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A COMPACTIFICATION OF THE MODULI SPACE OF SELF-MAPS OF $\mathbb{CP}^1$ VIA STABLE MAPS

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ABSTRACT. We present a new compactification $M(d, n)$ of the moduli space of self-maps of $\mathbb{CP}^1$ of degree d with n markings. It is constructed via GIT from the stable maps moduli space $\overline{M}_{0,n}(\mathbb{CP}^1 \times \mathbb{CP}^1, (1, d))$. We show that it is the coarse moduli space of a smooth Deligne-Mumford stack and we compute its rational Picard group.

CONTENTS

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1. Introduction

1.1. **The moduli space of self-maps.** Self-maps of the projective line constitute a rich subject with connections to many branches of mathematics, including, among others, complex and arithmetic dynamics ([Mil06], [Sil07]), Hurwitz theory ([Hur01]) and enumerative geometry ([OP06]).

Recall that a degree d morphism $\varphi : \mathbb{CP}^1 \to \mathbb{CP}^1$ is given by two homogeneous polynomials of degree d with no common zeros. A natural parameter space is thus the complement Rat_d of the resultant hypersurface in $\mathbb{CP}^{2d+1}$. The automorphism group $\mathrm{Aut}(\mathbb{CP}^1) = \mathrm{PGL}_2(\mathbb{C})$ acts on Rat_d by conjugation and we can form the GIT quotient $M_d = \mathrm{Rat}_d /\!\!/ \mathrm{PGL}_2(\mathbb{C})$. This quotient M_d is a moduli space of degree d self-maps $C \to C$ of curves C (abstractly) isomorphic to $\mathbb{CP}^1$.

To study how such maps degenerate in families, when the two defining polynomials obtain common zeroes, it is necessary to compactify M_d. In [Sil98], Silverman obtains a compactification M_d^{ss} of M_d as a GIT quotient of $\mathbb{CP}^{2d+1}$ by $\mathrm{PGL}_2(\mathbb{C})$. The boundary consists of self-maps $\tilde{\varphi} : C \to C$ of curves $C \cong \mathbb{CP}^1$ with degree $d' < d$ together with an effective divisor $D = \sum_i k_i[q_i]$ on C of degree $d - d'$, where we ask that $k_i \leq (d+1)/2$ and, moreover, $k_i \leq (d-1)/2$ if q_i is a fixed point of $\tilde{\varphi}$. Such a point $(\tilde{\varphi}, D)$ is the limit of a family of self-maps $\varphi : C \to C$ where the defining polynomials acquire common zeroes q_i of multiplicities k_i.

1.2. **A new compactification.** Building on the work of Silverman, we construct a different compactification $M(d,0)$ (for $d \geq 2$ even), where we allow C to degenerate into a nodal curve. More precisely, the points of $M(d,0)$ correspond to stable self-maps $\varphi : C \to C$, where C is a connected genus 0 curve with at worst nodal singularities, such that φ has image in a unique irreducible component $C_0 \subset C$ and the total degree of φ (summed over the components of the domain) is d. Here φ is called stable if

- all components $C_i \neq C_0$ of C map with degree at most $(d+1)/2$. If in addition C_i meets C_0 and $C_i \cap C_0$ is a fixed point of $\varphi|_{C_0}$, we require the degree of φ on C_i to be at most $(d-1)/2$;
- if a component $C_i \neq C_0$ of C is contracted by φ, it contains at least three nodes.

By allowing the curve C to have n marked smooth points $p_1, \ldots, p_n$, we define the space $M(d,n)$ similarly.

To construct $M(d,n)$, we consider the stable maps moduli space

$$Y_{d,n} = \overline{M}_{0,n}(\mathbb{CP}^1 \times \mathbb{CP}^1, (1,d)).$$

It parametrizes maps

$$f = (\pi, \varphi) : (C; p_1, \ldots, p_n) \to \mathbb{CP}^1 \times \mathbb{CP}^1$$

of degree $(1,d)$ from an n-marked at worst nodal genus 0 curve C, such that each component of C contracted by f contains at least three special points, i.e., nodes or markings. Observe that there is exactly one component C_0 of C on which the map π has degree 1, so restricted to C_0 it gives an isomorphism $C_0 \cong \mathbb{CP}^1$. The inverse of this isomorphism gives a closed embedding $\pi^{-1} : \mathbb{CP}^1 \hookrightarrow C$ with image C_0.

The data of f is equivalent to a degree d self-map

$$C \xrightarrow{f} \mathbb{CP}^1 \times \mathbb{CP}^1 \xrightarrow{\mathrm{pr}_2} \mathbb{CP}^1 \xrightarrow{\pi^{-1}} C_0 \subset C \tag{1}$$

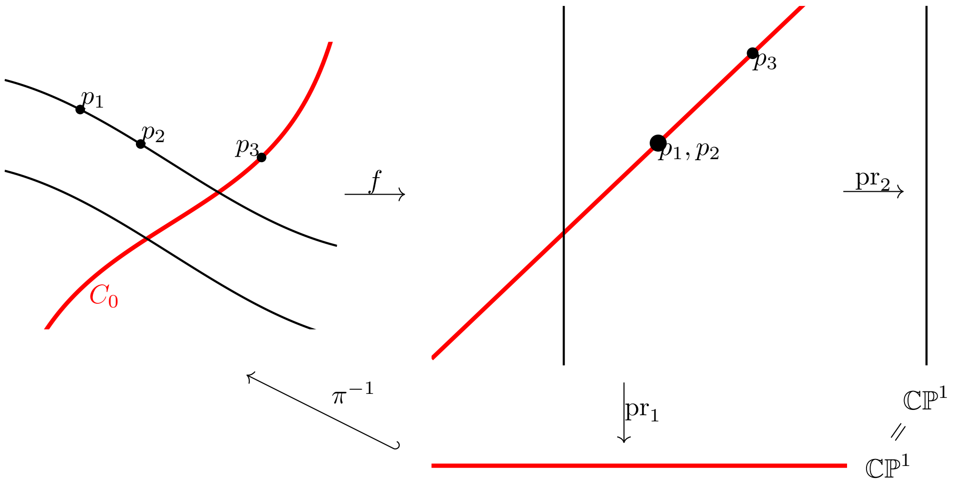

FIGURE 1. A parametrized graph in $Y_{2,3}$ with contracted component.

of C with image contained in a single component C_0, together with an isomorphism $\pi : C_0 \xrightarrow{\sim} \mathbb{CP}^1$. In Figure 1 we illustrate how to obtain the self-map of C from the map f. Note that when C is irreducible, we have $C = C_0$ and this identifies $\mathrm{Rat}_d \subset Y_{d,0}$ as the open subset of points f with smooth source curve. We can thus see $Y_{d,0}$ as a compactification of Rat_d.

In order to forget the additional information of the isomorphism $\pi : C_0 \to \mathbb{CP}^1$ above to obtain the space $M(d,n)$, we let $\mathrm{PGL}_2(\mathbb{C})$ act on $Y_{d,n}$ by postcomposing f with $\psi \in \mathrm{PGL}_2(\mathbb{C})$ acting diagonally on $\mathbb{CP}^1 \times \mathbb{CP}^1$. As

$$(\psi \circ \pi)^{-1} \circ (\psi \circ \varphi) = \pi^{-1} \circ \psi^{-1} \circ \psi \circ \varphi = \pi^{-1} \circ \varphi,$$

the self-map $C \to C$ in (1) is invariant under this action. The orbits of $\mathrm{PGL}_2(\mathbb{C})$ parametrize exactly all possible ways to choose the identification $\pi : C_0 \xrightarrow{\sim} \mathbb{CP}^1$. Then we can take the quotient using GIT.

A priori, for technical reasons, this GIT quotient only works well for d even (see Lemma 2.6 and Corollary 2.2). However, if we allow ourselves to put nonnegative rational weights d_i on the points p_i, which affect the GIT stability condition, we are able to define quotient spaces $M(d|d_1, \ldots, d_n)$ also in the case d odd (provided we have at least one marking). More precisely, we ask that there exists $k \in \mathbb{Z}_{\geq 1}$ such that the numbers $\tilde{d}_i = kd_i$ are integers and such that $k(d+1) + \sum_i \tilde{d}_i$ is odd. The corresponding tuples $(d|d_1, \ldots, d_n)$ are called admissible. When d is even and all $d_i = 0$, we recover the previous description of $M(d,n)$.

Theorem (see Corollary 2.11)**.** *For $(d|d_1, \ldots, d_n)$ admissible, there exists a GIT quotient $M(d|d_1, \ldots, d_n)$ of $Y_{d,n}$ by $PGL_2(\mathbb{C})$ which is a normal projective variety.*

For later use we mention here that the natural map $Y_{d,n} \to \overline{M}_{0,n}$ forgetting the map f and stabilizing the curve $(C; p_1, \ldots, p_n)$ is $\mathrm{PGL}_2(\mathbb{C})$-invariant and thus induces forgetful maps $M(d|d_1, \ldots, d_n) \to \overline{M}_{0,n}$.

It is a natural question to ask how the new compactification $M(d,n)$ compares to Silverman's compactification M_d^{ss}. From our construction it follows that there is a natural map $\tilde{j} : M(d,n) \to M_d^{ss}$ (for $d \geq 2$ even, $n \geq 0$), which is an isomorphism over $M_d \subset M_d^{ss}$ for $n = 0$ (see Corollary 2.8).

Let us point out some differences between the compactifications $M(d,0)$ and M_d^{ss} of M_d. When a map $\varphi \in M_d$ degenerates, instead of introducing a base point q_i and recording its multiplicity k_i as in M_d^{ss}, we insert a new component C' of C over the point q_i together with a map $C' \to C$ of degree k_i. Thus in this map $C' \to C$ we can hope to preserve information about the behaviour of φ around q_i as it degenerates. Also note that the locus of maps $\tilde{\varphi}$ in M_d^{ss} with a base point of multiplicity k has codimension $2k-1$. On the other hand, the corresponding locus of $\varphi : C \to C$ in $M(d,0)$ having a component $C' \neq C_0$ with degree k under φ is a divisor. Thus it seems easier to study such degenerations in $M(d,0)$.

1.3. **Modular interpretation.** We can also give a rigorous modular interpretation for $M(d|d_1,\ldots,d_n)$. It is the coarse moduli space of a smooth Deligne-Mumford stack $\mathcal{M}(d|d_1,\ldots,d_n)$, which is an open substack of the quotient stack $\overline{\mathcal{M}}_{0,n}(\mathbb{CP}^1 \times \mathbb{CP}^1,(1,d))/\mathrm{PGL}_2(\mathbb{C})$. While every single geometric point of this stack can be interpreted as a self-map of an n-pointed nodal curve C as described above, this does not generalize well to self-maps of families of nodal curves. Instead, it is necessary to work with two different families of curves.

Theorem (see Theorem 3.8)**.** *Given a scheme S, the objects of the stack $\overline{\mathcal{M}}_{0,n}(\mathbb{CP}^1 \times \mathbb{CP}^1,(1,d))/PGL_2(\mathbb{C})$ over S are diagrams*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\varphi"'] \arrow[d, shift right, "\pi"'] & \tilde{\mathcal{C}} \arrow[dl, "\tilde{\pi}"] \\ S, \arrow[u, shift right, "\sigma_i"'] & \end{tikzcd}$$

where $\pi,\tilde{\pi}$ are flat, projective families of quasi-stable genus 0 curves, all geometric fibres of $\tilde{\pi}$ are isomorphic to $\mathbb{CP}^1$ and the maps μ,φ satisfy $\mu_*([\mathcal{C}_s]) = [\tilde{\mathcal{C}}_s]$ and $\varphi_*([\mathcal{C}_s]) = d[\tilde{\mathcal{C}}_s]$ for all geometric points $s \in S$ (plus the stable maps condition for the map $(\mu_s,\phi_s) : (\mathcal{C}_s;\sigma_1(s),\ldots,\sigma_n(s)) \to \tilde{\mathcal{C}}_s \times \tilde{\mathcal{C}}_s$).

Over the points s with $\mathcal{C}_s \cong \mathbb{CP}^1$, the morphism $\mu^{-1} \circ \phi$ becomes a self-map of the family $\mathcal{C}$ of degree d as expected. While it might seem that the use of two different families of curves no longer allows an interpretation as a self-map, the second family $\tilde{C}$ will be an important ingredient in defining self-composition and iteration maps on $M(d|d_1,\ldots,d_n)$ in [Sch].

1.4. **General properties and the Picard group of $M(d|d_1,\ldots,d_n)$.** From a result of Levy ([Lev11]), we conclude that the space $M(d|d_1,\ldots,d_n)$ is rational. As it is the coarse moduli space of smooth Deligne-Mumford stack, it has finite quotient singularities. This allows us to compute its rational Picard group.

