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THE SPACE OF FUNCTIONS OF BOUNDED VARIATION ON CURVES IN METRIC MEASURE SPACES

O. MARTIO

Dedicated to Jan Malý on his 60^{th} birthday

Abstract. The approximation modulus, AM–modulus for short, was defined in an earlier paper by the author. In this paper it is shown that an $L^1(X)$–function u in a metric measure space (X, d, ν) can be defined to be of bounded variation on AM–a.e. curve in X without an approximation of u by Lipschitz or Newtonian functions. The essential variation of u on AM almost every curve is bounded by a sequence of non-negative Borel functions in $L^1(X)$. The space of such functions is a Banach space and there is a Borel measure associated with u. For $X = \mathbf{R}^n$ this gives the classical space of BV functions.

1. Introduction

Since the pioneering work of Ahlfors and Beurling the modulus method has been widely used in conformal geometry in $\mathbf{R}^n$, $n \geq 2$, and, more recently, in metric measure spaces; see [10, 11]. Fuglede [8] showed that the method is useful in the theory of function spaces and in the study of regularity properties of functions in these spaces. The modulus method has also been applied to function spaces in metric measure spaces (see [4, 5, 15]), where it has a more fundamental role than in $\mathbf{R}^n$.

In this paper a new modulus, the AM–modulus, is used to construct a function space of functions of bounded variation (BV) in a metric measure space X. The construction corresponds to the construction of the so-called Newtonian spaces $N^{1,p}(X)$ in X; see [5, 15]. Instead of the concept of a (weak) upper gradient we use the concept of a BV upper bound which consists of a sequence of non-negative Borel functions in $L^1(X)$.

An $N^{1,p}(X)$ function need not be absolutely continuous on every curve; it is necessary to omit the family of curves of M_p–modulus zero. Similarly a BV function need not be of bounded variation on every curve. This is already evident in $\mathbf{R}$ where the function $u = \chi_{\{0\}}$ is not BV on every curve. For example, u is not BV on the curve which crosses 0 infinitely often. The AM–modulus takes care of this situation.

It is customary to consider BV functions in X defined only almost everywhere. From the approximation point of view $L^1(X)$ is a good choice for the base space and this approach is used both in $\mathbf{R}^n$ (see [7]) and in X (see [14]). In [1, 2, 6] several alternative definitions for BV functions in X based on approximation by Lipschitz functions are discussed. As in the case of the Newtonian spaces our approach does not use approximation by smooth functions. It turns out, as in the Newtonian

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space case, that in $\mathbf{R}^n$ our approach and the approximation approach lead to the same space.

The AM–modulus can be applied to other base spaces than $L^1(X)$ (see [13, Section 4]), but we consider $L^1(X)$ since it naturally leads to a Banach space of BV functions in X. The space $L^1(X)$ also has the advantage that its functions are well defined on M_1–a.e. curve and thus on AM–a.e. curve.

In a forthcoming paper with V. Honzlová–Exernova and J. Malý we show that there exist curve families Γ in $\mathbf{R}^n$, $n \geq 1$, such that $M_1(\Gamma) = \infty$ but $AM(\Gamma) = 0$. Since $AM(\Gamma) \leq M_1(\Gamma)$ for every curve family Γ in X, this shows that the AM–modulus is essentially weaker than the M_1–modulus.

After preliminaries in Section 2 we prove the Banach property of the space $BV_{AM}(X)$ of BV functions in Section 3. Section 4 is devoted to the construction of a Borel measure associated with u. The construction resembles the construction due to Miranda in [14] who used approximation by Lipschitz functions with L^1 bounded upper gradients for u. In Section 5 the case $X = \mathbf{R}^n$ is considered. We use minimal assumptions on the space (X, d) and the measure ν.

Our notation is standard.

2. Preliminaries

We recall the basic properties of functions of bounded variation and the AM–modulus and introduce the concept of a BV_{AM} upper bound for a function $u \in L^1(X)$.

Let (X, d, ν) be a metric measure space where d is a metric and ν is a regular Borel measure in X, We use the definition in [5, I.1.1] for regularity and, in particular, ν is an outer measure in X. In addition we assume that ν satisfies

$$0 < \nu(B(x, r)) < \infty \tag{2.1}$$

for each ball $B(x, r)$ in X. In Section 4 we make some additional assumptions on the space (X, d).

Functions of bounded variation. Suppose that $u : [a, b] \to \mathbf{R} \cup \{\infty, -\infty\}$ and there is a set $C \subset [a, b]$ such that $m_1([a, b] \setminus C) = 0$ and $u|C$ is of bounded variation on C. The latter means that

$$V(u, C) = \sup \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})| < \infty$$

where the supremum is taken over all finite sequences $x_0 < x_1 < \cdots < x_n$ of points in C. Such a function u is said to be of essentially bounded variation in $[a, b]$. Then (see [13, Section 2]), there is a function $\hat{u} : [a, b] \to \mathbf{R}$ such that $\hat{u} = u$ a.e. in $[a, b]$ and

$$V(\hat{u}, [a, b]) = \inf\{V(v, [a, b]) : v = u \text{ a.e. in } [a, b]\}.$$

Moreover, the function $\hat{u}$ can be chosen to be right continuous in $[a, b)$ and continuous at b. With these properties $\hat{u}$ is uniquely determined by u, i.e. if $u = v$ a.e. in $[a, b]$, then $\hat{u} = \hat{v}$. Note also that

$$V(\hat{u}, [a, b]) = V(\hat{u}, C) \tag{2.2}$$

for every $C \subset [a,b]$ of full measure, since if $V(\hat{u},[a,b]) > V(\hat{u},C)$, then by [13, Lemma 2.1] there is a function v in $[a,b]$ such that $v = \hat{u}$ in C and

$$V(v,[a,b]) = V(v,C) = V(\hat{u},C)$$

so that $\hat{u}$ is not of minimal variation.

