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THE HEISENBERG GROUP ACTS ON A STRICTLY CONVEX DOMAIN

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ABSTRACT. This paper gives the first example of a unipotent group that is not virtually abelian and preserves a strictly convex domain.

Every linear group acts by isometries on some properly convex domain in real projective space. This follows from the fact that action of $\mathrm{SL}(n,\mathbb{R})$ on the space of quadratic forms in n variables preserves the projectivization, $\mathrm{Pos}(n)$, of the properly convex cone consisting of positive definite forms. If Γ is the holonomy of a properly convex orbifold of finite volume, then every virtually nilpotent subgroup of Γ is virtually abelian; moreover, every unipotent subgroup of Γ is conjugate into $\mathrm{PO}(n,1)$. A reference for all of this is [1]. This paper gives the first example of a unipotent group that is not virtually abelian and preserves a strictly convex domain. It answers a question asked by Misha Kapovich.

The *Heisenberg group* is the subgroup $H \subset \mathrm{SL}(3,\mathbb{R})$ of unipotent upper-triangular matrices. Define $\theta : H \to \mathrm{SL}(10,\mathbb{R})$ and $G = \theta(H)$ where

$$\theta\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}$$
$$= \begin{pmatrix}
1 & 2a & 2c & a & a^2/2 & a^3/6 & b & 2a^2+b^2/2 & b^3/6+2ac & (a^4+b^4)/24+c^2\\
0 & 1 & b & 0 & 0 & 0 & 0 & 2a & ab+c & bc\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & a & c\\
0 & 0 & 0 & 1 & a & a^2/2 & 0 & 0 & 0 & a^3/6\\
0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & a^2/2\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 & b^3/6\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

It is clear that θ is injective, and it is easy to check that it is a homomorphism.

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Theorem 0.1. *There is a strictly convex domain $\Omega \subset \mathbb{RP}^9$ that is preserved by G. This is an effective action of the Heisenberg group on Ω by parabolic isometries that are unipotent.*

Proof. The group G acts affinely on the affine patch $[x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7 : x_8 : x_9 : x_{10} = 1]$ that we identify with $\mathbb{R}^9$. Let $p \in \mathbb{R}^9$ be the origin. Then $G \cdot p$ is

$$((a^4 + b^4)/24 + c^2, bc, c, a^3/6, a^2/2, a, b^3/6, b^2/2, b)$$

This orbit is an algebraic embedding $\mathbb{R}^3 \hookrightarrow \mathbb{R}^9$ which limits on the single point

$$q = [1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0] \in \mathbb{RP}^9$$

in the hyperplane at infinity, P_∞, given by $x_{10} = 0$. This follows from the fact that $(a^4 + b^4)/24 + c^2$ dominates all the other entries whenever at least one of $|a|, |b|, |c|$ is large.

Let $S \subset \mathbb{R}^9$ be this orbit. Choose 10 random points on $S \subset \mathbb{RP}^9$ and compute the determinant, d, of the corresponding 10 vectors in $\mathbb{R}^{10}$. Then $d \neq 0$ therefore the interior $\Omega^+ \subset \mathbb{R}^9$ of the convex hull of S has dimension 9.

Moreover, the closure, Ω', of Ω^+ in $\mathbb{RP}^9$ is disjoint from the closure of the affine hyperplane $x_1 = -1$ in $\mathbb{R}^9$, hence Ω^+ is properly convex. Since $\Omega' \cap \mathrm{cl}(P_\infty) = q$, and G preserves q and P_∞, and G is unipotent, it follows from (5.8) in [1] that G preserves some strictly convex domain $\Omega \subset \Omega'$. Moreover, P_∞ is a supporting hyperplane to Ω at q. $\square$

We sketch an argument that the extrinsic curvature of $\partial\Omega \setminus q$ is positive. A properly convex domain $\Omega_0 \subset \mathbb{RP}^n$ is the projectivization of a cone $\mathcal{C} \subset \mathbb{R}^{n+1}$. There is a Vinberg characteristic function $\phi : \mathcal{C} \to \mathbb{R}$ and the level sets have extrinsic positive curvature. Applying this to $\Omega_0 = \Omega^+$ in the construction of Ω in (5.8) of [1] implies the positive curvature of $\partial\Omega \setminus q$.

