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COMPACT NON-ORIENTABLE SURFACES OF GENUS 6 WITH EXTREMAL METRIC DISCS

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Dedicated to Professor Noriaki Suzuki on the occasion of his sixtieth birthday

ABSTRACT. A compact hyperbolic surface of genus g is said to be extremal if it admits an extremal disc, a disc of the largest radius determined only by g. We discuss how many extremal discs are embedded in non-orientable extremal surfaces of genus 6. This is the final genus in our interest because it is already known for $g = 3, 4, 5$, or $g > 6$. We show that non-orientable extremal surfaces of genus 6 admit at most two extremal discs. The locus of extremal discs is also obtained for each surface. Consequently non-orientable extremal surfaces of arbitrary genus $g \geqq 3$ admit at most two extremal discs. Furthermore we determine the groups of automorphisms of non-orientable extremal surfaces of genus 6 with two extremal discs.

1. INTRODUCTION

Let S be a compact hyperbolic surface S of genus g, where g denotes the number of handles ($g \geqq 2$) if S is orientable, or the number of cross caps ($g \geqq 3$) if S is non-orientable. The hyperbolic metric on S is the one induced by the hyperbolic metric defined by the differential $ds = 2|dz|/(1 - |z|^2)$ on the unit disc $\mathbb{D}$. In [2] C. Bavard showed that the radius r of a disc embedded in S satisfies the inequality

$$\cosh r \leqq \frac{1}{2\sin\frac{\pi}{6-6\chi_g}}, \tag{1.1}$$

where χ_g denotes the Euler characteristic, namely, $\chi_g = 2 - 2g$ in the orientable case and $\chi_g = 2 - g$ in the non-orientable case. For each case we denote by R_g the radius satisfying equality in (1.1). A compact surface S of genus g is called an extremal surface if it admits an extremal disc, a disc of radius R_g. The hyperbolic surfaces are Riemann surfaces or non-orientable Klein surfaces. (For a reference of Klein surfaces, see, e.g., [1], [3], [13].) Let $\mathcal{K}_g$ be the moduli space of compact non-orientable Klein surfaces of genus g. We can then rephrase the radius R_g of extremal discs in terms of injectivity radius $r_p(S)$ at a point p of a compact non-orientable Klein surface S as follows:

$$R_g = \max_{S\in\mathcal{K}_g}\max_{p\in S} r_p(S). \tag{1.2}$$

In the case of Riemann surfaces the expression of R_g is similar to (1.2) by using the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g.

Our problem here is to find how many extremal discs extremal surfaces can admit. For the orientable case, the problem is completely solved ([4], [5], [7], [8],

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[12]). For the non-orientable case, we know that extremal surfaces of genus $g > 6$ admit a unique extremal disc ([6]) and that those of genus 3, 4 or 5 admit at most two ([6], [9], [10], [11]). Those surfaces are presented by fundamental regions of their Fuchsian groups or non-Euclidean crystallographic groups (NEC groups). In the present paper we shall discuss non-orientable extremal surfaces of genus 6, the final genus in our interest.

2. Main results

First we present our main results on non-orientable extremal surfaces of genus 6.

Theorem 2.1. *There exist* 149, 279 *non-orientable extremal surfaces of genus* 6 *up to isomorphism. They admit at most two extremal discs, and* 107 *of them admit exactly two extremal discs.*

The first column of Table 1 shows the 107 surfaces. The surfaces S_j are derived from the hyperbolic polygons P_j in Figures 1-5, where lines and dotted lines in P_j indicate pairs of sides identified by the opposite direction and the same direction, respectively. Note that we can take a regular $(6g-6)$-gon as a fundamental region of an NEC group for a non-orientable extremal surface of genus g and that the inscribed disc corresponds to an extremal disc ([2]). The second column of Table 1

Table 1. Extremal surfaces with two extremal discs. (Note that $S_2 \cong S_3$, $S_5 \cong S_6$, $S_{10} \cong S_{11}$, $S_{13} \cong S_{16}$, $S_{21} \cong S_{22}$, $S_{24} \cong S_{25}$, $S_{38} \cong S_{39}$, $S_{69} \cong S_{70}$, $S_{76} \cong S_{77}$.)

Surface	Centres of extremal discs	$\mathrm{Aut}^{\pm}$
S_2, S_5, S_{10}, S_{13}, S_{21}, S_{38}, S_{69}, S_{76}	$\pi(0)$, $\pi\left(\frac{2i\sin 4\beta}{\tanh R}\right)$	$\{1\}$
S_{17}	$\pi(0)$, $\pi(\zeta_{17})$	$\mathbb{Z}_2$
S_{28}, S_{29}, S_{34}, S_{35}, S_{80}, S_{81}, S_{82}	$\pi(0)$, $\pi\left(\frac{i\sin 2\beta}{\sin 3\beta\tanh R}\right)$	$\mathbb{Z}_2\times\mathbb{Z}_2$
S_{30}, S_{31}, S_{49}, S_{50}, S_{83}, S_{85}, S_{88}	$\pi(0)$, $\pi\left(\frac{1}{2\sin 7\beta\tanh R}\right)$	$\mathbb{Z}_2\times\mathbb{Z}_2$
S_{32}, S_{33}, S_{44}, S_{45}, S_{72}, S_{86}, S_{87}	$\pi(0)$, $\pi\left(\frac{i}{2\cos\beta\tanh R}\right)$	$\mathbb{Z}_2\times\mathbb{Z}_2$
S_{36}, S_{37}, S_{38}, S_{39}, S_{46}, S_{47}, S_{48}, S_{52}, S_{53}, S_{54}, S_{57}, S_{58}, S_{59}, S_{64}, S_{66}, S_{67}, S_{69}, S_{70}, S_{78}, S_{79}, S_{84}, S_{89}, S_{90}, S_{94}, S_{95}, S_{102}, S_{104}, S_{112}, S_{114}	$\pi(0)$, $\pi\left(\frac{2i\sin 4\beta}{\tanh R}\right)$	$\mathbb{Z}_2\times\mathbb{Z}_2$
S_{63}, S_{101}	$\pi(0)$, $\pi\left(\frac{2i\sin 4\beta}{\tanh R}\right)$	D_3
S_{68}, S_{103}	$\pi(0)$, $\pi\left(\frac{2i\sin 4\beta}{\tanh R}\right)$	$D_3\times\mathbb{Z}_2$
S_{91}, S_{92}, S_{96}, S_{97}, S_{98}, S_{111}, S_{115}	$\pi(0)$, $\pi\left(\frac{2\sin 3\beta}{\tanh R}\right)$	$\mathbb{Z}_2\times\mathbb{Z}_2$
The other 37 surfaces	$\pi(0)$, $\pi\left(\frac{2i\sin 4\beta}{\tanh R}\right)$	$\mathbb{Z}_2$

$$\zeta_{17} = \frac{e^{18\beta i}\sin 3\beta + e^{11\beta i}\tanh^2 R\sin 2\beta}{\sin 7\beta\tanh R}.$$