Lemma 2.1. *Suppose that $u_i \to u$ in $L^1([a,b])$. Then*

$$V(\hat{u},[a,b]) \le \liminf_{i\to\infty} V(\hat{u}_i,[a,b]). \tag{2.3}$$

Proof. Note that $\hat{u}_i \to \hat{u}$ in $L^1([a,b])$. Choose a subsequence $(\hat{u}_{i_j})$ of $(\hat{u}_i)$ such that

$$\lim_{j\to\infty} V(\hat{u}_{i_j},[a,b]) = \liminf_{i\to\infty} V(\hat{u}_i,[a,b])$$

and then a subsequence of $(\hat{u}_{i_j})$, denoted again by $(\hat{u}_{i_j})$, such that $\hat{u}_{i_j} \to \hat{u}$ in C where $m_1([a,b]\setminus C) = 0$. From (2.2) we obtain

$$\begin{aligned} V(\hat{u},[a,b]) = V(\hat{u},C) &\le \liminf_{j\to\infty} V(\hat{u}_{i_j},C) \\ = \liminf_{j\to\infty} V(\hat{u}_{i_j},[a,b]) &= \liminf_{i\to\infty} V(\hat{u}_i,[a,b]) \end{aligned}$$

as required. □

M_1– and AM–modulus. A curve in X is a continuous mapping $\gamma : [a,b] \to X$. We can always assume that a curve is non-constant and rectifiable and then we parametrize γ by arc length, i.e. $\gamma : [0,l(\gamma)] \to X$ where $l(\gamma)$ is the length of γ. This is due to the fact that both the M_1– and the AM–modulus of the family of all non-rectifiable curves in X is zero. Note that it is essential that we consider curves which are defined on compact intervals. The M_1– and AM–modulus can be defined on families of curves which are defined on open or half open intervals and which are only locally rectifiable. Then it turns out that there are families of non-rectifiable curves whose M_1– and the AM–modulus are not zero.

Let Γ be a family of curves in X. We first recall the concept of M_p–modulus, $p \ge 1$. A non–negative Borel function ρ is *M_p–admissible* for Γ if

$$\int_\gamma \rho\, ds \ge 1$$

for every $\gamma \in \Gamma$. The *M_p–modulus* of Γ is defined as

$$M_p(\Gamma) = \inf \int_X \rho^p\, d\nu$$

where the infimum is taken over all admissible functions ρ. For the theory of M_p–modulus see [5,8]. In this paper we only use the M_1–modulus.

A sequence of non-negative Borel functions ρ_i, $i = 1, 2, \ldots$, is *AM–admissible* for Γ if

$$\liminf_{i\to\infty} \int_\gamma \rho_i\, ds \ge 1 \tag{2.4}$$

for every $\gamma \in \Gamma$. The *approximation modulus*, AM–modulus for short, of Γ is defined as

$$AM(\Gamma) = \inf_{(\rho_i)} \{\liminf_{i\to\infty} \int_X \rho_i\, d\nu\} \tag{2.5}$$

where the infimum is taken over all AM–admissibe sequences (ρ_i) for Γ.

We say that a property holds on M_p–almost every (a.e.) curve or on AM–a.e. curve if it holds for every curve except for a family Γ with $M_p(\Gamma) = 0$ or $AM(\Gamma) = 0$, respectively. Since $AM(\Gamma) \le M_1(\Gamma)$ for each curve family Γ in X, the property which holds on M_1–a.e. curve holds on AM–a.e. curve as well. For the properties of the AM–modulus see [13].

BV_{AM} upper bounds. Let γ be a curve in X and u a function such that $u\circ\gamma$ is a.e. defined in $[0, l(\gamma)]$. If $V(\widehat{u\circ\gamma}, [0, l(\gamma)]) < \infty$, we set $V(u,\gamma) = V(\widehat{u\circ\gamma}, [0, l(\gamma)])$ and if there is no set $C \subset [0, l(\gamma)]$ such that $m_1([0, l(\gamma)] \setminus C) = 0$ and $V(u\circ\gamma, C) < \infty$, then we set $V(u,\gamma) = \infty$.

Suppose that $u \in L^1(X)$. By [5, Lemma 1.43 and Proposition 1.37] the integral

$$\int_\gamma u\, ds$$

is well defined and finite on M_1–a.e. curve γ. In particular the function $u\circ\gamma$ is a.e. defined on $[0, l(\gamma)]$ and hence $V(u,\gamma)$ is defined on M_1– and hence on AM–a.e. curve γ in X.

A sequence (g_i) of non-negative Borel functions is called a *BV_{AM} upper bound* for $u \in L^1(X)$ if

$$V(u,\gamma) \le \liminf_{i\to\infty} \int_\gamma g_i\, ds \tag{2.6}$$

on AM–a.e. curve γ in X. Note that if (g_i) is a BV_{AM} upper bound for u, then every subsequence of (g_i) is also a BV_{AM} upper bound for u.

The next lemma gives alternative characterizations for (2.6).

Lemma 2.2. *Let $u \in L^1(X)$ and (g_i) a sequence of non-negative Borel functions in X. Then the following conditions are equivalent:*

$$(g_i) \textit{ is a } BV_{AM} \textit{ upper bound for } u, \tag{2.7}$$

$$V(u,\gamma|[t_1,t_2]) \le \liminf_{i\to\infty} \int_{\gamma|[t_1,t_2]} g_i\, ds \textit{ for all } 0 \le t_1 < t_2 \le l(\gamma), \tag{2.8}$$

$$|u(\gamma(t_2)) - u(\gamma(t_1))| \le \liminf_{i\to\infty} \int_{\gamma|[t_1,t_2]} g_i\, ds \textit{ for a.e. } t_1, t_2 \in [0, l(\gamma)], \tag{2.9}$$

where (2.8) *and* (2.9) *are supposed to hold on AM–a.e. curve γ in X.*

Proof. Suppose that (2.7) holds. Since (2.6) holds on every curve γ, it holds on every subcurve of γ as well and so (2.8) follows.

Clearly (2.8) implies (2.9) and it remains to show (2.9) $\Rightarrow$ (2.7). Suppose that (2.9) holds on a curve γ. Let $C \subset [0, l(\gamma]$ be a set of full measure and (2.9) holds for all $t_1, t_2 \in C$, $t_1 < t_2$ and let $a_1 < a_2 < \cdots < a_n$ be points in C. Then

$$\sum_{j=1}^{n-1} |u(\gamma(a_{j+1}) - u(\gamma(a_j))| \le \sum_{j=1}^{n-1} \liminf_{i\to\infty} \int_{\gamma|[a_j,a_{j+1}]} g_i\, ds$$

$$\le \liminf_{i\to\infty} \sum_{j=1}^{n-1} \int_{\gamma|[a_j,a_{j+1}]} g_i\, ds \le \liminf_{i\to\infty} \int_\gamma g_i\, ds$$

and this shows that (g_i) is a BV_{AM} upper bound for u. The lemma follows. $\square$

Lemma 2.3. *If $u_i \to u$ in $L^1(X)$, then the sequence (u_i) has a subsequence (u_{i_j}) such that*

$$V(u,\gamma) \le \liminf_{j\to\infty} V(u_{i_j},\gamma)$$

for AM–a.e. curve γ in X.