Theorem (see Theorem 4.4, Theorem 4.9 and Corollary 4.17)**.** *For $d \geq 2$ even and $n \geq 0$, the rational Picard group of $M(d,n)$ is generated by boundary divisors together with the divisor class $\mathcal{H}$ descending from the evaluation class $ev_1^*\mathcal{O}_{\mathbb{CP}^1\times\mathbb{CP}^1}(1,0)$ on $Y_{d,n}$ for $n = 1,2$. The relations between these generators are generated by the* *pullbacks of the relations*

$$\sum_{\substack{i,j\in A\\ k,l\in B}} D(A;B) = \sum_{\substack{i,k\in A\\ j,l\in B}} D(A;B)$$

between boundary divisors in $\overline{M}_{0,n}$ *(for* $\{i,j,k,l\} \subset \{1,\ldots,n\}$*) under the forgetful map* $M(d,n) \to \overline{M}_{0,n}$.

This solves the case when all $d_i = 0$. If some of them are not, we have for $d = 0$ an additional generator $\mathcal{G}$ descending from $\mathrm{ev}_1^*\mathcal{O}_{\mathbb{CP}^1\times\mathbb{CP}^1}(0,1)$. On the other hand, we can have additional relations: we need to divide by the classes of all codimension one loci which are not GIT-stable with respect to $(d|d_1,\ldots,d_n)$ (see Corollary 4.17).

The relations above are determined using a method adapted from [Pan99], which is based on intersecting possible relations with test curves $C_{B,k}$. In Proposition 4.12, we give an algorithm for computing the class of a divisor D in terms of the generators above from its intersection numbers with such test curves. We also determine the classes of some interesting divisors, such as the locus $D_{i=\mathrm{fix}}$, where the i-th marking is a fixed point of the self-map (4.15).

1.5. **Forthcoming work.** While the present paper focuses on the definition and basic properties of $M(d|d_1,\ldots,d_n)$, the paper [Sch] will study iteration maps and the intersection theory of $M(d|d_1,\ldots,d_n)$. The iteration maps $\mathfrak{sc}_m$ are the natural extensions of the functions taking a degree d self map $\varphi : C \to C$ of a smooth curve C to its m-fold self-composition

$$\varphi^{\circ m} = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{m \text{ copies}} : C \to C,$$

which is then a degree d^m self-map of C. We give a geometric interpretation for the extension of the corresponding rational map $\mathfrak{sc}_m : M(d,0) \dashrightarrow M(d^m,0)$ to parts of the boundary and also show how to deal with markings. This will allow us to compute pullbacks of divisors via $\mathfrak{sc}_m$.

One of the main advantages of using quotients of moduli spaces of stable maps $Y_{d,n}$ to compactify the spaces of self-maps of $\mathbb{CP}^1$ is that many properties of the very well-developed intersection theory on the stable map spaces carry over to their quotients. Indeed, using the recursive boundary structure of the self-maps moduli spaces, we are able to give an explicit algorithm for computing top-intersection numbers of divisors on the spaces $M(d|d_1,\ldots,d_n)$.

1.6. **Connection to dynamics and possible applications.** In the study of dynamics of rational maps on $\mathbb{CP}^1$, compactifications of moduli spaces of such maps have arisen naturally in the past [Mil93, Mil00, DeM07]. One of the advantages of a compact moduli space is the possibility to use intersection theory on it. Many geometrically interesting loci of rational maps (e.g., the set $\mathrm{Per}_m(\lambda)$ of maps $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ with a point p of period m and multiplier λ) have natural extensions to some compactification. Then one can hope to find the number of maps f satisfying certain conditions by computing the intersection number of the corresponding loci.

By constructing the moduli spaces $M(d|d_1,\ldots,d_n)$ and studying their Picard group (many geometric loci like $\mathrm{Per}_m(\lambda)$ are divisorial), we start working towards such applications. As an example, let us mention that in [Sch] we will use the self-composition morphisms from above to compute the divisor class of the closure

$\overline{\mathrm{Per}_m(\lambda)} \subset M(d,0)$. For $d = 2$ this allows a new proof of a result of Milnor [Mil93, Theorem 4.2], computing the degree of the curve $\overline{\mathrm{Per}_m(\lambda)}$ inside $M_2^{ss} \cong M(2,0) \cong \mathbb{CP}^2$. This degree has an interpretation in dynamics as the number of hyperbolic components of period m in the Mandelbrot set (see [Mil93, Theorem 4.2]).

1.7. **Plan of the paper.** The paper is organized as follows: in section 2 we present the construction of the spaces $M(d|d_1,\dots,d_n)$. Their general properties as well as some key examples for small d and n are studied in section 3. The rational Picard groups of $Y_{d,n}$ and $M(d|d_1,\dots,d_n)$ are computed in section 4.

Appendix A contains a list of notations used throughout the text. A collection of small technical results, which we include due to the lack of a good reference, can be found in Appendix B. In Appendix C we treat group actions on stacks and the corresponding quotient stacks, based on the work of Romagny ([Rom05]).

Conventions

Throughout the paper we will work over the complex numbers. Hence in the future, notations such as $\mathbb{P}^1, \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n$ will always denote the corresponding objects defined over $\mathbb{C}$.

By a coarse moduli space of a stack $\mathcal{X}$ we mean a morphism $\mathcal{X} \to X$ from $\mathcal{X}$ to an algebraic space X, which is initial among morphisms to algebraic spaces and induces a bijection on geometric points.

2. Construction of the moduli space of self-maps

We want to study self-maps of $\mathbb{P}^1$ of a fixed degree d modulo the conjugation action of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$. For this we will start with a moduli space which was constructed by Silverman in [Sil98] and choose a different way to compactify it, also allowing markings now.

2.1. **The space of degree d maps.** An algebraic morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d is uniquely determined by its graph $\Gamma_\varphi \subset \mathbb{P}^1 \times \mathbb{P}^1$, which is a curve of degree $(1,d)$. Such curves are the vanishing loci of sections of the line bundle $\mathcal{O}(d,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and thus they are classified by the points of

$$Z_d = \mathbb{P}(H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d,1))) \cong \mathbb{P}^{2d+1}.$$

Conversely any smooth curve of degree $(1,d)$ is the graph of a morphism φ and inside Z_d the space of smooth curves Rat_d is the complement of the resultant hypersurface. The elements of $Z_d \backslash \mathrm{Rat}_d$ correspond to curves C with components of degrees $(0,k)$, $k > 0$, which we will call *vertical components.* The unique component of C of degree $(1,\tilde{d})$ will be the *horizontal component.*

The group PGL_2 acts on Rat_d via conjugation

$$(g.\varphi)(x) = g\varphi(g^{-1}x) \text{ for } g \in \mathrm{PGL}_2, \varphi \in \mathrm{Rat}_d.$$

At the level of graphs $\Gamma_\varphi \subset \mathbb{P}^1 \times \mathbb{P}^1$, this action is induced by the diagonal action of PGL_2 on $\mathbb{P}^1 \times \mathbb{P}^1$ and it extends to a linear action of PGL_2 on Z_d.

Now in order to define a quotient of Z_d by PGL_2, we use Geometric Invariant Theory as described by Mumford in [MFK94]. Here it is more convenient to work with the induced action of $G = \mathrm{SL}_2$. As $Z_d \cong \mathbb{P}^{2d+1}$, it has a natural line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{2d+1}}(1)$, which is ample and carries a canonical G-linearization (see [MFK94, 1.§3]). In [Sil98, Proposition 2.2], Silverman describes the set of (semi)stable points Z_d^{ss}, Z_d^s corresponding to the action above. His results have the following convenient interpretation in terms of vertical sections.

Lemma 2.1 (see Proposition 2.2 in [Sil98]). *An element $\Gamma \in Z_d$ consisting of a horizontal section h and vertical sections $v_i = \{p_i\} \times \mathbb{P}^1$ of multiplicities e_i ($i = 1, \ldots, k$) is*

- *semistable, iff $e_i \leq \frac{d-1}{2}$ or we have $e_i \leq \frac{d+1}{2}$ and p_i is not a fixed point of the induced morphism $\mathbb{P}^1 \to \mathbb{P}^1$ of lower degree for $i = 1, \ldots, k$;*
- *stable, iff $e_i < \frac{d-1}{2}$ or we have $e_i < \frac{d+1}{2}$ and p_i is not a fixed point of the induced morphism $\mathbb{P}^1 \to \mathbb{P}^1$ of lower degree for $i = 1, \ldots, k$.*

Corollary 2.2. *For d even we have $Z_d^{ss} = Z_d^s$.*

It is also obvious that $\mathrm{Rat}_d \subset Z_d^s$. By [Sil98, Theorem 2.1] we can now define quotients $M_d = \mathrm{Rat}_d /\!\!/ G$, $M_d^{ss} = Z_d^{ss} /\!\!/ G$, $M_d^s = Z_d^s /\!\!/ G$. We will not work with those in the future, but mention them for completeness.

2.2. **Parametrized graphs.** We have already interpreted Z_d as the space of (generalized) graphs of rational self-maps of $\mathbb{P}^1$ of degree d. Now we will consider parametrizations of these graphs by trees of $\mathbb{P}^1$s, that is, nodal curves of genus 0.

To be more precise, consider the stable maps space

$$Y_{d,n} = \overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1, d)).$$

The notation $Y_{d,n}$ is not standard but as the space will be used frequently in the future, we use $Y_{d,n}$ for brevity. From [FP97, Theorem 2] we see that it is a normal, projective variety of pure dimension $2d + 1 + n$ and it is locally the quotient of a nonsingular variety by a finite group.

We now show that there is a natural action of PGL_2 on $Y_{d,n}$. At this point it will be advantageous to construct this action as an action of the group scheme PGL_2 on the moduli stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1, d))$ inducing an action of PGL_2 on the coarse moduli space $Y_{d,n}$. Here we use the definition of [Rom05] for a group action on a stack (see also Appendix C).

Corollary 2.3. *There exists a natural strict action of the group PGL_2 on the stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1, d))$ induced by the diagonal action of PGL_2 on $\mathbb{P}^1 \times \mathbb{P}^1$. For $g \in PGL_2$ and $(f : C \to \mathbb{P}^1 \times \mathbb{P}^1; p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1, d))$ a pair of $\mathbb{C}$-valued points, we have*

$$g(f; p_1, \ldots, p_n) = (g_* \circ f; p_1, \ldots, p_n),$$

where $g_* : \mathbb{P}^1 \times \mathbb{P}^1, (p, q) \mapsto (gp, gq)$.

Proof. This follows from Lemma C.6 applied to the diagonal action of PGL_2 on $X = \mathbb{P}^1 \times \mathbb{P}^1$. □

We immediately obtain an induced action of PGL_2 on the coarse moduli space $Y_{d,n}$ of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1, d))$. Again, for taking the GIT quotient, in the following we will work with the induced action of $G = \mathrm{SL}_2$.

Lemma 2.4. *There exists a natural G-equivariant morphism $j : Y_{d,n} \to Z_d$ uniquely defined by requiring that the image*

$$s = j((f; p_1, \ldots, p_n)) \in Z_d = \mathbb{P}(H^0(\mathbb{P}^1 \times \mathbb{P}^1, (d, 1)))$$

of the class of the morphism $f : C \to \mathbb{P}^1 \times \mathbb{P}^1$ *satisfies*

$$div(s) = f_*[C].$$

In particular, we see that j *does not depend on the marked points.*

Proof. The map j was constructed by Givental in [Giv96, The Main Lemma] for the case $n = 0$. We define it for arbitrary n by precomposing with the forgetful map of all marked points, which is clearly G-equivariant. As the G-action on Z_d is given by the componentwise action of G on $\mathbb{P}^1 \times \mathbb{P}^1$ when considering elements of Z_d as their corresponding vanishing schemes in $\mathbb{P}^1 \times \mathbb{P}^1$, one verifies that the map j is G-equivariant too. $\square$

In the case $n \geq 1$, the map j loses information by forgetting the marked points. However, by using the evaluation maps, we can preserve this information in many cases.

