CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 20, Pages 43–57 (March 18, 2016)
http://dx.doi.org/10.1090/ecgd/290

BÖTTCHER COORDINATES AT SUPERATTRACTING FIXED POINTS OF HOLOMORPHIC SKEW PRODUCTS

KOHEI UENO

Abstract. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a germ of holomorphic skew product with a superattracting fixed point at the origin. If it has a suitable weight, then we can construct a Böttcher coordinate which conjugates f to the associated monomial map. This Böttcher coordinate is defined on an invariant open set whose interior or boundary contains the origin.

1. Introduction

Let $p : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ with a superattracting fixed point at the origin. Taking an affine conjugate, we may write $p(z) = z^\delta + O(z^{\delta+1})$, where $\delta \geq 2$. Let $p_0(z) = z^\delta$. Böttcher's theorem [2] asserts that there is a conformal function φ_p defined on a neighborhood of the origin, with $\varphi_p \sim id$, that conjugates p to p_0. Here $\varphi_p \sim id$ means that the ratio of φ_p and id converges to 1 as $z \to 0$. This function is called the Böttcher coordinate for p at the origin, and obtained as the limit of the compositions of p_0^{-n} and p^n, where p^n denotes the n-th iterate of p. The branch of p_0^{-n} is taken such that $p_0^{-n} \circ p_0^n = id$.

Böttcher's theorem does not extend to higher dimensions entirely as stated in [6]. For example, let $f(z, w) = (z^2, w^2 + z^4)$. Then it has a superattracting fixed point at the origin, but there is no neighborhood of the origin on which f is conjugate to $f_0(z, w) = (z^2, w^2)$ because the critical orbits of f and f_0 behave differently. However, we can completely understand the dynamics of f because it is semiconjugate to $g(z, w) = (z^2, w^2 + 1)$ by $\pi(z, w) = (z, z^2 w)$: $\pi \circ g = f \circ \pi$. In particular, from the one-dimensional Böttcher coordinate for $w \to w^2 + 1$ near infinity, one can construct a biholomorphic map defined on $\{|z| < r|w|^2\}$ for small r that conjugates f to f_0. This domain is not a neighborhood of the origin, but its boundary contains the origin. In this paper we analyze such phenomena for holomorphic skew products with superattracting fixed points at the origin in $\mathbb{C}^2$. By assigning suitable weights, we obtain an analogue of the one-dimensional Böttcher coordinates; see Theorems 1.2 and 1.4 below. The idea of this study is the same as that of our previous study [12], in which we obtained similar results on Böttcher coordinates for polynomial skew products near infinity. Moreover, our results are closely related to Theorem 5.1 in [5], which is obtained by Theorem C in [5] and the result in [4]. Favre and Jonsson [5] have established a systematic way to study the dynamics of all holomorphic germs with superattracting fixed points in dimension two; see also Section 8 in the survey article [7]. Favre [4] has classified contracting rigid germs in

Received by the editors April 18, 2015 and, in revised form, December 31, 2015 and January 14, 2016.

2010 *Mathematics Subject Classification.* Primary 32H50.

Key words and phrases. Complex dynamics, Böttcher coordinates, skew products.

dimension two; a germ is called rigid if the union of the critical set of all its iterates is a divisor with normal crossing and forward invariant. See also [8] and [9].

For other studies on Böttcher's theorem in higher dimensions, we refer to [13], [10] and [3]; they dealt with holomorphic germs with superattracting fixed points at the origin in dimension two or more. Ushiki [13] and Ueda [10] gave different classes of germs that have the Böttcher coordinates on neighborhoods of the origin. Buff, Epstein and Koch [3] gave criteria, in terms of vector fields, for a certain class of germs to have the Böttcher coordinates on neighborhoods of the origin. The germs in [13] are rigid and conjugate to monomial maps, whereas the germs in [10] or [3] are conjugate to homogeneous or quasihomogeneous maps. In addition, we refer to the survey article [1]. Besides theorems for the superattracting case, Abate [1] collected major theorems on local dynamics of holomorphic germs with fixed points of several types in one and higher dimensions.

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a holomorphic germ of the form $f(z, w) = (p(z), q(z, w))$, which is called a holomorphic skew product in this paper. We assume that it has a superattracting fixed point at the origin; that is, $f(0) = 0$ and $Df(0)$ is the zero matrix. Then we may write $p(z) = z^\delta + O(z^{\delta+1})$, where $\delta \geq 2$. On the other hand, let

$$q(z, w) = bz^\gamma w^d + \sum b_j z^{n_j} w^{m_j},$$

where $b \neq 0$, $n_j \geq \gamma$, and $m_j > d$ if $n_j = \gamma$. In other words, (γ, d) is the minimal exponent with respect to the lexicographic order that appears in the power series expansion of q. Since the origin is superattracting, $\gamma + d \geq 2$ and $n_j + m_j \geq 2$. If $d \geq 2$, then we may assume that $b = 1$. In this paper we say that f is *trivial* if $m_j \geq d$ for any j. For this case, we prove that the Böttcher coordinate for f exists on a neighborhood of the origin, and the proof is rather easy. As a remark, f is rigid if it is *trivial* or $d = 1$. Moreover, f belongs to class 6 or class 4 in Favre's classification [4] if it is *trivial* and $d \geq 2$ or if $d = 1$, and the result follows. On the other hand, we say that f is *non-trivial* if $m_j < d$ for some j. This case is the difficult part, in which we need the idea of assigning a suitable weight.

We define the rational number α associated with f as

$$\alpha = \min \left\{ a \geq 0 \;\middle|\; \begin{array}{l} a\gamma + d \leq \delta \text{ and } a\gamma + d \leq an_j + m_j \\ \text{for any } j \text{ such that } b_j \neq 0 \end{array} \right\}$$

if f is *non-trivial*, and as 0 if f is *trivial*. When there is at least one a satisfying all conditions, then α is a well-defined non-negative real number. If there are no such a, we say that α is not well-defined. Let $U_r = U_r^\alpha = \{|z| < r|w|^\alpha, |w| < r\}$. The benefit of α is presented in the following lemma.

Lemma 1.1. *Let $d \geq 2$. If α is well-defined, then $f(z, w) \sim (z^\delta, z^\gamma w^d)$ on U_r^α as $r \to 0$, and $f(U_r^\alpha) \subset U_r^\alpha$ for small r.*

The notation $f \sim f_0$ means that the ratios of the first and second components of f and f_0 converge to 1 on U_r^α as $r \to 0$. Hence Lemma 1.1 says that the asymptotic behavior of f on U_r^α when $r \to 0$ coincides with f_0, where $f_0(z, w) = (z^\delta, z^\gamma w^d)$. With the next theorem, we get a stronger result, the existence of a conjugacy between f and f_0.

Theorem 1.2. *Let $d \geq 2$. If α is well-defined, then there is a biholomorphic map ϕ defined on U_r^α, with $\phi \sim id$ on U_r^α as $r \to 0$, that conjugates f to $(z, w) \to (z^\delta, z^\gamma w^d)$.*

We call ϕ the Böttcher coordinate for f in this paper. As in the one-dimensional case, it is obtained as the limit of the compositions of f_0^{-n} and f^n.

For $d = 1$ we need the additional condition $\alpha < (\delta - 1)/\gamma$ to get again Böttcher coordinates.

Lemma 1.3. *Let $d = 1$. If α is well-defined and $\alpha < (\delta - 1)/\gamma$, then $f(z, w) \sim (z^\delta, bz^\gamma w)$ on U_r^α as $r \to 0$, and $f(U_r^\alpha) \subset U_r^\alpha$ for small r.*

Theorem 1.4. *Let $d = 1$. If α is well-defined and $\alpha < (\delta - 1)/\gamma$, then there is a biholomorphic map ϕ defined on U_r^α, with $\phi \sim id$ on U_r^α as $r \to 0$, that conjugates f to $(z, w) \to (z^\delta, bz^\gamma w)$.*

As remarked above, f is rigid if $d = 1$ and this theorem follows from [4]. More strongly, it follows from our conditions on α that f is conjugate to the monomial map on a neighborhood of the origin, not only on the wedge.

