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NIELSEN EQUIVALENCE IN MAPPING TORI OVER THE TORUS

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ABSTRACT. We use the geometry of the Farey graph to give an alternative proof of the fact that if $A \in GL_2\mathbb{Z}$ and if $G_A = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is generated by two elements, then there is a single Nielsen equivalence class of 2-element generating sets for G_A unless A is conjugate to $\pm\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, in which case there are two.

1. INTRODUCTION

Let G be a finitely generated group. Two ordered n-element generating sets S, T for G are *Nielsen equivalent* if the associated surjections $F_n \longrightarrow G$ differ by precomposition with a free group automorphism. This is equivalent to requiring that S, T are related by a sequence of *Nielsen moves*:

(1) if $a \neq b$ are generators, replace a with ab,
(2) if $a \neq b$ are generators, switch their places in the ordering,
(3) if a is a generator, replace it with a^{-1},

as the associated automorphisms generate $\mathrm{Aut}(F_n)$; see [5, Chap. I, Prop. 4.1].

In [4], Levitt–Metaftsis studied Nielsen equivalence within groups of the form $G_A = \mathbb{Z}^d \rtimes_A \mathbb{Z}$, where $A \in GL_d\mathbb{Z}$. Using the Cayley–Hamilton theorem, they show that G_A is 2-generated exactly when there is a vector $v \in \mathbb{Z}^d$ such that $\langle v, Av\rangle = \mathbb{Z}^d$. They also show that the number of Nielsen equivalence classes of 2-element generating sets is the index of $\langle A, -Id\rangle$ in its $GL_d\mathbb{Z}$-centralizer.

When $d = 2$, one can combine this with an observation of Cooper–Scharlemann [2, Lemma 5.1] to prove the following theorem.

Theorem 1.1. *If $A \in GL_2\mathbb{Z}$ and if $G_A = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is 2-generated, then there is a single Nielsen equivalence class of 2-element generating sets for G_A unless A is conjugate to $\pm\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, in which case there are two.*

Note that when $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, G_A is 2-generated, since $\langle \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)\rangle = \mathbb{Z}^2$.

Our goal here is not to prove anything new, but rather to understand how to prove Theorem 1.1 using the geometry of the *Farey graph* $\mathcal{F}$. Algebraically, vertices of $\mathcal{F}$ are primitive elements $v = (p, q) \in \mathbb{Z}^2$ up to negation, and vertices v, w are connected by an edge if together they generate $\mathbb{Z}^2$. Any matrix $A \in GL_2\mathbb{Z}$ acts on $\mathcal{F}$, and it turns out that Nielsen equivalence classes of 2-element generating sets of G_A correspond to geoesics in $\mathcal{F}$, on which A acts as a unit translation; see §2.

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Using this perspective, one can then prove Theorem 1.1 just using separation properties of geodesics in $\mathcal{F}$.

In [2], Cooper–Scharlemann were interested in an analogue of Theorem 1.1 in the world of Heegaard splittings. Recall that a closed surface S in a closed, orientable 3-manifold is a *Heegaard splitting* if $M \setminus H$ has two components, each of which are (open) handlebodies. They showed that there is a unique minimal genus Heegaard splitting of M_A up to isotopy unless A is conjugate to $\pm\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$, in which case there are two.

Any Heegaard splitting gives a pair of generating sets for $\pi_1 M$ just by taking free bases for the fundamental groups of the two handlebodies. These generating sets are well-defined up to Nielsen equivalence, and their Nielsen types certainly do not change if the Heegaard splitting S is isotoped in M. However, in general it is hard to say when a generating set for $\pi_1 M$ is "geometric", i.e., when its Nielsen class comes from a Heegaard splitting, and when two (say, nonisotopic) Heegaard splittings give the same Nielsen class; see, e.g., Johnson [3].

However, inspired by the fact that the Cooper–Scharlemann result also applies when the minimal genus of a Heegaard splitting is 3, we ask:

Question 1. *Is it true that if* $\operatorname{rank}(G_A) = 3$*, then there is a single Nielsen equivalence class of* 3*-element generating sets?*

Here, *rank* is the minimal size of a generating set. In [1], the author and Souto studied rank and Nielsen equivalence for mapping tori M_ϕ, where $\phi : S \longrightarrow S$ is a pseudo-Anosov homeomorphism of a closed orientable surface of genus $g \geq 2$. We showed that *as long as ϕ has large translation distance in the curve complex $C(S)$*, the group $\pi_1 M_\phi$ has rank $2g+1$ and all minimal size generating sets are Nielsen equivalent.