Proof. By the Fuglede lemma [5, Lemma 2.1] the sequence (u_i) has a subsequence (u_{i_j}) such that

$$\int_\gamma |u_{i_j} - u|\, ds \to 0$$

as $j \to \infty$ for M_1–a.e., and thus for AM–a.e., curve γ in X. The conclusion now follows from Lemma 2.1. □

If (g_i) and (h_i) are BV_{AM} upper bounds of u in X, then $(\max(g_i, h_i))$ is clearly a BV_{AM} upper bound for u but, as easy examples show, $(\min(g_i, h_i))$ is not. In the following lemma we consider the situation where the BV_{AM} upper bounds are defined on different open sets.

Lemma 2.4. *Let Ω_j, $j = 1, 2$, be open sets in X and (g_i^j) a BV_{AM} upper bound for a function $u \in L^1(\Omega_1 \cup \Omega_2)$ in Ω_j, $j = 1, 2$. Then the sequence (g_i),*

$$g_i(x) = \begin{cases} g_i^1(x),\ x \in \Omega_1 \setminus \Omega_2, \\ \max(g_i^1(x), g_i^2(x)),\ x \in \Omega_1 \cap \Omega_2, \\ g_i^2(x), x \in \Omega_2 \setminus \Omega_1, \end{cases}$$

is a BV_{AM} upper bound for u in $\Omega_1 \cup \Omega_2$.

Proof. Note that (g_i) is a BV_{AM} upper bound for u both in Ω_1 and Ω_2 and hence

$$V(u,\gamma') \le \liminf_{i\to\infty} \int_{\gamma'} g_i\, ds \tag{2.10}$$

whenever γ' is a subcurve of γ with $\gamma' \subset \Omega_j$, $j = 1$ or 2.

Fix a curve γ in $\Omega_1 \cup \Omega_2$. Since $\gamma([0, l(\gamma)])$ is compact in $\Omega_1 \cup \Omega_2$, there is $\delta > 0$ such that γ' lies in Ω_1 or in Ω_2 whenever γ' is a subcurve of γ with $l(\gamma') < \delta$. By (2.10) and Lemma 2.2 there is a set $C \subset [0, l(\gamma)]$ with $m_1([0, l(\gamma)] \setminus C) = 0$ and

$$|u(\gamma(t_2)) - u(\gamma(t_1))| \le \liminf_{i\to\infty} \int_{\gamma|[t_1,t_2]} g_i\, ds \tag{2.11}$$

whenever t_1, $t_2 \in C$, $0 < t_2 - t_1 < \delta$. Let $0 \le t_0 < t_1 < \cdots < t_n \le l(\gamma)$, $t_k \in C$, $k = 0, 1, \dots, n$. Adding a finite number of points from C to the sequence (t_k) we can assume that $t_k - t_{k-1} < \delta$ for each k. Now (2.11) yields

$$\sum_{k=1}^n |u(\gamma(t_k)) - u(\gamma(t_{k-1}))| \le \sum_{k=1}^n \liminf_{i\to\infty} \int_{\gamma|[t_k,t_{k-1}]} g_i\, ds$$

$$\le \liminf_{i\to\infty} \sum_{k=1}^n \int_{\gamma|[t_k,t_{k-1}]} g_i\, ds \le \liminf_{i\to\infty} \int_\gamma g_i\, ds$$

and the lemma follows. □

3. The space $BV_{AM}(X)$

The space, abreviated as $BV_{AM}(X)$, of the functions of bounded variation on AM–a.e. curve in X consists of all functions $u \in L^1(X)$ which have a BV_{AM} upper bound (g_i) such that

$$\liminf_{i\to\infty} \int_X g_i \, d\nu < \infty. \tag{3.1}$$

We call

$$\| u \|_{BV(X)} = \| u \|_{L^1(X)} + \| D_{BV} u \| (X).$$

where

$$\| D_{BV} u \| (X) = \inf\{\liminf_{i\to\infty} \int_X g_i \, d\nu \, : \, (g_i) \text{ is a } BV_{AM} \text{ upper bound for } u\}$$

the BV_{AM} *norm* of u in X. The essential implication of (3.1) is the following; see [13]. For completeness we give a proof.

Lemma 3.1. *Suppose that $u \in L^1(X)$ has a BV_{AM} upper bound (g_i) in X as in (3.1). Then*

$$V(u,\gamma) \le \liminf_{i\to\infty} \int_\gamma g_i \, ds < \infty \tag{3.2}$$

for AM–a.e. curve γ in X.

Proof. For each $k = 1, 2, \dots$ let Γ_k be the family of all curves γ in X such that

$$\liminf_{i\to\infty} \int_\gamma g_i \, ds \ge k.$$

Then the sequence (g_i/k) is AM–admissible for Γ_k and hence

$$AM(\Gamma_k) \le \liminf_{i\to\infty} \int_X \frac{g_i}{k} \, d\nu = \frac{M}{k}$$

where

$$M = \liminf_{i\to\infty} \int_X g_i \, d\nu.$$

Since

$$\Gamma_\infty = \{\gamma \, : \, \liminf_{i\to\infty} \int_\gamma g_i \, ds = \infty\}$$

satisfies $\Gamma_\infty \subset \Gamma_k$ for each k, it follows that $AM(\Gamma_\infty) = 0$ and thus (3.2) holds as required. $\square$

The following property of a function $u \in BV_{AM}(X)$ is often useful. For each $\varepsilon > 0$ there is a BV_{AM} upper bound (g_i) for u in X such that

$$\int_X g_i \, d\nu \le \| D_{BV} u \| (X) + \varepsilon \tag{3.3}$$

for all i and (3.2) holds for AM–a.e. curve γ in X.

Remark 3.2. A locally Lipschitz function $u \in L^1(X)$ with an upper gradient also in $L^1(X)$ and a function u in the Newtonian space $N^{1,1}(X)$ is in the space $BV_{AM}(X)$ since for such a function u the sequence (g_i), where $g_i = g$ and g is an upper gradient or a 1–weak upper gradient of u, respectively, is a BV_{AM} upper bound for u. For the concept of an upper gradient and a 1–weak upper gradient see [4,5,6,15].

