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METRICS ρ, QUASIMETRICS ρ^s AND PSEUDOMETRICS $\inf \rho^s$

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Abstract. Let ρ be a metric on a space X and let $s \geq 1$. The function $\rho^s(a, b) = \rho(a, b)^s$ is a quasimetric (it need not satisfy the triangle inequality). The function $\inf \rho^s(a, b)$ defined by the condition $\inf \rho^s(a, b) = \inf\{\sum_0^n \rho^s(z_i, z_{i+1})\ z_0 = a, z_n = b\}$ is a pseudometric (i.e., satisfies the triangle inequality but can be degenerate). We show how this degeneracy can be connected with the Hausdorff dimension of the space (X, ρ). We also give some examples showing how the topology of the space $(X, \inf \rho^s)$ can change as s changes.

Introduction

Let $\rho(x, y) : X \times X \to [0, \infty)$ be a metric in a space X and let $s > 0$. For $s > 1$ the function ρ^s may fail to be a metric because it need not satisfy the triangle inequality. However, it is a q-quasimetric due to fulfillment of the "q-triangle inequality" for $q = 2^{s-1}$:

Definition. Refer to a function $\rho : X \to [0, \infty)$ as q-quasimetric if the "q-triangle inequality" holds (see also [2], eq. (2.5)):

$$\forall x, y, z \in X \quad \rho(x, z) \leq q \cdot (\rho(x, y) + \rho(y, z)).$$

Such q-quasimetrics are a particular case of f-quasimetrics.

Definition. Refer to a function $\rho : X \to [0, \infty)$ as f-quasimetric if the "f-triangle inequality"

$$\rho(x, z) \leq f(\rho(x, y) + \rho(y, z)),$$

holds, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $f(t) \underset{t \to 0}{\to} f(0) = 0$.

The quantity $\max(\rho(x, y), \rho(y, z))$ is considered in some works instead of the sums but this difference is insignificant for our purposes.

The topology on an f-quasimetric space is defined in the same manner as on a metric space: a set U is regarded as open whenever, for each $x \in U$, there exists $\varepsilon > 0$ such that the ε-ball $B(x, \varepsilon) = \{y \mid \rho(x, y) < \varepsilon\}$ is included in U. The metrizability of the spaces with nondegenerate symmetric f-quasimetric was established by Chittenden [7].

Given an f-quasimetric ρ, denote by $\inf \rho(A, B)$ the infimum of the lengths of polygonal chains that connect two points $A, B \in (X, \rho)$. These polygonal chains are understood as ordered sets of points $\{A = z_0, \dots, z_n = B\} \subset (X, \rho)$; the length of a

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polygonal chain L is the sum of the lengths of its segments: $|L| = \sum_{i=1}^{n} \rho(z_{i-1}, z_i)$. Thus,

$$\inf \rho(A, B) = \inf\{\sum_{i=1}^{n} \rho(z_{i-1}, z_i) \mid n < \infty, z_0, \dots, z_n \in X, z_0 = A, z_n = B\}.$$

Definition. A pseudometric is a function ρ admitting $\rho(a, b) = 0$ for some $a \neq b$ but satisfying all the other properties of a metric.

The function $\inf \rho$ satisfies the conventional triangle inequality, that is, it is a pseudometric. In addition, $\inf \rho \leq \rho$, and $\inf \rho = \rho$ if and only if the symmetric f-quasimetric ρ is a metric. This construction was used for metrization by Birkhoff [4] in his proof of the metrizability of any Abelian topological group with a countable base of topology. Frink [9] used it for a proof of the metrizability of an f-quasimetric, which was simpler than in [7]. It is convenient to express some asymptotic properties of the metrics in the Gromov hyperbolic spaces ([5, 6, 13]) in terms of the behavior of the function $\inf \rho^s$ in its dependence on the parameter s.