Our results also hold for the nilpotent case. We say that the germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is nilpotent if $f(0) = 0$ and the eigenvalues of $Df(0)$ are both zero. If f is nilpotent, then f^2 is superattracting. Hence Lemmas 1.1 and 1.3 hold for f^2; these lemmas hold even for f on $U_r^\alpha \cap \{|z| \le r_1, |w| \le r_2\}$, where r_1 is enough smaller than r_2. Consequently, Theorems 1.2 and 1.4 hold for f itself.

Moreover, we can perturb f slightly so that it is not a skew product but our results hold. Let $\tilde{p}(z, w) = z^\delta + \sum a_l z^{n_l} w^{m_l}$, where $n_l \ge \delta$, and $m_l \ge 1$ if $n_l = \delta$, and let q be the same as above. Then, for the holomorphic germ of the form $f = (\tilde{p}, q)$, we have the same lemma and theorem as in the skew product case.

The organization of the paper is as follows. In Section 2 we study the properties of the weight α, and prove Lemmas 1.1 and 1.3. Assuming $d \ge 2$, we prove that $\phi_n = f_0^{-n} \circ f^n$ is well-defined and converges uniformly to ϕ on U_r^α in Section 3, and that ϕ is injective in Section 4. An optimality of α is shown by an example at the end of Section 4. The case $d = 1$ is studied in Section 5. Finally, we slightly generalize our results to holomorphic germs in Section 6.

2. Weights

We now describe how to associate to a germ f, as above, an interval $\mathcal{I}_f \subset \mathbb{R}$ so that when α is well-defined it is given by $\alpha = \max\{\inf \mathcal{I}_f, 0\}$. The interval $\mathcal{I}_f$ provides a wider class of weights for which all our results hold, although it does not appear directly in the final conclusions in the introduction.

We define the interval $\mathcal{I}_f$ associated with f as

$$\mathcal{I}_f = \left\{ a \in \mathbb{R} \;\middle|\; \begin{array}{l} a(a\gamma + d) \le a\delta \text{ and } a\gamma + d \le an_j + m_j \\ \text{for any } j \text{ such that } b_j \ne 0 \end{array} \right\}.$$

Let $U_{r_1,r_2}^a = \{|z| < r_1|w|^a, |w| < r_2\} \cap \{|z| < r_2\}$. We remark that, unlike the definition of U_r^α in the introduction, this set needs to be intersected with $\{|z| < r_2\}$ because a can be negative.

Lemma 2.1. *Let $d \ge 2$. For any number a in $\mathcal{I}_f$, it follows that $q(z, w) \sim z^\gamma w^d$ on U_{r_1,r_2}^a as r_1, $r_2 \to 0$, and $f(U_{r_1,r_2}^a) \subset U_{r_1,r_2}^a$ for small r_1 and r_2.*

Proof. We first define $\eta(z, w) = (q(z, w) - z^\gamma w^d)/z^\gamma w^d$ and show that $\eta \to 0$ on U_{r_1,r_2}^a as r_1, $r_2 \to 0$, which implies that $q(z, w) \sim z^\gamma w^d$ on U_{r_1,r_2}^a as r_1, $r_2 \to 0$.

Let $|z| = |cw^a|$ for any a in $\mathcal{I}_f$. Then $U^a_{r_1,r_2} \subset \{|c| < r_1, |w| < r_2\}$ and

$$|\eta| = \left|\sum \frac{b_j z^{n_j} w^{m_j}}{z^\gamma w^d}\right| = \left|\sum \frac{b_j (cw^a)^{n_j} w^{m_j}}{(cw^a)^\gamma w^d}\right| = \left|\sum \frac{b_j c^{n_j} w^{an_j+m_j}}{c^\gamma w^{a\gamma+d}}\right|$$
$$\leq \sum |b_j||c|^{n_j-\gamma}|w|^{(an_j+m_j)-(a\gamma+d)}.$$

The conditions $n_j \geq \gamma$ and $an_j + m_j \geq a\gamma + d$ ensure that the left-hand side is a power series in $|c|$ and $|w|$, and so converges on $\{|c| < r_1, |w| < r_2\}$. Moreover, at least one of the inequalities $n_j > \gamma$ or $an_j + m_j > a\gamma + d$ holds since $n_j \geq \gamma$, and $m_j > d$ if $n_j = \gamma$. In other words, $n_j - \gamma \geq 1$ or $(an_j + m_j) - (a\gamma + d) \geq 1$ holds. Therefore, $\eta \to 0$ on $U^a_{r_1,r_2}$ as r_1, $r_2 \to 0$.

For the invariance of $U^a_{r_1,r_2}$, it is enough to show that $|p(z)| < r_1|q(z,w)|^a$ for any (z,w) in $U^a_{r_1,r_2}$. Since

$$\left|\frac{p(z)}{q(z,w)^a}\right| \sim \left|\frac{z^\delta}{(z^\gamma w^d)^a}\right| = \left|\frac{(cw^a)^\delta}{((cw^a)^\gamma w^d)^a}\right| = |c|^{\delta-a\gamma}|w|^{a\delta-a(a\gamma+d)}$$

on $U^a_{r_1,r_2}$, we need the conditions $\delta - a\gamma \geq 0$ and $a\delta \geq a(a\gamma + d)$. However, the condition $\delta - a\gamma \geq 0$ follows from the condition $a\delta \geq a(a\gamma + d)$ because $d \geq 2$. In fact, it follows that $\delta - a\gamma \geq 2$; if $a \leq 0$, then $\delta - a\gamma \geq \delta \geq 2$, and if $a > 0$, then $\delta - a\gamma \geq d \geq 2$. Hence $|p(z)/q(z,w)^a| \leq C \cdot |c|^2 \leq |c| < r_1$ for some constant C and sufficiently small r_1. □

Lemma 2.2. *Let $d = 1$. For any number a in $\mathcal{I}_f$, if $a < (\delta - 1)/\gamma$, then $q(z,w) \sim bz^\gamma w$ on $U^a_{r_1,r_2}$ as r_1, $r_2 \to 0$, and $f(U^a_{r_1,r_2}) \subset U^a_{r_1,r_2}$ for small r_1 and r_2.*

Proof. The proof of the asymptotic behavior of q is similar to the proof of Lemma 2.1. To prove the invariance of $U^a_{r_1,r_2}$, we need to check that $\delta - a\gamma \geq 0$. In fact, the additional condition $a < (\delta - 1)/\gamma$ implies that $\delta - a\gamma > 1$. Hence $|p/q^a| \leq C \cdot |c|^{1+\varepsilon} \leq |c| < r_1$ for some constant C and small r_1, where $\varepsilon = \delta - a\gamma - 1 > 0$. □

We show that Lemmas 2.1 and 2.2 induce Lemmas 1.1 and 1.3, respectively, at the end of this section.

Let us describe $\mathcal{I}_f$ more practically. Let $\alpha_0 = (\delta - d)/\gamma$, which is derived from the first condition in the definition of $\mathcal{I}_f$. The second condition $a\gamma + d \leq an_j + m_j$ implies that

$$a \geq \frac{d - m_j}{n_j - \gamma}$$

if $n_j > \gamma$. We define m_f as

$$\sup\left\{\frac{d - m_j}{n_j - \gamma} \;\middle|\; b_j \neq 0 \text{ and } n_j > \gamma\right\},$$

where this value is set as $-\infty$ when the supremum is taken over the empty set. Note that $\mathcal{I}_f \subset [m_f, \infty)$. If f is *trivial*, then $m_f \leq 0$. If f is *non-trivial*, then $m_f > 0$ and we can replace the supremum to the maximum in the definition of m_f.

If f is *trivial*, then we can describe $\mathcal{I}_f$ as follows, where $m_f \leq 0$.

f trivial	$\gamma = 0$	$\gamma \neq 0$
$\delta > d$	$[0, \infty)$	$[0, \alpha_0]$
$\delta = d$	$[m_f, \infty)$	$\{0\}$
$\delta < d$	$[m_f, 0]$	$[\max\{m_f, \alpha_0\}, 0]$

In particular, $\mathcal{I}_f$ is always non-empty if f is *trivial*. If f is *non-trivial*, then we can describe $\mathcal{I}_f$ as follows, where $m_f > 0$.

f non-trivial	$\gamma = 0$	$\gamma \neq 0$
$\delta > d$	$[m_f, \infty)$	$[m_f, \alpha_0]$ or $\emptyset$
$\delta = d$	$[m_f, \infty)$	$\emptyset$
$\delta < d$	$\emptyset$	$\emptyset$

Note that $\mathcal{I}_f$ can be empty if f is *non-trivial*. For the case $\delta > d$ and $\gamma \neq 0$, the interval $\mathcal{I}_f$ is equal to $[m_f, \alpha_0]$ if $m_f \leq \alpha_0$ and is empty if $m_f > \alpha_0$.