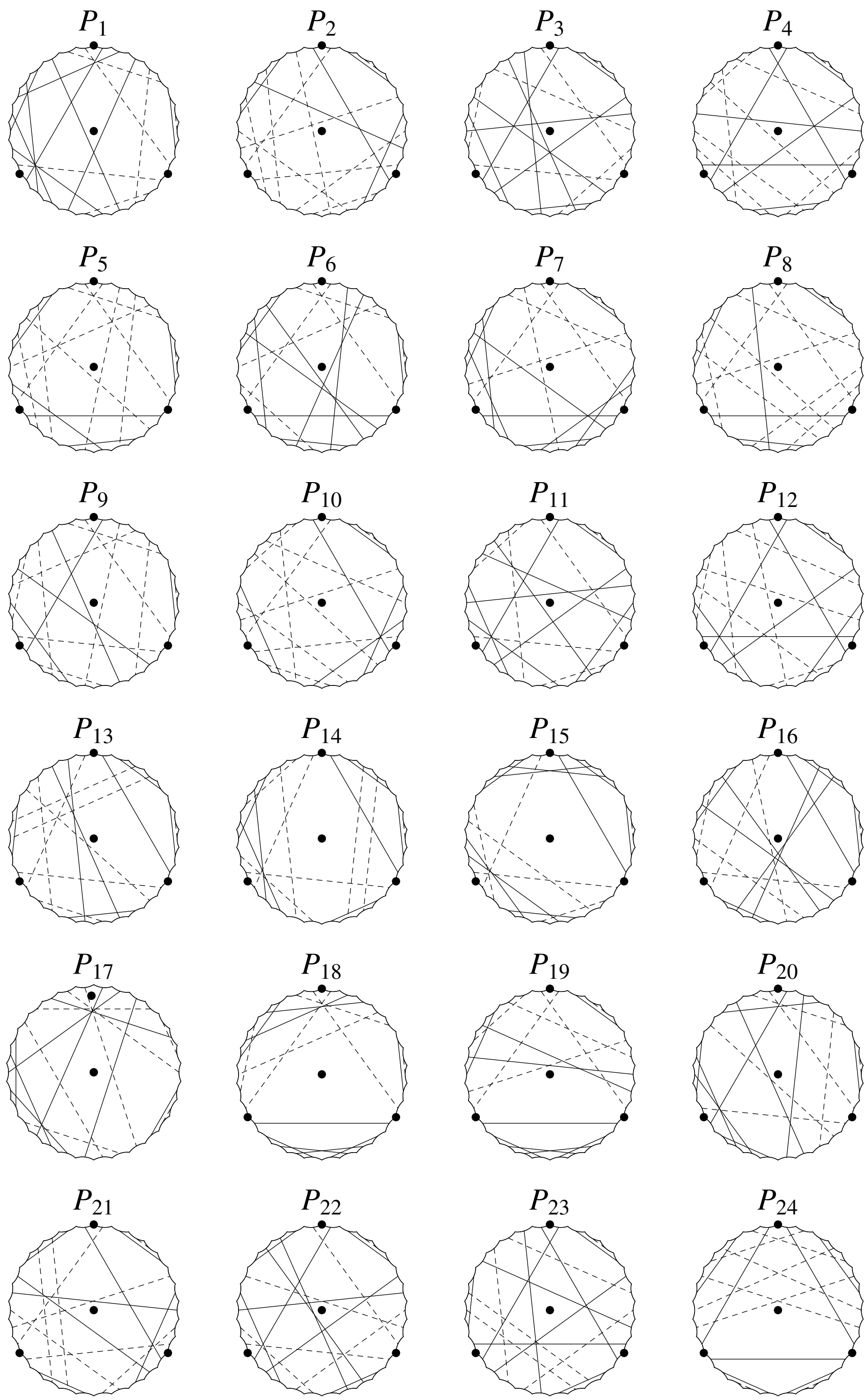

FIGURE 1. Side-pairing patterns and the centres of extremal discs (•)

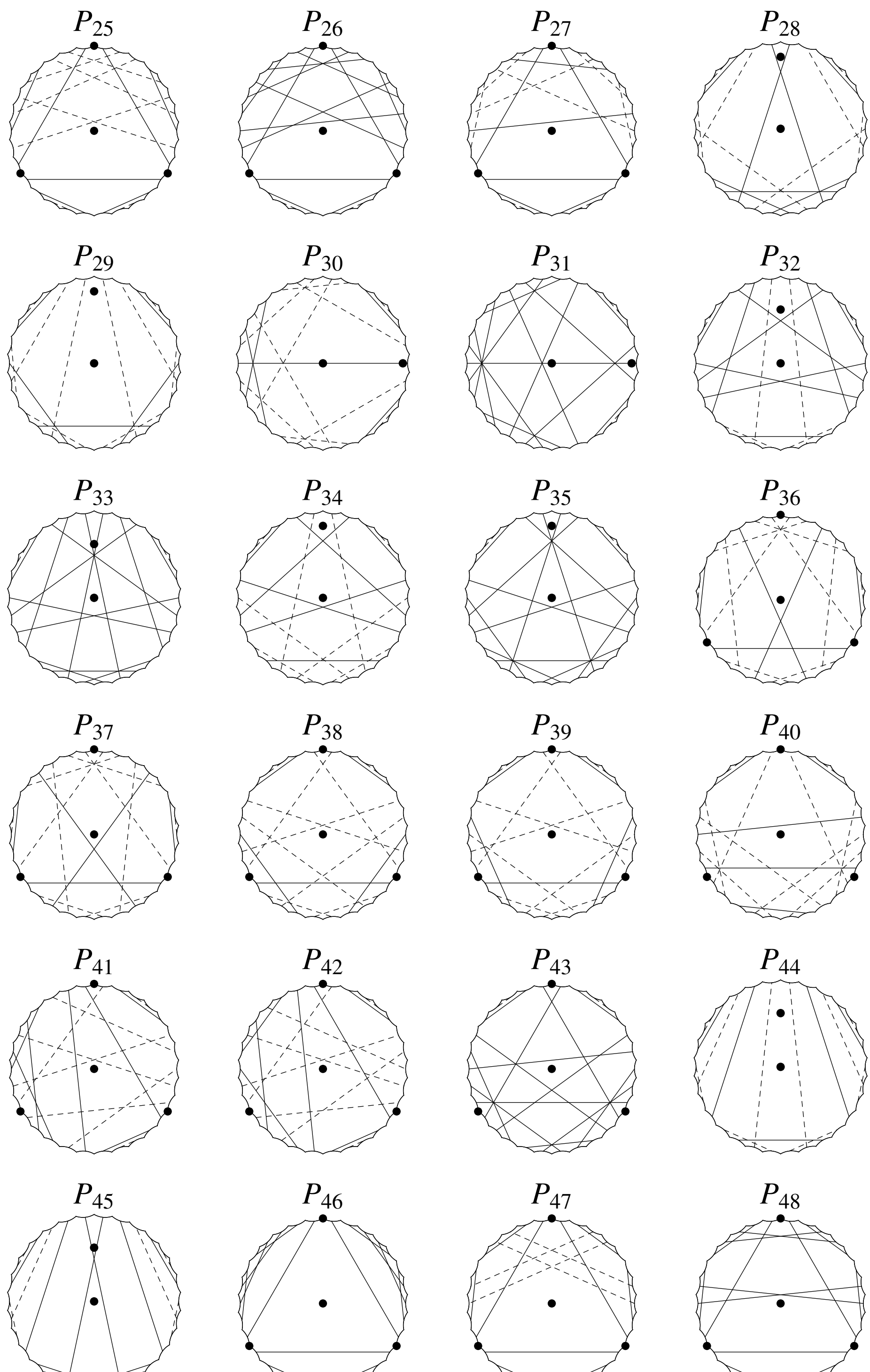

FIGURE 2. Side-pairing patterns and the centres of extremal discs (•)

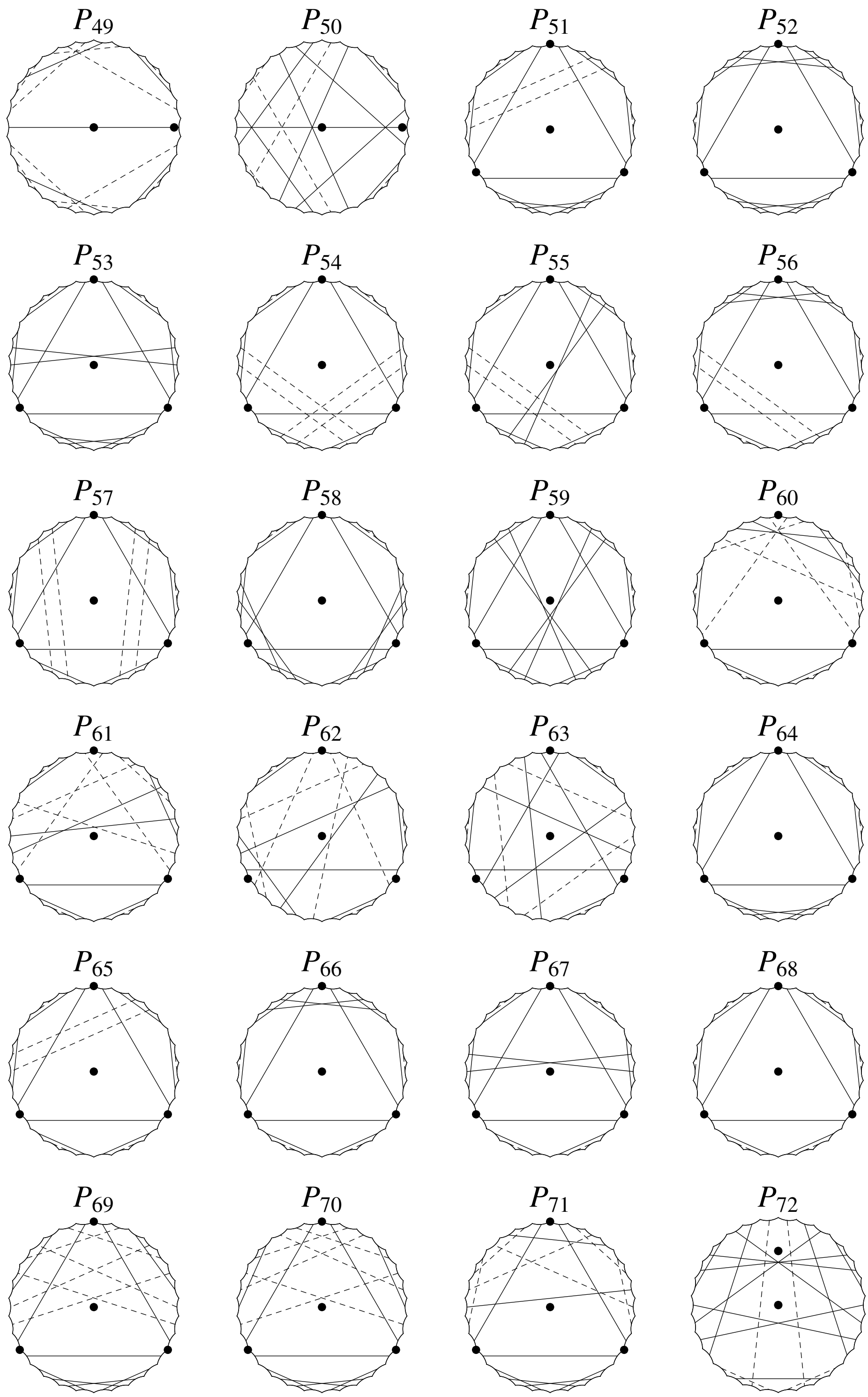

FIGURE 3. Side-pairing patterns and the centres of extremal discs (•)

