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $j \geq 1$ are polynomials in x of degree $j-1$ and j with leading coefficients

$$\begin{pmatrix} -\alpha & 1 \\ -1 & 0 \end{pmatrix}^j \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively. Hence $\deg_x P = m-1$ and $\deg_x Q = m$. From (5.4) we see that

$$\deg_x F_{0\,n+1} = 2m = 2^{n-2}.$$

From (5.4) the leading coefficient of $\deg_x F_{0\,n+1}$ in the variable x is

$$(0,1) \begin{pmatrix} -\alpha & -1 \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} 1 & -\alpha/2 \\ -\alpha/2 & 1 \end{pmatrix} \begin{pmatrix} -\alpha & 1 \\ -1 & 0 \end{pmatrix}^m \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1.$$

Here we used the fact that

$$\begin{pmatrix} -\alpha & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\alpha/2 \\ -\alpha/2 & 1 \end{pmatrix} \begin{pmatrix} -\alpha & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -\alpha/2 \\ -\alpha/2 & 1 \end{pmatrix}.$$

Now we conclude $F_{0\,n+1} = F_{0\,n+1}(\Sigma_{0,n+1})$ is monic polynomial of degree 2^{n-2} in variables a_n, a_{n+1} and also in x_n and y_n. By the induction hypothesis F_{0n} is monic and has degree 2^{n-3} in $a_2, \ldots, a_{n-1}$. The term α_0^2 in Q^2 shows that $F_{0\,n+1}$ is monic and has degree 2^{n-2} in $a_2, \ldots, a_{n-1}$. □

We introduce auxiliary traces $z_j = -\mathrm{tr} A_1 A_2 \cdots A_{j-1}$ (so $b = z_n$). Then, from a series of tuples,

$$(A_1, (A_2 \cdots A_{j-2}), A_{j-1}, (A_j \cdots A_{n+1})),\ j = 4, \ldots n+1,$$

of type $(0,4)$, we obtain the equations

$$F_{04}(x_{j-1}, y_{j-1}, z_{j-1}, a_1, x_{j-2}, a_{j-1}, z_j) = 0,\ j = 4, \ldots, n$$

with $x_2 = a_2$, $z_3 = z$ and

$$F_{04}(x_n, y_n, z_n, a_1, x_{n-1}, a_n, a_{n+1}) = 0.$$

Starting with $\mathscr{R} = \mathscr{R}(\Sigma_{0,n+1} \setminus \{x_n\})$, we have a sequence of quadratic extensions $\mathscr{R} \subset \mathscr{R}(z_4) \subset \mathscr{R}(z_4, z_5) \subset \cdots \subset \mathscr{R}(z_4, \ldots, z_n) \subset \mathscr{R}(z_4, \ldots, z_n, x_n)$, and hence, x_n is in an extension of degree 2^{n-2} of the rational function field $\mathscr{R}(\Sigma_{0,n+1} \setminus \{x_n\})$. Thus $F_{0\,n+1}(\Sigma_{0,n+1})$ in (5.4) is irreducible in x_n. Then the first equation in (5.4) shows that, as polynomials in x_n, P and Q are relatively prime. Therefore $P = Q = 0$ occurs on a proper subvariety in the $2n-4$ dimensional $(z, x_3, \ldots, x_{n-1}, y_3, \ldots, y_n)$-space. Since b is continuous,

$$b = -\mathrm{tr} A_n A_{n+1} = -Q/P \tag{5.7}$$

throughout the Teichmüller space. Equation (5.7) shows that b belongs to $\mathscr{R}(\Sigma_{0n+1})$ and hence $\mathscr{R}(\Sigma_{0n}) \subset \mathscr{R}(\Sigma_{0n+1})$ (see (5.1)).

In order to conclude the proof, by Lemma 2.3 it suffices to show that

$$\mathrm{tr} A_i A_j,\ (1 \leq i < j \leq n), \quad \mathrm{tr} A_i A_j A_k\ (1 \leq i < j < k \leq n) \tag{5.8}$$

are rational functions in $\Sigma_{0\,n+1} = \{z, x_3, \ldots, x_n, y_3, \ldots, y_n, a_1, \ldots, a_{n+1}\}$. This is true for $j < n$ or $k < n$ because $\mathscr{R}(\Sigma_{0n}) \subset \mathscr{R}(\Sigma_{0n+1})$. We consider only $\mathrm{tr} A_i A_j A_n$, because $\mathrm{tr}A_iA_n$ can be treated in a similar way. Let $X = A_iA_j$ with $i < j < n$. By a repeated use of Lemma 3.2, the tuple $\mathscr{S}_0 = (A_1, X, Y, A_n, A_{n+1})$ with