From above, when $A \in GL_2\mathbb{Z}$, the group G_A has rank 2 exactly when there was some $v \in \mathbb{Z}^d$ such that $\langle v, Av\rangle = \mathbb{Z}^d$. The Farey graph is the curve graph of T^2, and $\langle v, Av\rangle = \mathbb{Z}^d$ exactly when $v, Av \in \mathcal{F}$ are adjacent, so in the Euclidean setting the analogue of the rank part of our theorem in [1] still holds, and says that $\operatorname{rank}(G_A) = 3$ if the translation distance of A on $\mathcal{F}$ is at least two. The analogue of the Nielsen equivalence part is (a weaker version of) Question 1.

2. The proof

We will first show that for a general $A \in GL_2\mathbb{Z}$, there can be at most two Nielsen equivalence classes of 2-element generating sets for G_A. We will then show that the conjugates of $\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$ are the only A that realize this bound.

The beginning of this argument overlaps with that of Levitt–Metaftsis [4], so we will just outline it and give citations when necessary. Suppose that G_A is 2-generated. By [4, Proposition 4.1], every minimal size generating set for G_A is Nielsen equivalent to a generating set of the form

$$x = (v, 0)\,,\ y = (0, 1)\,,\ \text{where } v \in \mathbb{Z}^2.$$

Set $\mathcal{S}_A = \{v \in \mathbb{Z}^2 \mid \langle v, Av\rangle = \mathbb{Z}^2\}$. Again by [4, Proposition 4.1], if $v, v' \in \mathcal{S}_A$, then $\{(v,0),(0,1)\}$ and $\{(v',0),(0,1)\}$ are Nielsen equivalent if and only if v, v' lie in the same $\langle A\rangle \times \mathbb{Z}/2\mathbb{Z}$-orbit on $\mathcal{S}_A$, where $\mathbb{Z}/2\mathbb{Z}$ acts by$v \mapsto -v$.

We now reinterpret this in terms of the Farey graph $\mathcal{F}$. Recall from the Introduction that the vertex set of $\mathcal{F}$ consists of primitive elements of $\mathbb{Z}^2$ up to negation, so can be identified with $\mathbb{Q} \cup \{\infty\}$ through the map

$$\mathcal{F} \longrightarrow \mathbb{Q} \cup \{\infty\}, \quad \pm \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \frac{a}{b}.$$

Below, we will regard $\mathbb{Q} \cup \{\infty\}$ as a subset of $\mathbb{R} \cong \partial_\infty \mathbb{H}^2$, where $\mathbb{H}^2$ is considered in the upper half plane model, and we will identify edges of $\mathcal{F}$ with the corresponding geodesics in $\mathbb{H}^2$. (See Figures 1–3 below.) This embedding of $\mathcal{F}$ has some convenient properties. All edges of $\mathcal{F}$ separate $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$, and also $\mathcal{F}$, into two connected components. Every component of $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2 \setminus \mathcal{F}$ is an ideal hyperbolic triangle, which we will call a *complementary triangle* below. Finally, the action of $A \in GL_2\mathbb{Z}$ on $\mathcal{F}$ is the restriction of its action on $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ as a fractional linear transformation.

Returning to the proof, vertices $v, w \in \mathcal{F}$ are adjacent if $\langle v, w \rangle = \mathbb{Z}^2$, so $\mathcal{S}_A$ is exactly the set of vertices in $\mathcal{F}$ that A translates a distance of 1. Also, in the Farey graph we have identified primitive pairs up to negation, so the action of $\langle A \rangle \times \mathbb{Z}/2\mathbb{Z}$ on $\mathcal{S}_A$ is just the A-action on the corresponding set of vertices of $\mathcal{F}$. Define a *1-orbit* of $A \circlearrowright \mathcal{F}$ to be an orbit all of whose points are translated a distance of 1 by A. Theorem 1.1 then becomes the following lemma.

Lemma 2.1. *The action of $A \circlearrowright \mathcal{F}$ has a single 1-orbit unless A is conjugate to $\pm \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, in which case it has two.*

Fix a matrix $A \in GL_2\mathbb{Z}$ and let ℓ be a 1-orbit of A. Adding in edges connecting each $v \in \ell$ to Av, we will regard ℓ as an oriented path in $C(T^2)$. At each of its vertices v, a path ℓ has a *turning number*, whose absolute value is one more than the number of Farey graph edges that separate the two edges of ℓ incident to v. The turning number at v is positive if the turn is counterclockwise when ℓ is traversed positively, and negative when the turn is clockwise. (Remember that we are viewing $\mathcal{F}$ as a subset of the upper half plane in $\mathbb{R}^2$.) When $v = \infty$, the turning number is just $A(v) - A^{-1}(v)$. For instance, in Figure 2 all turning numbers on the red 1-orbit are 3, and on the blue 1-orbit they are -3.