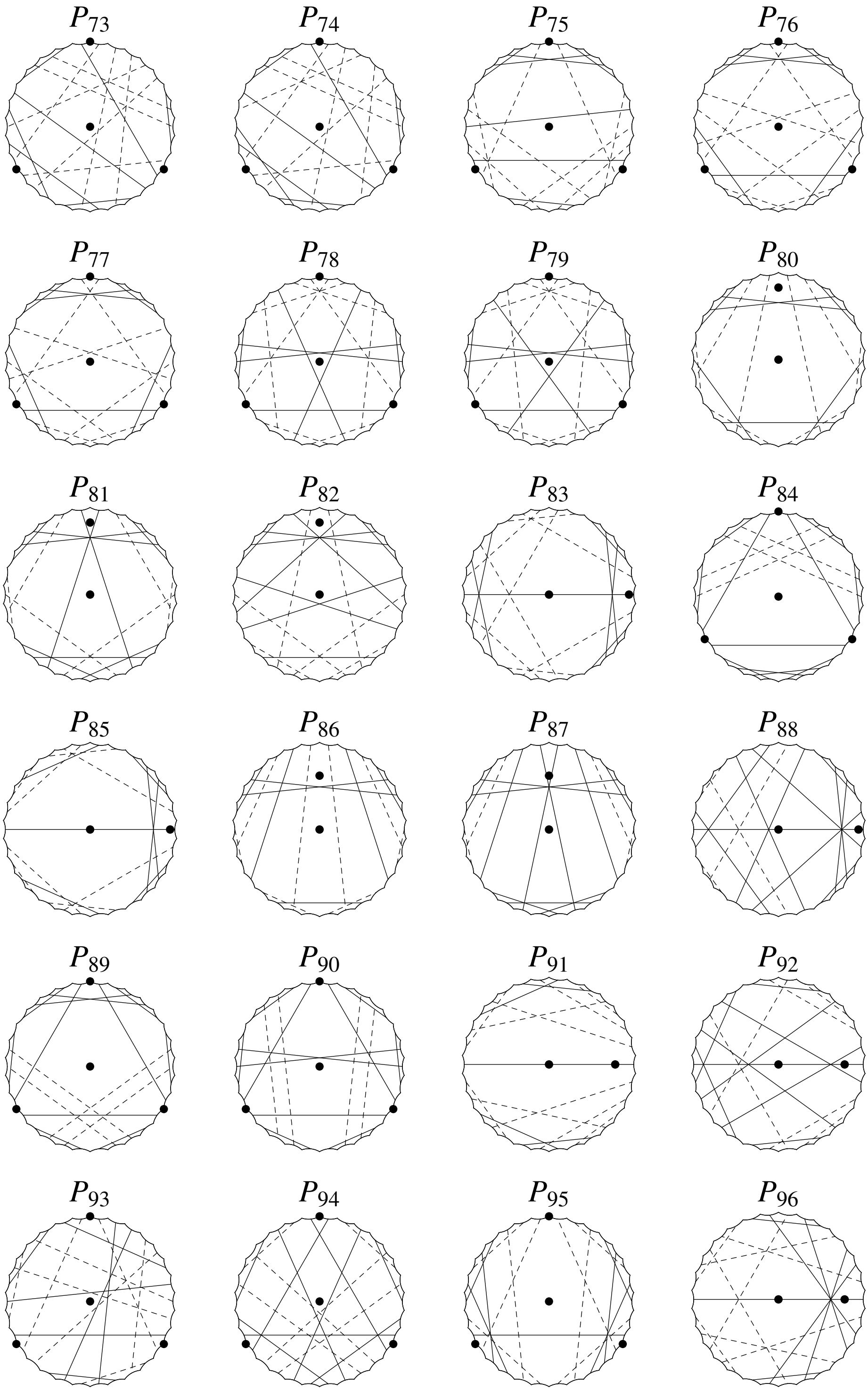

FIGURE 4. Side-pairing patterns and the centres of extremal discs (•)

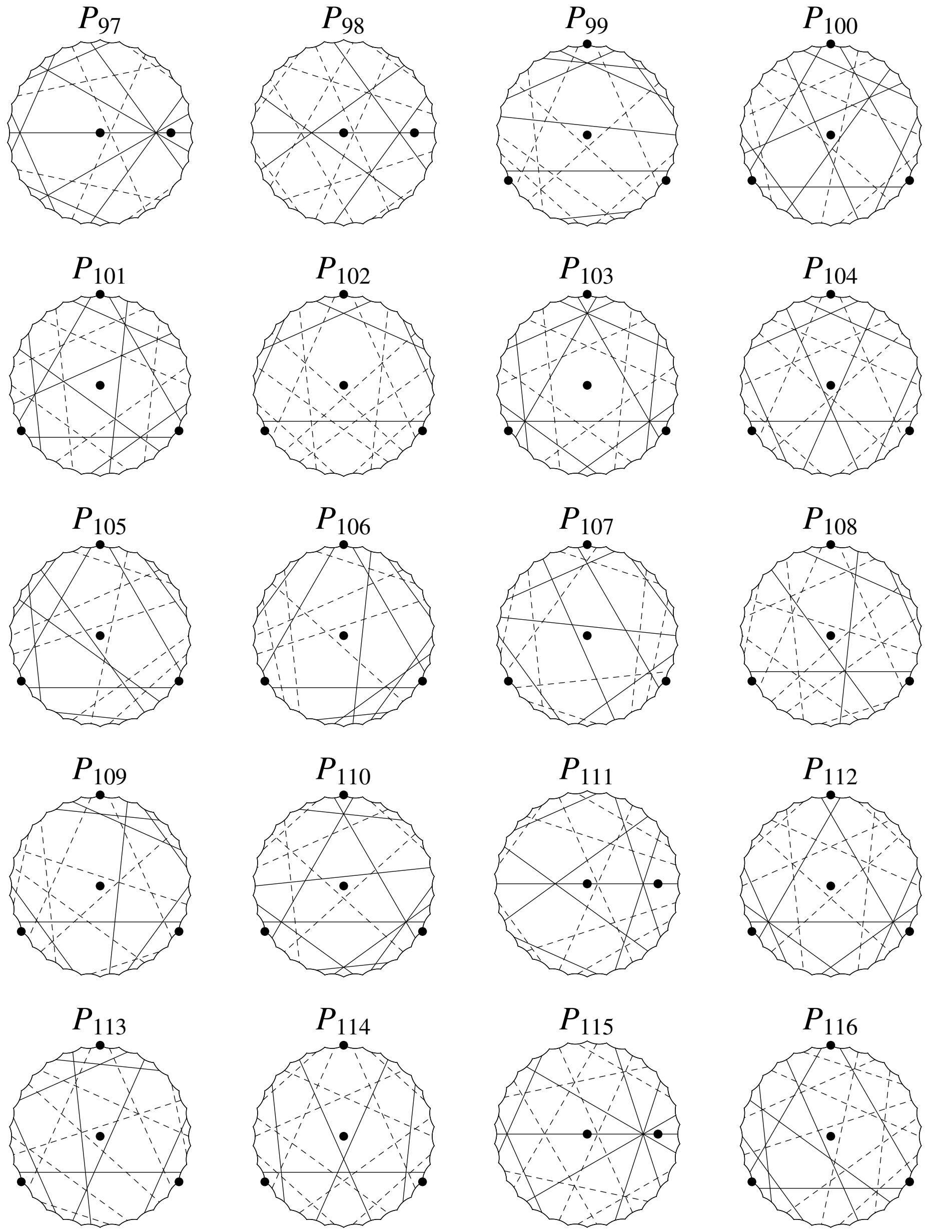

FIGURE 5. Side-pairing patterns and the centres of extremal discs (•)

shows the loci of the centres of extremal discs, where $\pi(=\pi_j)$ denotes the natural projection from the unit disc $\mathbb{D}$ onto each surface S_j, $R = R_6$ and $\beta = \pi/30$. To describe the centres we assume that the fundamental region is centrally located in $\mathbb{D}$ such that the vertices v_n, $n = 1, 2, \ldots, 30$, satisfy $\arg v_n = (2n-1)\pi/30$. The bullets in Figures 1-5 correspond to the centres of extremal discs. For example, P_2 has three bullets on the boundary. Since they project to the same point of S_2, a value $2i \sin 4\beta / \tanh R$ is selected as a representative and listed in Table 1. The third column of Table 1 shows the group of automorphisms $\mathrm{Aut}^{\pm}(S_j)$ of S_j in the category of Klein surfaces.