Lemma 2.5. *Let* $ev_i : Y_{d,n} = \overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1, d)) \to \mathbb{P}^1 \times \mathbb{P}^1$ *be the evaluation associated to the i-th marked point, and let* $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ *be the projection to the first component. Let* $\Delta \subset (\mathbb{P}^1)^n$ *be the union of all the diagonals, that is,*

$$\Delta = \{(p_1, \ldots, p_n) \in (\mathbb{P}^1)^n; \exists i \neq j \text{ such that } p_i = p_j\}.$$

Then the map

$$J : Y_{d,n} \to Z_d \times (\mathbb{P}^1)^n, J = j \times (\pi_1 \circ ev_1) \times \ldots \times (\pi_1 \circ ev_n) \tag{2}$$

is equivariant with respect to the diagonal action of G *on* $(\mathbb{P}^1)^n$*. Furthermore, it is an isomorphism over* $Rat_d \times ((\mathbb{P}^1)^n \setminus \Delta)$*.*

Proof. The G-equivariance follows directly from the equivariance of the map j and the definition of the G-action on $Y_{d,n}$. If a closed point $(f : C \to \mathbb{P}^1 \times \mathbb{P}^1; p_1, \ldots, p_n) \in Y_{d,n}$ maps to $([s], q_1, \ldots, q_n) \in \mathrm{Rat}_d \times ((\mathbb{P}^1)^n \setminus \Delta)$, then all components of C must map with degrees $(1, d)$ or $(0, 0)$, as s has no vertical sections. But as the marked points $p_1, \ldots, p_n$ also map to distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$, there can be no contracted components for stability reasons. Hence C is irreducible and $\pi_1 \circ f : C \to \mathbb{P}^1$ is an isomorphism. But then we can uniquely reconstruct the positions of the markings $p_i \in C$ from the points $q_i = \pi_1(f(p_i)) \in \mathbb{P}^1$. All in all, this shows that the restriction of J to the preimage of $\mathrm{Rat}_d \times ((\mathbb{P}^1)^n \setminus \Delta)$ induces a bijection on closed points. Thus it is an isomorphism by Zariski's main theorem. $\square$

For an algebraic group H acting on a scheme X, we denote by $\mathrm{Pic}^H(X)$ the group of H-linearized line bundles on X (cf. [MFK94, 1.§3]). We want to obtain an ample G-linearized line bundle $\mathcal{M}'$ on $Y_{d,n}$ allowing us to form a quotient of $Y_{d,n}$ by G. Additionally, we want the stable and semistable loci of $\mathcal{L}'$ to coincide and to be given by the preimage of the stable locus of $\mathcal{L}$ in Z_d via the map j. For this, we use the following construction adapted from [MFK94, Proposition 2.18].

Lemma 2.6. *Let* $G = SL_2$ *act on projective varieties* Y, Z *and let* $j : Y \to Z$ *be a* G*-equivariant map. Assume we have* $\mathcal{L} \in Pic^G(Z)$*,* $\mathcal{M} \in Pic^G(Y)$ *both ample,* G*-linearized line bundles and assume* $Z^{ss} = Z^s$ *((semi)stability with respect to* $\mathcal{L}$*). Then there exists* $N > 0$ *such that for all* $n \geq N$ *we have that* $\mathcal{M}' = \mathcal{M} \otimes (j^*\mathcal{L})^{\otimes n}$ *is an ample* G*-linearized line bundle on* Y *with* $Y^{ss} = Y^s = j^{-1}(Z^s)$*.*

Proof. As $\mathcal{M}$ is ample and $\mathcal{L}$ is base-point free, it is clear that $\mathcal{M}'$ is ample and G-linearized. To identify (semi)stable points, we want to use the Hilbert-Mumford numerical criterion (see [MFK94, Theorem 2.1]). This criterion says that there exists an explicit function μ taking a G-linearized line bundle $\mathcal{N}$ on Y, a point $y \in Y$ and a one-parameter subgroup $\lambda : \mathbb{C}^* \to G$ and associating to this a number $\mu^{\mathcal{N}}(y, \lambda)$ such that y is semistable (respectively stable) with respect to $\mathcal{N}$ iff $\mu^{\mathcal{N}}(y, \lambda) \geq 0$ (respectively > 0) for all one-parameter subgroups λ. Given $y \in Y$ and a one-parameter subgroup λ in G they satisfy the functoriality property

$$\mu^{\mathcal{M}\otimes(j^*\mathcal{L})^{\otimes n}}(y, \lambda) = \mu^{\mathcal{M}}(y, \lambda) + n\mu^{\mathcal{L}}(j(y), \lambda).$$

Now all one-parameter subgroups of SL_2 are conjugate to a multiple of the standard diagonal one-parameter subgroup

$$T : \mathbb{G}_m \to \mathrm{SL}_2, t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

By the further functoriality properties

$$\mu^{\mathcal{N}}(g.y, \lambda) = \mu^{\mathcal{N}}(y, g^{-1}\lambda g) \text{ for } g \in G,\ \mu^{\mathcal{N}}(y, \lambda^k) = k\mu^{\mathcal{N}}(y, \lambda)$$

we may restrict ourselves to considering $\mu^{\mathcal{N}}(y, \lambda)$ for $\lambda = T$ or $\lambda = T^{-1}$ for studying stability. Note further that as $Z^{ss} = Z^s$, we either have $\mu^{\mathcal{L}}(j(y), \lambda) > 0$ if $y \in j^{-1}(Z^s)$ or $\mu^{\mathcal{L}}(j(y), \lambda) < 0$ otherwise. Thus, if we can uniformly bound $\mu^{\mathcal{M}}(y, \lambda)$ over all $y \in Y$ and $\lambda = T, T^{-1}$, we can choose N larger and then the stability of y is only determined by the stability of $j(y)$ as desired. Note that here we use $\mu^L(z, g^{-1}Tg) = \mu^L(gz, T)$. But that such a bound exists follows immediately from [MFK94, Proposition 2.14]. □

We now apply this result to the map j constructed in Lemma 2.4. In order to achieve $Z_d^{ss} = Z_d^s$, we have to restrict to the case d even.

Corollary 2.7. *Let $\mathcal{M}$ be any G-linearized ample line bundle on $Y_{d,n}$, let $N > 0$ as in Lemma* 2.6 *and $\mathcal{M}' = \mathcal{M} \otimes (j^*\mathcal{L})^{\otimes N}$. Then $Y_{d,n}^{ss} = Y_{d,n}^s$ is independent of the choice of $\mathcal{M}$ and it admits a uniform geometric quotient $\phi : Y_{d,n}^s \to M(d, n)$. Here ϕ is affine and universally submersive and $M(d, n)$ is a normal projective variety over $\mathbb{C}$.*

Proof. Let $\mathcal{M}_0$ be an ample line bundle on the projective, normal variety $Y_{d,n}$, then by [MFK94, Corollary 1.6] some power $\mathcal{M}$ is G-linearizable. It is clear from Lemma 2.6 that $Y_{d,n}^{ss} = Y_{d,n}^s = j^{-1}(Z_d^s)$ is independent of the constructed $\mathcal{M}'$. By [MFK94, Theorem 1.10] we conclude the existence of the affine, universally submersive uniform geometric quotient ϕ, reducedness and normality of $M(d, n)$ follow from [MFK94, 0.§2 (2)] and projectivity from the remark above [MFK94, Converse 1.12]. □

The spaces $M(d, n)$ (and their generalizations $M(d|d_1, \ldots, d_n)$ defined in the following section) will now be our main object of study. Note that we have a natural map to the compactification M_d^{ss} constructed by Silverman.

Corollary 2.8. *Let $d \geq 2$ be even and $n \geq 0$, then the G-equivariant map $j : Y_{d,n} \to Z_d$ descends to a regular map $\tilde{j} : M(d, n) \to M_d^s = M_d^{ss}$. For $n = 0$ this map is an isomorphism over $M_d \subset M_d^{ss}$.*

Proof. By Lemma 2.6 we have $Y^{ss}_{d,n} = j^{-1}(Z^{ss}_d)$, so we have a solid diagram of maps

$$\begin{tikzcd} Y^{ss}_{d,n} \arrow[r, "j"] \arrow[d, "\phi"] & Z^{ss}_d \arrow[d, "\psi"] \\ M(d,n) \arrow[r, dashed, "\exists ! \tilde{j}"] & M^{ss}_d, \end{tikzcd}$$

where the vertical maps are the respective G-quotients. Since j is G-equivariant and ψ is G-invariant, the composition $\psi \circ j$ is also invariant and since ϕ is a categorical quotient, there exists a unique map $\tilde{j}$ completing the diagram as above. For $n = 0$, Lemma 2.5 implies that j is an isomorphism over $\mathrm{Rat}_d \subset Z^{ss}_d$, which implies that $\tilde{j}$ is an isomorphism over the dense open set $M_d \subset M^{ss}_d$. □

2.3. **Weighted points.** In order to analyze the recursive boundary structure of $M(d,n)$, it will be necessary to generalize the construction above. We will do so by attributing nonnegative rational weights to the marked points in $\overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1,d))$. This will not change the action, but it will affect the set of (semi)stable points and hence the geometric quotient.

Let $d, n \geq 0$, $k \geq 1$ and $\tilde{d}_1, \ldots, \tilde{d}_n \in \mathbb{Z}_{\geq 0}$ nonnegative integers such that $d_S = k(d+1) + \sum_{i=1}^n \tilde{d}_i$ is positive and odd. We write $\mathbf{d}^\sim = (d, k|\tilde{d}_1, \ldots, \tilde{d}_n)$. Consider the map

$$J : Y_{d,n} \to Z_d \times (\mathbb{P}^1)^n, J = j \times (\pi_1 \circ \mathrm{ev}_1) \times \ldots \times (\pi_1 \circ \mathrm{ev}_n)$$

defined in Lemma 2.5. On the target

$$Z_d \times (\mathbb{P}^1)^n = \mathbb{P}(H^0(\mathbb{P}^1 \times \mathbb{P}^1, (d,1))) \times (\mathbb{P}^1)^n$$

we have the base-point free line bundle

$$\mathcal{L}_{\mathbf{d}^\sim} = \mathcal{O}_Z(k) \boxtimes \mathcal{O}(\tilde{d}_1) \boxtimes \ldots \boxtimes \mathcal{O}(\tilde{d}_n).$$

We want to apply Lemma 2.6 to obtain an ample linearized line bundle $\mathcal{M}'$ on $Y_{d,n}$ such that a point in $Y_{d,n}$ is (semi)stable iff it maps to a (semi)stable point in $Z_d \times (\mathbb{P}^1)^n$ with respect to the line bundle $\mathcal{L}_{\mathbf{d}^\sim}$[1]. Therefore we need to analyze (semi)stability for the action of SL_2 on $Z_d \times (\mathbb{P}^1)^n$ with respect to $\mathcal{L}_{\mathbf{d}^\sim}$.

Lemma 2.9. *A point $q = ([s], p_1, \ldots, p_n) \in Z_d \times (\mathbb{P}^1)^n$ is semistable with respect to $\mathcal{L}_{\mathbf{d}^\sim}$ iff for all $p \in \mathbb{P}^1$ we have*

$$\nu_p([s]) + \delta_{p=fix}([s]) + \sum_{i:p_i=p} \frac{\tilde{d}_i}{k} \leq \frac{d+1+\sum_i \frac{\tilde{d}_i}{k}}{2}, \tag{3}$$

Here $\nu_p([s])$ is the order of vanishing of s on the cycle $\{p\} \times \mathbb{P}^1$ or equivalently the multiplicity of a potential vertical section over p. The number $\delta_{p=fix}([s])$ is 1 if p is a fixed point of the underlying map $\tilde{\varphi}$ from the horizontal section of s and 0 otherwise. The point q is stable iff the inequality above is strict for all p.

[1]Note in the following that Mumford's numerical criterion and its consequence Lemma 2.6 were formulated for ample line bundles. Therefore if some of the numbers d or d_i are zero, we have to modify the map J to leave out the corresponding factors Z_d or $\mathbb{P}^1$ in the target to make the modified line bundle $\mathcal{L}_{\mathbf{d}^\sim}$ ample. However, the analysis will not be affected by this so we will ignore this technicality henceforth.

Proof. The lemma follows by applying the Hilbert-Mumford numerical criterion [MFK94, Theorem 2.1]. As in the proof of Lemma 2.1, we can restrict to compute $\mu = \mu^{\mathcal{L}_{\mathbf{d}^\sim}}(q, \lambda)$ for the diagonal one-parameter subgroup $\lambda \subset \mathrm{SL}_2$. From the proof of [Sil98, Proposition 2.2], we see that the line bundle $\mathcal{O}_{Z_d}(k)$ on the factor Z_d contributes a summand

$$k\left(d + 1 - 2\nu_{[1:0]}([s]) + 2\delta_{[1:0]=\mathrm{fix}}([s])\right)$$

to μ. On the other hand, the line bundle $\mathcal{O}(\tilde{d}_i)$ on the i-th factor $\mathbb{P}^1$ contributes $\tilde{d}_i$ for $p_i \neq [1:0]$ and $-\tilde{d}_i$ for $p_i = [1:0]$. This can be seen by using [MFK94, Proposition 2.3]. The conditions $\mu \geq 0$ for semistability and $\mu > 0$ for stability then translate to the claimed result by rearranging the terms and dividing by $2k$. □

We remark that multiplying the inequality (3) by k, the left side is an integer. So for $k(d+1) + \sum_i \tilde{d}_i$ odd, the inequality is satisfied iff it is satisfied strictly. Thus semistability is the same as stability. Then we can apply Lemma 2.6 as described above to obtain an ample linearized line bundle $\mathcal{M}'$ on $Y_{d,n}$ to define semistability. For brevity in the later text we make the following definition.