Lemma 3.3. *The space $BV_{AM}(X)$ is a linear subspace of $L^1(X)$ and $\| u \|_{BV(X)}$ is a norm in $BV_{AM}(X)$.*

Proof. If $u \in BV_{AM}(X)$ and $\lambda \in \mathbf{R}$, then obviously $\lambda u \in BV_{AM}(X)$ and $\| \lambda u \|_{BV(X)} = |\lambda| \, \| u \|_{BV(X)}$. If $u = 0$ in $L^1(X)$, then $u \circ \gamma = 0$ a.e. in $[0, l(\gamma)]$ on M_1-a.e. curve γ in X; see Section 2. Hence (g_i), $g_i = 0$, $i = 1, 2, \ldots$, is a BV_{AM} upper bound for u and thus $\| u \|_{BV(X)} = 0$.

It remains to show that $u + v \in BV_{AM}(X)$ provided that $u, v \in BV_{AM}(X)$ and that the triangle inequality holds. Let $u, v \in BV_{AM}(X)$ and let (g_i) and (h_i) be BV_{AM} upper bounds for u and v, respectively. Now $(g_i + h_i)$ is a BV_{AM} upper bound for $u + v$. Indeed, for AM–a.e. curve γ in X

$$V(u+v,\gamma) \le V(u,\gamma) + V(v,\gamma)$$

$$\le \liminf_{i\to\infty} \int_\gamma g_i \, ds + \liminf_{i\to\infty} \int_\gamma h_i \, ds \le \liminf_{i\to\infty} \int_\gamma (g_i + h_i) \, ds$$

and thus $u+v \in BV_{AM}(X)$. For the triangle inequality note that the BV_{AM} upper bounds (g_i) and (h_i) can be chosen so that for $\varepsilon > 0$

$$\| D_{BV} u \| (X) \ge \int_X g_i \, d\nu - \varepsilon, \ \| D_{BV} v \| (X) \ge \int_X h_i \, d\nu - \varepsilon$$

for every i. Since $(g_i + h_i)$ is a BV_{AM} upper bound for $u+v$, the triangle inequality easily follows from the above property of (g_i) and (h_i). $\square$

Theorem 3.4. *The space $BV_{AM}(X)$ with the norm $\| u \|_{BV(X)}$ is a Banach space.*

Proof. It remains to show the Cauchy property. Let u_i, $i = 1, 2, \ldots$, be a Cauchy sequence in $BV_{AM}(X)$. Since $u_i \to u$ in $L^1(X)$, it suffices to show that a subsequence, denoted again by (u_i), satisfies $\| D_{BV}(u_i - u) \| (X) \to 0$ as $i \to \infty$ and $u \in BV_{AM}(X)$. By the Fuglede lemma (see [5, Lemma 2.1]), we may assume that

$$\int_\gamma |u_i - u| \, ds \to 0 \tag{3.4}$$

as $i \to \infty$ for M_1–a.e. curve γ in X and

$$\| D_{BV}(u_{i+1} - u_i) \| (X) < 2^{-i} \tag{3.5}$$

for each i. From (3.5) it follows that for every i there is a BV_{AM} upper bound (g_j^i) for $u_{i+1} - u_i$ such that

$$\| D_{BV}(u_{i+1} - u_i) \| (X) \le \int_X g_j^i \, d\nu < 2^{-i} \tag{3.6}$$

for all $j = 1, 2, \ldots$.

For $k = 1, 2, \ldots$ consider the sequence

$$\tilde{g}_j^k = \sum_{l=k}^{\infty} g_j^l, \ j = 1, 2, \ldots .$$

Since $\tilde g_j^k \ge g_j^i$ for each $i \ge k$, $(\tilde g_j^k)$ is a BV_{AM} upper bound for $u_{i+1} - u_i$, $i \ge k$. Now $\tilde g_j^k$ is a BV_{AM} upper bound for $u_{i+1} - u_k$, $i \ge k$ as well, because

$$
\begin{aligned}
V(u_{i+1} - u_k, \gamma) &\le \sum_{l=k}^{i} V(u_{l+1} - u_l, \gamma) \le \sum_{l=k}^{i} \liminf_{j\to\infty} \int_\gamma g_j^l \, ds \\
&\le \liminf_{j\to\infty} \sum_{l=k}^{i} \int_\gamma g_j^l \, ds \le \liminf_{j\to\infty} \int_\gamma \tilde g_j^k \, ds,
\end{aligned}
$$

and this holds for AM–a.e. curve γ by the subadditivity of the AM–modulus.

Next we show that $(\tilde g_j^k)$ is a BV_{AM} upper bound for $u - u_k$. For this fix a curve γ such that (3.4) holds and that

$$
V(u_{i+1} - u_k, \gamma) \le \liminf_{j\to\infty} \int_\gamma \tilde g_j^k \, ds
$$

for every $i \ge k$. Note that AM–a.e. curve γ satisfies these conditions. By Lemma 2.1

$$
\begin{aligned}
V(u - u_k, \gamma) &\le \liminf_{i\to\infty} V(u_{i+1} - u_k, \gamma) \\
&\le \liminf_{i\to\infty} \liminf_{j\to\infty} \int_\gamma \tilde g_j^k \, ds = \liminf_{j\to\infty} \int_\gamma \tilde g_j^k \, ds
\end{aligned}
$$

and hence $(\tilde g_j^k)$ is a BV_{AM} upper bound for $u - u_k$. From (3.6) it follows that

$$
\| D_{BV}(u - u_k) \| (X) \le \liminf_{j\to\infty} \int_X \tilde g_j^k \, d\nu
$$

$$
= \liminf_{j\to\infty} \sum_{l=k}^{\infty} \int_X g_j^l \, d\nu \le \liminf_{j\to\infty} \sum_{l=k}^{\infty} \frac{1}{2^l} = \frac{2}{2^k}\,. \tag{3.7}
$$

Thus $u - u_k \in BV_{AM}(X)$ and so $u \in BV_{AM}(X)$. Inequality (3.7) also shows that $\| D_{BV}(u - u_k) \| (X) \to 0$ as $k \to \infty$. This completes the proof. □

In [14] Miranda considered functions $u \in L^1(X)$ which can be approximated in $L^1(X)$ by locally Lipschitz functions whose upper gradients belong to $L^1(X)$. Set

$$
\| D_L u \| (X) = \inf\{\liminf_{i\to\infty} \int_X g_i \, d\nu : u_i \to u \text{ in } L^1(X),\ u_i \text{ loc. Lipschitz}\} < \infty \tag{3.8}
$$

where g_i is an upper gradient of u_i and the first infimum is taken over all sequences of locally Lipschitz functions (u_i) in X such that $u_i \to u$ in $L^1(X)$ and all upper gradients g_i of u_i. The space of all functions such that

$$
\| u \|_L (X) = \| u \|_{L^1(X)} + \| D_L u \| (X) < \infty
$$

is a Banach space and the space is a generalization of the ordinary space of functions of bounded variation in $\mathbf{R}^n$; see [7].