As a result of Theorem 2.1 and the results of $g \geqq 3$ ($g \neq 6$) ([6], [9], [10], [11]) we have the following:

Theorem 2.2. *Compact non-orientable hyperbolic surfaces of arbitrary genus admit at most* 2 *extremal discs.*

3. Side-pairing patterns of 30-gon

As mentioned in §1, a non-orientable extremal surface of genus $g \geqq 3$ is represented by an NEC group of which a fundamental region is a regular $(6g-6)$-gon in $\mathbb{D}$. We shall therefore consider a regular 30-gon for $g = 6$. Since every angle of the polygon is equal to $2\pi/3$, three vertices project to one point on the surface. Therefore the boundary of the polygon projects to a trivalent graph with 10 vertices and 15 edges. A walk around the boundary of the polygon in one direction corresponds to a walk on the trivalent graph going through every edge twice. Conversely, for a given trivalent graph G, if we can walk continuously on every edge of G twice, we obtain a side-pairing pattern of the polygon. Handling a computer, we see that there are 388 trivalent graphs with 10 vertices and 15 edges. Considering such a walk on each of those graphs, we see that there are 149,288 side-pairing patterns for the 30-gon which becomes a non-orientable surface of genus 6; besides we obtain 927 side-pairing patterns which becomes an *orientable* surface of genus *three*.

Let $\mathcal{G}$ be the set of all trivalent graphs with 10 vertices and 15 edges. Since there are a lot of graphs, we shall classify $\mathcal{G}$ by introducing the terminology “loop” and “double” to a subgraph as depicted in Figure 6. Let lndm be the subset of $\mathcal{G}$ with n loops and m doubles. For example, l1d4 consists of 4 graphs (Figure 7).

In Table 2 let TG denote the number of trivalent graphs; SP^{-} the number of side-pairing patterns which make a non-orientable surface; and SP^{+} the number of side-pairing patterns which make an orientable surface. In addition to that, we give the number of extremal surfaces derived from those side-pairing patterns: let E^{-} denote the number of non-orientable extremal surfaces with more than one extremal disc; and E^{+} the number of orientable extremal surfaces with more than one extremal disc.

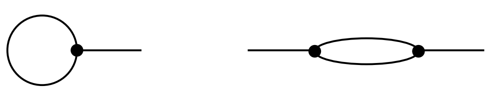

Figure 6. A loop and a double

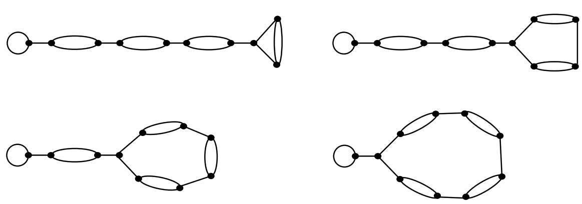

Figure 7. l1d4 (one loop and 4 doubles)

Table 2. Trivalent graphs, induced side-pairing patterns, and induced extremal surfaces with more than one extremal disc.

	TG	SP^-	SP^+	E^-	E^+
l1d0	19	20110	0	12	0
l1d1	33	18760	0	11	0
l1d2	32	7596	0	0	0
l1d3	18	1668	0	0	0
l1d4	4	114	0	0	0
l2d0	23	6050	0	20	0
l2d1	34	5178	0	0	0
l2d2	31	1912	0	16	0
l2d3	13	306	0	0	0
l2d4	1	6	0	0	0
l3d0	15	1086	0	4	0
l3d1	21	844	0	0	0
l3d2	16	270	0	0	0
l3d3	3	14	0	0	0
l4d0	9	162	0	0	0
l4d1	11	110	0	4	0
l4d2	5	19	0	0	0
l5d0	4	15	0	0	0
l5d1	3	7	0	0	0
l6d0	2	3	0	1	0
l0d0	19	33365	437	26	5
l0d1	23	32780	338	15	5
l0d2	25	14655	123	0	0
l0d3	16	3707	29	7	4
l0d4	7	535	0	0	0
l0d5	1	16	0	0	0
Total	388	149288	927	116	14

4. Loci of extremal discs

As mentioned in §1, we shall normalize the hyperbolic regular 30-gon P such that the centre is the origin and the vertices v_n satisfy $\arg v_n = (2n-1)\beta$ ($n = 1, \ldots, 30$), where $\beta = \pi/30$. We denote by C_n the side between v_n and v_{n+1} and by w_n the middle point of C_n, where subscripts are regarded as modulo 30. By elementary calculation the two hyperbolic distances $l = d(0, v_1)$ and $s = d(v_1, v_2)$ are $l = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\sinh R\right) \approx 2.388$ and $s = 2\sinh^{-1}\left(\frac{2}{\sqrt{3}}\sin\beta\sinh R\right) \approx 1.076$, where $R = R_6$ denotes the maximal radius satisfying (1.1) in §1.

Lemma 4.1. *Let S be a non-orientable extremal surface of genus 6 and $\pi : \mathbb{D} \to S$ the natural projection. If $p \in S$ is the centre of an extremal disc, then the hyperbolic distances between two points in the set $\pi^{-1}(p)$ are listed in increasing order such as $d_1 = 2R \approx 4.494$, $d_2 \approx 5.852$, $d_3 \approx 6.642$, and so on.*

Proof. The elements of the list are calculated by considering the tessellation of $\mathbb{D}$ by hyperbolic regular 30-gons. For example, $d_2 = 2\sinh^{-1}(\sinh 2R \sin 2\beta)$ and $d_3 = 2\sinh^{-1}(\sinh 2R \sin 3\beta)$. □

Remark 4.2. The distance d_2 has another expression: $d_2 = 2l + s$ ([9, Figure 2]).

Let $K_n \subset P$ $(n = 1, \dots, 30)$ be the pentagon with vertices at w_{n-1}, v_n, v_{n+1}, w_{n+1}, and the origin.

Lemma 4.3. *If a point z in K_n projects to the centre of an extremal disc, then it follows that $d(z, t_n(z)) = d_1$, where $t_n = t_{n,m}$ denotes an orientation preserving or reversing side-pairing mapping from C_n onto the other side C_m.*

Proof. Since $d(z, t_n(z)) \leqq \max\{d(v, t_n(v)) \mid v \text{ is a vertex of } K_n\}$, we shall estimate $d(v, t_n(v))$. It is clear that $d(0, t_n(0)) = 2R$; $d(v, t_n(v)) \leqq 2l$ $(v = v_{n-1}, v_n)$ because a vertex maps to the other vertex of P; and $d(v, t_n(v)) \leqq 2R$ $(v = w_{n-1}, w_n)$ because the midpoint of a side maps to the midpoint of the other side of P. Therefore $d(z, t_n(z)) \leqq 2l < 2l + s = d_2$. Hence we have $d(z, t_n(z)) = d_1$. □

If a point z in K_n projects to the centre of an extremal disc, then the equation $d(z, t_{n,m}(z)) = 2R$ implies that z is on the curve $L_{n,m}$ or $M_{n,m}$ (resp. $L'_{n,m}$ or $M'_{n,m}$) if $t_{n,m}$ is orientation preserving (resp. reversing) (cf. [6, Lemma 5.5]):

$$L_n = L_{n,m} : \left| z - \frac{\tanh R\, e^{i(n+m)\beta}}{2\cos(n-m)\beta} \right| = \frac{\tanh R}{2|\cos(n-m)\beta|} \ (n - m \not\equiv 15 \ (\mathrm{mod} 30)),$$

$$M_n = M_{n,m} : z = \coth R\, e^{2in\beta} - te^{i(n+m+15)\beta} \ (t \in \mathbb{R}),$$

$$L'_n = L'_{n,m} : \left| z - \frac{\coth R\, e^{i(n+m)\beta}}{2\cos(n-m)\beta} \right| = \frac{\coth R}{2|\cos(n-m)\beta|} \ (n - m \not\equiv 15 \ (\mathrm{mod} 30)),$$

$$M'_n = M'_{n,m} : z = \tanh R\, e^{2in\beta} - te^{i(n+m+15)\beta} \ (t \in \mathbb{R}).$$

We shall describe our process to find the centres of extremal discs for a surface S_j:

(1) Take a polygon P_j. If a side-pairing mapping t_n of P_j is orientation preserving, then draw curves L_n and M_n on K_n, otherwise draw L'_n and M'_n on K_n for every $n = 1, \dots, 30$.
(2) Find intersections of these curves on the subregion $K_n \cap K_{n+1}$ for every n.
(3) Choose every point ζ in the intersections such that the hyperbolic distance $d(\zeta, t_k(\zeta))$ belongs to the list of Lemma 4.1 for every side-pairing mapping t_k of P_j.