$$Y = X^{-1}A_2 \cdots A_{n-1} = (A_iA_j)^{-1}A_2(A_iA_j)\cdots(A_iA_j)^{-1}A_{i-1}(A_iA_j) \\ \times(A_j^{-1}A_{i+1}A_j)\cdots(A_j^{-1}A_{j-1}A_j)A_{j+1}\cdots A_{n-1}$$

is of type $(0,5)$. Its corresponding Σ_{05} in (4.1) is

$$\{x_{n-1}, u, v, x_n, y_n, a_1, x, y, a_n, a_{n+1}\},$$

where $u = -\mathrm{tr}A_1Y$, $v = -\mathrm{tr}A_1X$. Since all of its elements are rational functions in $\Sigma_{0\,n+1}$, so is $\mathrm{tr}XA_n = \mathrm{tr}A_iA_jA_n$ by Proposition 4.1. □

5.2. **Case of type** $(1,n)$**.** Let $\mathscr{S} = (A, B, A_1, \ldots, A_n)$ represent a point of the Teichmüller space $\mathscr{T}(1,n)(a_1,\ldots,a_n)$ with $n \geq 2$. The matrices A, B, A_1,…, A_n in $SL(2,\mathbb{R})$ satisfy $ABA^{-1}B^{-1}A_1A_2\cdots A_n = 1$ and $a_j = -\mathrm{tr}A_j$ for $j = 1,\ldots,n$. The group $G(\mathscr{S})$ is generated by A, B, A_1, A_2,…, A_{n-1}. We define

$$\Sigma_{1n} = \{a, b, z, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, a_1, \ldots, a_n\},$$

where

$$a = \mathrm{tr}A, \quad b = \mathrm{tr}B, \quad z = \mathrm{tr}AB,$$
$$x_j = \mathrm{tr}BA^{-1}B^{-1}A_1\cdots A_j, \quad y_j = \mathrm{tr}AA_j, \quad j = 1,\ldots,n-1.$$

Proposition 5.2. *Traces of all elements in $G(\mathscr{S})$ are rational functions in Σ_{1n}. The traces in Σ_{1n} satisfy a polynomial identity*

$$F_{1n}(\Sigma_{1n}) = F_{1n}(a, b, z, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, a_1, \ldots, a_n) = 0, \tag{5.9}$$

where $F_{1n}(\Sigma_{1n})$, in a_k, $k = 2,\ldots,n$, x_{n-1} and y_{n-1}, is monic and has degree 2^{n-1}.

Proof. Let $k = -\mathrm{tr}ABA^{-1}B^{-1}$. Since $a = \mathrm{tr}A$, we apply Proposition 5.1 to the tuple $\mathscr{S}_0 = (-A, -BA^{-1}B^{-1}, A_1, \ldots, A_n)$ of type $(0, n+2)$. Then we see that traces of all elements in the group $G(\mathscr{S}_0)$ are rational functions in

$$\{k, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, a, a_1, \ldots, a_n\},$$

and hence in Σ_{1n}, because $k = abz - a^2 - b^2 - z^2 + 2$. The traces in Σ_{1n} satisfy

$$F_{0\,n+2}(abz - a^2 - b^2 - z^2 + 2, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, a, a, a_1, \ldots, a_n) = 0.$$

We rewrite this identity as

$$F_{1n}(\Sigma_{1n}) = F_{1n}(a, b, z, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, a_1, \ldots, a_n) = 0.$$

Then $F_{1n}(\Sigma_{1n})$ is monic and has degree $2^{n-1} = 2^{(n+2)-3}$ in a_2, …, a_n, x_{n-1} and y_{n-1}. If $X = A_i$ or $X = A_iA_j$, where $1 \leq i, j < n$ and $i < j$, then the traces AX, $ABA^{-1}X$, $ABA^{-1}B^{-1}X$ are rational functions in Σ_{1n}. That $\mathrm{tr}BX$ and $\mathrm{tr}ABX$ are also in $\mathscr{R}(\Sigma_{1n})$ follows from Proposition 4.2. □