The functions in the above space are also of bounded variation on AM–a.e. curve in X (see [13, Theorem 5.4]) in the sense that $V(u, \gamma) < \infty$ on AM–a.e. curve in X. More precisely, if (u_i) is a sequence of locally Lipschitz functions with upper gradients g_i in X and $u_i \to u$ in $L^1(X)$, then there is a subsequence (u_{i_j}) such that

$$
\int_\gamma |u_{i_j} - u| \, ds \to 0
$$

on M_1–a.e. curve γ in X and (see Lemma 2.3) that

$$V(u,\gamma) \le \liminf_{j\to\infty} V(u_{i_j},\gamma) \le \liminf_{j\to\infty} \int_\gamma g_{i_j}\, ds$$

where g_{i_j} is an upper gradient of u_{i_j}. Thus the sequence (g_{i_j}) is a BV_{AM} upper bound for u. Hence

$$\| D_{BV}u \| (X) \le \| D_L u \| (X) \tag{3.9}$$

holds for every function $u \in L^1(X)$.

Remark 3.5. It would be interesting to know if the inequality

$$\| D_{BV}u \| (\Omega) < \| D_L u \| (\Omega)$$

could hold in some open set $\Omega \subset X$ in the case where functions in $L^1(X)$ can be approximated by locally Lipschitz functions.

4. The measure associated with $\| D_{BV}u \|$

We show that, under some additional assumptions on X, for each $u \in BV_{AM}(X)$ there is a Borel measure $\mu = \mu_u$ defined initially on open subsets Ω of X as $\mu(\Omega) = \| D_{BV}u \| (\Omega)$ and then extended to arbitrary sets $A \subset X$ as

$$\mu(A) = \inf\{\mu(\Omega) :\ A \subset \Omega,\ \Omega\, \text{open}\}.$$

This procedure (see [9]) requires that the space (X, d) is separable and locally compact. These assumptions imply that the metric space X is σ–compact and hence open sets Ω in X can be exhausted by compact sets and, in particular, by open sets with compact closure in Ω. If the measure ν is doubling and the space X is complete, then X is proper, i.e. closed and bounded sets are compact, so that these assumptions imply separability and local compactness. For the discussion on these conditions see [5]. Miranda [14] also used this construction for the corresponding measure based on $\| D_L u \| (\Omega)$.

In this section we assume that X is separable and locally compact. The measure ν in X is supposed to satisfy (2.1).

Theorem 4.1. *If $u \in BV_{AM}(X)$, then the set function $\mu(\Omega) = \| D_{BV}u \| (\Omega)$, Ω open in X, defines a Borel measure in X.*

Proof. By [9, Theorem 5.1] it suffices to show that for open sets Ω, Ω_1 and Ω_2 in X it holds that

$$\mu(\Omega_1) \le \mu(\Omega_2),\ \text{if}\ \Omega_1 \subset \Omega_2, \tag{4.1}$$

$$\mu(\Omega_1 \cup \Omega_2) = \mu(\Omega_1) + \mu(\Omega_2),\ \text{if}\ \Omega_1 \cap \Omega_2 = \emptyset, \tag{4.2}$$

$$\mu(\Omega_1 \cup \Omega_2) \le \mu(\Omega_1) + \mu(\Omega_2), \tag{4.3}$$

$$\mu(\Omega) = \sup\{\mu(\Omega_1) :\ \Omega_1 \subset\subset \Omega\}. \tag{4.4}$$

Here $A \subset\subset B$ means that the closure of A is a compact subset of B.

Now (4.1) is obvious and (4.2) follows from the fact that each curve in $\Omega_1 \cup \Omega_2$ lies either in Ω_1 or in Ω_2. It remains to verify (4.3) and (4.4).

To prove (4.3) let $\varepsilon > 0$ and choose BV_{AM} upper bounds (g_i^1) and (g_i^2) for u in Ω_1 and Ω_2, respectively, so that

$$\| D_{BV}u \| (\Omega_j) \geq \int_{\Omega_j} g_i^j \, d\nu - \varepsilon \tag{4.5}$$

for all i and $j = 1, 2$. Define $g_i = g_i^1$ in $\Omega_1 \setminus \Omega_2$, $g_i = g_i^2$ in $\Omega_2 \setminus \Omega_1$ and $g_i = \max(g_i^1, g_i^2)$ in $\Omega_1 \cap \Omega_2$. Then by Lemma 2.4, (g_i) is a BV_{AM} upper bound for u in $\Omega_1 \cup \Omega_2$ and we obtain from (4.5)

$$\begin{aligned} \| D_{BV}u \| (\Omega_1 \cup \Omega_2) &\leq \liminf_{i\to\infty} \int_{\Omega_1\cup\Omega_2} g_i \, d\nu \\ &\leq \liminf_{i\to\infty} \left(\int_{\Omega_1\setminus\Omega_2} g_i^1 \, d\nu + \int_{\Omega_2\setminus\Omega_1} g_i^2 \, d\nu + \int_{\Omega_1\cap\Omega_2} (g_i^1 + g_i^2) \, d\nu \right) \\ &\leq \| D_{BV}u \| (\Omega_1) + \| D_{BV}u \| (\Omega_2) + 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we complete the proof for (4.3).