Applying this process to the 149,288 side-pairing patterns by computer, we see that only 116 of them, denoted by P_j $(j = 1, \dots, 116)$, yield two points (the origin and $\zeta \neq 0$) and that the others yield a unique point (the origin). For each of the patterns, except for 18 patterns P_2, P_3, P_5, P_6, P_{10}, P_{11}, P_{13}, P_{16}, P_{21}, P_{22}, P_{24}, P_{25}, P_{38}, P_{39}, P_{69}, P_{70}, P_{76}, P_{77}, we can show that the two points are transitive by a certain isometry f of $\mathbb{D}$ which is compatible with the side-pairing mappings. Since we know that the origin projects to the centre of an extremal disc, so does the other point ζ.

The following result holds for the 98 surfaces which are derived from the 116 patterns except for the 18 surfaces above.

Theorem 4.4. *The* 98 *surfaces admit exactly two extremal discs.*

Though there are a lot of P_j to be considered in this process, we shall give only two examples.

Example 4.5. We shall apply the process to P_1 (Figure 8). Then we get two

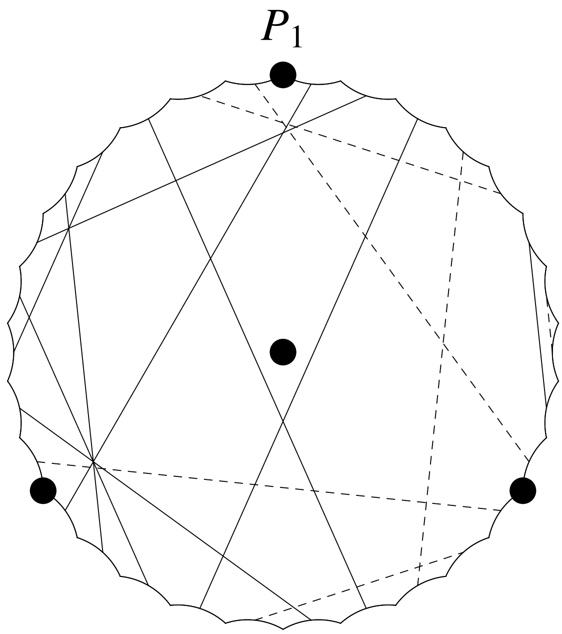

FIGURE 8. Side-pairing pattern P_1

points, the origin and $\zeta = 2i \sin 4\beta / \tanh R$ (two more equivalent points $e^{2\pi i/3}\zeta$ and $e^{4\pi i/3}\zeta$ to be precise), where ζ is a solution of the simultaneous equations $M_{7,18}$ and $M'_{8,28}$. Let $\alpha_{n,m}$ (resp. $\gamma_{n,m}$) denote the orientation preserving (resp. reversing) side-pairing mapping from C_n onto C_m, that is,

$$\alpha_{n,m}(z) = \frac{i \cosh R e^{i(m-n)\beta} z - i \sinh R e^{i(m+n)\beta}}{i \sinh R e^{-i(m+n)\beta} z - i \cosh R e^{-i(m-n)\beta}}$$

and $\gamma_{n,m}(z) = \alpha_{-n,m}(\bar{z})$. Put $f(z) := (\zeta - z)/(1 - \bar{\zeta} z)$; then we can verify that f is compatible with the side-pairing mappings of P_1 as follows:

$$\begin{array}{ll}
f\,\gamma_{1,30}\,f^{-1} = \gamma_{28,8}\,\gamma_{30,1}\,\gamma_{8,28}, & f\,\alpha_{2,29}\,f^{-1} = \gamma_{28,8}\,\gamma_{9,3}, \\
f\,\gamma_{3,9}\,f^{-1} = \gamma_{9,3}, & f\,\gamma_{4,25}\,f^{-1} = \gamma_{28,8}\,\alpha_{10,24}, \\
f\,\alpha_{5,21}\,f^{-1} = \alpha_{18,7}\,\alpha_{11,15}, & f\,\alpha_{6,13}\,f^{-1} = \alpha_{13,6}, \\
f\,\alpha_{7,18}\,f^{-1} = \alpha_{18,7}, & f\,\gamma_{8,28}\,f^{-1} = \gamma_{28,8}, \\
f\,\alpha_{10,24}\,f^{-1} = \gamma_{28,8}\,\gamma_{4,25}, & f\,\alpha_{11,15}\,f^{-1} = \alpha_{18,7}\,\alpha_{5,21}, \\
f\,\alpha_{12,19}\,f^{-1} = \alpha_{18,7}\,\alpha_{6,13}, & f\,\alpha_{14,20}\,f^{-1} = \alpha_{18,7}\,\alpha_{11,15}\,\alpha_{6,13}, \\
f\,\alpha_{16,23}\,f^{-1} = \gamma_{28,8}\,\gamma_{22,26}\,\alpha_{7,18}, & f\,\gamma_{17,27}\,f^{-1} = \gamma_{28,8}\,\alpha_{7,18}, \\
f\,\gamma_{22,26}\,f^{-1} = \gamma_{28,8}\,\alpha_{16,23}\,\alpha_{7,18}. &
\end{array}$$

Consequently the projection $\pi(\zeta)$ of ζ is the centre of an extremal disc of the surface S_1, and S_1 admits exactly two extremal discs centred at $\pi(0)$ and $\pi(\zeta)$.

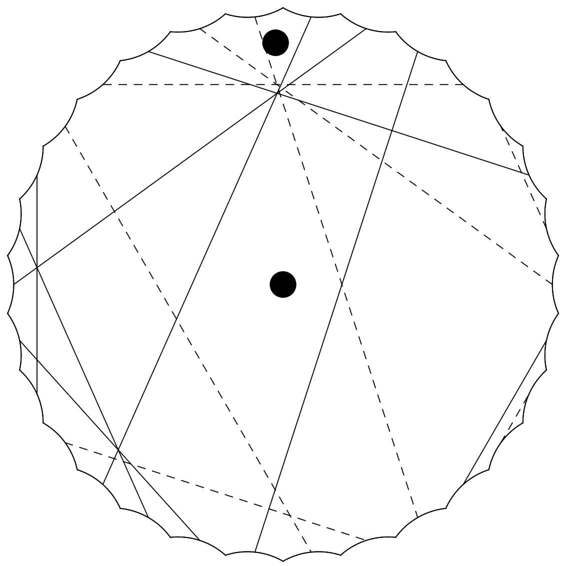

FIGURE 9. Side-pairing pattern P_{17}

Example 4.6. We shall apply the process to P_{17} (Figure 9). Then we get two points, the origin and ζ_{17}, where

$$\zeta_{17} = \frac{e^{18\beta i}\sin 3\beta + e^{11\beta i}\tanh^2 R\sin 2\beta}{\sin 7\beta\tanh R}$$

and ζ_{17} is a solution of the simultaneous equations $M_{7,19}$ and $M'_{8,25}$. Put $f(z) := (\zeta_{17} - z)/(1 - \overline{\zeta_{17}}z)$; then we can verify that f is compatible with the side-pairing mappings of P_{17} as follows:

$f\,\gamma_{1,3}\,f^{-1} = \alpha_{2,10}\,\gamma_{9,0}$, $f\,\alpha_{2,10}\,f^{-1} = \alpha_{10,2}$,
$f\,\gamma_{4,11}\,f^{-1} = \gamma_{11,4}$, $f\,\alpha_{5,22}\,f^{-1} = \gamma_{25,8}\,\gamma_{12,23}$,
$f\,\alpha_{6,15}\,f^{-1} = \alpha_{15,6}$, $f\,\alpha_{7,19}\,f^{-1} = \alpha_{19,7}$,
$f\,\gamma_{8,25}\,f^{-1} = \gamma_{25,8}$, $f\,\gamma_{9,30}\,f^{-1} = \gamma_{30,9}$,
$f\,\gamma_{12,23}\,f^{-1} = \gamma_{25,8}\,\alpha_{5,22}$, $f\,\alpha_{13,17}\,f^{-1} = \alpha_{19,7}\,\alpha_{5,22}$,
$f\,\alpha_{14,20}\,f^{-1} = \alpha_{19,7}\,\alpha_{6,15}$, $f\,\alpha_{16,21}\,f^{-1} = \alpha_{19,7}\,\alpha_{13,17}\,\alpha_{6,15}$,
$f\,\gamma_{18,24}\,f^{-1} = \gamma_{25,8}\,\alpha_{7,19}$, $f\,\alpha_{26,29}\,f^{-1} = \gamma_{30,9}\,\gamma_{8,25}$,
$f\,\gamma_{27,28}\,f^{-1} = \gamma_{30,9}\,\gamma_{27,28}\,\gamma_{8,25}$.

Consequently $\pi(\zeta_{17})$ is the centre of an extremal disc of the surface S_{17}, and S_{17} admits exactly two extremal discs centred at $\pi(0)$ and $\pi(\zeta_{17})$.

For the exceptional cases, we have the following result.