The main objective of this paper is to present some examples concerning the dependence of the geometry and topology of the space $(X, \inf \rho^s)$ on the index s. Our interest in this question has arisen in connection with the work [13], where a rather subtle example is constructed of a space X with degenerate metric $\inf \rho^s$, $s > 1$. In particular, in Basic Example 1, we present a simpler construction. Next, we establish natural connections between the Hausdorff dimension of the space (X, ρ) and the condition of the nondegeneracy of the function $\inf \rho^s$. In particular, it turns out that, for quasiconformal curves G, the function $\inf \rho^s$ ($s \geq 1$) is degenerate if and only if the s-Hausdorff measure of the set G equals zero. In the last section, we construct a set $X \subset \mathbb{R}^2$, on which there is a continuum of pairwise different topologies induced by the metrics $\inf d^s$, $s \geq 1$.

1. For some s, $\inf \rho^s$ is degenerate, for others, nondegenerate

In what follows, if $G \subset (X, \rho)$, then $\inf_G \rho$ stands for the metric of the space $(G, \inf \rho)$. Note that the function $\inf_G \rho$ does not coincide with the restriction to $G \times G$ of the function $\inf \rho : X \times X \to \mathbb{R}_+$. The symbol d denotes the Euclidean metric.

Basic Example 1 ([3]; see also [14], Example 2.2 for $s = 2$)**.** Consider the segment $G = [0, 1]$. Then

$$\inf_G d^s = \begin{cases} d^s \text{ (the standard topology)}, & 0 < s \leq 1, \\ 0 \text{ (the trivial topology)}, & s > 1. \end{cases}$$

Indeed, if $0 < s \leq 1$, then the function d^s satisfies the triangle inequality, i.e., is a metric itself and $\inf_G d^s = d^s$. Suppose that $s > 1$. Let $N \in \mathbb{N}$. Partition the segment G into equal segments $I_1, \dots, I_N$, the d^s-length of each segment being equal to N^{-s}. Thus, the d^s-length of the entire polygonal chain (that is, the sum of the sth powers of the lengths of all its segments) is equal to N^{1-s}. If N is large, then the d^s-length of the entire polygonal chain is small. □

It is easy to see that, instead of a segment, one can take any rectifiable curve in a metric space as well as a dense set on such curve.

However, there are curves for which the metric $\inf_G d^s$ is nondegenerate for some $s > 1$. Informally speaking, such curves (1) must have as many turns as possible; (2) should not have too many narrow bridges.

These properties are possessed, for example, by the quasiconformal curves whose Hausdorff dimension is greater than one. More specifically, the condition of large dimension is geometrically responsible for condition (1), whereas the quasiconformality condition is responsible for condition (2).

A curve G is called quasiconformal if there exists a neighborhood $I \subset U \subset \mathbb{R}^2$ (I is a segment or a circle) and a quasiconformal map $\gamma : U \to \mathbb{R}^2$ such that $G = \gamma(I)$.

Recall also the definition of the Hausdorff dimension. Let G be a metric space. Then the Hausdorff dimension $\dim_H(G)$ is the infimum of the numbers $s > 0$ such that the Hausdorff s-measure $\mu_s(G)$ is zero. The condition $\mu_s(G) = 0$ means that the set G can be covered by subsets $U_1, \dots, U_n, \dots$, where the sum $\sum(\operatorname{diam} U_i)^s$ can be made arbitrarily small.

Theorem 1. *Let $s \geq 1$ and let (X, ρ) be a metric space.*

1. If $G \subset (X, \rho)$ is a connected compact set and the s-Hausdorff measure $\mu_s(G) = 0$, then the function $\inf_G \rho$ is degenerate.

2. If $G \subset (X, \rho)$ is a curve with endpoints A, B satisfying the condition

$$\text{(1)} \qquad \textit{There exists } C < \infty \textit{ such that } \operatorname{diam} G[x, y] \leq C\rho(x, y) \textit{ for all } x, y \in G$$

(here $G[x, y]$ stands for the arc G between x and y), then $\mu_s(G) = 0$ if and only if $\inf \rho^s(A, B) = 0$.