To prove (4.4) write

$$\alpha = \sup\{\mu(\Omega_1) : \Omega_1 \subset\subset \Omega\}.$$

We may assume that $\alpha < \infty$. Let $\varepsilon > 0$ and pick an open set $U_0 \subset\subset \Omega$ such that

$$\| D_{BV}u \| (U_0) > \alpha - \varepsilon. \tag{4.6}$$

Next pick open subsets U_i and V_i, $i = 1, 2, \ldots$, such that

$$U_0 \subset\subset V_1 \subset\subset U_1 \subset\subset V_2 \subset\subset U_2 \subset\subset \cdots \subset\subset \Omega$$

and $\bigcup_{i=1}^{\infty} U_i = \Omega$. Note that by (4.6) and (4.1) we have

$$\| D_{BV}u \| (U_1) > \alpha - \varepsilon. \tag{4.7}$$

The open sets $U_i \setminus \overline{U}_{i-1}$, $i \geq 2$, are disjoint and lie in $\Omega \setminus \overline{U}_1$. Hence by (4.2) for each $j \geq 2$,

$$\| D_{BV}u \| (\bigcup_{i=2}^{j} (U_i \setminus \overline{U}_{i-1})) = \sum_{i=2}^{j} \| D_{BV}u \| (U_i \setminus \overline{U}_{i-1})$$

and we obtain

$$\| D_{BV}u \| (U_1) + \| D_{BV}u \| (\bigcup_{i=2}^{j} (U_i \setminus \overline{U}_{i-1}))) \leq \alpha \tag{4.8}$$

and thus (4.7) yields

$$\sum_{i=2}^{j} \| D_{BV}u \| (U_i \setminus \overline{U}_{i-1}) < \varepsilon \tag{4.9}$$

for every $j \geq 2$. Similarly we obtain

$$\sum_{i=1}^{j} \| D_{BV}u \| (V_{i+1} \setminus \overline{V}_i) < \varepsilon \tag{4.10}$$

for every $j \geq 1$.

Choose a BV_{AM} upper bound (g_k^1) for u in U_1 such that

$$\int_{U_1} g_k^1 \, d\nu < \alpha + \varepsilon \tag{4.11}$$

and for every $i = 2, 3, \ldots$, a BV_{AM} upper bound (g_k^i) for u in $U_i \setminus \overline{U}_{i-1}$ such that

$$\int_{U_i \setminus \overline{U}_{i-1}} g_k^i \, d\nu < \| D_{BV} u \| (U_i \setminus \overline{U}_{i-1}) + \frac{\varepsilon}{2^i} \tag{4.12}$$

for all $k = 1, 2, \ldots$. Similarly for $i = 1, 2, \ldots$ we choose BV_{AM} upper bounds (h_k^i) for u in $V_{i+1} \setminus \overline{V}_i$ such that

$$\int_{V_{i+1} \setminus \overline{V}_i} h_k^i \, d\nu < \| D_{BV} u \| (V_{i+1} \setminus \overline{V}_i) + \frac{\varepsilon}{2^i} \tag{4.13}$$

for each k. Since (4.9) and (4.10) hold for each j, (4.12) and (4.13) yield

$$\sum_{i=2}^{\infty} \int_{U_i \setminus \overline{U}_{i-1}} g_k^i \, d\nu \le \sum_{i=2}^{\infty} \| D_{BV} u \| (U_i \setminus \overline{U}_{i-1}) + \varepsilon \le 2\varepsilon \tag{4.14}$$

and

$$\sum_{i=1}^{\infty} \int_{V_{i+1} \setminus \overline{V}_i} h_k^i \, d\nu \le \sum_{i=1}^{\infty} \| D_{BV} u \| (V_{i+1} \setminus \overline{V}_i) + \varepsilon \le 2\varepsilon. \tag{4.15}$$

Extend each of the functions g_k^i and h_k^i as zero outside their domains of definition and let

$$\tilde{g}_k = \sum_{i=1}^{\infty} (g_k^i + h_k^i), \; k = 1, 2, \ldots .$$

Now $(\tilde{g}_k)$ is a BV_{AM} upper bound for u in Ω. To see this note that for a fixed k the sum $\sum_{i=1}^{\infty} (g_k^i(x) + h_k^i(x))$ contains at most two non-zero terms for each $x \in \Omega$ and that

$$\Omega = U_1 \cup \bigcup_{i=2}^{\infty} (U_i \setminus \overline{U}_{i-1}) \cup \bigcup_{i=1}^{\infty} (V_{i+1} \setminus \overline{V}_i).$$

Fix a curve γ in Ω. Since $\gamma([0, l(\gamma)])$ is a compact subset of Ω, there is j such that

$$\gamma([0, l(\gamma)]) \subset U_1 \cup \bigcup_{i=2}^{j} (U_i \setminus \overline{U}_{i-1}) \cup \bigcup_{i=1}^{j} (V_{i+1} \setminus \overline{V}_i) = V_{j+1}.$$

On the other hand,

$$\tilde{g}_k(x) \ge g_k^1(x) + \max_{2 \le i \le j} g_k^i(x) + \max_{1 \le i \le j} h_k^i(x) = w_k^j(x)$$

for each $x \in V_{j+1}$ and every k and, since by Lemma 2.4, (w_k^j) is a BV_{AM} upper bound for u in V_{j+1}, $(\tilde{g}_k)$ is a BV_{AM} upper bound for u in Ω as required. Consequently,

$$\begin{aligned} \| D_{BV} u \| (\Omega) &\le \liminf_{k \to \infty} \int_{\Omega} \tilde{g}_k \, d\nu \\ &\le \liminf_{k \to \infty} \left(\int_{U_1} g_k^1 \, d\nu + \sum_{i=2}^{\infty} \int_{U_i \setminus \overline{U}_{i-1}} g_k^i \, d\nu + \sum_{i=1}^{\infty} \int_{V_i \setminus \overline{V}_{i-1}} h_k^i \, d\nu \right) \\ &\le \liminf_{k \to \infty} \left(\int_{U_1} g_k^1 \, d\nu + \sum_{i=2}^{\infty} \| D_{BV} u \| (V_i \setminus \overline{V}_{i-1}) + \sum_{i=1}^{\infty} \| D_{BV} u \| (V_i \setminus \overline{V}_{i-1}) \right) \\ &\le \alpha + \varepsilon + 2\varepsilon + 2\varepsilon = \alpha + 5\varepsilon \end{aligned}$$

where we have used (4.11), (4.14) and (4.15). Since $\varepsilon > 0$ is arbitrary, $\| D_{BV} u \| (\Omega) \le \alpha$ and the proof for (4.4) follows and completes the proof for the theorem. $\square$

5. The measure $\| D_{BV} u \|$ in $\mathbf{R}^n$

We show that for $X = \mathbf{R}^n$ the space $BV_{AM}(X)$ is the ordinary space of BV functions. The simplest domain Ω in $\mathbf{R}^n$ from the modulus and the Fubini theorem point of view is the Cartesian product $\Omega = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$. We let $\Omega' = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_{n-1}, b_{n-1})$ and $(a, b) = (a_n, b_n)$ so that $\Omega = \Omega' \times (a, b)$. There is nothing particular in the choice of the last coordinate.