When A is orientation preserving, all the turning numbers on a given 1-orbit coincide. On the other hand, if A is orientation reversing, then the turning numbers on a 1-orbit all have the same absolute value and alternate sign. As $GL_2\mathbb{Z}$ acts edge transitively on $\mathcal{F}$, any 1-orbit of A may be translated to pass through $\infty, 0$, which conjugates A so that it has the form

$$A = \begin{pmatrix} 0 & \epsilon \\ 1 & x \end{pmatrix}, \quad x \in \mathbb{Z}, \ \epsilon = \pm 1. \tag{1}$$

When A is as above, the turning number at 0 is $-\epsilon x$. Checking eigenvalues, two matrices $\left(\begin{smallmatrix} 0 & \epsilon_i \\ 1 & x_i \end{smallmatrix}\right)$, where $i = 1, 2$, are conjugate in $PGL_2\mathbb{Z}$ if and only if $\epsilon_1 = \epsilon_2$ and $|x_1| = |x_2|$. This implies that the turning numbers of all the 1-orbits of a matrix A have the same absolute value.

It suffices to prove the lemma when $A = \left(\begin{smallmatrix} 0 & \epsilon \\ 1 & x \end{smallmatrix}\right)$ as above. Here, the conjugacy classes of $\pm\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ correspond to the cases $\epsilon = -1,\ x = \pm 3$, so the goal is to prove that there are two 1-orbits in those cases, and one otherwise.

- If $x = 0$, then $A^2 = \pm 1$, and one can check directly that the only 1-orbit of A is the edge connecting $\infty, 0$.
- If $\epsilon = -1$ and $x = \pm 1$, then A is orientation preserving and $A^3 = \pm 1$. Each of its 1-orbits has turning number either 1 or -1, so bounds a complementary triangle in $\mathcal{F}$. But then A is a rotation around the barycenter of this triangle in $\mathbb{H}^2$, so this 1-orbit is the only one.
- If $\epsilon = -1$ and $x = \pm 2$, then A is parabolic. Its 1-orbit has turning number ± 2, so consists of all vertices in the $\mathcal{F}$-link of the fixed point of A.

When A is hyperbolic, its 1-orbits are simple, biinfinite paths in $\mathcal{F}$ that accumulate onto the attracting and repelling fixed points $\lambda_+(A), \lambda_-(A)$.

- If $\epsilon = 1$ and $|x| \geq 1$, then A is hyperbolic and orientation reversing. The turning numbers on a 1-orbit ℓ alternate sign, so there is an edge of ℓ that separates $\lambda_+(A)$ from $\lambda_-(A)$ in the upper half plane. Any other 1-orbit would then have to intersect ℓ, which is impossible, so A has a single 1-orbit. See Figure 1 for an illustration of the case $\epsilon = 1,\ x = 1$.
- If $\epsilon = -1$ and $x = \pm 3$, then A is orientation preserving, hyperbolic, and conjugate to $\pm\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. When $x = 3$, the orbits of -1 and 0 are distinct, since they have opposite turning numbers (see Figure 2). Since the edge from -1 to 0 in $\mathcal{F}$ separates the attracting and repelling fixed points of A, any 1-orbit of A must pass through either -1 or 0. So, the orbits of ∞ and -1 are the only 1-orbits. The argument when $x = -3$ is similar.

It remains to deal with the case $\epsilon = -1$, $|x| \geq 4$, in which case A is again orientation preserving and hyperbolic. We claim that any biinfinite path ℓ whose turning numbers are all at least 3 in absolute value is a geodesic in $\mathcal{F}$, and that if the turning numbers are all at least 4 in absolute value, then ℓ is the unique geodesic in $C(T^2)$ connecting its endpoints. This will imply that when $|x| \geq 4$, the matrix A has only a single 1-orbit.