3. For each $s < 2$, there exist quasiconformal curves $G \subset \mathbb{R}^2$ with nondegenerate metric $\inf_G d^s$.

Proof. 1. Let $\varepsilon > 0$. It follows from the condition $\mu_s(G) = 0$ that there exists a covering of G by sets $U_1, \dots, U_n, \dots$ such that

$$\sum(\operatorname{diam} U_i)^s < \varepsilon. \tag{2}$$

It is possible to expand these sets to open ones, preserving inequality (2), and then to proceed to a finite subcovering $G = U_1 \cup U_2 \cup \dots \cup U_N$. Suppose that $A, B \in G$, $A \in U_i, B \in U_j$ (possibly, $i = j$). The open sets U_i and U_j can be joined by a chain of successively intersecting subsets from the system $\{U_1, \dots, U_n\}$, $n \leq N$: this can be easily deduced from the connectedness of G. Having numbered the sets U_i in accordance with the order of appearance in this chain, choose points

$$A = z_0 \in U_{i_1},\ z_1 \in U_{i_1} \cap U_{i_2}, \dots, z_{k-1} \in U_{i_{k-1}} \cap U_{i_k},\ z_k = B \in U_{i_k}.$$

The distance between the neighboring points z_i, z_{i+1} does not exceed the diameter of the corresponding set U_{i+1}. Therefore, the ρ^s-length of the corresponding polygonal chain does not exceed the sum (2), that is, it is less than ε. In view of the arbitrariness of the choice of ε, we infer that $\inf \rho^s(A, B) = 0$. The first claim of the theorem is proved.

2. Let $\varepsilon > 0$. It follows from the equality $\inf_G \rho^s(A, B) = 0$ that there are points $A = z_0, z_1, z_2, \dots, z_N = B$ on the curve G such that

$$\sum_{i=0}^{N-1} d(z_i, z_{i+1})^s \leq \frac{\varepsilon}{C^s}. \tag{3}$$

In accordance with condition (1), for every $i \in \{0, \dots, N\}$, we have: $\operatorname{diam} G[z_i, z_{i+1}] \leq C \cdot d(z_i, z_{i+1})$. The points z_i are not necessarily ordered along the curve, but nevertheless the union of the arcs $G[z_i, z_{i+1}]$ covers the whole G. From (3) we obtain

$$\sum_{i=0}^{N-1} (\operatorname{diam} G[z_i, z_{i+1}])^s \le \varepsilon.$$

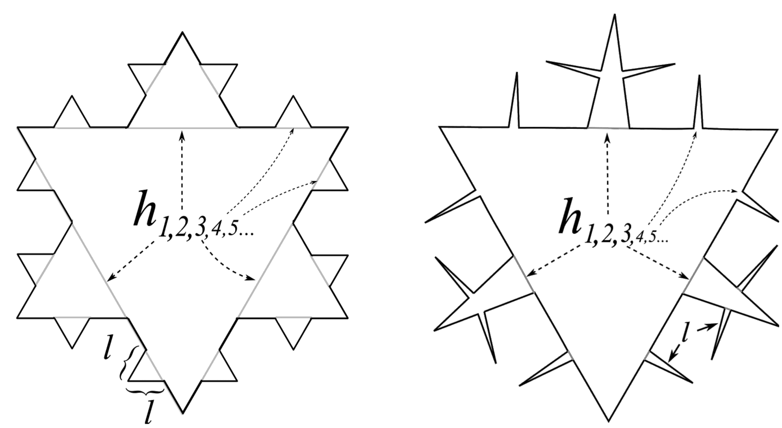
Thus, the s-Hausdorff measure of G is less than ε. Since ε is arbitrarily small, $\mu_s(G) = 0$. The implication $\mu_s(G) = 0 \ \Rightarrow\ \inf \rho^s(A, B) = 0$ follows from item 1.