We may restrict our attention to non-negative weights for our theorems, although negative weights make sense as in Lemmas 2.1 and 2.2. Then the assumption $a \geq 0$ reduces the condition $a(a\gamma+d) \leq a\delta$ to the condition $a\gamma+d \leq \delta$ unless $a = 0$, which induces the definition of α. The interval of non-negative numbers that satisfy the conditions in the definition of α, coincides with $\mathcal{I}_f \cap [0, \infty)$ if $\delta \geq d$. For any case, it follows that α is well-defined if and only if $\mathcal{I}_f$ is not empty, and that

$$\alpha = \min \mathcal{I}_f \cap [0, \infty) = \max\{\inf \mathcal{I}_f, 0\}$$

if it is well-defined. If f is *trivial*, then $\alpha = 0$. The next table summarizes the relations between α and m_f in the *non-trivial* case.

f non-trivial	$\gamma = 0$	$\gamma \neq 0$
$\delta > d$	m_f	m_f or $\nexists$
$\delta = d$	m_f	$\nexists$
$\delta < d$	$\nexists$	$\nexists$

The notation m_f in the table means that α is well-defined and coincides with m_f. The notation $\nexists$ means that α is not well-defined.

We are now ready to show Lemmas 1.1 and 1.3.

Proof of Lemmas 1.1 *and* 1.3. We may assume that $\mathcal{I}_f \neq \emptyset$ since α is well-defined. If f is *trivial*, then $\alpha = 0 \in \mathcal{I}_f$. If f is *non-trivial*, then $\alpha = m_f = \min \mathcal{I}_f > 0$. Therefore, Lemmas 2.1 and 2.2 imply Lemmas 1.1 and 1.3, respectively, by taking r as $\min\{r_1, r_2\}$. □

3. Existence of the limit ϕ for the case $d \geq 2$

In this section we show that ϕ_n is well-defined and converges uniformly to ϕ on U_r for the case $d \geq 2$, where $\phi_n = f_0^{-n} \circ f^n$. The proof is similar to [12]. In particular, the estimate of $\|\Phi_{n+1} - \Phi_n\|$ is almost the same, where Φ_n is a lift of ϕ_n. However, we give a much more detailed description of ϕ_n and an explicit estimate of $\|\Phi - id\|$ with proofs in this paper, where Φ is the limit of Φ_n. The injectivity of ϕ will be proved in the next section, which completes the proof of Theorem 1.2. The proof is different from the proof in [12].

Before going into the proofs, we remark on similarities and differences between this paper and [12]. Although the idea of assigning suitable weights and the style of the main theorems are the same, the choices of the major term of q, the definitions of weights and invariant open sets are different. There are also several differences between settings and results in this paper and [12]. The main theorems in this paper

do not follow immediately from those in [12] because the two situations cannot be connected with a simple conjugacy even if we can extend the results for polynomial skew products in [12] to holomorphic skew products defined near infinity.

Let us prove that ϕ_n is well-defined, assuming that $d \geq 2$ and that α is well-defined. Let $p(z) = z^\delta(1+\zeta(z))$ and $q(z,w) = z^\gamma w^d(1+\eta(z,w))$; Lemma 1.1 implies that ζ and η are holomorphic on U_r and converge to 0 as $r \to 0$. Then the first and second components of f^n are written as

$$z^{\delta^n} \prod_{j=1}^{n} (1+\zeta(p^{j-1}(z)))^{\delta^{n-j}} \text{ and}$$

$$z^{\gamma_n} w^{d^n} \prod_{j=1}^{n-1} (1+\zeta(p^{j-1}(z)))^{\gamma_{n-j}} \prod_{j=1}^{n} (1+\eta(f^{j-1}(z,w)))^{d^{n-j}},$$

where $\gamma_n = \sum_{j=1}^{n} \delta^{n-j} d^{j-1}\gamma$. Using ζ and η, we can also describe ϕ_n explicitly.

Proposition 3.1. *We can define ϕ_n as follows:*

$$\phi_n(z,w) = \left(z \cdot \prod_{j=1}^{n} \sqrt[\delta^j]{1+\zeta(p^{j-1}(z))}, w \cdot \prod_{j=1}^{n} \frac{\sqrt[d^j]{1+\eta(f^{j-1}(z,w))}}{\sqrt[(\delta d)^j]{\{1+\zeta(p^{j-1}(z))\}^{\gamma_j}}} \right),$$

which is well-defined and so holomorphic on U_r.

Proof. Formally, $f_0^{-n}(z,w) = (z^{1/\delta^n}, z^{-\gamma_n/\delta^n d^n} w^{1/d^n})$ and we can define the first and second components of ϕ_n as

$$\left\{ z^{\delta^n} \prod_{j=1}^{n} (1+\zeta(p^{j-1}(z)))^{\delta^{n-j}} \right\}^{1/\delta^n} = z \cdot \prod_{j=1}^{n} \sqrt[\delta^j]{1+\zeta(p^{j-1}(z))} \quad \text{and}$$

$$\left\{ \frac{z^{\gamma_n} w^{d^n} \prod_{j=1}^{n-1} (1+\zeta(p^{j-1}(z)))^{\gamma_{n-j}} \prod_{j=1}^{n} (1+\eta(f^{j-1}(z,w)))^{d^{n-j}}}{\left\{ z^{\delta^n} \prod_{j=1}^{n} (1+\zeta(p^{j-1}(z)))^{\delta^{n-j}} \right\}^{\gamma_n/\delta^n}} \right\}^{1/d^n}$$

$$= w \cdot \left\{ \frac{\prod_{j=1}^{n} (1+\eta(f^{j-1}(z,w)))^{d^{n-j}}}{\left\{ \prod_{j=1}^{n-1} (1+\zeta(p^{j-1}(z)))^{\gamma_n/\delta^j - \gamma_{n-j}} \right\} \left(1+\zeta(p^{n-1}(z))\right)^{\gamma_n/\delta^n}} \right\}^{1/d^n}.$$

Lemma 3.2 below gives the explicit formula of ϕ_n above, and Lemma 1.1 ensures that ϕ_n is well-defined and so holomorphic on U_r. □

Lemma 3.2. *For any $1 \leq j \leq n-1$, it follows that*

$$\frac{\gamma_n}{\delta^j d^n} - \frac{\gamma_{n-j}}{d^n} = \frac{\gamma_j}{(\delta d)^j}.$$

Proof. If $\delta \neq d$, then $\gamma_n = (\delta^n - d^n)\gamma/(\delta - d)$ and so

$$\frac{\gamma_n}{\delta^j d^n} - \frac{\gamma_{n-j}}{d^n} = \frac{\delta^n - d^n}{\delta^j d^n} \cdot \frac{\gamma}{\delta - d} - \frac{\delta^{n-j} - d^{n-j}}{d^n} \cdot \frac{\gamma}{\delta - d}$$

$$= \left(\frac{1}{d^j} - \frac{1}{\delta^j} \right) \cdot \frac{\gamma}{\delta - d} = \frac{\delta^j - d^j}{(\delta d)^j} \cdot \frac{\gamma}{\delta - d} = \frac{\gamma_j}{(\delta d)^j}.$$

If $\delta = d$, then $\gamma_n = nd^{n-1}\gamma$ and so

$$\frac{\gamma_n}{\delta^j d^n} - \frac{\gamma_{n-j}}{d^n} = \frac{n\gamma}{d^{j+1}} - \frac{(n-j)\gamma}{d^{j+1}} = \frac{j\gamma}{d^{j+1}} = \frac{jd^{j-1}\gamma}{d^{2j}} = \frac{\gamma_j}{(\delta d)^j}. \qquad \square$$