The classical space $BV(\Omega)$ of BV functions in Ω consists of all functions $u \in L^1(\Omega)$ such that

$$\| Du \| (\Omega) = \sup\{ \int_\Omega u \nabla \cdot \varphi \, dm_n : \varphi \in C_0^1(\Omega, \mathbf{R}^n), \ |\varphi| \le 1 \} < \infty; \tag{5.1}$$

see [7, Section 5.1]. Note that Lipschitz functions with compact support can be considered as well as functions φ in $C_0^1(\Omega, \mathbf{R}^n)$. Now $\| Du \| (U)$ and U an open subset of Ω, defines a Borel measure in Ω.

In the following we let $X = \Omega$ and $\nu = m_n$ be the Lebesgue measure and consider the behavior of $V(u, \gamma)$ on line segments and the measure $\mu = \mu_u = \| D_{BV} u \|$, $u \in L^1(\Omega)$, defined in Section 4.

For $y \in \Omega'$ let $\gamma_y(t) = y + (a + t)e_n$, $t \in [0, b - a]$, where $e_n = (0, 0, \dots, 1)$. Then γ_y is a straight line segment parallel to the x_n–axis. Note that γ_y is a curve in $\Omega' \times [a, b]$ but not in Ω. However, for $u \in L^1(\Omega)$, $V(u, \gamma_y)$ is well defined; see Section 2. The following lemma shows that the behavior of u is similar on γ_y as on curves in Ω.

Lemma 5.1. *Let $u \in L^1(\Omega)$, U an open subset of Ω' and (g_i) a BV_{AM} upper bound for u in $U \times (a, b)$ with*

$$\liminf_{i \to \infty} \int_{U \times (a,b)} g_i \, dm_n < \infty.$$

Then for m_{n-1}–a.e. $y \in U$ we have

$$V(u, \gamma_y) \le \liminf_{i \to \infty} \int_{\gamma_y} g_i \, ds < \infty.$$

Proof. For $y \in U$ and $j > (b-a)/2$ let $\gamma_y^j(t) = y + (a + 1/j + t)e_n$, $t \in [0, b - a - 2/j]$. Then each curve in the family $\Gamma^j = \{\gamma_y^j\}$ lies in $U \times (a, b)$ and hence for AM–a.e. $\gamma_y^j \in \Gamma^j$ we have

$$V(u, \gamma_y^j) \le \liminf_{i \to \infty} \int_{\gamma_y^j} g_i \, ds < \infty. \tag{5.2}$$

Now (5.2) holds for m_{n-1}–a.e. $y \in U$ because $AM(\Gamma^j) = m_{n-1}(U)$; see [13, Example 3.9].

The Fatou lemma yields

$$\int_U (\liminf_{i \to \infty} \int_{\gamma_y} g_i \, ds) \, dm_{n-1}(y) \le \liminf_{i \to \infty} \int_{U \times (a,b)} g_i \, dm_n < \infty$$

and hence

$$V(u, \gamma_y^j) \le \liminf_{i \to \infty} \int_{\gamma_y} g_i \, ds < \infty$$

for m_{n-1}–a.e. $y \in U$ for each j. From this it follows that

$$V(u,\gamma_y) = \lim_{j\to\infty} V(u,\gamma_y^j) \le \liminf_{i\to\infty} \int_{\gamma_y} g_i \, ds < \infty$$

as required. $\square$

Lemma 5.2. *Suppose that $u \in BV_{AM}(\Omega)$. Then the function $y \mapsto V(u,\gamma_y)$ belongs to $L^1(\Omega')$ with respect to the m_{n-1} measure and*

$$\int_{\Omega'} V(u,\gamma_y)\, dm_{n-1} \le \| D_{BV} u \| (\Omega). \tag{5.3}$$

Proof. We first show that the function $V(u,\gamma_y) = V(\widehat{u\circ\gamma_y},[0,b-a])$ is measurable in Ω'. To see this note that $\widehat{u\circ\gamma_y}(t) = u(y,\gamma(t))$ for a.e. $(y,t) \in \Omega' \times [0,b-a]$ and thus $\widehat{u\circ\gamma_y}$ is a measurable function in $\Omega' \times [0,b-a]$ and, moreover, an $L^1(\Omega' \times [0,b-a])$–function. Pick a set $C \subset [0,b-a]$ such that $m_1([0,b-a]\setminus C) = 0$ and for every $t \in C$ the function $y \mapsto \widehat{u\circ\gamma_y}(t)$ is m_{n-1}–measurable in Ω'. This is possible by the Fubini theorem. Thus if the points $t_1 < t_2 < \cdots < t_m$ lie in C, then the function

$$y \mapsto \sum_{i=2}^{m} |\widehat{u\circ\gamma_y}(t_i) - \widehat{u\circ\gamma_y}(t_{i-1})|$$

is measurable in Ω'. For each $j = 2, 3, \ldots$ choose a finite increasing sequence (t_i^j), $i = 1, 2, \ldots, j$ of points in C such that $\max_i |t_i^j - t_{i-1}^j| \to 0$ as $j \to \infty$. Now the continuity properties of the function $\widehat{u\circ\gamma_y}$ imply that

$$V(\widehat{u\circ\gamma_y}, C) = \limsup_{j\to\infty} \sum_{i=2}^{j} |\widehat{u\circ\gamma_y}(t_i) - \widehat{u\circ\gamma_y}(t_{i-1})| \tag{5.4}$$

and thus $y \mapsto V(u,\gamma_y)$ is measurable in Ω'.