3. The condition of the quasiconformality of the curve $G \subset \mathbb{R}^2$ is equivalent to the so-called Ahlfors M-condition ([1, 12]): there exists $M < \infty$ such that

$$|x - z| + |z - y| \le M|x - y| \text{ for all } x, y \in G \text{ and } z \in G[x, y].$$

Clearly, this condition is equivalent to condition (1). It is known (Ponomarëv, [11]) that, for any $s < 2$, on the plane, there exist self-similar quasiconformal curves whose Hausdorff dimension is greater than s. On the other hand, the Hausdorff dimension of G is $\inf\{s \mid \mu_s(G) = 0\}$. The rest follows from item 2. The theorem is proved. $\square$

Examples 2, 3. The standard self-similar Koch snowflake (see Figure 1a) $*$ is quasiconformal, has Hausdorff dimension $s = \log_3(4)$ and positive Hausdorff s-measure. Theorem 1 implies that the metric $\inf_* d^s$ is nondegenerate. It follows from Lemma 1 formulated below that if we construct a snowflake G so that the lengths of the bridges h_i taken away at steps $i = 1, 2, 3, \ldots$ decrease rapidly enough (see Figure 1b), then the metric $\inf_G d^s$ may be degenerate also for $s < \dim_H G$. For example, if the sum of lengths of the bridges is finite, then the function $\inf_G d^s$ is degenerate for all $s > 1$. If, in addition, the heights of the isosceles triangles constructed on the bridges decrease slowly enough, then the Hausdorff dimension of G can be made arbitrarily close to two. Such a curve is not quasiconformal because it does not satisfy the Ahlfors M-condition.



(a) (b)

FIGURE 1

Lemma 1. *Let $s > 1$ and G be as in Figure* 1b. *If the sum of the lengths of the bridges satisfies the relation $\sum |h_i|^s < \infty$, then $\inf_G d^s$ is degenerate (regardless of the lengths of the sides $\{l_i\}$ being added).*

Proof. Let $A, B \in G$. The series $\sum_{i=1}^{\infty} |h_i|^s$ is convergent. Let $\varepsilon > 0$. Consider $N = N(\varepsilon)$ such that $\sum_{i>N}^{\infty} |h_i|^s < \varepsilon$. Consider the set G_N that is obtained by returning all bridges with the numbers larger than N to G. The points A, B can be joined by a rectifiable polygonal chain in G_N. Inscribe a polygonal chain L of an arbitrary small s-length in this chain (as in Basic Example 1):

$$L = \{A = z_0, z_1, z_2, \dots, z_{n+1} = B\},\ z_i \in G_N,\ \sum_{i=0}^{n} |z_i - z_{i+1}|^s < \varepsilon.$$

Move the vertices of this polygonal chain that are inside the bridges ($z_i \in G_N \backslash G$) by shifting them to one of the endpoints $z_i' \in G$ of these bridges. The other vertices will be just renamed: $z_j' = z_j \in G$. We obtain a closed polygonal chain L' whose all vertices z_i' lie in G, and, in addition, by Minkowski's inequality,

$$\sum_{i=0}^{n} |z_i' - z_{i+1}'|^s \le \sum_{i=0}^{n} (|z_i' - z_i| + |z_i - z_{i+1}| + |z_{i+1} - z_{i+1}'|)^s < 3^s \varepsilon.$$

The rest is obvious. The lemma is proved. □

Example 4. In an infinite-dimensional Hilbert space, there exists a curve Γ with the following property: For any three successive points A, B, C, the angle ABC is right (see Halmos, [10], Problem 4). It can be easily deduced from the Pythagorean theorem that the d^2-length of any polygonal chain inscribed in this curve is equal to the squared distance between its endpoints. It follows that the function $\inf_\Gamma d^2$ coincides with the function d^2 and hence the latter is a nondegenerate distance.