6. Case of type (g,n)

We consider tuples

$$\mathscr{S} = (A_1, B_1, \ldots, A_g, B_g, K_{g+1}, \ldots, K_{g+n})$$

of type (g,n) with $g \geq 1$. Even for $j = 1,\ldots,g$, we define $K_j = [A_j, B_j] = A_jB_jA_j^{-1}B_j^{-1}$. Then the matrices satisfy $K_1K_2\cdots K_{g+n} = 1$. For $j = 2,\ldots,g$, let

$$\mathscr{H}_j = (A_j, B_j, C_j, D_j),$$

where

$$C_j = K_{j+1}\cdots K_{g+n}A_1 \quad (j \geq 2),$$
$$D_j = B_1A_1^{-1}B_1^{-1}K_2\cdots K_{j-1} \quad (j \geq 3), \quad D_2 = B_1A_1^{-1}B_1^{-1}. \tag{6.1}$$

Then $\mathscr{H}_j$ is a tuple of type $(1, 2)$. We consider the following $6g + 3n - 5$ traces

$$\Sigma_{g,n} = \left\{ \begin{array}{l|l} a_1, b_1, z_1 & \\ x_i,\ y_i & i = 2, \ldots, g+n-1 \\ u_j,\ v_j,\ w_j,\ z_j & j = 2, \ldots, g \\ k_{g+1}, \ldots, k_{g+n} & \end{array} \right\},$$

where

$$\begin{aligned} & a_1 = \mathrm{tr}A_1, \quad b_1 = \mathrm{tr}B_1, \quad z_1 = \mathrm{tr}A_1B_1, \\ & x_j = \mathrm{tr}B_1A_1^{-1}B_1^{-1}K_2\cdots K_j, \quad y_j = \mathrm{tr}A_1K_j, \quad j = 2, \ldots, g+n-1, \\ & \left.\begin{array}{ll} u_j = \mathrm{tr}C_jA_jB_jA_j^{-1}, & v_j = \mathrm{tr}C_jA_jB_j^2 \\ w_j = \mathrm{tr}C_jA_jB_j, & z_j = \mathrm{tr}A_jB_j \end{array}\right\} \ j = 2, \ldots, g, \end{aligned} \tag{6.2}$$

and $k_j = -\mathrm{tr}K_j$ for $j = 1, \ldots, g+n$. We also let $x_1 = \mathrm{tr}B_1A_1^{-1}B_1^{-1} = a_1$.

Proposition 6.1. *Traces of all elements in $G(\mathscr{S})$ are rational functions in $\Sigma_{g,n}$. The parameters in $\Sigma_{g,n}$ satisfy a polynomial identity*

$$F_{g,n}(\Sigma_{g,n}) = 0, \tag{6.3}$$

where $F_{g,n}(\Sigma_{g,n})$ is monic and has degree 2^{2g+n-3} in $k_{g+1}, \ldots, k_{g+n}$, x_{g+n-1} and y_{g+n-1}.

Our proof of Proposition 6.1 is by induction on $j = 1, \ldots, g$. The case for $j = 1$ is treated in Section 5.2. At the g-th step we conclude the proof of the proposition. For $j \geq 1$, We define a tuple $\mathscr{S}_j$ of type $(j, g+n-j)$:

$$\mathscr{S}_j = (A_1, B_1, \ldots, A_{j-1}, B_{j-1}, A_j, B_j, K_{j+1}, \ldots, K_{g+n})$$

and

$$\Sigma_{j,g+n-j} = \left\{ \begin{array}{l|l} a_1, b_1, z_1 & \\ x_i,\ y_i & i = 2, \ldots, g+n-1 \\ u_k,\ v_k,\ w_k,\ z_k & k = 2, \ldots, j \\ k_l & l = j+1, \ldots, g+n \end{array} \right\}.$$

We assume that traces of all elements of the group generated by $\mathscr{S}_{j-1}$ are rational functions in $\Sigma_{j-1,g+n+1-j}$ and also that we have the polynomial identity

$$F_{j-1,g+n+1-j}(\Sigma_{j-1,g+n+1-j}) = 0, \tag{6.4}$$

where the left-hand side is monic and has degree $2^{g+n+j-4}$ in $k_j, \ldots, k_{g+n}$, x_{g+n-1} and y_{g+n-1}. By Proposition 4.3 traces of all elements in the group generated by $\mathscr{H}_j$, in particular, $a_j = \mathrm{tr}A_j$ and $b_j = \mathrm{tr}B_j$ are rational functions in u_j, v_j, w_j, z_j, k_j, x_j and x_{j-1}. These seven parameters satisfy (4.12), which can be described as

$$F_{12}(u_j, v_j, w_j, z_j, k_j, x_j, x_{j-1}) = k_j^2 + Ak_j + B = 0, \tag{6.5}$$