Let $\varepsilon > 0$ and choose a BV_{AM} upper bound (g_i) for u in Ω with

$$\liminf_{i\to\infty} \int_{\Omega} g_i \, dm_n \le \| D_{BV} u \| (\Omega) + \varepsilon.$$

By Lemma 5.1 and the Fatou lemma we have

$$\int_{\Omega'} V(u,\gamma_y)\, dm_{n-1} \le \int_{\Omega'} (\liminf_{i\to\infty} \int_{\gamma_y} g_i \, ds)\, dm_{n-1}$$

$$\le \liminf_{i\to\infty} \int_{\Omega} g_i \, dm_n \le \| D_{BV} u \| (\Omega) + \varepsilon,$$

and letting $\varepsilon \to 0$ we obtain (5.3). $\square$

The measure $\| D_{BV} u \|$ defines a Borel measure $\hat\mu$ in Ω' by the rule $\hat\mu(A) = \| D_{BV} u \| (A \times (a,b))$ for each Borel set $A \subset \Omega'$. The next theorem clarifies the roles of $V(u,\gamma_y)$ and $\hat\mu$.

Lemma 5.3. *Let $u \in BV_{AM}(\Omega)$. Then for m_{n-1}–a.e. $y \in \Omega'$*

$$V(u,\gamma_y) \le d\hat\mu(y) \tag{5.5}$$

where $d\hat\mu(y) = \lim_{r\to 0} \hat\mu(B^{n-1}(y,r))/(m_{n-1}(B^{n-1}(y,r))$ is the derivative of $\hat\mu$ with respect to the m_{n-1} measure at $y \in \Omega'$.

Proof. By Lemma 5.1 for m_{n-1}–a.e. $y \in \Omega'$,

$$V(u, \gamma_y) \le \liminf_{i\to\infty} \int_\gamma g_i \, ds \tag{5.6}$$

whenever (g_i) is a BV_{AM} upper bound for u in $B^{n-1}(z,r) \times (a,b)$, $B^{n-1}(z,r) \subset \Omega'$ and, by Lemma 5.2, the Lebesgue theorem yields

$$V(u, \gamma_y) = \lim_{r\to 0} m_{n-1}(B^{n-1}(y,r))^{-1} \int_{B^{n-1}(y,r)} V(u, \gamma_x) \, dm_{n-1}(x) \tag{5.7}$$

for m_{n-1}–a.e. $y \in \Omega'$.

Fix a point $y \in \Omega'$ such that (5.6) and (5.7) hold and the derivative $d\hat{\mu}(y)$ exists. Almost every point $y \in \Omega'$ satisfies these conditions. Choose radii $r_1 > r_2 > \dots$ with $\lim_{j\to\infty} r_j = 0$, set $B_j = B^{n-1}(y, r_j))$ and for each j pick a BV_{AM} upper bound (g_i^j) for u in $B_j \times (a,b)$ such that

$$\int_{B_j\times(a,b)} g_i^j \, dm_n \le \| D_{BV} u \| (B_j \times (a,b)) + \frac{r_j^{n-1}}{2^j} \tag{5.8}$$

for every $i = 1, 2, \dots$. The Fatou lemma and (5.8) yield

$$\begin{aligned} V(u,\gamma_y) &= \lim_{j\to\infty} m_{n-1}(B_j)^{-1} \int_{B_j} V(u,\gamma_x)\, dm_{n-1}(x) \\ &\le \liminf_{j\to\infty} m_{n-1}(B_j)^{-1} \int_{B_j} (\liminf_{i\to\infty} \int_{\gamma_x} g_i^j \, ds)\, dm_{n-1}(x) \\ &\le \liminf_{j\to\infty} m_{n-1}(B_j)^{-1} \liminf_{i\to\infty} \int_{B_j\times(a,b)} g_i^j \, dm_n \\ &\le \liminf_{j\to\infty} m_{n-1}(B_j)^{-1} (\| D_{BV} u \| (B_j \times (a,b)) + \frac{r_j^{n-1}}{2^j}) = d\hat{\mu}(y) \end{aligned}$$

and (5.5) follows. ☐

The next theorem shows that the spaces $BV_{AM}(\Omega)$ and $BV(\Omega)$ defined in Section 3 and in (5.1), respectively, are the same.

Theorem 5.4. *For each* $u \in L^1(\Omega)$,

$$\| D_{BV} u \| (\Omega) \le \| Du \| (\Omega) \le n \, \| D_{BV} u \| (\Omega). \tag{5.9}$$

Remark 5.5. It follows from the proof that the first inequality in (5.9) holds for all open sets $U \subset \Omega$.

Proof. Let $u \in BV(\Omega)$. By [7, 5.2.2, Theorem 2] there is a sequence of $C_0^\infty(\Omega)$–functions u_i such that $u_i \to u$ in $L^1(\Omega)$ and

$$\| Du \| (\Omega) = \lim_{i\to\infty} \int_\Omega |\nabla u_i| \, dm_n.$$

Then passing to a subsequence (see Lemma 2.3) we can assume that the sequence $(|\nabla u_i|)$ is a BV_{AM} upper bound for u in Ω. Hence

$$\| D_{BV} u \| (\Omega) \le \liminf_{i\to\infty} \int_\Omega |\nabla u_i| \, dm_n = \| Du \| (\Omega)$$

and the first inequality in (5.9) follows.

If $u \in BV(\Omega)$, then there exist measures $\| D_j u \|$, $j = 1, 2, \ldots, n$, defined on open sets $U \subset \Omega$ as

$$\| D_j u \| (U) = \sup\{ \int_U u \partial_j \varphi \, dm_n \}$$

where the supremum is taken over all $\varphi \in C_0^\infty(U)$ such that $|\varphi| \le 1$; see [7, 5.10]. By [7, the proof for Theorem 2 (7.10.2)] and Lemma 5.2 we have

$$\| D_n u \| (\Omega) = \int_{\Omega'} V(u, \gamma_y) \, dm_{n-1}(y) \le \| D_{BV} u \| (\Omega)$$

and a similar inequality holds for each $j = 1, 2, \ldots, n$. Let $\varepsilon > 0$ and choose a function $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ in $C_0^1(\Omega, \mathbf{R}^n)$ such that $|\varphi| \le 1$ and

$$\| Du \| (\Omega) \le \int_\Omega u \nabla \cdot \varphi \, dm_n + \varepsilon.$$

Now the above estimates yield

$$\| Du \| (\Omega) \le \sum_{j=1}^n \int_\Omega u \partial_j \varphi_j \, dm_n + \varepsilon \le \sum_{j=1}^n \| D_j u \| (\Omega) + \varepsilon \le n \| D_{BV} u \| (\Omega) + \varepsilon,$$

and letting $\varepsilon \to 0$ we obtain the second inequality in (5.9). □

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Department of Mathematics and Statistics, FI–00014 University of Helsinki, Finland
E-mail address: `olli.martio@helsinki.fi`