Let K be the 1-parameter subgroup of G given by $a = c = 0$. Assume, for simplicity only, that $p \in \mathrm{cl}(\Omega)$. Then $K \cdot p$ is the curve $[b^4/24 : 0 : 0 : 0 : 0 : 0 : b^3/6 : b^2/2 : b : 1]$. Omitting the 0-coordinates, and rescaling the remainder to remove the constants, this is $\gamma(t) = [1 : t : t^2 : t^3 : t^4]$ where $t = b^{-1}$. It follows that the height of $\gamma(t)$ above P_∞ is t^4, so $\gamma(t)$ has third-order contact with P_∞ at $q = \gamma(0)$, and therefore $\partial\Omega$ does not have strictly positive extrinsic curvature at q. Replacing p by a generic point gives a similar result.

Define $\phi(x_1, \cdots, x_9) = x_5$ then $\phi(G \cdot p) = a^2/2$ hence Ω^+ is disjoint from the affine hyperplane $P' = \phi^{-1}(-1) \subset \mathbb{R}^9$. Moreover, q is in the closure of P' in $\mathbb{RP}^9$. Since $\Omega \subset \Omega^+$, it follows that the closures in $\mathbb{RP}^9$ of P_∞ and P' are distinct supporting projective hyperplanes to Ω at q. Thus q is not a C^1 point of $\partial\Omega$.

Corollary 0.2. *There is a strictly convex real projective manifold Ω/Γ of dimension 9 with nilpotent fundamental group $\Gamma \cong \langle \alpha, \beta : [\alpha, [\alpha, \beta]], [\beta, [\alpha, \beta]] \rangle$ that is not virtually abelian. Moreover, Γ is unipotent.*

Proof. If Γ is a lattice in G, then Ω/Γ is a strictly convex manifold with unipotent holonomy and Γ is nilpotent but not virtually abelian. $\square$

The genesis of this example is as follows. The image of H in $\mathrm{SL}(6,\mathbb{R})$ under the irreducible representation $\mathrm{SL}(3,\mathbb{R}) \to \mathrm{SL}(6,\mathbb{R})$ is

$$\begin{pmatrix} 1 & 2a & a^2 & 2c & 2ac & c^2 \\ 0 & 1 & a & b & ab+c & bc \\ 0 & 0 & 1 & 0 & 2b & b^2 \\ 0 & 0 & 0 & 1 & a & c \\ 0 & 0 & 0 & 0 & 1 & b \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and preserves $Q := \mathrm{Pos}(3) \subset \mathbb{RP}^5$. The boundary of the closure of Q consists of semidefinite forms and contains flats, so Q is *not strictly convex*. Let $A, B, C \in \mathrm{SL}(6,\mathbb{R})$ be the elements corresponding to one of a, b, c being 1 and the others 0. Each of A, B, C has a parabolic fixed point in ∂Q corresponding to a rank 1 quadratic form. Every point in Q converges to this parabolic fixed point under iteration by the given group element. The fixed points for A and B are distinct and lie in a flat in ∂Q.

The idea is to increase the dimension of the representation and use the extra dimensions to add parabolic blocks of size 5 onto A (row 1 and rows 7-10) and onto B (row 1 and rows 11-14) that commute, and the parabolic fixed point of each block is the rank-1 form that is a fixed point of C. This gives a 14-dimensional representation of H:

$$\begin{pmatrix}
1 & 2a & a^2 & 2c & 2ac & c^2 & a & a^2/2 & a^3/6 & a^4/24 & b & b^2/2 & b^3/6 & b^4/24 \\
0 & 1 & a & b & ab+c & bc & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 2b & b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & a & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a^2/2 & a^3/6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a^2/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 & b^3/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

The top-left 6×6 block is the image of H in $\mathrm{SL}(6,\mathbb{R})$. The entries in A^n and B^n grow like n^2. This is less than the growth of some entries in the added blocks of size 5 which grow like n^4. The orbit of

$$[0:0:0:0:0:1:0:0:0:1:0:0:0:1]$$

is

$$[(a^4+b^4)/24 + c^2 : bc : b^2 : c : b : 1 : a^3/6 : a^2/2 : a : 1 : b^3/6 : b^2/2 : b : 1]$$

so there is a codimension-4 projective hyperplane that is preserved, and which is defined by

$$x_6 = x_{10} = x_{14} \qquad x_5 = x_{13} \qquad x_3 = 2x_{12}$$

The restriction to this hyperplane gives θ.

Reference

[1] D. Cooper, D. D. Long, and S. Tillmann, *On convex projective manifolds and cusps*, Adv. Math. **277** (2015), 181–251, DOI 10.1016/j.aim.2015.02.009. MR3336086

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