Theorem 4.7. *The surfaces S_2 and S_3 are isomorphic to each other. Also, S_5 and S_6; S_{10} and S_{11}; S_{13} and S_{16}; S_{21} and S_{22}; S_{24} and S_{25}; S_{38} and S_{39}; S_{69} and S_{70}; and S_{76} and S_{77} are isomorphic, respectively. Furthermore they admit exactly two extremal discs.*

Proof. We shall construct an isomorphism between S_2 and S_3. Let $f(z) := (|\zeta|^2 - \zeta\bar{z})/(\overline{\zeta} - |\zeta|^2\bar{z})$, where $\zeta = 2i\sin 4\beta/\tanh R$ is a solution of the simultaneous equations $M'_{7,17}$ and $M_{8,27}$. Then f induces an isomorphism from S_2 onto S_3 because f gives S_3 a regular 30-gon centred at ζ of which its side-pairing pattern is the same as that of P_2. In other words, the polygon $f(P_2)$ is a fundamental region of an NEC group of S_3 whose side-pairing mappings are generated by those of P_3. Therefore the projection $\pi(\zeta)$ is the centre of an extremal disc of S_2, and S_2 has exactly two extremal discs. We next show the relations between the side-pairing mappings of P_2 and those of P_3 (the right-hand sides are the compositions of side-pairing mappings of P_3):

$f\,\alpha_{1,25}\,f^{-1} = \alpha_{18,7}\,\alpha_{1,15}\,\alpha_{6,3}$, $\quad f\,\gamma_{2,16}\,f^{-1} = \gamma_{28,8}\,\alpha_{6,3}$,
$f\,\alpha_{3,6}\,f^{-1} = \alpha_{6,3}$, $\quad f\,\gamma_{4,5}\,f^{-1} = \gamma_{5,4}$,
$f\,\gamma_{7,17}\,f^{-1} = \gamma_{28,8}$, $\quad f\,\alpha_{8,27}\,f^{-1} = \alpha_{18,7}$,
$f\,\gamma_{9,23}\,f^{-1} = \gamma_{28,8}\,\alpha_{13,26}$, $\quad f\,\alpha_{10,14}\,f^{-1} = \alpha_{22,9}\,\alpha_{10,24}$,
$f\,\gamma_{11,20}\,f^{-1} = \gamma_{28,8}\,\alpha_{10,24}$, $\quad f\,\alpha_{12,29}\,f^{-1} = \alpha_{18,7}\,\alpha_{9,22}$,
$f\,\gamma_{13,19}\,f^{-1} = \gamma_{28,8}\,\alpha_{9,22}$, $\quad f\,\gamma_{15,24}\,f^{-1} = \alpha_{18,7}\,\alpha_{1,15}\,\gamma_{8,28}$,
$f\,\gamma_{18,28}\,f^{-1} = \alpha_{18,7}\,\gamma_{8,28}$, $\quad f\,\gamma_{21,30}\,f^{-1} = \alpha_{18,7}\,\alpha_{25,21}\,\gamma_{8,28}$,
$f\,\gamma_{22,26}\,f^{-1} = \alpha_{18,7}\,\gamma_{16,12}\,\alpha_{7,18}$.

The other cases are as follows:

$S_5 \to S_6$: $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$, $\zeta = 2i\sin 4\beta/\tanh R$.
$f\,\gamma_{1,30}\,f^{-1} = \gamma_{28,8}\,\gamma_{30,1}\,\gamma_{8,28}$, $\quad f\,\alpha_{2,29}\,f^{-1} = \gamma_{28,8}\,\gamma_{9,3}$,
$f\,\gamma_{3,9}\,f^{-1} = \gamma_{9,3}$, $\quad f\,\gamma_{4,15}\,f^{-1} = \gamma_{17,7}\,\alpha_{10,14}$,
$f\,\gamma_{5,24}\,f^{-1} = \gamma_{28,8}\,\alpha_{11,25}$, $\quad f\,\gamma_{6,22}\,f^{-1} = \gamma_{28,8}\,\alpha_{13,26}$,
$f\,\gamma_{7,17}\,f^{-1} = \gamma_{17,7}$, $\quad f\,\gamma_{8,28}\,f^{-1} = \gamma_{28,8}$,
$f\,\alpha_{10,14}\,f^{-1} = \alpha_{23,6}\,\alpha_{5,21}$, $\quad f\,\gamma_{11,20}\,f^{-1} = \gamma_{17,7}\,\alpha_{5,21}$,
$f\,\gamma_{12,26}\,f^{-1} = \gamma_{28,8}\,\alpha_{6,23}$, $\quad f\,\gamma_{13,19}\,f^{-1} = \gamma_{17,7}\,\alpha_{6,23}$,
$f\,\alpha_{16,23}\,f^{-1} = \gamma_{28,8}\,\alpha_{13,26}\,\gamma_{7,17}$, $\quad f\,\alpha_{18,27}\,f^{-1} = \gamma_{28,8}\,\gamma_{7,17}$,
$f\,\alpha_{21,25}\,f^{-1} = \gamma_{28,8}\,\alpha_{20,24}\,\gamma_{7,17}$.

$S_{10} \to S_{11}$: $f(z) = (|\zeta|^2 - \zeta\bar{z})/(\bar{\zeta} - |\zeta|^2\bar{z})$, $\zeta = 2i\sin 4\beta/\tanh R$.
$f\,\alpha_{1,25}\,f^{-1} = \alpha_{18,7}\,\alpha_{1,15}\,\alpha_{6,3}$, $\quad f\,\gamma_{2,16}\,f^{-1} = \gamma_{28,8}\,\alpha_{6,3}$,
$f\,\alpha_{3,6}\,f^{-1} = \alpha_{6,3}$, $\quad f\,\gamma_{4,5}\,f^{-1} = \gamma_{5,4}$,
$f\,\gamma_{7,17}\,f^{-1} = \gamma_{28,8}$, $\quad f\,\alpha_{8,27}\,f^{-1} = \alpha_{18,7}$,
$f\,\gamma_{9,13}\,f^{-1} = \gamma_{13,9}$, $\quad f\,\gamma_{10,21}\,f^{-1} = \gamma_{28,8}\,\alpha_{11,25}$,
$f\,\gamma_{11,30}\,f^{-1} = \alpha_{18,7}\,\gamma_{10,21}$, $\quad f\,\gamma_{12,26}\,f^{-1} = \alpha_{18,7}\,\gamma_{9,13}$,
$f\,\alpha_{14,20}\,f^{-1} = \gamma_{28,8}\,\alpha_{30,24}\,\gamma_{8,28}$, $\quad f\,\gamma_{15,24}\,f^{-1} = \alpha_{18,7}\,\alpha_{1,15}\,\gamma_{8,28}$,
$f\,\gamma_{18,28}\,f^{-1} = \alpha_{18,7}\,\gamma_{8,28}$, $\quad f\,\gamma_{19,23}\,f^{-1} = \alpha_{18,7}\,\alpha_{23,16}\,\gamma_{8,28}$,
$f\,\alpha_{22,29}\,f^{-1} = \alpha_{18,7}\,\alpha_{16,23}\,\alpha_{7,18}$.

$S_{13} \to S_{16}$: $f(z) = (|\zeta|^2 - \zeta\bar{z})/(\bar{\zeta} - |\zeta|^2\bar{z})$, $\zeta = 2i\sin 4\beta/\tanh R$.
$f\,\gamma_{1,30}\,f^{-1} = \alpha_{28,7}\,\gamma_{1,30}\,\alpha_{6,3}$, $\quad f\,\alpha_{2,29}\,f^{-1} = \alpha_{28,7}\,\alpha_{6,3}$,
$f\,\alpha_{3,6}\,f^{-1} = \alpha_{6,3}$, $\quad f\,\gamma_{4,15}\,f^{-1} = \gamma_{18,8}\,\alpha_{5,21}$,
$f\,\gamma_{5,14}\,f^{-1} = \gamma_{18,8}\,\alpha_{4,20}$, $\quad f\,\alpha_{7,28}\,f^{-1} = \alpha_{28,7}$,
$f\,\gamma_{8,18}\,f^{-1} = \gamma_{18,8}$, $\quad f\,\alpha_{9,22}\,f^{-1} = \gamma_{18,8}\,\gamma_{12,16}$,
$f\,\alpha_{10,24}\,f^{-1} = \alpha_{28,7}\,\alpha_{11,25}$, $\quad f\,\gamma_{11,20}\,f^{-1} = \gamma_{18,8}\,\alpha_{10,14}$,
$f\,\gamma_{12,26}\,f^{-1} = \alpha_{28,7}\,\gamma_{9,23}$, $\quad f\,\alpha_{13,16}\,f^{-1} = \gamma_{18,8}\,\gamma_{9,23}$,
$f\,\gamma_{17,27}\,f^{-1} = \alpha_{28,7}\,\gamma_{8,18}$, $\quad f\,\gamma_{19,23}\,f^{-1} = \alpha_{28,7}\,\alpha_{13,26}\,\gamma_{8,18}$,
$f\,\alpha_{21,25}\,f^{-1} = \alpha_{28,7}\,\gamma_{15,24}\,\gamma_{8,18}$.