Definition 2.10. A tuple $\mathbf{d} = (d|d_1, \ldots, d_n) \in \mathbb{Z}_{\geq 0} \times (\mathbb{Q}_{\geq 0})^n$ is called admissible if there exists an integer $k \geq 1$ such that all numbers $\tilde{d}_i = kd_i$ are integers and such that $k(d+1) + \sum_i \tilde{d}_i$ is odd.

We are now able to define the space $M(d|d_1, \ldots, d_n)$.

Corollary 2.11. *Let $\boldsymbol{d} = (d|d_1, \ldots, d_n)$ be admissible, then the set*

$$Y_{d,n}^{ss,\boldsymbol{d}} = Y_{d,n}^{s,\boldsymbol{d}} = J^{-1}((Z_d \times (\mathbb{P}^1)^n)^{s,\mathcal{L}_{d^\sim}})$$

admits a uniform geometric quotient $\phi : Y_{d,n}^{s,\boldsymbol{d}} \to M(d|d_1, \ldots, d_n)$. Here ϕ is affine and universally submersive and $M(d|d_1, \ldots, d_n)$ is a normal projective variety over $\mathbb{C}$.

Proof. This is exactly the same proof as for Corollary 2.7. □

Note that for nontrivial $d_1, \ldots, d_n$, the map $j : Y_{d,n} \to Z_d$ will in general no longer satisfy $j^{-1}(Z_d^{ss}) = Y_{d,n}^{ss,\mathbf{d}}$, so in contrast to Corollary 2.8 the map j will only descend to a rational map $M(d|d_1, \ldots, d_n) \dashrightarrow M_d^{ss}$.

3. Examples and properties

3.1. **Examples.** In the following we will study some of the spaces $\overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1, d))$ and the corresponding quotient spaces $M(d|d_1, \ldots, d_n)$ in more detail. For $B \subset \{1, \ldots, n\}$ and $0 \leq k \leq d$ with $k \geq 1$ or $|B| \geq 2$, we denote by

$$D_{B,k} = D(\{1, \ldots, n\} \setminus B, (1, d-k)|B, (0, k)) \subset Y_{d,n}$$

the boundary divisor with general point $(f : C \to \mathbb{P}^1 \times \mathbb{P}^1; p_1, \ldots, p_n)$ for C having two irreducible components C_1, C_2 carrying the markings $\{1, \ldots, n\} \setminus B$ and B and mapping with degrees $(1, d-k), (0, k)$, respectively.

3.1.1. $d = 0, n \leq 2$. It is clear that we have an isomorphism

$$\overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1,0)) \cong \mathbb{P}^1 \times \overline{M}_{0,n}(\mathbb{P}^1, 1).$$

For small n the space $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is easy to describe:

$$\overline{M}_{0,1}(\mathbb{P}^1, 1) \cong \mathbb{P}^1, \overline{M}_{0,2}(\mathbb{P}^1, 1) \cong (\mathbb{P}^1)^2,$$

both via the evaluation maps of the markings.

Concerning the spaces $M(0|d_1, \ldots, d_n)$ with $n \leq 2$, one checks using Lemma 2.9 that the only cases with nonempty semistable sets occur for $n = 2$ and the semistable set does not depend on the choice of d_1, d_2. Thus these nonempty spaces $M(0|d_1, d_2)$ are all isomorphic to $M(0|1, 1)$.

Lemma 3.1. *The space $M(0|1,1)$ is isomorphic to a single point Spec$(\mathbb{C})$, without isotropy.*

Proof. Using Lemma 2.9 we see that the semistable points in $\overline{M}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (0,1)) \cong (\mathbb{P}^1)^3$ are exactly the complement $(\mathbb{P}^1)^3 \setminus \Delta$ of the big diagonal, on which PGL_2 acts freely and transitively with quotient Spec$(\mathbb{C})$. □

3.1.2. $d = 1, n = 1$. First we note that the only nonempty moduli spaces $M(1|d_1)$ are those with $0 < d_1 < 2$ and all of them are isomorphic to $M(1|1)$ (by an analysis of stable loci using Lemma 2.9).

For $\overline{M}_{0,1}(\mathbb{P}^1 \times \mathbb{P}^1, (1,1))$ we first look at the locus of stable maps with smooth source curve and find

$$M_{0,1}(\mathbb{P}^1 \times \mathbb{P}^1, (1,1)) \cong \mathrm{PGL}_2 \times \mathbb{P}^1$$

by Lemma 2.5. Here PGL_2 acts on itself by conjugation and on $\mathbb{P}^1$ in the usual way. There are two boundary divisors on $Y_{1,1}$, namely $D_{\emptyset,1}$ and $D_{\{1\},1}$. Then we have the following.

Lemma 3.2. *Consider coordinates $[X : Y], [S : T]$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and the rational map*

$$\begin{aligned} \varphi : Z_1 = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)) &\dashrightarrow \mathbb{P}^1 \\ [aXS + bXT + cYS + dYT] &\mapsto [-bc + da : b^2 - 2bc + c^2]. \end{aligned}$$

It is PGL_2-invariant and the image of

$$j : \overline{M}_{0,1}(\mathbb{P}^1 \times \mathbb{P}^1, (1,1))^{ss,(1|1)} \to Z_1$$

lies in the domain of definition of φ. Hence $\varphi \circ j$ induces a map $\psi : M(1|1) \to \mathbb{P}^1$ and this map is an isomorphism. All points in $M(1|1)$ have trivial PGL_2-stabilizers, except for the point corresponding to the orbit of

$$f_0 = \left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, [1:1] \right) = ([z \mapsto -z], z = 1) \in PGL_2 \times \mathbb{P}^1,$$

which has a $\mathbb{Z}/2\mathbb{Z}$-isotropy given by

$$B_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [z \mapsto 1/z] \in PGL_2.$$

Proof. From the semistability condition in Lemma 2.9 it follows that

$$M_{0,1}(\mathbb{P}^1 \times \mathbb{P}^1, (1,1))^{ss,(1|1)} = \mathrm{PGL}_2 \times \mathbb{P}^1 \setminus \underbrace{\{([A], p) : Ap = p\}}_{=\mathrm{Fix}}.$$

We will first study the conjugation action of PGL_2 on itself. Here it is clear that in each orbit of a given $[A] \in \mathrm{PGL}_2$ there is a matrix in Jordan canonical form and by scaling we may pick the representative A' of this matrix in PGL_2 to have entries $\alpha, 1$ on the diagonal.

Now we determine the stabilizer under the conjugation action. If $B \in \mathrm{GL}_2$ satisfies $BA'B^{-1} = \lambda A'$ for some $\lambda \in \mathbb{C}^*$, then taking trace and determinant, we have

$$\alpha + 1 = \operatorname{tr}(A') = \operatorname{tr}(BA'B^{-1}) = \operatorname{tr}(\lambda A') = \lambda(\alpha+1),$$
$$\alpha = \det(BA'B^{-1}) = \det(\lambda A') = \lambda^2\alpha.$$

Thus, $\lambda = \pm 1$ and $\lambda = 1$ for $\alpha \neq -1$.

We first look for the solutions B of $BA' = A'B$, so the equation for $\lambda = 1$. If A is diagonalizable, the matrix A' has the form

$$A' = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that for $\alpha = 1$, the induced map $[A] \in \mathrm{PGL}_2$ is the identity. Therefore all points are fixed and thus this case does not occur in the semistable set above. Excluding this case let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $BA' = A'B$. Spelling this out means exactly $\alpha b = b$, $\alpha c = c$. So as $\alpha \neq 1$, we have $b = c = 0$, so B is a diagonal matrix. Note that the diagonal matrices in PGL_2 are isomorphic to $\mathbb{C}^*$. But now we also want to take into account the additional marked point above. The complement of the fixed points of A' are the points p in $\mathbb{P}^1 \setminus \{[0:1],[1:0]\}$. This eliminates the remaining stabilizing elements in $\mathbb{C}^*$, for instance, moving the marked point to $[1:1]$.

On the other hand, for $\alpha = -1$ we have to check the case $BA' = -A'B$. This amounts to $-a = a, -d = d$. Thus in this case the stabilizer of $[A']$ also contains antidiagonal matrices. As above, we can use the $\mathbb{C}^*$-part of the stabilizer to move the marking to $[1:1]$. Of the antidiagonal matrices, exactly the class of the element $B_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ fixes the marked point $[1:1]$. This gives precisely the stabilizer of the point f_0 above.

Finally, there is the case where A is not diagonalizable. Necessarily its eigenvalues have to coincide, amounting to $\alpha = 1$ (and thus $\lambda = 1$) above, and the matrix A' has the form

$$A' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The equation $BA' = A'B$ gives $c = 0$, $a = d$, so the stabilizer consists of matrices $B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. By scaling the diagonal to $a = 1$, we see that this is exactly isomorphic to $\mathbb{C}$. But now the only fixed point of A' is $[1:0]$, so we see that the marked point $p \in \mathbb{C} = \mathbb{P}^1 \setminus \{[1:0]\}$ eliminates the remaining stabilizer.

We also have to analyze the boundary. Note that by Lemma 2.9, the entire divisor $D_{\{1\},1}$ is unstable. On the other hand, we know from [FP97] that there is a bijective gluing morphism

$$\overline{M}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (1,0)) \times_{\mathbb{P}^1 \times \mathbb{P}^1} \underbrace{\overline{M}_{0,1}(\mathbb{P}^1 \times \mathbb{P}^1, (0,1))}_{=\mathbb{P}^1 \times \mathbb{P}^1} \to D_{\emptyset,1}.$$

We see that the left side is actually isomorphic to $\overline{M}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (1,0))$. One checks that the preimage of the (semi)stable locus $D_{\emptyset,1}^{ss,(1|1)}$ is precisely $\overline{M}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (1,0))^{ss,(0|1,1)}$. As seen in the proof of Lemma 3.1 this is exactly the variety PGL_2 and the corresponding action is given by left-multiplication. Hence the boundary divisor $D_{\emptyset,1}$ in $M(1|1)$ is isomorphic to a point (without stabilizer).

Now we are ready to prove that the map ψ is bijective on closed points. An element of the boundary divisor $D_{\emptyset,1}$ corresponds to a horizontal section $[e:f] \times \mathbb{P}^1$ and a vertical section $\mathbb{P}^1 \times [g:h]$ with $[e:f] \neq [g:h]$. Under j it maps to sections of the form

$$(Xf - Ye)(Sh - Tg) = fhXS - fgXT - ehYS + egYT \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)).$$

Thus the image under ψ is

$$\begin{aligned}&[-(-fg)(-eh) + (eg)(fh) : (fg)^2 - 2(-fg)(-eh) + (eh)^2]\\ &\quad = [0 : (fg - eh)^2] = [0:1],\end{aligned}$$

where we use $fg - eh \neq 0$ as $[e:f] \neq [g:h]$.

On the other hand, we look at $(\mathrm{PGL}_2 \times \mathbb{P}^1) \setminus \mathrm{Fix}$. Note that $[A] = [(a_{i,j})_{i,j=1}^2] \in \mathrm{PGL}_2$ corresponds to the map

$$\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1, [x:y] \mapsto ([x:y], [a_{1,1}x + a_{1,2}y : a_{2,1}x + a_{2,2}y]).$$

Under j the point $([A], p)$ maps to $[S(a_{2,1}X + a_{2,2}Y) - T(a_{1,1}X + a_{1,2}Y)]$. Hence our proposed map ψ here simply takes the form

$$\psi([A], p) = [a_{1,1}a_{2,2} - a_{2,1}a_{1,2} : a_{1,1}^2 + 2a_{1,1}a_{2,2} + a_{2,2}^2] = [\det(A) : \mathrm{tr}(A)^2].$$

Note that this map is independent of the choice of representative A in the class $[A] \in \mathrm{PGL}_2$ and also invariant under the conjugation action. Because j is dominant and PGL_2-equivariant, we also obtain that φ is PGL_2 invariant.

We know $\det(A) \neq 0$ and by scaling we may reach $\det(A) = 1$. The scaling constant is unique up to a sign. On the other hand, if we know $\psi([A], p)$, then $\mathrm{tr}(A)$ is determined up to a sign, giving us the eigenvalues of A up to a common factor ± 1. But this specifies the Jordan canonical form (as the identity matrix is excluded). Thus ψ in injective on closed points. On the other hand, one sees quickly that every point $[1:t]$ is in the image of ψ. Thus by Zariski's main theorem, the space $M(1|1)$ is isomorphic to $\mathbb{P}^1$ via the map ψ. □

3.1.3. $d = 2, n = 0$.