So, suppose that $\ell = (v_i)$ is a biinfinite path in $\mathcal{F}$ whose turning numbers are all at least 3 in absolute value. For each i, let m_i be the edge of $\mathcal{F}$ incident to v_i that lies between the edges $[v_{i-1}, v_i]$ and $[v_i, v_{i+1}]$, and shares a complementary triangle of $\mathcal{F}$ with $[v_i, v_{i+1}]$, as in Figure 3. Each m_i separates m_{i-1} from m_{i+1}, so by planarity all the m_i are disjoint. Two vertices v_i and v_j, with $i < j$, are disjoint from and separated by all the edges

$$m_{i+1}, \ldots, m_{j-1}.$$

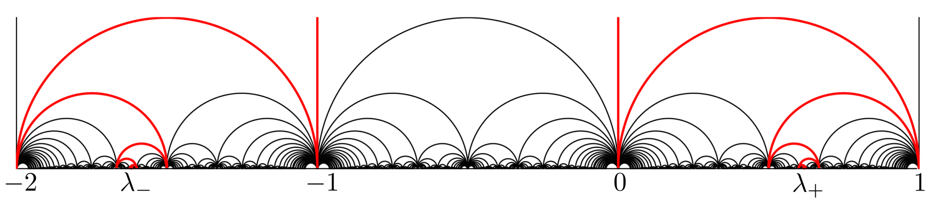

FIGURE 1. There is a single 1-orbit for the action $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \circlearrowright \mathcal{F}$, on which the turning numbers alternate between ± 1.

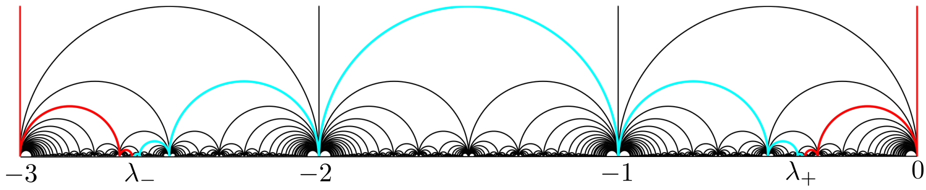

FIGURE 2. There are two orbits of $\left(\begin{smallmatrix}0&-1\\1&3\end{smallmatrix}\right) \circlearrowleft \mathcal{F}$, its *1-orbits*, on which the matrix acts as a translation by a distance of 1. The action is hyperbolic, with every forward orbit converging to $\lambda_+ \approx -0.38$ and every backwards orbit converging to $\lambda_- \approx -2.62$. Incidentally, the square of $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$ is a conjugate of $\left(\begin{smallmatrix}0&-1\\1&3\end{smallmatrix}\right)$, which is why the vertex set of the 1-orbit in Figure 1 is a translation of the union of the vertices of the two 1-orbits above.

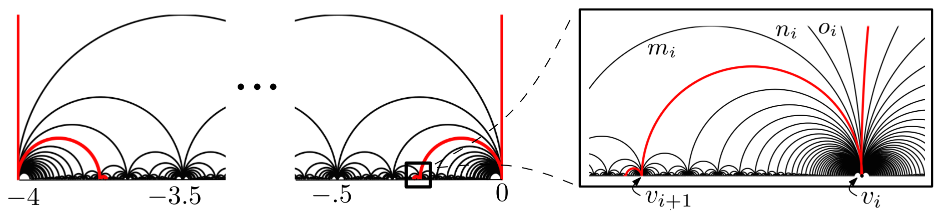

FIGURE 3. When $A = \left(\begin{smallmatrix}0&-1\\1&4\end{smallmatrix}\right)$, there is a single 1-orbit for the action $A \circlearrowleft \mathcal{F}$, which is the unique geodesic connecting the attracting and repelling fixed points of A in $\partial_\infty \mathbb{H}^2$.

Any path from v_i to v_j must go through all of these edges, so must have length at least $|i-j|$. Therefore, ℓ is a geodesic in $\mathcal{F}$.

Suppose now that all the turning numbers of $\ell = (v_i)$ are at least 4 in absolute value. Choose for each i two more edges n_i, o_i incident to v_i that lie between $[v_{i-1}, v_i]$ and $[v_i, v_{i+1}]$, as in Figure 3. All the edges m_i, n_i, o_i separate the forward and backward limits of ℓ, so any geodesic γ in $\mathcal{F}$ connecting these limits must pass through a vertex of each m_i, n_i, o_i. As γ cannot pass through all three of the non-v_i vertices of m_i, n_i, o_i, it must pass through v_i, so $\gamma = \ell$. Thus, ℓ is the unique geodesic in $\mathcal{F}$ connecting its endpoints. This concludes the proof of Lemma 2.1, and thus the proof of Theorem 1.1.

Remark 2.2. *The educated reader will note that some of the simple properties of $\mathcal{F}$ used above reflect (and probably inspired) deeper results about the curve complexes of higher genus surfaces. For instance, the argument used to prove that a path whose turning numbers are all at least* 3 *in absolute value is a geodesic is a simple version of Masur–Minsky's bounded geodesic image theorem* [6].

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