In order to prove the uniform convergence of ϕ_n, we lift f and f_0 to F and F_0 by the exponential product $\pi(z,w) = (e^z, e^w)$; that is, $\pi \circ F = f \circ \pi$ and $\pi \circ F_0 = f_0 \circ \pi$. More precisely, we define

$$F(Z,W) = (\delta Z + \log(1+\zeta(e^Z)), \gamma Z + dW + \log(1+\eta(e^Z, e^W)))$$

and $F_0(Z,W) = (\delta Z, \gamma Z + dW)$; let $F_0 = (P_0, Q_0)$. By Lemma 1.1, we may assume

$$\|F - F_0\| < \varepsilon \text{ on } \pi^{-1}(U_r)$$

for any small $\varepsilon > 0$, taking r small enough. Similarly, we can lift ϕ_n to Φ_n so that the equation $\Phi_n = F_0^{-n} \circ F^n$ holds; thus $\Phi_0 = id$ and, for any $n \geq 1$,

$$\Phi_n(Z,W) = \left(\frac{1}{\delta^n} P_n(Z), \frac{1}{d^n} Q_n(Z,W) - \frac{\gamma_n}{\delta^n d^n} P_n(Z)\right),$$

where $(P_n(Z), Q_n(Z,W)) = F^n(Z,W)$. Let $\Phi_n = (\Phi_n^1, \Phi_n^2)$. Then

$$|\Phi_{n+1}^1 - \Phi_n^1| = \left|\frac{P_{n+1}}{\delta^{n+1}} - \frac{P_n}{\delta^n}\right| = \frac{|P_{n+1} - \delta P_n|}{\delta^{n+1}} < \frac{1}{\delta^{n+1}}\varepsilon$$

since $|P_{n+1} - \delta P_n| = |P(P_n) - P_0(P_n)| = |Z \circ (F - F_0)(F^n)| < \varepsilon$, and

$$\begin{aligned}
|\Phi_{n+1}^2 - \Phi_n^2| &= \left|\left\{\frac{Q_{n+1}}{d^{n+1}} - \frac{\gamma_{n+1}P_{n+1}}{\delta^{n+1}d^{n+1}}\right\} - \left\{\frac{Q_n}{d^n} - \frac{\gamma_n P_n}{\delta^n d^n}\right\}\right| \\
&= \left|\frac{Q_{n+1}}{d^{n+1}} - \frac{\gamma P_n}{d^{n+1}} - \frac{Q_n}{d^n}\right| + \left|\frac{\gamma_{n+1}P_{n+1}}{\delta^{n+1}d^{n+1}} - \frac{\gamma_n P_n}{\delta^n d^n} - \frac{\gamma P_n}{d^{n+1}}\right| \\
&= \frac{|Q_{n+1} - (\gamma P_n + dQ_n)|}{d^{n+1}} + \frac{\gamma_{n+1}|P_{n+1} - \delta P_n|}{\delta^{n+1}d^{n+1}} < \frac{1}{d^{n+1}}\varepsilon + \frac{\gamma_{n+1}}{\delta^{n+1}d^{n+1}}\varepsilon
\end{aligned}$$

since $|Q_{n+1} - (\gamma P_n + dQ_n)| = |Q(F^n) - Q_0(F^n)| = |W \circ (F - F_0)(F^n)| < \varepsilon$. Hence Φ_n converges uniformly to Φ. In particular, we can estimate $\|\Phi - id\|$ as follows.

Lemma 3.3. *It follows that*

$$\|\Phi - id\| < \max\left\{\frac{1}{\delta - 1}, \frac{1}{d-1} + \frac{\gamma}{\delta - d}\left(\frac{1}{d-1} - \frac{1}{\delta - 1}\right)\right\}\varepsilon \text{ if } \delta \neq d, \text{ and}$$

$$\|\Phi - id\| < \left\{\frac{1}{d-1} + \frac{\gamma}{(d-1)^2}\right\}\varepsilon \text{ if } \delta = d.$$

Proof. Since $\|\Phi - id\| = \max\{|\Phi^1 - Z|, |\Phi^2 - W|\}$, where $\Phi = (\Phi^1, \Phi^2)$,

$$\begin{aligned}
\|\Phi - id\| &\leq \max\left\{\sum_{n=0}^{\infty} |\Phi_{n+1}^1 - \Phi_n^1|, \sum_{n=0}^{\infty} |\Phi_{n+1}^2 - \Phi_n^2|\right\} \\
&< \max\left\{\sum_{n=0}^{\infty} \frac{1}{\delta^{n+1}}, \sum_{n=0}^{\infty} \frac{1}{d^{n+1}} + \sum_{n=0}^{\infty} \frac{\gamma_{n+1}}{\delta^{n+1}d^{n+1}}\right\}\varepsilon.
\end{aligned}$$

If $\delta \neq d$, then $\gamma_n = (\delta^n - d^n)\gamma/(\delta - d)$ and so

$$\sum_{n=0}^{\infty} \frac{\gamma_{n+1}}{\delta^{n+1}d^{n+1}} = \sum_{n=1}^{\infty} \frac{\gamma_n}{\delta^n d^n} = \sum_{n=1}^{\infty} \frac{\gamma}{\delta - d}\left(\frac{1}{d^n} - \frac{1}{\delta^n}\right) = \frac{\gamma}{\delta - d}\left(\frac{1}{d-1} - \frac{1}{\delta - 1}\right).$$

If $\delta = d$, then $\gamma_n = nd^{n-1}\gamma$ and so

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{\delta^n d^n} = \sum_{n=1}^{\infty} \frac{nd^{n-1}\gamma}{d^{2n}} = \frac{\gamma}{d} \cdot \sum_{n=1}^{\infty} \frac{n}{d^n} = \frac{\gamma}{d} \cdot \frac{d}{(d-1)^2} = \frac{\gamma}{(d-1)^2}. \qquad \square$$

By the inequality $|e^z/e^w - 1| \leq |z-w|e^{|z-w|}$, the uniform convergence of Φ_n translates into that of ϕ_n. Therefore, ϕ is holomorphic on $U_r - \{z = 0\}$, which extends to U_r by Riemann's removable singularity theorem. In particular, if $|\Phi - id| < \varepsilon$, then $|\phi - id| < \varepsilon e^{\varepsilon}|id|$. Therefore, $\phi \sim id$ on U_r as $r \to 0$.

4. Injectivity of ϕ and optimality of α for the case $d \geq 2$

We continue the proof of Theorem 1.2. In the previous section we showed that ϕ is well-defined and so holomorphic on U_r. However, unlike the one-dimensional case, the injectivity of ϕ does not follow immediately because the domain U_r may not be a neighborhood of the origin. In this section we prove that, after shrinking r if necessary, the map ϕ is actually injective on U_r. More precisely, the property $\phi \sim id$ suggests the injectivity of ϕ, which is ensured by Rouché's theorem.

Let $\phi = (\phi_1, \phi_2)$ and $U_{r_1,r_2} = \{|z| < r_1|w|^{\alpha}, |w| < r_2\}$. For simplicity, we may assume that $\alpha > 0$ and that the function ϕ_1 in z is injective because it is conformal at the origin. Let us fix small ε, r_1 and r_2 such that $|\zeta|$, $|\eta| < \varepsilon$ on U_{r_1,r_2} and $f(U_{r_1,r_2}) \subset U_{r_1,r_2}$. Then $\|F - F_0\| < \log(1+\varepsilon)$ on $\pi^{-1}(U_{r_1,r_2})$, where F is the lift of f by $\pi(Z,W) = (e^Z, e^W)$ and

$$\pi^{-1}(U_{r_1,r_2}) = \{\mathrm{Re}(Z - \alpha W) < \log r_1, \mathrm{Re}W < \log r_2\}.$$

Let $\Phi(Z,W) = (\Phi_1(Z), \Phi_Z(W))$ be the lift of ϕ, which is holomorphic on $\pi^{-1}(U_{r_1,r_2})$. The injectivity of ϕ_1 derives that of Φ_1 because $\Phi_1 \sim id$. We prove the injectivity of Φ_Z in Proposition 4.1 below; then the injectivity of Φ derives that of ϕ because $\Phi \sim id$. Recall that $|\Phi_Z - id| < C\tilde{\varepsilon}$, where $\tilde{\varepsilon} = \log(1+\varepsilon)$ and

$$C = \frac{1}{d-1} + \frac{\gamma}{\delta - d}\left(\frac{1}{d-1} - \frac{1}{\delta - 1}\right) \text{ or } C = \frac{1}{d-1} + \frac{\gamma}{(d-1)^2}$$

if $\delta \neq d$ or $\delta = d$. Let $V_Z = V \cap (\{Z\} \times \mathbb{C})$ and $V'_Z = V' \cap (\{Z\} \times \mathbb{C})$, where

$$V = \pi^{-1}(U_{r_1,r_2}) = \left\{\frac{\mathrm{Re}Z}{\alpha} - \frac{\log r_1}{\alpha} < \mathrm{Re}W < \log r_2\right\} \text{ and}$$

$$V' = \left\{\frac{\mathrm{Re}Z}{\alpha} - \frac{\log r_1}{\alpha} + 2C\tilde{\varepsilon} < \mathrm{Re}W < \log r_2 - 2C\tilde{\varepsilon}\right\} \subset V.$$