where A and B are polynomials in $\{u_j, v_j, w_j, z_j, x_j, x_{j-1}\}$. Now we proceed just as in the proof of Proposition 5.1. By dividing (6.4) by this polynomial (6.5) we obtain $Pk_j + Q = 0$, where P and Q are polynomials in

$$\Sigma_{j,g+n-j} = (\Sigma_{j-1,g+n+1-j} \setminus \{k_j\}) \cup \{u_j, v_j, w_j, z_j\}$$

and Q is monic and has degree $2^{g+n+j-4}$ in $k_{j+1},\ldots, k_{g+n}$, x_{g+n-1} and y_{g+n-1}. We obtain also the polynomial identity

$$F_{j,g+n-j}(\Sigma_{j,g+n-j}) = BP^2 - APQ + Q^2 = 0. \tag{6.6}$$

The left-hand side is monic and has degree degree $2^{g+n+j-3}$ in $k_{j+1},\ldots, k_{g+n}$, x_{g+n-1} and y_{g+n-1}. Let

$$\mathscr{R} = \mathscr{R}(a_1, b_1, z_1, x_2, \ldots, x_{g+n-1}, y_2, \ldots, y_{g+n-2}, k_2, \ldots, k_{g+n}). \tag{6.7}$$

(Note that y_{g+n-1} drops out of the variables.) We introduce

$$\tau_j = -\mathrm{tr}K_jK_{j+1}\cdots K_{g+n} = -\mathrm{tr}K_1\cdots K_{j-1},\ j = 3,\ldots, g+n-1.$$

We have a series of $g+n-2$ tuples

$$\begin{aligned}&(A_1, (B_1A_1^{-1}B_1^{-1}), K_2, (K_3\cdots K_{g+n})),\\ &(A_1, (B_1A_1^{-1}B_1^{-1}K_2\cdots K_{j-1}), K_j, (K_{j+1}\cdots K_{g+n})), \quad j = 3,\ldots, g+n-2,\\ &(A_1, (B_1A_1^{-1}B_1^{-1}K_2\cdots K_{g+n-2}), K_{g+n-1}, K_{g+n})\end{aligned}$$

of type $(0, 4)$ which yield

$$F_{04}(x_2, y_2, k_1, a_1, a_1, k_2, \tau_3) = 0$$

with $k_1 = -\mathrm{tr}K_1 = a_1b_1z_1 - a_1^2 - b_1^2 - z_1^2 + 2$,

$$F_{04}(x_j, y_j, \tau_j, a_1, x_{j-1}, k_j, \tau_{j+1}) = 0,\ j = 3,\ldots, g+n-2,$$

and

$$F_{04}(x_{g+n-1}, y_{g+n-1}, \tau_{g+n-1}, a_1, x_{g+n-2}, k_{g+n-1}, k_{g+n}) = 0.$$

Then, starting with $\mathscr{R}$ in (6.7), we have a sequence of quadratic extensions $\mathscr{R} \subset \mathscr{R}(\tau_3) \subset \mathscr{R}(\tau_3, \tau_4) \subset \cdots \subset \mathscr{R}(\tau_3,\ldots, \tau_{g+n-1}) \subset \mathscr{R}(\tau_3,\ldots, \tau_{g+n-1}, y_{g+n-1})$, and hence, y_{g+n-1} is in an extension of degree 2^{g+n-2} of the rational function field $\mathscr{R}$. By $j-1$ quadratic equations as in (6.5) with j replaced by $2,\ldots, j$, we see that y_{g+n-1} is in an extension of degree $2^{g+n+j-3}$ of $\mathscr{R}(\Sigma_{j,g+n-j})$. Hence the polynomial in (6.6) is irreducible in y_{g+n-1} and $P = Q = 0$ occurs on a proper subvariety of $6j + 2(g+n-j) - 6$ dimensional $\Sigma_{j,g+n-j} \setminus \{y_{g+n-1}\}$-space. By continuity of k_j, $k_j = -Q/P$ holds throughout the Teichmüller space. Now we conclude that k_j is a rational function in $\Sigma_{j,g+n-j}$.