Here is a convenient parametrization γ of this curve: $t \mapsto \xi_{[0,t]}$, where $t \in [0,1]$ and $\xi_{[0,t]} \in L_2[0,1]$ is the characteristic function of the segment $[0,t]$. Namely, $\gamma(t)(x) = \xi_{[0,t]}(x) = \begin{cases} 1, & 0 \le x \le t, \\ 0, & t < x \le 1. \end{cases}$

If we consider the last curve Γ in the space $L_p[0,1]$, then for $p \in [1, \infty)$ the metrics d^p and $\inf_\Gamma d^p$ coincide. Moreover, the curve (Γ, d^p) is isometric to the segment $[0,1]$, while the Hausdorff dimension of the curve $\Gamma \subset L_p[0,1]$ is equal to p. Note that for $p = \infty$ the set Γ becomes discrete, and its points are pairwise separated by the distance 1.

2. The metrics $(X, \inf d^s)$ are nondegenerate, but the topology depends on s

In the previous section, we showed that there exist compact arcs G on the plane with nondegenerate metric $\inf_G d^s$ for $s > 1$. Due to the compactness of G, this metric defines the standard topology on this arc. In general, if an f-quasimetric space (X, ρ) is compact and the metric $\inf \rho$ is nondegenerate, then the space $(X, \inf \rho)$ is Hausdorff and, by the Weierstrass theorem (see, for example, [8], theorem 3.1.13), the identity map $Id : (X, \rho) \to (X, \inf \rho)$ is a homeomorphism.

The spaces that change the topology under the transition $d^s \mapsto \inf d^s$ will be sought for among disconnected noncompact sets on the plane.

A particular case of the next example can be found in [3]. Here the space (X, d^s) is not compact, whereas the space $(X, \inf d^s)$ is compact.

Example 5. Let (r, φ) be the polar coordinates on the plane and let S^1 be the unit circle $r = 1$. Consider a sequence z_n on the plane "densely winding up" on S^1. More precisely, let $z_n = (r_n, \varphi_n)$, where the radii r_n tend to one, whereas the

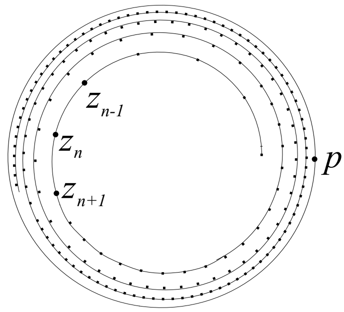


FIGURE 2

set of arguments $\varphi_n \mod 2\pi$ is everywhere dense in S^1. Let X be the union of the set $\{z_n\}$ with a certain point $p \in S^1$. Let us show that if $s > 1$, then the space $(X, \inf d^s)$ has the topological type of a sequence converging to p, and hence this space is compact.

Let $N < \infty$. If n and m are large, then z_m and z_n lie near the limiting circle, and they can be joined by an N-segment polygonal chain, in which the d-lengths of the segments are of order $O(\frac{1}{N})$ (compare with Basic Example 1). Then the d^s-length of this polygonal chain is at most $N \cdot (O(\frac{1}{N})^s)$. The last expression is small for large N. Thus, $\{z_n\}$ is a Cauchy sequence in the metric $\inf d^s$ for $s > 1$. Now it is easy to understand that p is the limit of this sequence in the metric $\inf d^s$ (for instance, because there is a subsequence $z_{n_k} \to p$).

Example 6 (A comb)**.** Let us construct a set $X \subset \mathbb{R}^2$ on which there is a continuum of pairwise different topologies induced by the metrics $\inf d^s$, $s \geq 1$. For each $n = 0, 1, 2, \ldots$, consider a dashed ray $L_n = \{A_{n0}, A_{n1}, \ldots\}$ in $\mathbb{R}^2$, which goes upward from the point $A_n = A_{n0} = (n, 0) \in \mathbb{R}^2$,

$$A_{nk} = (n, k \cdot 2^{-n}) \in \mathbb{R}^2, k = 0, 1, 2, \ldots$$

(each next ray is two times "denser" than the previous one). The base A_n of each ray L_n is joined with the point $z = (0, -1)$ by a dashed segment R_n consisting of 2^n equal parts. Define

$$X = \{z\} \cup \bigcup_{n \geq 1} (L_n \cup R_n).$$

All the points of X, except for z, are isolated.