Lemma 3.3. *For $d = 2, n = 0$ the map $j : Y_{d,n} \to Z_d$ is an isomorphism over Z_d^s and thus the map $\tilde{j} : M(2,0) \to M_2^s$ is also an isomorphism. Hence, the space $M(2,0)$ is isomorphic to $\mathbb{P}^2$.*

Proof. We show that the map $j : Y_{d,n}^s \to Z_d^s$ is bijective on closed points and thus an isomorphism by Zariski's main theorem. We already know that j is an isomorphism over $\mathrm{Rat}_d \subset Z_d^s$ by Lemma 2.5. By Lemma 2.1 all remaining points in $Z_d^s \setminus \mathrm{Rat}_d$ are classes $[s]$ of sections $s \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, (d,1))$ corresponding to graphs $\Gamma = V(s) \subset \mathbb{P}^1 \times \mathbb{P}^1$ with one or two vertical sections v_i over $p_i \in \mathbb{P}^1$ of multiplicity exactly 1 such that p_i is not a fixed point of the induced map $\tilde{\varphi} : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 1 or 0. But one sees easily that the inclusion $\Gamma \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a stable map of genus 0 curves corresponding to a point in $Y_{d,n}^s$ mapping to $[s]$. Thus $j : Y_{d,n}^s \to Z_d^s$ is surjective. On the other hand, if $f : C \to \mathbb{P}^1 \times \mathbb{P}^1$ is a stable map such that

$j(f) = [s]$, one sees that it defines an isomorphism $C \to \Gamma$. Thus up to isomorphism of stable maps, the inclusion $C = \Gamma \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is the only curve in the preimage of $[s]$, which shows injectivity. □

3.2. **Isotropy and singularities.** In this section we first show that the spaces $M(d|d_1, \ldots, d_n)$ we constructed have nice singularities. This will be important for analyzing their Picard group. In the following, let $\mathbf{d} = (d|d_1, \ldots, d_n)$ be admissible.

Lemma 3.4. *Let $d \geq 0, n \geq 0$ with $(d,n) \neq (0,0), (0,1), (1,0)$. Then for the action of $\tilde{G} = PGL_2$ let $A_{d,n} \subset \overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1,d))$ be the locus of points where $\tilde{G}$ acts with nontrivial stabilizer. Then $A_{d,n}$ is of codimension at least 1 for all d, n as above. Even more, for $d \geq 1$ and $(d,n) \neq (1,1), (2,0)$ the set $A_{d,n}$ is of codimension at least 2 and for $d = 0$ we have that*

$$A_{d,n} \setminus D_{\{1,\ldots,n\},0} \subset \overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1,d)) \setminus D_{\{1,\ldots,n\},0}$$

is of codimension at least 2.

Proof. For fixed d, our proof will basically be an induction on n. Consider the forgetful map

$$F : \overline{M}_{0,n+1}(\mathbb{P}^1 \times \mathbb{P}^1, (1,d)) \to \overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1,d)).$$

The fibre over a closed point point $[(f : C \to \mathbb{P}^1 \times \mathbb{P}^1; p_1, \ldots, p_n)]$ is exactly isomorphic to $C/\mathrm{Aut}(C)$, corresponding to the possibilites to add another marked point, possibly having to add additional components to C afterwards. It is clear that $F(A_{d,n+1}) \subset A_{d,n}$. By standard theorems on fibre dimensions this shows that

$$\mathrm{codim}(A_{d,n+1}) \leq \mathrm{codim}(A_{d,n})$$

with a strict inequality if for some (d,n) only a finite number of points in the fibre over a general point $[(f : C \to \mathbb{P}^1 \times \mathbb{P}^1; p_1, \ldots, p_n)] \in A_{d,n}$ have a PGL_2-isotropy.

Using this we see that for $d \geq 1$ it suffices to prove the assertions above if

$$(d,n) = (1,1), (1,2), (2,0), (2,1), (3,0), (4,0) \ldots.$$

Our analysis of $A_{d,n}$ will be split in analyzing the interior of the moduli space $\overline{M}_{0,n+1}(\mathbb{P}^1 \times \mathbb{P}^1, (1,d))$ and its boundary. By [FP97, Lemma 12] the boundary divisor $D_{B,k}$ admits a surjective, birational map from

$$\overline{M}_{0,n-|B|+1}(\mathbb{P}^1 \times \mathbb{P}^1, (1, d-k)) \times_{\mathbb{P}^1 \times \mathbb{P}^1} \overline{M}_{0,|B|+1}(\mathbb{P}^1 \times \mathbb{P}^1, (0,k)), \tag{4}$$

which amounts to gluing two curves and maps at a single point. One sees easily that this gluing map is PGL_2-equivariant with respect to the natural actions. Note that $D_{B,k}$ is irreducible by [KP01, Corollary 2] and already has codimension 1. Thus in order to show that the boundary part of $A_{d,n}$ has codimension at least 2, we only have to show that a general point in $D_{B,k}$ has trivial stabilizer. But this immediately follows if we know that a general point of $\overline{M}_{0,n-|B|+1}(\mathbb{P}^1 \times \mathbb{P}^1, (d-k, 1))$ has trivial stabilizer. For almost all B, k this will be implied by our inductive argument below, where we first induce over d and then for fixed d over n. The only exceptional case is $B = \{1, \ldots, n\}$ and $k = d$. Here we have

$$\begin{aligned} D_{B,k} &\cong \overline{M}_{0,1}(\mathbb{P}^1 \times \mathbb{P}^1, (1,0)) \times_{\mathbb{P}^1 \times \mathbb{P}^1} \overline{M}_{0,n+1}(\mathbb{P}^1 \times \mathbb{P}^1, (0,d)) \\ &\cong (\mathbb{P}^1 \times \mathbb{P}^1) \times_{\mathbb{P}^1 \times \mathbb{P}^1} (\mathbb{P}^1 \times \overline{M}_{0,n+1}(\mathbb{P}^1, d)) \\ &\cong \mathbb{P}^1 \times \overline{M}_{0,n+1}(\mathbb{P}^1, d). \end{aligned}$$

Under these identifications, PGL_2 acts on the first factor in the usual way and on the second factor by postcomposition. By another inductive argument using a forgetful morphism, it is enough to consider the case $n = 0$. We claim that for $d \geq 2$, a general point of the space above has trivial stabilizer.

Indeed, let $(q, [\varphi : \mathbb{P}^1 \to \mathbb{P}^1; p])$ be a general element of $\mathbb{P}^1 \times M_{0,n+1}(\mathbb{P}^1, d)$. Then we may assume that $q \neq \varphi(p)$ and we can use our PGL_2 action to move those points to $[0 : 1], [1 : 0]$, respectively. Now assume $\psi \in \mathrm{PGL}_2$ fixes $[0 : 1], [1 : 0]$ and satisfies $[\psi \circ \varphi] = [\varphi] \in M_{0,0}(\mathbb{P}^1, d)$. In other words, there exists $B \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\psi \circ \varphi = \varphi \circ B$. Then B must fix the preimages $\varphi^{-1}([0 : 1])$ and $\varphi^{-1}([1 : 0])$. As φ was supposed to be general, these are two sets of d points, all distinct. For $d \geq 2$ one sees that this implies $B = \mathrm{id}$ so $\psi = \mathrm{id}$ as desired.

To conclude, we will now show the claimed results in the following order:

$$(d, n) = (0, 2), (0, 3), (0, 4), \ldots; (1, 1), (1, 2), (2, 0), (2, 1), (3, 0), (4, 0), \ldots,$$

where for $d \geq 2$ we only have to consider the points of $A_{d,n}$ in the interior $M_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1, d))$ of $Y_{d,n}$.

$\underline{d = 0 \text{ and } n \geq 2}$

It is clear that

$$M_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1, 0)) \cong \mathbb{P}^1 \times ((\mathbb{P}^1)^n \setminus \Delta),$$

where PGL_2 acts in the usual way on each component. Hence for $n \geq 2$, a general point of this space has trivial stabilizer. For $n \geq 3$ we even have that no point in this open locus has any stabilizer, so it remains to consider the boundary components. For a proper subset $B \subsetneq \{1, \ldots, n\}$ with $|B| \geq 2$, a general element of $D_{B,0}$ will consist of a horizontal inclusion $\mathbb{P}^1 \to \mathbb{P}^1 \times \{q\}$ with at least one marking from $\{1, \ldots, n\} \setminus B$ mapping to the point (p, q) and a component of the source curve containing the marks B contracted to a point (p', q). For q, p, p' pairwise distinct this element has trivial stabilizer as desired.

$\underline{d = 1 \text{ and } n = 1}$

This case was analyzed in great detail in Lemma 3.2, where in particular it is shown that a general point of $M_{0,1}(\mathbb{P}^1 \times \mathbb{P}^1, (1, 1))$ has trivial stabilizer.

$\underline{d = 1 \text{ and } n = 2}$

Here we will look at the fibres of the forgetful map

$$F : \overline{M}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (1, 1)) \to \overline{M}_{0,1}(\mathbb{P}^1 \times \mathbb{P}^1, (1, 1)).$$

On $M_{0,1}(\mathbb{P}^1 \times \mathbb{P}^1, (1, 1)) = \mathrm{PGL}_2 \times \mathbb{P}^1$ we have seen that away from the locus $\mathrm{Fix} \subset \mathrm{PGL}_2 \times \mathbb{P}^1$, the only orbit with nontrivial stabilizer is the orbit of $f_0 = [(z \mapsto -z, 1)]$, which has finite stabilizer. Therefore all but finitely many points in the fibre $\mathbb{P}^1/\mathrm{Aut}(f_0)$ are not fixed points of this stabilizer as desired.

In the closed subset Fix there are two types of elements: first we have $([A], p)$ with $[A] \neq [\mathrm{id}]$ and p one of the finitely many fixed points of A. Here as above we see that adding an additional marked point in a sufficiently general position (i.e., different from the fixed points) removes the remaining stabilizer. The other remaining case is the points $([\mathrm{id}, p])$ with $p \in \mathbb{P}^1$ arbitrary. But this locus is already itself of codimension 3 so its preimage under F also has codimension 3.

In Lemma 3.2 we have already seen that a general point on the boundary divisor $D_{\emptyset,1}$ has no isotropy and similarly it is easy to see the same for $D_{\{1\},1}$. Hence $A_{1,1}$ intersected the boundary of $\overline{M}_{0,1}(\mathbb{P}^1 \times \mathbb{P}^1, (1, 1))$ has codimension at least 2.

$\underline{d \geq 2 \text{ and } n = 0}$

By our preparations, we need to show that $A_{d,n} \cap M_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1,d))$ has codimension at least 1 for $d = 2$ and at least 2 for $d \geq 3$. But by [MSW14, Corollary 4] $A_{d,0}$ has codimension $d-1$ in $M_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1,d))$. Here we use that j is an isomorphism over Rat_d by Lemma 2.4.

$\underline{d = 2 \text{ and } n = 1}$

All points in $M_{0,0}(\mathbb{P}^1 \times \mathbb{P}^1, (1,2)) \cong \mathrm{Rat}_2$ have finite PGL_2-isotropy by [Sil07, Proposition 4.65]. Hence as above we only have finitely many points in each fibre of F with nontrivial isotropy. This finishes the proof. □

Corollary 3.5. *Let $\boldsymbol{d} = (d|d_1, \dots, d_n)$ be admissible. Then $\tilde{G} = PGL_2$ acts on $Y^{ss,\boldsymbol{d}}_{d,n}$ with finite stabilizers at geometric points and for $(d,n) \neq (2,0),(1,1)$, the action is free on an invariant open set with complement of codimension at least* 2.

Proof. By [MFK94, 1.§4 (1)] the function $y \mapsto \dim(\mathrm{Stab}(y))$ is locally constant on $Y_{d,n}$. But note that $Y_{d,n}$ is connected (see [KP01, Corollary 1]). Hence for showing that the action has finite stabilizers on the semistable set, it suffices to find any semistable point with finite stabilizer. One sees quickly that for all $\mathbf{d}$ as above such that there are semistable points at all, the locus $Y^{ss,\mathbf{d}}_{d,n} \setminus A_{d,n}$, where $\tilde{G}$ acts freely, is nonempty by Lemma 3.4. Thus every geometric point has finite stabilizer.

Moreover, Lemma 3.4 immediately implies that $A_{d,n} \cap Y^{ss,\mathbf{d}}_{d,n}$ is of codimension at least 2 for $(d,n) \neq (2,0),(1,1)$ and $d \geq 1$. Finally, for $d = 0$ we see that $D_{\{1,\dots,n\},0}$ is always disjoint from the locus of semistable points by Lemma 2.9, so again we can apply Lemma 3.4. □

Denote by

$$\mathcal{Y}_{d,n} = \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1,d))$$

the moduli stack of stable maps to $\mathbb{P}^1 \times \mathbb{P}^1$ with coarse moduli space $\mathcal{Y}_{d,n} \to Y_{d,n}$. Let $\mathcal{Y}^{ss,\mathbf{d}}_{d,n}$ be the preimage of the locus of semistable points $Y^{ss,\mathbf{d}}_{d,n}$.

Lemma 3.6. *The space $M(d|d_1, \dots, d_n)$ is the coarse moduli space of the smooth Deligne-Mumford stack*

$$\mathcal{M}(d|d_1, \dots, d_n) = \mathcal{Y}^{ss,\boldsymbol{d}}_{d,n}/PGL_2.$$

In particular, it has finite quotient singularities.