The traces of the elements in $T(\mathscr{S}_j)$ which are not yet known to be rational functions in $\Sigma_{j,g+n-j}$ are included in

$$\mathrm{tr}XA_j, \quad \mathrm{tr}XB_j, \quad \mathrm{tr}XA_jB_j \tag{6.8}$$

where X runs over

(i)	A_i,	$1 \le i < j$,
(ii)	B_i, A_iB_i,	$1 \le i < j$,
(iii)	$A_{i_1}A_{i_2}$,	$1 \le i_1 < i_2 < j$,
(iv)	$A_{i_1}B_{i_2},\ B_{i_1}A_{i_2},\ B_{i_1}B_{i_2}$,	$1 \le i_1 < i_2 < j$,
(v)	K_i,	$j < i \le g+n$,
(vi)	$K_{i_1}K_{i_2}$,	$j < i_1 < i_2 \le g+n$,
(vii)	$K_{i_2}A_{i_1}$,	$1 \le i_1 < j < i_2 \le g+n$,
(viii)	$K_{i_2}B_{i_1}$,	$1 \le i_1 < j < i_2 \le g+n$.

(X in $\mathrm{tr}XA_jB_j$ may run over (i), B_i in (ii) and (v).) Let $\mathscr{R}_j = \mathscr{R}(\Sigma_{j,g+n-j})$. Our goal is to show that traces in (6.8) are in $\mathscr{R}_j$. Let $J_1 = g+n-j+1$ and $J_2 = g+n+j-2$. We define $(M_1, M_2, \ldots, M_{g+n+j})$ by

$$(M_1, \ldots, M_{J_1}) = (K_{j+1}, \ldots, K_{g+n}, A_1)$$

and

$$(M_{J_1+1}, \ldots, M_{g+n+j}) = (B_1A_1^{-1}B_1^{-1}, A_2, B_2A_2^{-1}B_2^{-1}, \ldots, A_j, B_jA_j^{-1}B_j^{-1}).$$

Then

$$M_1M_2\cdots M_{J_1} = C_j, \quad M_{J_1+1}\cdots M_{J_2} = D_j.$$

The tuple $\mathscr{S}' = (M_1, \ldots, M_{g+n+j})$ is of type $(0, g+n+j)$, because $G(\mathscr{S}')$ is the fundamental group of the surface obtained by cutting $S_{j,g+n-j}$ along j loops determined by $A_1, \ldots, A_j$.

First we see that, if U runs over

(6.9) $$M_i, \quad M_{i_1}M_{i_2}, \quad (1 \le i, i_1, i_2 \le J_2,\ i_1 < i_2)$$

and if

(6.10) $$t_1 = \mathrm{tr}UA_j, \quad t_2 = \mathrm{tr}UB_jA_j^{-1}B_j^{-1}$$

belong to $\mathscr{R}_j$, then so do the matrices in (6.8). The tuple (A_j, B_j, U, V), where $V = U^{-1}[B_j, A_j]$, is of type $(1, 2)$ by a repeated use of Lemma 3.2. Since the trace of $UA_jB_jA_j^{-1}B_j^{-1} = UK_j \in G(\mathscr{S}_{j-1})$ belongs to $\mathscr{R}_j$, Proposition 4.2 claims (#) *all traces of the elements in the group generated by* (A_j, B_j, U)*; in particular,* $\mathrm{tr}UB_j$ *and* $\mathrm{tr}UA_jB_j$ *are rational functions in* $\Sigma_{j-1,g+n-j+1} \cup \{t_1, t_2\}$*. Therefore, if* t_1 *and* t_2 *belong to* $\mathscr{R}_j$*, then so do* $\mathrm{tr}UB_j$ *and* $\mathrm{tr}UA_jB_j$. Matrices X in (i), (iii), (v), (vi) and (vii) are included in (6.9) and hence $\mathrm{tr}XB_j$ and $\mathrm{tr}XA_jB_j$ belong to $\mathscr{R}_j$.

The argument that we will repeat is as above: *knowing that all traces of*

$$U,\ A_iB_iA_i^{-1}B_i^{-1}U = K_iU,\ A_iU = M_{J_1+2i-2}U,\ B_iA_i^{-1}B_i^{-1}U = M_{J_1+2i-1}U$$