Proposition 4.1. *Let $\alpha > 0$. Then Φ_Z is injective on V'_Z for any fixed Z.*

Proof. Let W_1 and W_2 be two points in V'_Z such that $\Phi_Z(W_1) = \Phi_Z(W_2)$, and show that $W_1 = W_2$. Define $g(W) = \Phi_Z(W) - \Phi_Z(W_1)$ and $h(W) = W - \Phi_Z(W_1)$. Then $|g - h| = |\Phi_Z - id| < C\tilde{\varepsilon}$ on V_Z. By the definition of V_Z and V'_Z, there is a smooth, simply closed curve Γ in V_Z whose distances from W_1 and W_2 are greater than $C\tilde{\varepsilon}$. Hence $|h| \geq \mathrm{dist}(\Phi_Z(\Gamma), \partial V_Z) \geq 2C\tilde{\varepsilon} - C\tilde{\varepsilon} = C\tilde{\varepsilon}$ on Γ. Therefore, $|g - h| < |h|$ on Γ. Rouché's theorem implies that the number of zero points of g is exactly one in the region surrounded by Γ; thus $W_1 = W_2$. $\square$

Proposition 4.2. *Let $\alpha > 0$. Then ϕ is injective on*

$$\left\{\frac{|z|}{|w|^{\alpha}} < \frac{r_1}{(1+\varepsilon)^{2\alpha C}},\ |w| < \frac{r_2}{(1+\varepsilon)^{2C}}\right\}.$$

Proof. Since Φ_1 and Φ_Z are injective for any Z by Proposition 4.1, we deduce that Φ is injective on V'. Hence ϕ is injective on $\pi(V')$ because $\Phi \sim id$, where $\pi(V') = \{|z/w^\alpha| < r_1', |w| < r_2'\}$ for some constants r_1' and r_2'. Indeed, $r_1' = r_1/(1+\varepsilon)^{2\alpha C}$ and $r_2' = r_2/(1+\varepsilon)^{2C}$ since $(\log r_1')/\alpha = (\log r_1)/\alpha - 2C\tilde{\varepsilon}$ and $\log r_2' = \log r_2 - 2C\tilde{\varepsilon}$. $\square$

Remark 4.3. By similar arguments, it follows that F is injective on

$$\left\{ \frac{ReZ}{\alpha} - \frac{\log r_1}{\alpha} + \frac{2\tilde{\varepsilon}}{d} < ReW < \log r_2 - \frac{2\tilde{\varepsilon}}{d} \right\}.$$

Hence F^n, Φ_n and Φ are injective on the same region. This region is bigger than V' since $C \geq 1/(d-1) > 1/d$. Therefore, we have a bigger region that ensures the injectivity of ϕ.

We next provide an example which indicates an optimality of α. It is a family of polynomial skew products that are semiconjugate to polynomial products, which contains the example $f(z,w) = (z^2, w^2 + z^4)$ in the introduction. See also [11] and [12, Section 10] for such maps.

Example 4.4. Let $f(z,w) = (z^d, w^d + cz^{ld})$ and $l = A/B$, where $d \geq 2$, $A \geq 1$ and B is a divisor of d. Then f is semiconjugate to a product $g(z,w) = (z^d, w^d + c)$ by $\pi(z,w) = (z^B, z^A w) : \pi \circ g = f \circ \pi$. We can construct the Böttcher coordinate that conjugates f to $f_0(z,w) = (z^d, w^d)$ as follows. Let φ_g be the Böttcher coordinate for $w \to w^d + c$ near infinity; it is defined on $\{|w| > R\}$ for large R and conjugates $w \to w^d + c$ to $w \to w^d$. Then $\phi_g(z,w) = (z, \varphi_g(w))$ is a biholomorphic map that conjugates g to $g_0(z,w) = (z^d, w^d)$. Consequently, $\phi_f = \pi \circ \phi_g \circ \pi^{-1}$ or, equivalently, $\phi_f(z,w) = \big(z, z^l \varphi_g\left(w/z^l\right)\big)$ is the required map; it is a well-defined biholomorphic map defined on $\{|w| > R|z|^l\}$ that conjugates f to f_0.

Let us explain an optimality of α, using this example. Note that $\alpha = 1/l = B/A$ and $\mathcal{I}_f = \{a \geq \alpha\}$. As stated above, the Böttcher coordinate ϕ_f exists on U_r^α for small r. This also follows from Theorem 1.2. Whereas we can replace α in Theorem 1.2 with any $a \geq \alpha$ that belongs to $\mathcal{I}_f$, we cannot replace it with any $a < \alpha$; that is, ϕ_f cannot extend from U_r^α to U_r^a for any $a < \alpha$. In fact, if ϕ_f extended to U_r^a for some $a < \alpha$, then φ_g could extend to $\mathbb{C}$, because the closure of $\pi^{-1}(U_r^a)$ includes the w-axis. However, φ_g cannot extend to a region larger than the attracting basin of infinity, except the special case $c = 0$. Moreover, it seems that the invariance of U_r^a does not hold for any $a < \alpha$. We remark that such an optimality of α does not hold for the case $d = 1$.

5. The case $d = 1$

We extend our ideas and results for the case $d \geq 2$ to the case $d = 1$; we prove Theorem 1.4. Because this theorem is a corollary of Favre's results [4], we first explain how Favre's results induce this theorem. After that, two examples show that the condition $\alpha < (\delta - 1)/\gamma$ is necessary for Theorem 1.4. We then give a proof of Theorem 1.4 using the techniques of this paper. The proof of the uniform convergence of ϕ_n is different from the previous case because the sum of d^{-n} does not converge anymore. We remark that the case $d = 0$ exists, but it is not treated in this paper because the map $f_0(z,w) = (z^\delta, bz^\gamma)$ is not dominant, and hence it cannot be conjugate, not even in a wedge, to a dominant map.

Let us apply Favre's results to our situation. If $d = 1$, then f is rigid of class 4 in [4] and hence conjugate to $(z, w) \to (z^\delta, bz^\gamma w + R(z))$ for some polynomial R on a neighborhood of the origin. More precisely, we have the following two cases:

(1) if $\gamma/(\delta - 1) \notin \mathbb{N}$ or $b \neq 1$, then f is "non-special", and $\deg R \leq \gamma$,
(2) if $\gamma/(\delta - 1) \in \mathbb{N}$ and $b = 1$, then f is "special"(meaning resonant), and R is the sum of a polynomial of degree $\leq \gamma$ and a term of the form $az^{\gamma/(\delta-1)+\gamma}$.

In the first case, it follows from the definition of f that $R \equiv 0$, and no conditions on α are needed. In the second case, the condition $\alpha < (\delta - 1)/\gamma$ allows us to avoid the term of degree $\gamma/(\delta - 1) + \gamma$ in the normal form, and again we get $R \equiv 0$. In particular, we obtain Theorem 1.4.

We next exhibit the following two examples. The first example satisfies all the conditions of Theorem 1.4, and the second one does not.

Example 5.1. Let $f(z, w) = (z^2, bzw + z^3)$. Then $\alpha = 1/2 < (\delta - 1)/\gamma = 1$. By Theorem 1.4, there exists the Böttcher coordinate on U_r for small r, that conjugates f to $f_0(z, w) = (z^\delta, bzw)$.

Note that if α is well-defined, then $\alpha \leq (\delta - 1)/\gamma$. The second example satisfies the equation $\alpha = (\delta - 1)/\gamma$.

Example 5.2. Let $f(z, w) = (z^2, bzw + z^2)$. Then $\alpha = (\delta - 1)/\gamma = 1$, and f is semiconjugate to $g(z, w) = (z^2, bw + 1)$ by $\pi(z, w) = (z, zw) : \pi \circ g = f \circ \pi$. Moreover, if $b \neq 1$, then f is conjugate to $f_0(z, w) = (z^2, bzw)$ by h_f, and g is conjugate to $g_0(z, w) = (z^2, bw)$ by h_g, where $h_f(z, w) = (z, w + z/(1 - b))$ and $h_g(z, w) = (z, w + 1/(1 - b))$.