Proof. As $M(d|d_1, \dots, d_n)$ is a GIT quotient of $Y^{ss,\mathbf{d}}_{d,n} = Y^{s,\mathbf{d}}_{d,n}$ and as $Y^{ss,\mathbf{d}}_{d,n}$ is a coarse moduli space for $\mathcal{Y}^{ss,\mathbf{d}}_{d,n}$, we obtain that $M(d|d_1, \dots, d_n)$ is a coarse moduli space for the quotient stack $\mathcal{Y}^{ss,\mathbf{d}}_{d,n}/\mathrm{PGL}_2$ by Lemma C.4 and Remark C.5. To show that $\mathcal{M}(d|d_1, \dots, d_n)$ is a smooth Deligne-Mumford stack, we want to apply Proposition C.3. The fact that $\mathcal{Y}_{d,n}$ is an orbifold is proved in [FP97] and [KP01]. The action of PGL_2 on $\mathcal{Y}^{ss,\mathbf{d}}_{d,n}$ has finite stabilizers by Corollary 3.5. Thus, the conditions of Proposition C.3 are verified and the proof of the first part is finished. Finally, $M(d|d_1, \dots, d_n)$ has finite quotient singularities as it is normal and the coarse moduli space of a smooth Deligne-Mumford stack (see [Vis89, Proposition 2.8]). □

Corollary 3.7. *Every Weil divisor on $M(d|d_1, \dots, d_n)$ is $\mathbb{Q}$-Cartier.*

Proof. This follows as $M(d|d_1, \dots, d_n)$ has at most finite quotient singularities (see [KM98, Proposition 5.15]). □

3.3. **Modular interpretations.** Let $\mathbf{d} = (d|d_1, \dots, d_n)$ still be admissible. We want to find a way to interpret the quotient $M(d|d_1, \dots, d_n)$ as a moduli space for a functor of stable self-maps, in a sense yet to be defined. Our strategy is as follows: first we identify the stack $\mathcal{Y}_{d,n}/\mathrm{PGL}_2$ with an explicit category fibred in groupoids. Its objects over a scheme S are a natural generalization of families with fibres $\mathbb{P}^1$ together with n markings and a degree d self-map of the family. We can identify $\mathcal{M}(d|d_1, \dots, d_n)$ as an open substack defined by requiring that the self-maps satisfy a stability condition. By passing from this stack to its coarse moduli space $M(d|d_1, \dots, d_n)$, we obtain the functor from schemes to sets, which associates to S the set of isomorphism classes of families in $\mathcal{M}(d|d_1, \dots, d_n)(S)$. Finally, we see that over the locus M^* of points $[p] \in M(d|d_1, \dots, d_n)$ with smooth source curve and without PGL_2-isotropy or automorphisms (of $p \in \mathcal{Y}_{d,n}$), we have a universal family of curves with a self-map (in the category of schemes). Let $\mathcal{M}^{d,n}$ be the category fibred in groupoids over $\mathrm{Sch}_{\mathbb{C}}$, whose objects over a scheme S are diagrams

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\varphi"'] \arrow[d, shift right, "\pi"'] & \tilde{\mathcal{C}} \arrow[dl, "\tilde{\pi}"] \\ S, \arrow[u, shift right, "\sigma_i"'] & \end{tikzcd} \tag{5}$$

where

- $\pi, \tilde{\pi}$ are flat, projective families of quasi-stable genus 0 curves, all geometric fibres of $\tilde{\pi}$ are isomorphic to $\mathbb{P}^1$;
- μ, φ satisfy $\mu_*([\mathcal{C}_s]) = [\tilde{\mathcal{C}}_s]$ and $\varphi_*([\mathcal{C}_s]) = d[\tilde{\mathcal{C}}_s]$ for all geometric points $s \in S$;
- every component of a geometric fibre $\mathcal{C}_s$, which is contracted by both μ_s, φ_s contains at least three special points (nodes or markings).

The morphisms between $\mathcal{F} \in \mathcal{M}^{d,n}(S)$ and $\mathcal{F}' \in \mathcal{M}^{d,n}(S')$ lying over a morphism $\psi : S \to S'$ of schemes are exactly the pullback-diagrams identifying $\mathcal{F}$ as the base change of the diagram $\mathcal{F}'$ by ψ.

Theorem 3.8. *There is a natural isomorphism between the quotient stack $\mathcal{Y}_{d,n}/PGL_2$ and $\mathcal{M}^{d,n}$. Here, an element of $(\mathcal{Y}_{d,n}/PGL_2)(S)$ coming from a family*

$$(\pi : \mathcal{C} \to S; \sigma_1, \dots, \sigma_n : S \to \mathcal{C}; (f_1, f_2) : \mathcal{C} \to \mathbb{P}^1 \times \mathbb{P}^1) \in \mathcal{Y}_{d,n}(S)$$

is identified with the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\pi \times f_1"] \arrow[r, shift right, "\pi \times f_2"'] \arrow[d, shift right, "\pi"'] & S \times \mathbb{P}^1 \arrow[dl] \\ S \arrow[u, shift right, "\sigma_i"'] & \end{tikzcd}$$

in $\mathcal{M}^{d,n}(S)$.

Proof. We give a natural identification of the objects of $\mathcal{Y}_{d,n}/\mathrm{PGL}_2$ and $\mathcal{M}^{d,n}$ over a scheme S, which respects isomorphisms of these objects. Let $G = \mathrm{PGL}_2$ for the remainder of the proof.

Using [Rom05, Theorem 4.1], we know that an object of $\mathcal{Y}_{d,n}/\mathrm{PGL}_2$ over S is nothing but a G-torsor $\psi : T \to S$ over S together with a morphism $(f, \sigma) : T \to \mathcal{Y}_{d,n}$ of G-stacks. Note that as G is quasi-affine, the torsor T is actually a scheme (not an algebraic space); see [sga71, VII, Corollaire 7.9]. The map $f : T \to \mathcal{Y}_{d,n}$ is just a usual (1)-morphism, so it is equivalent to specifying the data

$$\begin{tikzcd} \mathcal{C} \arrow[r, "{(f_1, f_2)}"] \arrow[d, shift right, "\pi"'] & \mathbb{P}^1 \times \mathbb{P}^1 \\ T \arrow[u, shift right, "s_i"'] & \end{tikzcd} \tag{6}$$

of a family of stable maps to $\mathbb{P}^1 \times \mathbb{P}^1$ with degree $(1, d)$ and n markings. The 2-morphism σ relates the two paths in the commutative diagram

$$\begin{tikzcd} G \times T \arrow[r, "\mathrm{id}_G \times \psi"] \arrow[d, "\mu_T"'] & G \times \mathcal{Y}_{d,n} \arrow[d, "\mu_Y"] \\ T \arrow[r, "\psi"] & \mathcal{Y}_{d,n}, \end{tikzcd}$$

where the two vertical arrows are the corresponding actions. Now the morphism $\psi \circ \mu_T$ corresponds to the data of the family

$$(\mu_T^*\pi : \mu_T^*\mathcal{C} \to G \times T; \mu_T^* s_i : G \times T \to \mu_T^*\mathcal{C}; (f_1, f_2) : \mu_T^*\mathcal{C} \to \mathbb{P}^1 \times \mathbb{P}^1)$$

of stable maps. On the other hand, $\mu_Y \circ (\mathrm{id}_G \times \psi)$ corresponds to the data

$$(\mathrm{id}_G \times \pi : G \times \mathcal{C} \to G \times T; \mathrm{id}_G \times s_i : G \times T \to G \times \mathcal{C}; (g, c) \mapsto (g.f_1(c), g.f_2(c))).$$

Giving a 2-morphism σ between $\psi \circ \mu_T$ and $\mu_Y \circ (\mathrm{id}_G \times \psi)$ is thus equivalent to an isomorphism between these two families. One sees that the necessary data is a morphism

$$\eta : G \times \mathcal{C} \to \mathcal{C}$$

making the diagram

$$\begin{tikzcd} G \times \mathcal{C} \arrow[r, "\eta"] \arrow[d, "\mathrm{id}_G \times \pi"'] & \mathcal{C} \arrow[d, "\pi"] \\ G \times T \arrow[r, "\mu_T"] & T \end{tikzcd} \tag{7}$$

cartesian. Moreover, the map η should be compatible with the sections s_i and the maps to $\mathbb{P}^1 \times \mathbb{P}^1$.

Now for (f, σ) to be a 1-morphism of G-stacks, the map σ needs to satisfy a compatibility condition. For any scheme Z and $g, h \in G(Z), x \in T(Z)$ let

$$\sigma_g^x : g.\psi(x) \to \psi(g.x)$$

be the isomorphism of families of stable maps over Z given by σ. Then we require

$$\sigma_g^{h.x} \circ (g.\sigma_h^x) = \sigma_{gh}^x.$$

Unwinding the definitions, one sees that this is exactly equivalent to the map η satisfying $\eta(g, \eta(h, x)) = \eta(gh, x)$. As the diagram (7) is cartesian, we know that the identity element of G acts by an isomorphism. Hence, the compatibility condition exactly requires η to be a group action. Before we summarize, we note that assuming η is a group action, the diagram (7) is automatically cartesian if it is commutative.

We have seen above that an object of $\mathcal{Y}_{d,n}/\mathrm{PGL}_2$ over S is equivalent to a G-torsor $\psi : T \to S$, a family (6) of stable maps and an action of G on $\mathcal{C}$ making all the arrows in the diagram (6) G-equivariant. We now rearrange this data slightly as the following diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "\pi \times f_1"] \arrow[r, shift right, "\pi \times f_2"'] \arrow[d, shift right, "\pi"'] & T \times \mathbb{P}^1 \arrow[dl, "\tilde{\pi}"] \\
T \arrow[u, shift right, "s_i"']
\end{tikzcd} \tag{8}$$

and then take the quotient by G at every point of the diagram. As all G-actions are free, the quotients are again algebraic spaces. Also, as T is a G-torsor over S, we have $T/G = S$. Thus the quotient diagram looks like:

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d, shift right, "\pi'"'] & \tilde{\mathcal{C}} \arrow[dl, "\tilde{\pi}'"] \\
S, \arrow[u, shift right, "s_i'"']
\end{tikzcd} \tag{9}$$

and all maps between the quotients keep their respective fppf-local properties (flat, proper, finite type, etc.) because of Remark C.2. To prove that $\pi', \tilde{\pi}'$ are still projective, we will construct G-linearized relatively ample line bundles on $\mathcal{C}, T \times \mathbb{P}^1$ and then use a descent argument. The ingredients for these line bundles are the following:

- The relative dualizing sheaf $\omega_{\pi'}$ carries a natural G-action (this follows easily from the compatibility of the formation of relative dualizing sheaves with base change).
- The line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$ has a natural G-linearization and hence its pullbacks via f_1, f_2 and via the projection $T \times \mathbb{P}^1 \mapsto \mathbb{P}^1$ have induced G-linearizations.
- On $\mathcal{C}$ we also want to have line bundles corresponding to the Weyl divisors $s_i(T)$. To assure that they carry a G-linearization, we will make a detour via the quotient $\mathcal{C}' = \mathcal{C}/G$, because $\mathrm{Pic}_G(\mathcal{C}) = \mathrm{Pic}(\mathcal{C}')$. Now we have that $s_i' : S \to \mathcal{C}'$ is a section of the finite type, flat, separated morphism π'. If we knew that $\mathcal{C}'$ was a scheme, then Proposition B.2 would imply that s_i' is a Cartier divisor, hence it would give rise to our desired G-linearized line bundle $\mathcal{O}(s_i)$ on $\mathcal{C}$. However, being a Cartier divisor can be checked fpqc-locally. This is because a Cartier divisor is the same as a Koszul-regular immersion of codimension 1 (see [Sta14, Tag 061T] for a definition) and this notion is fpqc-local on the target (see [Sta14, Tag 0694]). But to check

this, we again pull back via the map $\mathcal{C} \to \mathcal{C}'$ and now Proposition B.2 really shows that s_i, which are the pullbacks of the maps s_i', are Cartier divisors. Clearly $\mathcal{O}_{\mathbb{P}^1}(2)$ is $\tilde{\pi}$-relatively ample on $T \times \mathbb{P}^1$. On the other hand, as the family $\pi : \mathcal{C} \to T$ was supposed to be stable, the bundle $\omega_\pi(s_1 + \ldots + s_n) \otimes f_1^* \mathcal{O}(l) \otimes f_2^* \mathcal{O}(m)$ is π-ample for l, m sufficiently large.

But then by [sga71, VIII,Proposition 7.8], the spaces $\mathcal{C}', \tilde{\mathcal{C}}$ are actually schemes and the line bundles above descend to relatively ample line bundles on them for the morphisms $\pi', \tilde{\pi}'$. As these morphisms are also of finite type and proper, they are projective.