belong to $\mathscr{R}_j$*, we apply Proposition* 4.2 *to the triple* (A_i, B_i, U) *to conclude that all traces of the elements in the group generated by* (A_i, B_i, U)*, in particular,* $\mathrm{tr}B_iU$ *and* $\mathrm{tr}A_iB_iU$ *belong to* $\mathscr{R}_j$. Let $Z \in \{A_j, B_j, A_jB_j\}$. For matrices in (ii), the claim (#) enables us to apply this argument to (A_i, B_i, Z). The conclusion is that $\mathrm{tr}B_iZ$ and $\mathrm{tr}A_iB_iZ \in \mathscr{R}_j$. The argument applied to (A_j, B_j, W_1W_2) for $W_1 \in \{A_{i_1}, B_{i_1}A_{i_1}^{-1}B_{i_1}^{-1}, K_{i_1}\}$ and $W_2 \in \{A_{i_2}, B_{i_2}A_{i_2}^{-1}B_{i_2}^{-1}, K_{i_2}\}$ concludes that $\mathrm{tr}W_1W_2Z \in \mathscr{R}_j$. Then, for matrices in (iv) the argument applied to the triples $(A_{i_2}, B_{i_2}, ZA_{i_1})$ and $(A_{i_1}, B_{i_1}, A_{i_2}Z)$ concludes $\mathrm{tr}A_{i_1}B_{i_2}Z$ and $\mathrm{tr}B_{i_1}A_{i_2}Z \in \mathscr{R}_j$, respectively. Next apply the argument to (A_{i_2}, B_{i_2}, ZW), where W is A_{i_1}, $B_{i_1}A_{i_1}^{-1}B_{i_1}^{-1}$ and K_{i_1}. The conclusion is that $\mathrm{tr}WB_{i_2}Z \in \mathscr{R}_j$. This conclusion and the above argument applied to the triple $(A_{i_1}, B_{i_1}, B_{i_2}Z)$ conclude that $\mathrm{tr}B_{i_1}B_{i_2}Z \in \mathscr{R}_j$. Finally for matrices in (viii) we can conclude likewise that $\mathrm{tr}K_{i_2}B_{i_1}Z \in \mathscr{R}_j$ from the triple $(A_{i_1}, B_{i_1}, ZK_{i_2})$.

Now we show that t_1 and t_2 in (6.10) belong to $\mathscr{R}_j$. In what follows we write (A, B, C, D) instead of (A_j, B_j, C_j, D_j). We will repeat the same argument in the last part of the proof of Proposition 5.1. By a repeated use of Lemma 3.2, we see the following tuples are of type $(0, 5)$:

(1) For $X = M_i$ or $X = M_{i_1}M_{i_2}$, where $J_1 < i, i_1, i_2 \le J_2$, $i_1 < i_2$,

$$\mathscr{S}_1^* = (C, DX^{-1}, X, A, BA^{-1}B^{-1}).$$

(2) For $X = M_i$ or $X = M_{i_1}M_{i_2}$, where $1 \le i, i_1, i_2 \le J_1$, $i_1 < i_2$,

$$\mathscr{S}_2^* = (D, D^{-1}XD, D^{-1}X^{-1}CD, A, BA^{-1}B^{-1}).$$

(3) For $X = M_{i_1}M_{i_2}$, where $1 \le i_1 \le J_1 < i_2 \le J_2$,

$$\mathscr{S}_3^* = (CM_{i_1}^{-1}, M_{i_1}DM_{i_2}^{-1}M_{i_1}^{-1}, X, A, BA^{-1}B^{-1}).$$

Case (1). We consider Σ_{05} for $\mathscr{S}_1^*$ up to sign, which is

$$\{\mathrm{tr}D, \mathrm{tr}CX, \mathrm{tr}CDX^{-1}, \mathrm{tr}AD, \mathrm{tr}AC, \mathrm{tr}C, \mathrm{tr}DX^{-1}, \mathrm{tr}X, \mathrm{tr}A, \mathrm{tr}A\}.$$

Since C, D, $X \in G(\mathscr{S}_{j-1})$ and A, AC, $AD \in G(\mathscr{H}_j)$, all traces in Σ_{05} belong to $\mathscr{R}_j$. Then, by Proposition 4.1, $\mathrm{tr}XA$ and $\mathrm{tr}XBA^{-1}B^{-1}$ belong to $\mathscr{R}_j$.

Case (2). The tuple Σ_{05} for $\mathscr{S}_2^*$ is

$$\{\mathrm{tr}C, \mathrm{tr}X^{-1}CD, \mathrm{tr}XD, \mathrm{tr}D^{-1}CDA, \mathrm{tr}AD, \mathrm{tr}D, \mathrm{tr}X, \mathrm{tr}CX^{-1}, \mathrm{tr}A, \mathrm{tr}A\},$$

up to sign. In this case again all traces in Σ_{05} are in $\mathscr{R}_j$. As

$$\begin{aligned} XA &= D(D^{-1}XD)D^{-1}A, & XBA^{-1}B^{-1} &= D(D^{-1}XD)D^{-1}BA^{-1}B^{-1}, \\ XDA &= D(D^{-1}XD)A, & CX^{-1}A &= D(D^{-1}CX^{-1}D)D^{-1}A \end{aligned}$$

are elements of $G(\mathscr{S}_2^*)$, by Proposition 4.1 their traces belong to $\mathscr{R}_j$.