For this example, Theorem 1.4 does not hold at least if $b = 1$. In fact, if we had a Böttcher coordinate that conjugates f to $f_0(z, w) = (z^2, zw)$, then g should be conjugate to $g_0(z, w) = (z^2, w)$. However, the translation $w \to w + 1$ cannot be conjugate to the identity $w \to w$. This case corresponds to the special case (1) above and shows that Böttcher coordinates may not exist even if α is well-defined. Although a Böttcher coordinate h_f exists if $b \neq 1$, the dynamics is different from our case. In particular, the second component of g_0 in this example is affine, whereas the second component of g_0 in Example 4.4 is w^d, where $d \geq 2$, and so it has a superattracting fixed point at infinity. We can slightly generalize this example to $f(z, w) = (z^\delta, bz^{\delta-1}w + z^\delta)$. Since $\gamma = \delta - 1$, again $\alpha = (\delta - 1)/\gamma = 1$, and f is semiconjugate to $g(z, w) = (z^\delta, bw + 1)$ by $\pi(z, w) = (z^{\delta-1}, z^\gamma w)$.

Let us begin the proof of Theorem 1.4 by the techniques of this paper. Since the investigation of the second components of maps is the essential part for proofs, we sometimes omit the expressions of the first components hereafter. In a similar fashion to the case $d \geq 2$, let $\eta = (q - bz^\gamma w)/bz^\gamma w$. Then $|Q - Q_0| = |\log(1 + \eta \circ \pi)|$. Since we may assume that $|\eta| < 1$ on U_r or, equivalently, $|\eta \circ \pi| < 1$ on $\pi^{-1}(U_r)$,

$$|Q - Q_0| \leq \log(1 + |\eta \circ \pi|) \leq |\eta \circ \pi| \text{ and so } |Q(F^n) - Q_0(F^n)| \leq |\eta \circ \pi(F^n)|.$$

To prove the uniform convergence of ϕ_n, we show that $|\eta \circ \pi(F^n)|$ or, equivalently, $|\eta(f^n)|$ decreases rapidly as $n \to \infty$ in Lemma 5.4. First, we claim that f^n contracts U_r rapidly. Since the origin is superattracting, it is clear that f^n contracts a small bidisk rapidly; e.g., $f^n(\{|z| < r, |w| < r\}) \subset \{|z| < r/2^n, |w| < r/2^n\}$. Moreover, the same contraction holds for U_r, where $U_r = \{|z| < r|w|^\alpha, |w| < r\}$.

Lemma 5.3. *Let $d = 1$. If α is well-defined and $\alpha < (\delta - 1)/\gamma$, then $f^n(U_r) \subset U_{r/2^n}$ for small r.*

Proof. By Lemma 1.3, for any small ε there is r such that

$$|p(z)| < (1+\varepsilon)|z^\delta| \text{ and } (1-\varepsilon)|bz^\gamma w| < |q(z,w)| < (1+\varepsilon)|bz^\gamma w|$$

on U_r. Let $|z| = |cw^\alpha|$. Then $U_r \subset \{|c| < r, |w| < r\}$ and

$$\left|\frac{p(z)}{q(z,w)^\alpha}\right| < \frac{1+\varepsilon}{(1-\varepsilon)^\alpha} \cdot \left|\frac{z^\delta}{(bz^\gamma w)^\alpha}\right| = \frac{1+\varepsilon}{(1-\varepsilon)^\alpha} \cdot \left|\frac{(cw^\alpha)^\alpha}{\{b(cw^\alpha)^\gamma w\}^\alpha}\right|$$
$$= \frac{1+\varepsilon}{(1-\varepsilon)^\alpha} \cdot \frac{1}{|b|^\alpha} \cdot |c|^{\delta-\alpha\gamma}|w|^{\alpha\delta-\alpha(\alpha\gamma+1)}.$$

By assumption, $\delta - \alpha\gamma > 1$ and $\alpha\{\delta - (\alpha\gamma+1)\} \geq 0$. Therefore, shrinking r so that $(1+\varepsilon)r^{\delta-\alpha\gamma-1}/(1-\varepsilon)^\alpha|b|^\alpha < 1/2$, we obtain that

$$|p/q^\alpha| < |c|/2 < r/2.$$

In addition, since $|q(z,w)| < (1+\varepsilon)|b(cw^\alpha)^\gamma w| = (1+\varepsilon)|b||c|^\gamma|w|^{\alpha\gamma} \cdot |w|$,

$$|q| < |w|/2 < r/2$$

for r such that $(1+\varepsilon)|b|r^{\gamma(\alpha+1)} < 1/2$. This implies that

$$f(\{|c| < r, |w| < r\}) \subset \{|c| < r/2, |w| < r/2\}; \text{ that is, } f(U_r) \subset U_{r/2}.$$

By repeating this calculation, it follows that

$$f^n(\{|c| < r, |w| < r\}) \subset \{|c| < r/2^n, |w| < r/2^n\}; \text{ that is, } f^n(U_r) \subset U_{r/2^n}. \qquad \square$$

From Lemma 5.3 we derive the uniform estimate of $|\eta(f^n)|$ on U_r.

Lemma 5.4. *Let $d = 1$. If α is well-defined and $\alpha < (\delta-1)/\gamma$, then*

$$|\zeta(p^n)| \leq \frac{C_1 r}{2^n} \text{ and } |\eta(f^n)| \leq \frac{C_2 r}{2^n}$$

on U_r for some constants C_1 and C_2.

Proof. Let $|z| = |cw^\alpha|$. Then

$$|\eta| = \left|\sum \frac{b_j z^{n_j} w^{m_j}}{bz^\gamma w}\right| \leq \sum \left|\frac{b_j}{b}\right| \cdot |c|^{n_j-\gamma}|w|^{(\alpha n_j + m_j)-(\alpha\gamma+1)}.$$

By assumption, $n_j \geq \gamma$ and $\alpha n_j + m_j \geq \alpha\gamma + 1$. Moreover, at least one of the inequalities $n_j - \gamma \geq 1$ or $(\alpha n_j + m_j) - (\alpha\gamma+1) \geq 1$ holds. Hence there exist constants A and B such that $|\eta| \leq A|c| + B|w|$. It then follows from Lemma 5.3 that $|\eta(f^n)| \leq Ar/2^n + Br/2^n = (A+B)r/2^n$ on U_r. $\square$

Now we are ready to prove the uniform convergence of ϕ_n.

Proposition 5.5. *Let $d = 1$. If α is well-defined and $\alpha < (\delta-1)/\gamma$, then ϕ_n converges uniformly to ϕ on U_r, and $\phi \sim id$ on U_r as $r \to 0$.*

Proof. Let Φ_n be the lift of ϕ_n and $\Phi_n = (\Phi_n^1, \Phi_n^2)$ as in Section 3. It is enough to show the uniform convergence of Φ_n^2. By Lemma 5.4,

$$|\Phi_{n+1}^2 - \Phi_n^2| \leq \frac{|Q(F^n) - Q_0(F^n)|}{d^{n+1}} + \frac{\gamma_{n+1}|P(P^n) - P_0(P^n)|}{\delta^{n+1}d^{n+1}}$$
$$\leq |\eta \circ \pi(F^n)| + \frac{\gamma}{\delta-1}|\zeta \circ \pi(P^n)| < \left(C_2 + \frac{\gamma}{\delta-1}C_1\right)\frac{r}{2^n}. \qquad \square$$

The proof of the injectivity of ϕ is the same as the case $d \geq 2$.