Note further that the geometric fibres of $\tilde{\pi}'$ are isomorphic to $\mathbb{P}^1$. Hence we have obtained a family in $\mathcal{M}^{d,n}(S)$. Conversely, for such a family as above, we can consider the morphism

$$\psi : T = \mathrm{Isom}_S(\tilde{\mathcal{C}}, S \times \mathbb{P}^1) \to S.$$

Here for schemes X, Y over S, the scheme $\mathrm{Isom}_S(X, Y)$ represents the functor

$$\mathrm{Sch}_S \to \mathrm{Sets}, S' \mapsto \{\psi : X_{S'} \to Y_{S'} : \psi_s \text{ isomorphism } \forall s \in S'\}.$$

The existence of this scheme follows from the fact that $\pi', \tilde{\pi}'$ are proper and that π' is flat. We want to show that ψ is a G-torsor. It clearly has a G-action by postcomposition. On the other hand, the map $\tilde{\pi}'$ is smooth. Thus it has sections étale locally and hence is a trivial $\mathbb{P}^1$-bundle after an étale base change $S' \to S$. But then the pullback of T to S' is simply $S' \times \mathrm{PGL}_2$, a trivial G-torsor.

Now after pulling back the diagram (9) above by the map ψ, we can clearly trivialize the $\mathbb{P}^1$-bundle $\psi^* \tilde{\mathcal{C}}$. Indeed, by definition the scheme T parametrizes all possible ways to perform this trivialization fibrewise. Now we have again a diagram as in (8) and the action of G on T lifts to an action of the other spaces under the pullback $T \to S$. This is not canonical, but it exactly mirrors the choice we had when translating from the 2-morphism σ above to an action map $\eta : G \times \mathcal{C} \to \mathcal{C}$.

One verifies that the two operations we described are inverse to one another and one checks, that they respect the isomorphisms inherent in the objects of $\mathcal{Y}_{d,n}/\mathrm{PGL}_2$ and $\mathcal{M}^{d,n}$ over S. Here one uses that starting with a diagram (8), the G-torsors T and $\mathrm{Isom}_S(\tilde{\mathcal{C}}, S \times \mathbb{P}^1)$ over S are isomorphic. This is because there exists a G-equivariant map $T \to \mathrm{Isom}_S(\tilde{\mathcal{C}}, S \times \mathbb{P}^1)$ by the universal property of $\mathrm{Isom}_S(\tilde{\mathcal{C}}, S \times \mathbb{P}^1)$.

The explicit description of the image of a family in $\mathcal{Y}_{d,n}/\mathrm{PGL}_2$ coming from a family in $\mathcal{Y}_{d,n}$ is obvious from the above construction. Hence, the proof is finished. □

Corollary 3.9. *For $\boldsymbol{d} = (d|d_1, \ldots, d_n)$ admissible, we have that the scheme $M(d|d_1, \ldots, d_n)$ is a coarse moduli space for the functor $Sch_{\mathbb{C}} \to Sets$, associating to a scheme S the set of isomorphism classes of diagrams* (5) *in $\mathcal{M}^{d,n}$ such that for all $s \in S$ the map*

$$(\mathcal{C}_s; \sigma_1(s), \ldots, \sigma_n(s)) \to \tilde{\mathcal{C}}_s \times \tilde{\mathcal{C}}_s$$

is semistable with respect to $\boldsymbol{d}$ as in Lemma 2.9.

Proof. The substack $\mathcal{Y}_{d,n}^{ss,\mathbf{d}} \subset \mathcal{Y}_{d,n}$ is open and G-invariant. By Lemma C.4, the coarse moduli space of its quotient $\mathcal{Y}_{d,n}^{ss,\mathbf{d}}/G$ is exactly $M(d|d_1, \ldots, d_n)$. On the other hand, Theorem 3.8 identifies this quotient with the open substack of $\mathcal{M}^{d,n}$ whose objects are exactly those described above. But then the functor sending S

to the set of isomorphism classes of such objects naturally has $M(d|d_1,\dots,d_n)$ as a coarse moduli space. □

Remark 3.10. If in the diagram (5), the fibres $\mathcal{C}_s$ of $\mathcal{C}$ are smooth, the map μ is an isomorphism and thus we have $\mathcal{C} \cong \tilde{\mathcal{C}}$. Hence in this case, the scheme S really parametrizes a family of self-maps $\mathcal{C}_s \to \mathcal{C}_s$ of degree d and the locus $M(d|d_1,\dots,d_n)^o$ of classes of maps with smooth source curve is a coarse moduli space for the functor

$$\mathrm{Sch}_{\mathbb{C}} \to \mathrm{Sets}$$

$$S \mapsto \left\{ \begin{array}{c} \mathcal{C} \circlearrowleft \phi \\ \pi \Big\downarrow \Big\uparrow \sigma_i \\ S \end{array} : \begin{array}{l} \pi \text{ flat, projective family of} \\ \text{smooth genus 0 curves,} \\ \sigma_1,\dots,\sigma_n \text{ disjoint sections of } \pi \\ \phi \text{ self-map over } S \text{ with} \\ \phi_*[\mathcal{C}_s] = d[\mathcal{C}_s] \text{ for all } s \in S \end{array} \right\} /\text{iso}.$$

Here, an isomorphism of families as above is an isomorphism $\mathcal{C}_1 \to \mathcal{C}_2$ making all diagrams commute.

Note that if some of the d_i are greater than $(d-1+\sum_{j=1}^n d_i)/2$, we need to require above that $\sigma_i(s)$ is not a fixed point of ϕ_2 for all $s \in S$.

Remark 3.11. The forgetful map $F : Y_{d,n+1} \to Y_{d,n}$ of the last marked point is G-equivariant and the preimage of $Y_{d,n}^{ss,\mathbf{d}}$ is $Y_{d,n+1}^{ss,(\mathbf{d},0)}$, with $(\mathbf{d},0) = (d|d_1,\dots,d_n,0)$. Thus F induces a map

$$F : M(d|d_1,\dots,d_n,0) \to M(d|d_1,\dots,d_n).$$

Similarly one constructs sections

$$\sigma_i : M(d|d_1,\dots,d_n) \to M(d|d_1,\dots,d_n,0)$$

of F. We note that over the locus M^* of points $[f] \in M(d|d_1,\dots,d_n)^o$ with trivial G-isotropy and such that $f \in Y_{d,n}$ has no automorphisms, the family F and the sections σ_i are exactly the universal family $\pi : \mathcal{C}|_{M^*} \to M^*$ from above. This follows because by [FP97], the locus $Y_{d,n}^*$ of points without automorphisms in $Y_{d,n}$ carries the restriction of the forgetful map from $Y_{d,n+1}$ as a universal family. On the other hand, the geometric quotient of a scheme by a free action is isomorphic to the corresponding quotient stack.

3.4. **Rationality.** In this section, we show that the spaces $M(d|d_1,\dots,d_n)$ are rational. First note that for different weights $d_1,\dots,d_n$, all (nonempty) spaces $M(d|d_1,\dots,d_n)$ are canonically birational. Indeed, they are categorical quotients for varying open, invariant subsets of $Y_{d,n}$, hence they are isomorphic over the image of the intersection of those subsets. Thus, given d,n we can choose the weights arbitrarily without changing the birational class of $M(d|d_1,\dots,d_n)$. This also shows that they are birational to a categorical quotient of some nonempty, $G = \mathrm{PGL}_2$-invariant subset of $\mathrm{Rat}_d \times ((\mathbb{P}^1)^n \setminus \Delta)$ by G, if this quotient exists.

Theorem 3.12. *The spaces $M(d|d_1,\dots,d_n)$ are rational.*

Proof. Our argument will basically be an induction on n for every fixed $d \geq 0$. In the case $n = 0$, Levy shows that $\mathrm{Rat}_d /\!\!/ G$ is rational for $d \geq 2$ (see [Lev11, Theorem 4.1]).

For $n = 1$ we must now also cover the case $d = 1$, but then we have seen that all nonempty moduli spaces $M(1|d_1)$ are isomorphic to $M(1|1) \cong \mathbb{P}^1$, which is rational.

For $d \geq 2$ consider the open subset

$$\mathcal{U} = \{(f, p) : p \neq f(p) \neq f(f(p)) \neq p\} \subset \mathrm{Rat}_d \times \mathbb{P}^1.$$

By the definition of the action of G, this subset is invariant. But on $\mathcal{U}$ we can use the action of G to move p to $0 = [0:1]$, $f(p)$ to $\infty = [1:0]$ and $f(f(p))$ to $1 = [1:1]$ in a unique way. More precisely, let

$$\mathcal{U}_0 = \{(f, 0) : f(0) = \infty, f(\infty) = 1\} \subset \mathcal{U}.$$

Then the action map restricted to $G \times \mathcal{U}_0$ is an isomorphism onto $\mathcal{U}$. Thus, $\mathcal{U}/\!\!/G = \mathcal{U}_0$. But the conditions $f(0) = \infty, f(\infty) = 1$ give linear conditions on the coefficients of $f \in \mathrm{Rat}_d \subset \mathbb{P}^{2d+1}$ and hence $\mathcal{U}_0$ is rational.

Finally, for $n \geq 2$ we note that the case $d = 0, n = 2$ is clear, as the only nonempty moduli spaces here are isomorphic to $M(0|1,1)$ (which is a point by Lemma 3.1). By the discussion preceding the theorem, we may restrict to the cases $M = M(0|1,1,0,\dots,0)$, $M = M(d|0,\dots,0)$ for d even and $M = M(d|1,0,\dots,0)$ for d odd. But by the forgetful map $F : M \to \tilde{M}$ of the last marked point (which carries weight 0), these spaces map to the corresponding spaces $\tilde{M}$ with one mark less. These are rational by induction. Using Remark 3.11, the forgetful maps are flat, projective $\mathbb{P}^1$-fibrations over the locus in $\tilde{M}$ parametrizing self-maps of smooth curves without isotropy or automorphisms. Moreover, as $n \geq 2$, the map F has a section σ_1 (whose image is the divisor $D_{\{1,n\},0}$). But any flat $\mathbb{P}^1$-fibration with a section is actually locally trivial (see [Har10, Proposition 25.3]), so indeed M is rational. ☐

4. The Picard group of $M(d|d_1, \dots, d_n)$

Lemma 4.1. *Let $\tilde{G} = PGL_2$ act freely on a normal variety X and let $\phi : X \to X'$ be a geometric quotient. Then the map*

$$\phi^* : Pic(X') \otimes_{\mathbb{Z}} \mathbb{Q} \to Pic(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism.

Proof. The map ϕ^* factors as the composition

$$\mathrm{Pic}(X') \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Pic}^{\tilde{G}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By [MFK94, 1.§3] the first map is an isomorphism (even before tensoring with $\mathbb{Q}$). By [Dol03, Thereom 7.1, Exercise 7.2] the second map is injective (using that PGL_2 has only trivial characters as the same is true for SL_2). Finally, [Dol03, Corollary 7.2] implies that the right map is surjective. ☐

Corollary 4.2. *Let $\boldsymbol{d} = (d|d_1, \dots, d_n)$ be admissible with $(d, n) \neq (2, 0)$, $(1, 1)$. Then the quotient map*

$$\phi : Y_{d,n}^{ss,\boldsymbol{d}} \to M(d|d_1, \dots, d_n)$$

induces an isomorphism

$$\phi^* : Pic(M(d|d_1, \dots, d_n)) \otimes_{\mathbb{Z}} \mathbb{Q} \to Pic(Y_{d,n}^{ss,\boldsymbol{d}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and we have $Pic(Y_{d,n}^{ss,\boldsymbol{d}}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong Cl(Y_{d,n}^{ss,\boldsymbol{d}}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. By Corollary 3.5 we know that there is an open, $\tilde{G}$-invariant subset $Y^f \subset Y_{d,n}^{ss,\mathbf{d}}$ with complement of codimension at least 2, on which $\tilde{G}$ acts freely. Let $M^f \subset M(d|d_1,\dots,d_n)$ be the image of Y^f under ϕ. Then as ϕ is a geometric quotient, this set is open with $Y^f = \phi^{-1}(M^f)$. But this implies that M^f has complement of codimension at least 2, too. As $Y_{d,n}$ and $M(d,n)$ only have finite quotient singularities, their corresponding rational Class groups and Picard groups coincide, and using that the Class group does not change when removing sets of codimension at least 2 we obtain

$$\begin{aligned}\mathrm{Pic}(Y_{d,n}^{ss,\mathbf{d}})\otimes\mathbb{Q} &\cong \mathrm{Cl}(Y_{d,n}^{ss,\mathbf{d}})\otimes\mathbb{Q}\cong \mathrm{Cl}(Y^f)\otimes\mathbb{Q}\cong \mathrm{Pic}(Y^f)\otimes\mathbb{Q},\\ \mathrm{Pic}(M(d|d_1,\dots,d_n))\otimes\mathbb{Q} &\cong \mathrm{Cl}(M(d|d_1,\dots,d_n))\otimes\mathbb{Q}\cong \mathrm{Cl}(M^f)\otimes\mathbb{Q},\\ &\cong \mathrm{Pic}(M^f)\otimes\mathbb{Q}.\end{aligned}$$

Note further that the restricted map $\phi : Y^f \to M^f$ is not only a geometric quotient for the action of $G = \mathrm{SL}_2$, but also for the induced action of $\tilde{G}$ on Y^f. Using Lemma 4.1 we conclude the desired statement. $\square$

We see that we can reduce the computation of $\mathrm{Pic}(M(d|d_1,\dots,d_n))\otimes\mathbb{Q}$ to the computation of $\mathrm{Cl}(Y_{d,n}^{ss,\mathbf{d}})\otimes\mathbb{Q}$. As $Y_{d,n}^{ss,\mathbf{d}} \subset Y_{d,n}$ is an open set whose complement is a union of divisors, we will first compute generators and relations for the whole group $\mathrm{Cl}(\overline{M}_{0,n}(\mathbb{P}^1\times\mathbb{P}^1,(1,d)))\otimes\mathbb{Q}$ and then obtain the desired group as a quotient by the classes coming from the unstable loci.