Theorem 2. *Let $\alpha > 1$. Put $z_n = (n, [2^{(\alpha-1)n}])$ (here the square brackets denote the integer part of a number). The sequence z_n converges to z in the metric $\inf d^s$ if and only if $s > \alpha$. Therefore, for different s, the topologies defined by the metric $\inf d^s$ on X are different: if $1 \leq s < \tilde{s}$, then the identity map $Id : (X, \inf d^s) \to (X, \inf d^{\tilde{s}})$ is continuous but the inverse map is discontinuous.*

Proof. Let us first show that, for each $s > 1$, the distance $\inf d^s(z, A_n)$ converges to zero as $n \to \infty$. The usual length of the segment R_n does not exceed $2n$; therefore, the d^s-length of each of its 2^n parts does not exceed $(\frac{2n}{2^n})^s$. Thus,

$$\inf d^s(A_n, z) \leq 2^n \cdot \left(\frac{2n}{2^n}\right)^s \to 0.$$

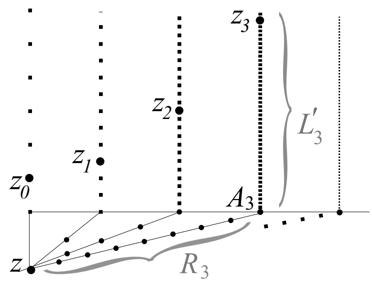


FIGURE 3

Denote by $L'_n = \{A_n, A_{n1}, A_{n2}, \dots, z_n\}$ the vertical polygonal path connecting A_n and z_n. One can go from the point z_n to z by moving the first top down along this polygonal path and then to the left along the shallow descent R_n.

Let us estimate the d^s-length of the polygonal chain L'_n. The number of segments on it equals $[2^{(\alpha-1)n}]\cdot 2^n$; this number lies between $\frac{2^{\alpha n}}{2}$ and $2^{\alpha n}$. Multiplying these numbers by the d^s-length of an individual segment, equal to $(2^{-n})^s$, we obtain:

$$\frac{2^{n(\alpha-s)}}{2} \le |L'_n| \le 2^{n(\alpha-s)}. \tag{4}$$

Hence, the d^s-length $|L'_n|$ converges to zero if and only if $s > \alpha$.

In view of the triangle inequality and the fact that $\inf d^s(z, A_n) \to 0$, it suffices to show that $\inf d^s(A_n, z_n) \to 0$ if and only if $s > \alpha$.

If $s > \alpha$, then the distance $\inf d^s(A_n, z_n)$ converges to zero as $n \to \infty$ since it does not exceed the d^s-length of the polygonal chains L'_n, while these lengths converge to zero by (4). If $s \le \alpha$, then one can show that $\inf d^s(A_n, z_n) \ge \frac{1}{2}$ for large n. To demonstrate this, suppose that $\inf d^s(A_n, z_n) < 1$ (otherwise, there is nothing to prove). It is clear that, in this case, the shortest path from A_n to z_n passes through all the points of the segment L'_n and the inequality $\inf d^s(A_n, z_n) \ge \frac{1}{2}$ also follows from (4). The rest of the theorem is obvious. □

Thus, the topology of $(X, \inf d^s)$ becomes "strictly weaker" as s grows. It is somewhat unexpected that for different s the spaces $(X, \inf d^s)$ are homeomorphic to one another (understandably, the homeomorphisms are not the identity map). Let us demonstrate this, but first we introduce a technical definition:

Definition. A metric space X has property (*) if it is countable, all the points except for one point z are isolated, whereas the point z has a base of neighborhoods $X = V_0 \supset V_1 \supset V_2 \supset \dots$ such that the set $F_n = V_n \setminus V_{n+1}$ is infinite for each n (the presence of this base is equivalent to the local noncompactness of X).