Proposition 5.6. *Let $d = 1$. If α is well-defined and $\alpha < (\delta-1)/\gamma$, then ϕ is injective on U_r for small r.*

6. A generalization to holomorphic germs

Until now we have dealt with a germ of holomorphic skew product of the form $f(z,w) = (p(z), q(z,w))$ such that $p(z) = z^\delta + a_{\delta+1}z^{\delta+1} + \cdots$ and

$$q(z,w) = bz^\gamma w^d + \sum b_j z^{n_j} w^{m_j},$$

where $b \neq 0$, $\gamma \leq n_j$, and $d < m_j$ if $\gamma = n_j$. Since the origin is a superattracting fixed point, $\delta \geq 2$, $\gamma + d \geq 2$ and $n_j + m_j \geq 2$. In this section we perturb p to a holomorphic germ $\tilde{p}$ in z and w such that $\tilde{p}(z,w) = a(w)z^\delta + a_{\delta+1}(w)z^{\delta+1} + \cdots$, where $a(0) = 1$. In other words,

$$\tilde{p}(z,w) = z^\delta + \sum a_l z^{n_l} w^{m_l},$$

where $n_l \geq \delta$, and $m_l \geq 1$ if $n_l = \delta$. Let $f(z,w) = (\tilde{p}(z,w), q(z,w))$ and let us call it a perturbed skew product hereafter.

We first construct a biholomorphic map ϕ that conjugates f to f_0 by arguments similar to the skew product case, where $f_0(z,w) = (z^\delta, bz^\gamma w^d)$. It is more difficult to prove the injectivity of ϕ because f no longer preserves the family of fibers. We then give another proof of f being conjugate to f_0. In fact, it follows from [9] that f is conjugate to a skew product of the form $\tilde{f}(z,w) = (z^\delta, \tilde{q}(z,w))$ for some holomorphic germ $\tilde{q}$.

In a similar fashion to the skew product case, we define the rational number α associated with f as

$$\alpha = \min\left\{ a \geq 0 \;\middle|\; \begin{array}{l} a\gamma + d \leq \delta,\ a\delta \leq an_l + m_l \text{ and } a\gamma + d \leq an_j + m_j \\ \text{for any } l \text{ and } j \text{ such that } a_l \neq 0 \text{ and } b_j \neq 0 \end{array} \right\}$$

if f is *non-trivial*, and as 0 if f is *trivial*. We remark that the condition $a\delta \leq an_l + m_l$ is trivial and can be removed since $n_l \geq \delta$ and $a \geq 0$, although the interval $\mathcal{I}_f$ may differ whether or not we add the condition. Hence the weights of the skew product (p,q) and the holomorphic germ $f = (\tilde{p}, q)$ are the same. Moreover, the weights of f and $\tilde{f}$ are also the same, as stated in Lemma 6.6 below.

Let us construct a Böttcher coordinate for f. Since $\tilde{p}(z,w) \sim z^\delta$ on the neighborhood $\{|z| < r, |w| < r\}$ as $r \to 0$, we have the following lemma.

Lemma 6.1. *Let f be a perturbed skew product and α well-defined. If $d \geq 2$ or if $d = 1$ and $\alpha < (\delta - 1)/\gamma$, then $f(z,w) \sim (z^\delta, bz^\gamma w^d)$ on U_r as $r \to 0$, and $f(U_r) \subset U_r$ for small r.*

This lemma induces the existence of the limit of the compositions of f_0^{-n} and f^n as previous cases, where $f_0(z,w) = (z^\delta, bz^\gamma w^d)$.

Theorem 6.2. *Let f be a perturbed skew product and α well-defined. If $d \geq 2$ or if $d = 1$ and $\alpha < (\delta - 1)/\gamma$, then there is a biholomorphic map ϕ defined on U_r, with $\phi \sim id$ on U_r as $r \to 0$, that conjugates f to $(z,w) \to (z^\delta, bz^\gamma w^d)$.*

The proof of the existence of ϕ is similar to the skew product case. The difficult part of the proof is the injectivity of ϕ. Since ϕ is clearly injective if $\alpha = 0$, we may assume that $\alpha > 0$ hereafter. Let us state the idea of the proof of the injectivity of ϕ. As in Section 4, we prove that the lift Φ of ϕ is injective, which implies the injectivity of ϕ because $\Phi \sim id$. For the skew product case, we applied Rouché's theorem to Φ restricted to a vertical line in order to show that Φ_Z is injective,

where $\Phi = (\Phi_1, \Phi_Z)$. Since we may assume that Φ_1 is injective, this implies that Φ is injective. On the other hand, in this section we apply Rouché's theorem to Φ restricted to a line, which may not be vertical, as follows. Let Φ be well-defined and holomorphic on V, and take a sufficiently small region V' in V. Let A_1 and A_2 be two points in V' such that $\Phi(A_1) = \Phi(A_2)$. Applying Rouché's theorem to Φ restricted to the intersection of V and the line L passing through A_1 and A_2, we can show that $A_1 = A_2$.

The point is taking a smaller region V' in V such that $L \cap (V \setminus V')$ has a suitable width for any line L intersecting V', as in Section 4. Recall that

$$V = \left\{ \frac{\mathrm{Re}Z}{\alpha} - \frac{\log r_1}{\alpha} < \mathrm{Re}W < \log r_2 \right\},$$

and let $\|\Phi - id\| < \varepsilon$. Then the following region is what we need:

$$V' = \left\{ \frac{\mathrm{Re}Z}{\alpha} - \frac{\log r_1}{\alpha} + \frac{1+\alpha}{\alpha} \cdot 2\varepsilon < \mathrm{Re}W < \log r_2 - 2\varepsilon \right\}.$$

Let us illustrate where the constant $(1+\alpha)/\alpha$ comes from. First, consider everything in $\mathbb{R}^2$. Let $L = \{y = mx\}$, $\mathcal{V} = \{y > x/\alpha\}$ and $\mathcal{V}' = \{y > x/\alpha + R \cdot 2\varepsilon\}$ for a constant R, where $(x, y) \in \mathbb{R}^2$ and $m \in \mathbb{R}$. If $|m| \geq 1$, then we take the projection π_2 to the second coordinate, and require that the length of the interval $\pi_2(L \cap (\mathcal{V} \setminus \mathcal{V}'))$ in $\mathbb{R}$ is greater than or equal to 2ε. It is enough to consider the case $m = -1$, since the length takes the minimum for this case. By an elementary calculation in terms of two right-angled triangles, it follows that, if $R = 1 + 1/\alpha$, then the length coincides with 2ε. If $|m| \leq 1$, then we take the projection π_1 to the first coordinate. By the same argument, it follows that, if $R = 1 + 1/\alpha$, then the length of $\pi_1(L \cap (\mathcal{V} \setminus \mathcal{V}'))$ is greater than or equal to 2ε. This sketch works for complex settings as well:

Lemma 6.3. *Let L be a line $\{W = mZ + n\}$ which intersects V'. Then*

$$\mathit{dist}(\pi_1^{-1}(L \cap V'), \partial\pi_1^{-1}(L \cap V)) \geq 2\varepsilon \text{ if } |m| \leq 1 \text{ and}$$

$$\mathit{dist}(\pi_2^{-1}(L \cap V'), \partial\pi_2^{-1}(L \cap V)) \geq 2\varepsilon \text{ if } |m| \geq 1,$$

where π_1 and π_2 are the projections to Z and W coordinates, respectively.

Proof. Let $n = 0$ for simplicity. We only prove the case $|m| \geq 1$. Note that

$$\pi_2^{-1}(L \cap V') = H \cap \left\{ \mathrm{Re}W < \frac{1}{\alpha}\mathrm{Re}\frac{W}{m} - \frac{\log r_1}{\alpha} + \frac{1+\alpha}{\alpha} \cdot 2\varepsilon \right\}$$

$$= H \cap \{\mathrm{Re}\{(\alpha - 1/m)W\} < -\log r_1 + (1+\alpha)2\varepsilon\},$$

where $H = \{\mathrm{Re}W < \log r_2 - 2\varepsilon\}$. It is enough to show that $\mathrm{dist}(l_0, l_\varepsilon) \geq 2\varepsilon$, where $l_0 : \{\mathrm{Re}\{(\alpha - 1/m)W\} = 0\}$ and $l_\varepsilon : \{\mathrm{Re}\{(\alpha - 1/m)W\} = (1+\alpha)2\varepsilon\}$. Actually,

$$\mathrm{dist}(l_0, l_\varepsilon) = \frac{(1+\alpha)2\varepsilon}{|\alpha - 1/m|} \geq 2\varepsilon \text{ since } \left|\alpha - \frac{1}{m}\right| \leq \alpha + \frac{1}{|m|} \leq \alpha + 1. \qquad \square$$

Now we are ready to prove the injectivity of Φ.