Case (3). Let $M = M_{i_1}$ and $N = M_{i_2}$. Then $X = MN$ and Σ_{05} for $\mathscr{S}_3^*$ is

$$\begin{aligned} \{&\mathrm{tr}MD,\ \mathrm{tr}CN,\ \mathrm{tr}CDN^{-1}M^{-1},\ \mathrm{tr}MDA,\ \mathrm{tr}CM^{-1}A, \\ &\mathrm{tr}CM^{-1},\ \mathrm{tr}DN^{-1},\ \mathrm{tr}MN,\ \mathrm{tr}A,\ \mathrm{tr}A\}, \end{aligned}$$

up to sign. Here MD, CN, $CDN^{-1}M^{-1}$, CM^{-1}, $N^{-1}D$, MN are elements of $G(\mathscr{S}_{j-1})$ and, as shown in Case (2), $\mathrm{tr}MDA$, $\mathrm{tr}CM^{-1}A$ belong to $\mathscr{R}_j$. Therefore, by Propostion 4.1, $\mathrm{tr}MNA$ and $\mathrm{tr}MNBA^{-1}B^{-1}$ belong to $\mathscr{R}_j$.

7. Action of the mapping class group

In this section we conclude the proof of Theorem 1.2. Let $\mathcal{M}(g,n)$ be the *mapping class group*, the group of isotopy classes of orientation-preserving homeomorphisms of the surface of type (g,n) which preserve every boundary component. Then $\mathcal{M}(g,n)$ acts on the Teichmüller space of type (g,n) by a change of marking. Let $\mathcal{N}$ denote the normal subgroup of $\mathcal{M}(g,n)$ of mapping classes which fixes all points of the Teichmüller space. Then $\mathrm{Mod}(g,n) = \mathcal{M}(g,n)/\mathcal{N}$ is called the *Teichmüller modular group*. With a finite number of exceptional cases of (g,n), $\mathcal{M}(g,n) = \mathrm{Mod}(g,n)$ (see p. 203 of [22]).

Let $\mathrm{Out}(G(g,n))$ be the group of outer automorphisms of $G(g,n)$. $\mathrm{Mod}(g,n)$ is embedded in $\mathrm{Out}(G(g,n))$ as a subgroup. This subgroup is characterized by Nielsen (see [22, 11.3]). Hence, if $\varphi \in \mathrm{Mod}(g,n)$ and if $\mathscr{S} = (A_1, \dots, A_m)$, $m = 2g+n$, represents a general point of $\mathscr{T} = \mathscr{T}(g,n)(c_1, \dots, c_n)$, then, up to simultaneous conjugations, the action of φ on the Teichmüller space can be written in the form

$$\varphi(A_1, \dots, A_m) = (\varphi_1(A_1, \dots, A_m), \dots, \varphi_N(A_1, \dots, A_m)),$$

where $\varphi_j(A_1, \dots, A_m) \in G(\mathscr{S})$. Then, by using the parameters $\chi = (x_1, \dots, x_N)$, $N = 6g + 2n - 5$, for $\mathscr{T}$ obtained so far, we can find rational functions

$$f_j(x_1, \dots, x_N) = \mathrm{tr}\varphi_j(A_1, \dots, A_m),\ j = 1, \dots, N,$$

with coefficients in $\mathbb{Q}(c_1, \dots, c_n)$. Thus φ induces a rational transformation

$$F_\varphi(x_1, \dots, x_N) = (f_1(x_1, \dots, x_N), \dots, f_N(x_1, \dots, x_N))$$

on $\mathscr{T}(g,n)(c_1, \dots, c_n)$ embedded in $\mathbb{R}^N$.

Remark. Let $Q(g,n)$ be the space of quasifuchsian structures on a Riemann surface of genus g with n punctures. Then $Q(g,n)$ is a fiber space over $\mathscr{T}(g,n)(2,2,\dots,2)$. The fiber over X is identified with the Bers embedding of the Teichmüller space of the Riemann surface represented by X. Hence $Q(g,n)$ is simply connected and every trace function on $\mathscr{T}(g,n)(2,2,\dots,2)$ extends to $Q(g,n)$. Consequently, Theorem 1.2 can be extended to $Q(g,n)$.

Acknowledgments

The authors thank the referee for many valuable comments on this paper.