$S_{21} \to S_{22}$: $f(z) = (|\zeta|^2 - \zeta\bar{z})/(\bar{\zeta} - |\zeta|^2\bar{z})$, $\zeta = 2i\sin 4\beta/\tanh R$.

$f\,\alpha_{1,25}\,f^{-1} = \alpha_{18,7}\,\alpha_{1,15}\,\alpha_{6,3}$, $f\,\gamma_{2,16}\,f^{-1} = \gamma_{28,8}\,\alpha_{6,3}$,
$f\,\alpha_{3,6}\,f^{-1} = \alpha_{6,3}$, $f\,\gamma_{4,5}\,f^{-1} = \gamma_{5,4}$,
$f\,\gamma_{7,17}\,f^{-1} = \gamma_{28,8}$, $f\,\alpha_{8,27}\,f^{-1} = \alpha_{18,7}$,
$f\,\alpha_{9,12}\,f^{-1} = \alpha_{12,9}$, $f\,\gamma_{10,21}\,f^{-1} = \gamma_{28,8}\,\alpha_{11,25}$,
$f\,\gamma_{11,20}\,f^{-1} = \gamma_{28,8}\,\alpha_{10,24}$, $f\,\alpha_{13,26}\,f^{-1} = \alpha_{18,7}\,\alpha_{9,12}$,
$f\,\alpha_{14,30}\,f^{-1} = \alpha_{18,7}\,\gamma_{30,21}\,\gamma_{8,28}$, $f\,\gamma_{15,24}\,f^{-1} = \alpha_{18,7}\,\alpha_{1,15}\,\gamma_{8,28}$,
$f\,\gamma_{18,28}\,f^{-1} = \alpha_{18,7}\,\gamma_{8,28}$, $f\,\alpha_{19,22}\,f^{-1} = \gamma_{28,8}\,\alpha_{23,26}\,\gamma_{8,28}$,
$f\,\gamma_{23,29}\,f^{-1} = \alpha_{18,7}\,\gamma_{16,22}\,\alpha_{7,18}$.

$S_{24} \to S_{25}$: $f(z) = (|\zeta|^2 - \zeta\bar{z})/(\bar{\zeta} - |\zeta|^2\bar{z})$, $\zeta = 2i\sin 4\beta/\tanh R$.

$f\,\alpha_{1,5}\,f^{-1} = \alpha_{30,4}\,\alpha_{7,28}$, $f\,\gamma_{2,16}\,f^{-1} = \alpha_{17,8}\,\gamma_{6,12}$,
$f\,\gamma_{3,9}\,f^{-1} = \gamma_{6,12}$, $f\,\gamma_{4,15}\,f^{-1} = \alpha_{17,8}\,\gamma_{5,14}$,
$f\,\gamma_{6,12}\,f^{-1} = \gamma_{3,9}$, $f\,\alpha_{7,28}\,f^{-1} = \alpha_{28,7}$,
$f\,\alpha_{8,17}\,f^{-1} = \alpha_{17,8}$, $f\,\alpha_{10,14}\,f^{-1} = \gamma_{3,9}\,\gamma_{10,1}$,
$f\,\gamma_{11,30}\,f^{-1} = \alpha_{28,7}\,\gamma_{10,1}$, $f\,\gamma_{13,29}\,f^{-1} = \alpha_{28,7}\,\gamma_{9,3}$,
$f\,\alpha_{18,27}\,f^{-1} = \alpha_{28,7}\,\alpha_{8,17}$, $f\,\alpha_{19,22}\,f^{-1} = \alpha_{17,8}\,\alpha_{22,19}\,\alpha_{8,17}$,
$f\,\gamma_{20,21}\,f^{-1} = \alpha_{17,8}\,\gamma_{21,20}\,\alpha_{8,17}$, $f\,\alpha_{23,26}\,f^{-1} = \alpha_{28,7}\,\alpha_{26,23}\,\alpha_{7,28}$,
$f\,\gamma_{24,25}\,f^{-1} = \alpha_{28,7}\,\gamma_{25,24}\,\alpha_{7,28}$.

$S_{38} \to S_{39}$: $f(z) = (|\zeta|^2 - \zeta\bar{z})/(\bar{\zeta} - |\zeta|^2\bar{z})$, $\zeta = 2i\sin 4\beta/\tanh R$.

$f\,\gamma_{1,20}\,f^{-1} = \gamma_{28,8}\,\alpha_{1,25}\,\alpha_{6,3}$, $f\,\gamma_{2,16}\,f^{-1} = \gamma_{28,8}\,\alpha_{6,3}$,
$f\,\alpha_{3,6}\,f^{-1} = \alpha_{6,3}$, $f\,\gamma_{4,5}\,f^{-1} = \gamma_{5,4}$,
$f\,\gamma_{7,17}\,f^{-1} = \gamma_{28,8}$, $f\,\gamma_{8,28}\,f^{-1} = \gamma_{17,7}$,
$f\,\alpha_{9,12}\,f^{-1} = \alpha_{12,9}$, $f\,\gamma_{10,11}\,f^{-1} = \gamma_{11,10}$,
$f\,\gamma_{13,29}\,f^{-1} = \gamma_{17,7}\,\alpha_{9,12}$, $f\,\gamma_{14,25}\,f^{-1} = \gamma_{17,7}\,\gamma_{30,21}\,\gamma_{8,28}$,
$f\,\alpha_{15,21}\,f^{-1} = \gamma_{28,8}\,\alpha_{1,25}\,\gamma_{8,28}$, $f\,\alpha_{18,27}\,f^{-1} = \gamma_{17,7}\,\gamma_{8,28}$,
$f\,\gamma_{19,23}\,f^{-1} = \gamma_{17,7}\,\gamma_{23,19}\,\gamma_{8,28}$, $f\,\gamma_{22,26}\,f^{-1} = \gamma_{17,7}\,\gamma_{19,23}\,\gamma_{7,17}$,
$f\,\alpha_{24,30}\,f^{-1} = \gamma_{17,7}\,\alpha_{20,14}\,\gamma_{7,17}$.

$S_{69} \to S_{70}$: $f(z) = (|\zeta|^2 - \zeta\bar{z})/(\bar{\zeta} - |\zeta|^2\bar{z})$, $\zeta = 2i\sin 4\beta/\tanh R$.

$f\,\alpha_{1,5}\,f^{-1} = \alpha_{30,4}\,\alpha_{7,28}$, $f\,\gamma_{2,16}\,f^{-1} = \alpha_{17,8}\,\gamma_{6,12}$,
$f\,\gamma_{3,9}\,f^{-1} = \gamma_{6,12}$, $f\,\gamma_{4,15}\,f^{-1} = \alpha_{17,8}\,\gamma_{5,14}$,
$f\,\gamma_{6,12}\,f^{-1} = \gamma_{3,9}$, $f\,\alpha_{7,28}\,f^{-1} = \alpha_{28,7}$,
$f\,\alpha_{8,17}\,f^{-1} = \alpha_{17,8}$, $f\,\alpha_{10,14}\,f^{-1} = \gamma_{3,9}\,\gamma_{10,1}$,
$f\,\gamma_{11,30}\,f^{-1} = \alpha_{28,7}\,\gamma_{10,1}$, $f\,\gamma_{13,29}\,f^{-1} = \alpha_{28,7}\,\gamma_{9,3}$,
$f\,\alpha_{18,27}\,f^{-1} = \alpha_{28,7}\,\alpha_{8,17}$, $f\,\alpha_{19,22}\,f^{-1} = \alpha_{17,8}\,\alpha_{22,19}\,\alpha_{8,17}$,
$f\,\alpha_{20,24}\,f^{-1} = \alpha_{28,7}\,\alpha_{21,25}\,\alpha_{8,17}$, $f\,\alpha_{21,25}\,f^{-1} = \alpha_{28,7}\,\alpha_{20,24}\,\alpha_{8,17}$,
$f\,\alpha_{23,26}\,f^{-1} = \alpha_{28,7}\,\alpha_{26,23}\,\alpha_{7,28}$.

$S_{76} \to S_{77}$: $f(z) = (|\zeta|^2 - \zeta\bar{z})/(\bar{\zeta} - |\zeta|^2\bar{z})$, $\zeta = 2i\sin 4\beta/\tanh R$.