Lemma 2. *The spaces with property (*) are homeomorphic.*

Proof. Let X and $\tilde{X}$ be two spaces possessing property (*). Let F_n and $\tilde{F}_n$ be infinite sets as above. For each n, there is a bijection $\varphi_n : F_n \to \tilde{F}_n$. It is continuous since both sets are discrete. Define the map $\varphi : X \to \tilde{X}$ by setting $\varphi(z) = \tilde{z}$ and $\varphi(x) = \varphi_n(x)$ for each point $x \in F_n$. It is easy to see that this map is a homeomorphism. The lemma is proved. □

Let us show that all $\inf d^s$-topologies on the comb have property (*).

For $s=1$, this is obvious. Let $s>1$. We will prove that if $1>r_1>r_2$, then the ball $B(z,r_1)\subset (X,\inf d^s)$ contains infinitely many points not belonging to the ball $B(z,r_2)$. Take n so large that $r_2+2^{-ns}<r_1$ and $\inf d^s(z,A_n)<r_2$. On the ray $L_n=\{A_n,A_{n1},A_{n2}\ldots\}$, there is a point A_{nk} with

$$r_2<\inf d^s(z,A_{nk})\le r_2+2^{-ns}<r_1. \tag{5}$$

Such point exists on each ray with sufficiently large number n. Thus, there are infinitely many points satisfying inequality (5), and the balls of the radii $r_1>r_2>\ldots$ form a base having property (*).

Remarks. The space X of Example 5 with the conventional metric also has property (*) and hence all "combs" of Example 6 are homeomorphic to it regardless of the number $s\ge 1$. Moreover, if we "break the teeth of the comb" above the points z_n, that is, eliminate all points A_{nk} for $k>[2^{(\alpha-1)n}]$ from the set X of Example 6, then the remaining set X_α is compact in the metric $\inf d^s$ if and only if $s>\alpha$.

If we do not require the embeddability of the initial space (X,ρ) into $\mathbb{R}^n$, then the idea of a "comb" can be carried out more transparently by taking as X a wedge sum of discrete lengthening segments.

Example 7 (A hedgehog)**.** Let $X_n=\{0,\frac{1}{n},\frac{2}{n},\ldots,\frac{n^2-1}{n},n\}$, $n=1,2,\ldots$. Denote the elements of X_n by $a_{n0},a_{n1},\ldots,a_{n,n^2}$. The space X is obtained by taking the disjoint union of all the X_n's and identifying them at their common point 0. We endow the hedgehog X with the conventional metric:

$$\rho(a_{ni},a_{nj})=|a_{ni}-a_{nj}|,\ \rho(a_{ni},a_{mj})=a_{ni}+a_{mj}$$

for $n\ne m$.

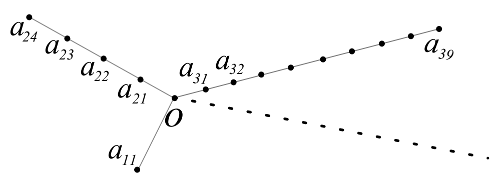


FIGURE 4

As in Theorem 2, for different s, $1\le s\le 2$, the topologies defined by the metric $\inf\rho^s$ on X are different. Namely, if $1<p<2$ and $k_n=[n^p]$, then the sequence a_{n,k_n} converges to 0 in $(X,\inf\rho^s)$ iff $s>p$ (cf. formula (4)). For example,

$$\lim_{n\to\infty}\inf\rho^s(0,a_{n,n^2})=\lim_{n\to\infty}n^2\cdot\left(\frac{1}{n}\right)^s=\begin{cases}0 & s>2,\\ 1 & s=2,\\ \infty & s<2.\end{cases}$$

The space $(X,\inf\rho^s)$ is compact for $s>2$. If we lengthen the "prickles" of the "hedgehog" so that the nth "prickle" contains n^n segments and not n^2, then the $\inf\rho^s$-topologies are different for all $s<\infty$ and not only for $s\le 2$. In addition, by the previous remark, all our noncompact hedgehogs are homeomorphic to each other.

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