References

[1] A. F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, vol. 91, Springer-Verlag, New York, 1983. MR698777

[2] M. Culler, *Lifting representations to covering groups*, Adv. in Math. **59** (1986), no. 1, 64–70, DOI 10.1016/0001-8708(86)90037-X. MR825087

[3] J. Gilman and B. Maskit, *An algorithm for* 2*-generator Fuchsian groups*, Michigan Math. J. **38** (1991), no. 1, 13–32, DOI 10.1307/mmj/1029004258. MR1091506

[4] L. Keen, *On Fricke moduli*, Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969), Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971, pp. 205–224. MR0288252

[5] I. Kra, *On lifting Kleinian groups to* $\mathrm{SL}(2,\mathbf{C})$, Differential geometry and complex analysis, Springer, Berlin, 1985, pp. 181–193. MR780044

[6] F. Luo, *Geodesic length functions and Teichmüller spaces*, J. Differential Geom. **48** (1998), no. 2, 275–317. MR1630186

[7] C. Maclachlan and A. W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Mathematics, vol. 219, Springer-Verlag, New York, 2003. MR1937957

[8] G. Nakamura and T. Nakanishi, *Parametrizations of some Teichmüller spaces by trace functions*, Conform. Geom. Dyn. **17** (2013), 47–57, DOI 10.1090/S1088-4173-2013-00254-3. MR3037875

[9] T. Nakanishi and M. Näätänen, *Parametrization of Teichmüller space by length parameters*, Analysis and topology, World Sci. Publ., River Edge, NJ, 1998, pp. 541–560. MR1667832

[10] T. Nakanishi and M. Näätänen, *Areas of two-dimensional moduli spaces*, Proc. Amer. Math. Soc. **129** (2001), no. 11, 3241–3252, DOI 10.1090/S0002-9939-01-06010-5. MR1844999

[11] Y. Okumura, *On the global real analytic coordinates for Teichmüller spaces*, J. Math. Soc. Japan **42** (1990), no. 1, 91–101, DOI 10.2969/jmsj/04210091. MR1027542

[12] Y. Okumura, *Global real analytic length parameters for Teichmüller spaces*, Hiroshima Math. J. **26** (1996), no. 1, 165–179. MR1380431

[13] R. C. Penner, *The decorated Teichmüller space of punctured surfaces*, Comm. Math. Phys. **113** (1987), no. 2, 299–339. MR919235

[14] N. Purzitsky, *Two-generator discrete free products*, Math. Z. **126** (1972), 209–223. MR0346070

[15] P. Schmutz, *Die Parametrisierung des Teichmüllerraumes durch geodätische Längenfunktionen* (German), Comment. Math. Helv. **68** (1993), no. 2, 278–288, DOI 10.1007/BF02565819. MR1214232

[16] M. Seppälä and T. Sorvali, *On geometric parametrization of Teichmüller spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 515–526, DOI 10.5186/aasfm.1985.1058. MR802516

[17] M. Seppälä and T. Sorvali, *Parametrization of Teichmüller spaces by geodesic length functions*, Holomorphic functions and moduli, Vol. II (Berkeley, CA, 1986), Math. Sci. Res. Inst. Publ., vol. 11, Springer, New York, 1988, pp. 267–284, DOI 10.1007/978-1-4613-9611-6_18. MR955845

[18] M. Seppälä and T. Sorvali, *Parametrization of Möbius groups acting in a disk*, Comment. Math. Helv. **61** (1986), no. 1, 149–160, DOI 10.1007/BF02621907. MR847525

[19] M. Seppälä and T. Sorvali, *Traces of commutators of Möbius transformations*, Math. Scand. **68** (1991), no. 1, 53–58. MR1124819

[20] M. Seppälä and T. Sorvali, *Geometry of Riemann surfaces and Teichmüller spaces*, North-Holland Mathematics Studies, vol. 169, North-Holland Publishing Co., Amsterdam, 1992. MR1202043

[21] S. A. Wolpert, *Geodesic length functions and the Nielsen problem*, J. Differential Geom. **25** (1987), no. 2, 275–296. MR880186

[22] H. Zieschang, *Finite groups of mapping classes of surfaces*, Lecture Notes in Mathematics, vol. 875, Springer-Verlag, Berlin, 1981. MR643627

Science Division, Center for General Education, Aichi Institute of Technology,1247 Yachigusa, Yakusa, Toyota, 470-0392, Japan
E-mail address: gou@aitech.ac.jp

Department of Mathematics, Shimane University, Matsue, 690-8504, Japan
E-mail address: tosihiro@riko.shimane-u.ac.jp