$f\,\gamma_{1,20}\,f^{-1} = \gamma_{28,8}\,\alpha_{1,25}\,\alpha_{6,3}$, $f\,\gamma_{2,16}\,f^{-1} = \gamma_{28,8}\,\alpha_{6,3}$,
$f\,\alpha_{3,6}\,f^{-1} = \alpha_{6,3}$, $f\,\alpha_{4,10}\,f^{-1} = \alpha_{5,11}$,
$f\,\alpha_{5,11}\,f^{-1} = \alpha_{4,10}$, $f\,\gamma_{7,17}\,f^{-1} = \gamma_{28,8}$,
$f\,\gamma_{8,28}\,f^{-1} = \gamma_{17,7}$, $f\,\alpha_{9,12}\,f^{-1} = \alpha_{12,9}$,
$f\,\gamma_{13,29}\,f^{-1} = \gamma_{17,7}\,\alpha_{9,12}$, $f\,\gamma_{14,25}\,f^{-1} = \gamma_{17,7}\,\gamma_{30,21}\,\gamma_{8,28}$,
$f\,\alpha_{15,21}\,f^{-1} = \gamma_{28,8}\,\alpha_{1,25}\,\gamma_{8,28}$, $f\,\alpha_{18,27}\,f^{-1} = \gamma_{17,7}\,\gamma_{8,28}$,
$f\,\gamma_{19,23}\,f^{-1} = \gamma_{17,7}\,\gamma_{23,19}\,\gamma_{8,28}$, $f\,\gamma_{22,26}\,f^{-1} = \gamma_{17,7}\,\gamma_{19,23}\,\gamma_{7,17}$,
$f\,\alpha_{24,30}\,f^{-1} = \gamma_{17,7}\,\alpha_{20,14}\,\gamma_{7,17}$.

Hence we are done. □

5. Isomorphicity of surfaces

Let $\mathcal{S}$ be the set of surfaces S_j derived from all P_j except for $j = 3, 6, 11, 16, 22, 25, 39, 70, 77$. Therefore the cardinality of $\mathcal{S}$ is $\#\mathcal{S} = 107$. We divide $\mathcal{S}$ into two parts $\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2$, where $\mathcal{S}_2$ consists of the surfaces appearing in Theorem 4.7, i.e.,

$$\mathcal{S}_2 = \{S_2, S_5, S_{10}, S_{13}, S_{21}, S_{24}, S_{38}, S_{69}, S_{76}\}$$

and $\mathcal{S}_1$ are the other surfaces in $\mathcal{S}$.

Theorem 5.1. *For every surface in $\mathcal{S}_1$ two centres of extremal discs are transitive, that is, there exists an automorphism which maps one to the other. On the other hand, no surface in $\mathcal{S}_2$ has this property.*

Proof. For the surfaces in $\mathcal{S}_1$, the statement is clear from our process to find that $\zeta(\neq 0)$ is the centre of an extremal disc. The isometry f of $\mathbb{D}$ appearing in the process induces the required automorphism.

Let S_j be a surface in $\mathcal{S}_2$ and let S_k be the isomorphic surface with a side-pairing pattern P_k different from P_j (for example, S_3 for S_2). If we suppose that there exists an automorphism of S_j which maps one centre to the other, then we can take an isomorphism $T : S_j \to S_k$ such that its lift $\tilde{T} : \mathbb{D} \to \mathbb{D}$ fixes the origin. Then $\tilde{T}$ must be a rotation around the origin or a reflection in a line passing through the origin. Such a mapping is incompatible with the side-pairing patterns P_j and P_k. Therefore we have a contradiction. □

Theorem 5.2. *No surfaces in $\mathcal{S}$ are isomorphic to each other.*

Proof. Case 1. Suppose that $S_j \in \mathcal{S}$ and $S_k \in \mathcal{S}_1$ are isomorphic ($j \neq k$). Then there exists an isomorphism $T : S_j \to S_k$ satisfying $T(\pi_j(0)) = \pi_k(0)$ because the centres of extremal discs of S_k are transitive. We can then take a lift $\tilde{T} : \mathbb{D} \to \mathbb{D}$ of T which fixes the origin, so that we arrive at a contradiction in the same way as in the proof of Theorem 5.1.

Case 2. Suppose that $S_j \in \mathcal{S}$ and $S_k \in \mathcal{S}_2$ are isomorphic ($j \neq k$). Let $T : S_j \to S_k$ be an isomorphism. If $T(\pi_j(0)) = \pi_k(0)$, then we have a lift $\tilde{T} : \mathbb{D} \to \mathbb{D}$ of T which fixes the origin, a contradiction. If $T(\pi_j(0)) = \pi_k(\zeta)$, then we take a surface S_l isomorphic to S_k such that the side-pairing pattern P_l is different from P_k. Let $F : S_k \to S_l$ be an isomorphism arising from the f appearing in the proof of Theorem 4.7 which interchanges 0 and ζ. Since $F(\pi_k(\zeta)) = \pi_l(0)$ hold, there exists an isomorphism $F \circ T : S_j \to S_l$ which maps $\pi_j(0)$ to $\pi_l(0)$. Hence we have a lift which fixes the origin, a contradiction again. □

Corollary 5.3. *There exist* 149,279 *non-orientable extremal surfaces of genus* 6 *up to isomorphism.*

Proof. There exist 149,288 side-pairing patterns. The 116 of them make the set $\mathcal{S}$ up to isomorphism, and $\#\mathcal{S} = 107$. The other 149,172 patterns correspond to the non-orientable surfaces of genus 6 admitting a unique extremal disc. It is clear that they are not isomorphic to each other and not isomorphic to a surface in $\mathcal{S}$. Therefore we have 149,279 surfaces up to isomorphism. □

6. The groups of automorphisms of surfaces

We shall consider the group of automorphisms for a surface in $\mathcal{S}$. In what follows, we use mappings $k : z \mapsto -\bar{z}$ and $r : z \mapsto e^{2\pi i/3} z$.

S_2: An automorphism T of S_2 fixes the centres of extremal discs because T does not interchange them (Theorem 5.1). We can take a lift $\tilde{T} : \mathbb{D} \to \mathbb{D}$ fixing the origin, so that it is a rotation around the origin or a reflection in a line passing through the origin. It is however incompatible with the side-pairing pattern of P_2 except for the identity mapping. Hence the group of automorphisms is trivial.

S_{17}: There exists an automorphism F induced by $f(z) = (\zeta_{17} - z)/(1 - \overline{\zeta_{17}}z)$. Considering the side-pairing pattern of P_{17} we see that an automorphism fixing $\pi(0)$ must be the identity mapping. Therefore F is a unique automorphism interchanging the centres. Hence

$$\mathrm{Aut}^{\pm}(S_{17}) = \mathbb{Z}_2 = \langle F \rangle.$$

S_{28}: Let $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$, where $\zeta = i \sin 2\beta/(\sin 3\beta \tanh R)$. Since f and k are compatible with the side-pairing pattern of P_{28}, they induce automorphisms F and K of S_{28}, respectively. All automorphisms are generated by F and K because each automorphism fixes or interchanges the centres of extremal discs. Since compositions of f and k satisfy $f^2 = k^2 = fkfk = id$, we have

$$\mathrm{Aut}^{\pm}(S_{28}) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle F \rangle \times \langle K \rangle.$$

S_{63}: Let $f(z) = (|\zeta|^2 - \zeta\bar{z})/(\bar{\zeta} - |\zeta|^2\bar{z})$, where $\zeta = 2i \sin 4\beta/ \tanh R$. Then f and r induce automorphisms F and σ of S_{63}, respectively. All automorphisms are generated by F and σ. Since $f^2 = r^3 = id$ and $frfr = \alpha_{27,8}$, we have

$$\mathrm{Aut}^{\pm}(S_{63}) = D_3 = \langle F, \sigma \mid F^2 = \sigma^3 = 1, F\sigma = \sigma^{-1}F \rangle.$$

S_{68}: Let $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$, where $\zeta = 2i \sin 4\beta/ \tanh R$. Then f, k, and r induce automorphisms F, K, and σ of S_{68}, respectively. All automorphisms are generated by F, K, and σ. Since $k^2 = r^3 = krkr = (fk)^2 = (fk)k(fk)k = id$ and $(fk)r(fk)r^{-1} = \alpha_{17,8}$, we have

$$\mathrm{Aut}^{\pm}(S_{68}) = D_3 \times \mathbb{Z}_2 = \langle K, \sigma \rangle \times \langle FK \rangle.$$

S_{101}: Let $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$, where $\zeta = 2i \sin 4\beta/ \tanh R$. Then f and r induce automorphisms F and σ of S_{101}, respectively. All automorphisms are generated by F and σ. Since $f^2 = r^3 = id$ and $frfr = \alpha_{28,7}$, we have

$$\mathrm{Aut}^{\pm}(S_{101}) = D_3 = \langle F, \sigma \rangle.$$

S_{103}: Let $f(z) = (\zeta - z)/(1 - \bar{\zeta}z)$, where $\zeta = 2i \sin 4\beta/ \tanh R$. Then f, k, and r induce automorphisms F, K, and σ of S_{103}, respectively. All automorphisms are generated by F, K, and σ. Since $k^2 = r^3 = krkr = f^2 = fkfk = id$ and $frfr^{-1} = \alpha_{18,7}$, we have

$$\mathrm{Aut}^{\pm}(S_{103}) = D_3 \times \mathbb{Z}_2 = \langle K, \sigma \rangle \times \langle F \rangle.$$

The groups of automorphisms for the other surfaces are determined in a similar way.

From the arguments in §§3–6 we have Theorem 2.1 and Table 1.

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