Proposition 6.4. *The map Φ is injective on V'.*

Proof. Let $\Phi(A_1) = \Phi(A_2)$ for points A_1 and A_2 in V'. Let L be the line passing through A_1 and A_2. It is enough to consider the case $L = \{W = mZ + n\}$.

Define $\tilde{\Phi}_1 = \pi_1 \circ \Phi \circ u$ and $\tilde{\Phi}_2 = \pi_2 \circ \Phi \circ v$, where $u(Z) = (Z, mZ + n)$ and $v(W) = (W/m, W + n)$:

$$\tilde{\Phi}_1(\text{or } \tilde{\Phi}_2) : \text{preimage in } \mathbb{C} \xrightarrow{u(\text{or } v)} L \cap V \xrightarrow{\Phi} \mathbb{C}^2 \xrightarrow{\pi_1(\text{or } \pi_2)} \mathbb{C}.$$

It then follows from Lemma 6.3 that $A_1 = A_2$, by applying Rouché's theorem to $\tilde{\Phi}_1$ or $\tilde{\Phi}_2$ if $|m| \leq 1$ or $|m| \geq 1$ as in Proposition 4.1. □

Finally, we give another proof of Theorem 6.2. The perturbed skew product f can be written as $(z^\delta(1 + \varepsilon(z, w)), q(z, w))$, where ε converges to 0 as z, $w \to 0$. Moreover, Theorem 1.3 in [9] induces the following.

Proposition 6.5. *The perturbed skew product f is conjugate to a skew product of the form $\tilde{f}(z, w) = (z^\delta, \tilde{q}(z, w))$ for some $\tilde{q}$.*

Proof. We briefly review the proof in [9] following a slightly different presentation. Define

$$\phi_n(z, w) = \left(z \cdot \prod_{j=1}^{n} \sqrt[\delta^j]{1 + \varepsilon(f^{j-1}(z, w))}, w \right).$$

Then ϕ_n is well-defined on a small neighborhood of the origin, and

$$\phi_n \circ f = \tilde{f}_n \circ \phi_{n+1}$$

holds, where $\tilde{f}_n(z, w) = (z^\delta, q(\phi_{n+1}^{-1}(z, w)))$. Since ϕ_n converges uniformly to ϕ_∞, it follows that $\phi_\infty \circ f = \tilde{f} \circ \phi_\infty$, where $\tilde{f}(z, w) = (z^\delta, q(\phi_\infty^{-1}(z, w)))$. □

Since $\tilde{f}$ is skew product, we can construct the Böttcher coordinate $\tilde{\phi}$ defined on $U_r^{\tilde{\alpha}}$ that conjugates $\tilde{f}$ to f_0 as previous sections, where $\tilde{\alpha}$ denotes the weight of $\tilde{f}$ and $f_0(z, w) = (z^\delta, bz^\gamma w^d)$. Moreover, the region $U_r^{\tilde{\alpha}}$ coincides with U_r^α:

Lemma 6.6. *The weights α and $\tilde{\alpha}$ of f and $\tilde{f}$ are the same.*

Proof. We may write $\phi_\infty(z, w) = (z(1 + u(z, w)), w)$ for a holomorphic germ u, and so $\tilde{q}(z, w) = q(z(1 + v(z, w)), w)$ for a holomorphic germ v since $\tilde{f} = \phi_\infty \circ f \circ \phi_\infty^{-1}$. Let $b_j z^{n_j} w^{m_j}$ be a term in q. Since v is holomorphic in a neighborhood of the origin, the power series expansion of the corresponding term $b_j\{z(1 + v(z, w))\}^{n_j} w^{m_j}$ in $\tilde{q}$ can be expressed as

$$b_j z^{n_j} w^{m_j} + \sum b_{ij} z^{n_{ij}} w^{m_{ij}},$$

where $n_{ij} \geq n_j$ and $m_{ij} \geq m_j$ for any i. In particular, $\tilde{q}$ has the same major term $bz^\gamma w^d$ as q. If f is *trivial*, then $\tilde{f}$ is also *trivial* and $\alpha = \tilde{\alpha} = 0$. Let f be *non-trivial*. Then $m_j < d$ and $n_j > \gamma$ for some j,

$$\alpha = \sup \left\{ \frac{d - m_j}{n_j - \gamma} \;\middle|\; b_j \neq 0 \text{ and } n_j > \gamma \right\} > 0 \text{ and}$$

$$\tilde{\alpha} = \sup \left\{ \frac{d - m_j}{n_j - \gamma}, \frac{d - m_{ij}}{n_{ij} - \gamma} \;\middle|\; b_j \neq 0, n_j > \gamma \text{ and } b_{ij} \neq 0 \right\} > 0.$$

Since $d - m_{ij} \leq d - m_j$ and $n_j - \gamma \leq n_{ij} - \gamma$, it follows that

$$\frac{d - m_{ij}}{n_{ij} - \gamma} \leq \frac{d - m_j}{n_j - \gamma}$$

if $n_j > \gamma$. Therefore, $\alpha = \tilde{\alpha}$. □

Consequently, the composition $\tilde{\phi} \circ \phi_\infty$ coincides with the Böttcher coordinate ϕ in Theorem 6.2, that is defined on U_r^α and conjugates f to f_0, where ϕ_∞ is the change of coordinates provided by the proof of Proposition 6.5.

Acknowledgments

The author would like to express his sincere gratitude to the referee for many helpful and valuable suggestions, which revised the proof of Lemma 5.4, and for instructions of the relationship between our works and Favre's for the case $d = 1$.

References

[1] M. Abate, *Open problems in local discrete holomorphic dynamics*, Anal. Math. Phys. **1** (2011), no. 4, 261–287, DOI 10.1007/s13324-011-0017-y. MR2887101

[2] L. E. Böttcher, *The principal laws of convergence of iterates and their application to analysis* (Russian), Izv. Kazan. Fiz.-Mat. Obshch. **14** (1904), 137-152.

[3] X. Buff, A. L. Epstein, and S. Koch, *Böttcher coordinates*, Indiana Univ. Math. J. **61** (2012), no. 5, 1765–1799, DOI 10.1512/iumj.2012.61.4981. MR3119600

[4] C. Favre, *Classification of 2-dimensional contracting rigid germs and Kato surfaces. I* (English, with English and French summaries), J. Math. Pures Appl. (9) **79** (2000), no. 5, 475–514, DOI 10.1016/S0021-7824(00)00162-8. MR1759437

[5] C. Favre and M. Jonsson, *Eigenvaluations*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 2, 309–349, DOI 10.1016/j.ansens.2007.01.002. MR2339287

[6] J. H. Hubbard and P. Papadopol, *Superattractive fixed points in* $\mathbf{C}^n$, Indiana Univ. Math. J. **43** (1994), no. 1, 321–365, DOI 10.1512/iumj.1994.43.43014. MR1275463

[7] M. Jonsson, *Dynamics of Berkovich spaces in low dimensions*, Berkovich spaces and applications, Lecture Notes in Math., vol. 2119, Springer, 2015, pp. 205–366, DOI 10.1007/978-3-319-11029-5_6. MR3330767

[8] M. Ruggiero, *Rigidification of holomorphic germs with noninvertible differential*, Michigan Math. J. **61** (2012), no. 1, 161–185, DOI 10.1307/mmj/1331222853. MR2904007

[9] M. Ruggiero, *Contracting rigid germs in higher dimensions* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **63** (2013), no. 5, 1913–1950, DOI 10.5802/aif.2818. MR3186512

[10] T. Ueda, *Complex dynamical systems on projective spaces*, Adv. Ser. Dynam. Systems, vol. 13, World Scientific, Singapore, 1993, pp. 120-138.

[11] K. Ueno, *Fiberwise Green functions of skew products semiconjugate to some polynomial products on* $\mathbf{C}^2$, Kodai Math. J. **35** (2012), no. 2, 345–357, DOI 10.2996/kmj/1341401055. MR2951261

[12] K. Ueno, *Böttcher coordinates for polynomial skew products*, to appear in Ergodic Theory Dynam. Systems. available online.

[13] S. Ushiki, *Böttcher's theorem and super-stable manifolds for multidimensional complex dynamical systems*, Adv. Ser. Dynam. Systems, vol. 11, World Scientific, Singapore, 1992, pp. 168-184.

Daido University, Nagoya 457-8530, Japan
E-mail address: k-ueno@daido-it.ac.jp