4.1. **The Picard group of $\overline{M}_{0,n}(\mathbb{P}^1\times\mathbb{P}^1,(1,d))$.** We will use methods adapted from [Pan99] to compute the Picard group of $Y_{d,n} = \overline{M}_{0,n}(\mathbb{P}^1\times\mathbb{P}^1,(1,d))$. First we find a set of divisors generating the group. Then we construct test curves, which we intersect with those divisors. As the intersection numbers only depend on the class of the divisors in the Picard group, we will be able to show linear independence for some subsets of our generators by comparing intersection numbers. We mention that in [Opr05], Oprea has computed a system of generators for the rational Picard group of $\overline{M}_{0,n}(X,\beta)$ for $X = G/P$ a projective homogeneous space, which of course also covers $X = \mathbb{P}^1\times\mathbb{P}^1$.

4.1.1. *Generators.* Consider the open set $Y_{d,n}^o = M_{0,n}(\mathbb{P}^1\times\mathbb{P}^1,(1,d)) \subset Y_{d,n}$ corresponding to maps with a smooth domain and its complement $Y_{d,n}^\partial = Y_{d,n}\setminus Y_{d,n}^o$, the boundary of $Y_{d,n}$. Then we have an exact sequence

$$A_{2d+n}Y_{d,n}^\partial \to A_{2d+n}Y_{d,n} \to A_{2d+n}Y_{d,n}^o \to 0 \tag{10}$$

of groups of algebraic $(2d+n)$-cycles modulo rational equivalence (see [Ful98, Proposition 1.8]). We note that as $\dim(Y_{d,n}) = \dim(Y_{d,n}^o) = 2d+1+n$, we have $A_{2d+n}Y_{d,n} = \mathrm{Cl}(Y_{d,n})$, $A_{2d+n}Y_{d,n}^o = \mathrm{Cl}(Y_{d,n}^o)$. Furthermore, by [FP97, §3.1], $Y_{d,n}^\partial$ is of pure dimension $2d+n$. Recall that by [KP01, Corollary 2] we know that its irreducible components are the divisors

$$D_{B,k} = D(\{1,\dots,n\}\setminus B,(1,d-k);B,(0,k)),$$

where $B \subset \{1,\dots,n\}$, $0 \le k \le d$ with $|B| \ge 2$ for $k = 0$. In our interpretation as pointed graphs of rational maps, this is the divisor of graphs, where the vertical section contains the markings B and maps with degree k. We conclude that $A_{2d+n}Y_{d,n}^\partial$ is the free abelian group generated by these divisors.

It remains to find generators for the Class group of $Y^o_{d,n}$. For this we use the fact that the map J defined in Lemma 2.5 is an isomorphism over this locus.

Corollary 4.3. *For $d \geq 1$, $n \geq 0$, the rational Picard group of $Y^o_{d,n}$ is given by*

$$Pic(Y^o_{d,n}) \otimes \mathbb{Q} = \begin{cases} 0 & \text{for } n = 0, \\ \mathbb{Q}\pi^*_{\mathbb{P}^1}\mathcal{O}(1) & \text{for } n = 1, 2, \\ 0 & \text{for } n \geq 3, \end{cases}$$

where for $n = 1, 2$ the map $\pi_{\mathbb{P}^1} : Y^o_{d,n} \cong Rat_d \times ((\mathbb{P}^1)^n \setminus \Delta) \to \mathbb{P}^1$ is the projection on the first factor $\mathbb{P}^1$.

In the case $d = 0$, we have $Y^o_{d,n} = M_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (0, 1)) = M_{0,n}(\mathbb{P}^1, 1) \times \mathbb{P}^1$ and an induced projection $p : Y^o_{d,n} \to \mathbb{P}^1$ on the last factor. This gives an additional direct summand $\mathbb{Q}\mathcal{G}$ for $\mathcal{G} = p^*(\mathcal{O}_{\mathbb{P}^1}(1))$ to the formula above.

Proof. From Lemma 2.5 we see that $Y^o_{d,n} \cong \mathrm{Rat}_d \times ((\mathbb{P}^1)^n \setminus \Delta)$. By [Isc74], the Picard group of the product of a rational variety with another variety is the direct sum of their corresponding Picard groups. Now for $d \geq 1$, the set Rat_d is the complement of a hypersurface in $\mathbb{P}^{2d+1}$ and thus has a finite Picard group, which vanishes after tensoring with $\mathbb{Q}$. For $d = 0$ the degree d maps from $\mathbb{P}^1$ to itself are exactly the constant maps, so $\mathrm{Rat}_d = \mathbb{P}^1$ giving the additional summand $\mathbb{Q}\mathcal{G}$.

For the other factor we note that the rational Picard group of $(\mathbb{P}^1)^n$ is freely generated by the classes

$$H_i = \pi_i^*\mathcal{O}_{\mathbb{P}^1}(1),$$

where $\pi_i : (\mathbb{P}^1)^n \to \mathbb{P}^1$ is the projection on the i-th factor. The set Δ we remove is the union of irreducible divisors

$$\Delta_{ij} = \{(p_1, \ldots, p_n); p_i = p_j\}.$$

Their divisor class equals $[\Delta_{ij}] = H_i + H_j$. By the excision exact sequence we have

$$\mathrm{Pic}((\mathbb{P}^1)^n \setminus \Delta) \otimes \mathbb{Q} = \bigoplus_{i=1}^{n} \mathbb{Q}H_i / \bigoplus_{i \neq j} \mathbb{Q}[\Delta_{ij}].$$

For $n = 0$ this is obviously trivial, for $n = 1$ we have exactly $\mathbb{Q}H_1$. For $n = 2$ we obtain $\mathbb{Q}H_1 \oplus \mathbb{Q}H_2/\mathbb{Q}(H_1 + H_2) \cong \mathbb{Q}H_1$. For $n \geq 3$ we note that

$$[\Delta_{1,2}] + [\Delta_{1,3}] - [\Delta_{2,3}] = H_1 + H_2 + H_1 + H_3 - H_2 - H_3 = 2H_1.$$

Similarly we can represent all other classes H_i and thus one sees that the Picard group of $(\mathbb{P}^1)^n \setminus \Delta$ is trivial in this case. □

Theorem 4.4. *For $d \geq 0$ and $n \geq 0$, the rational Picard group of $Y_{d,n}$ is generated by $D_{B,k}$ for $B \subset \{1, \ldots, n\}$, $0 \leq k \leq d$ and $|B| \geq 2$ if $k = 0$, together with the divisor class*

$$\mathcal{H} = (\pi_1 \circ ev_1)^*(\mathcal{O}_{\mathbb{P}^1}(1))$$

in the cases $n = 1, 2$ and the class $\mathcal{G} = \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$ for $d = 0$, where

$$\pi : \overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (0, 1)) = \overline{M}_{0,n}(\mathbb{P}^1, 1) \times \mathbb{P}^1 \to \mathbb{P}^1$$

is the projection on the factor $\mathbb{P}^1$.

Proof. The first part of the theorem is a combination of the discussion at the beginning of this section together with Corollary 4.3. Here we note that the divisor $\mathcal{H}$ above obviously restricts to $\pi^*_{\mathbb{P}^1}\mathcal{O}(1)$ on $Y^o_{d,n}$ and similarly for $\mathcal{G}$. □

In the following, we denote the set of generators above by $\mathfrak{G}_{d,n}$.

4.1.2. *Relations.* Now we want to find all relations between the generators of the rational Picard group of $Y_{d,n}$ found above. Using techniques adapted from [Pan99] we construct curves in $Y_{d,n}$ and intersect them with the divisors above. Relations among the classes of the divisors would imply relations between these intersection numbers. Hence using test curves for which the vectors of intersection numbers are linearly independent, we can show the linear independence of some of the generators above. The construction of the curves will rely on the following explicit family of stable maps.

Consider the variety $S = \mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $([z : w], [x_0 : x_1])$ and the projections π_1, π_2 to the two factors. Let $d \geq 0$, $0 \leq k \leq d$ and

$$\mathcal{N} = \mathcal{O}_S(d,k) = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(d)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(k)).$$

Two global sections $s_1, s_2 \in H^0(S, \mathcal{N})$ are given by

$$s_1 = (z - a_1 w)(z - a_2 w) \dots (z - a_{d-k} w)(x_1 z - b_1 x_0 w) \dots (x_1 z - b_k x_0 w),$$
$$s_2 = (z - c_1 w)(z - c_2 w) \dots (z - c_{d-k} w)(x_1 z - d_1 x_0 w) \dots (x_1 z - d_k x_0 w),$$

where $a_i, b_j, c_i, d_j \in \mathbb{C}^*$ for $i = 1, \dots, d-k$, $j = 1, \dots, k$ are sufficiently general. In the case $d = 0$ we have $\mathcal{N} = \mathcal{O}$ and we set $s_1 = a, s_2 = c$ with $(a,c) \neq (0,0)$. These sections define a rational map

$$\mu : S - \to \mathbb{P}^1 \times \mathbb{P}^1,$$
$$([z : w], [x_0 : x_1]) \mapsto ([z : w], [s_1 : s_2]).$$

For $k = 0$, this is a morphism and induces a constant map of degree d on the fibres of π_2. For $k \geq 1$ its *base points* are exactly

- the $k(d-k)$ points $([c_i : 1], [c_i/b_j : 1])$ for $i = 1, \dots, d-k$, $j = 1, \dots, k$,
- the $k(d-k)$ points $([a_i : 1], [a_i/d_j : 1])$ for $i = 1, \dots, d-k$, $j = 1, \dots, k$,
- the 2 points $([0 : 1], [0 : 1])$ and $([1 : 0], [1 : 0])$.

To illustrate this, consider Figure 2. We draw for $d = 4, k = 2$ the vanishing sets of s_1, s_2 on $\mathbb{P}^1 \times \mathbb{P}^1$. The horizontal lines of these sets come from the zeroes at $[z : w] = [a_i : 1]$ and $[z : w] = [c_i : 1]$, respectively. The curved parts depict the graphs of the linear maps $[x_0 : x_1] \mapsto [b_j x_0 : x_1]$ and $[x_0 : x_1] \mapsto [d_j x_0 : x_1]$, respectively. The base points are exactly the intersection points of $V(s_1)$ with $V(s_2)$. In particular, we see the two special intersection points over $[0 : 1]$ and $[1 : 0]$, where k zeroes of s_1 and s_2 come together. From the formulas above, we conclude that for sufficiently general a_i, b_j, c_i, d_j all base points have distinct $[x_0 : x_1]$-coordinates.

Now let $n \geq 0$ and assume we have sections $\sigma_1, \dots, \sigma_n : \mathbb{P}^1 \to S$ of $\pi_2 : S \to \mathbb{P}^1$. An *intersection point* is a point $p \in S$ lying in the image of at least two of the sections σ_i. The set of *special points* is the union of base points and intersection points.

Lemma 4.5. *Assume that there are finitely many intersection points and that at each of them, the sections σ_i meeting there have distinct tangent directions. Furthermore, assume that the $[x_0 : x_1]$-coordinates of all special points are pairwise distinct. Then the blow-up $\overline{S}$ of S at all special points resolves the indeterminacies of μ and the induced maps $\mu : \overline{S} \to \mathbb{P}^1 \times \mathbb{P}^1$, $\sigma_1, \dots, \sigma_n : \mathbb{P}^1 \to \overline{S}$ give a Kontsevich stable family of n-pointed genus 0 curves*

$$\mathcal{C} = (\pi_2 : \overline{S} \to \mathbb{P}^1; \sigma_1, \dots, \sigma_n; \mu : \overline{S} \to \mathbb{P}^1 \times \mathbb{P}^1)$$

over $\mathbb{P}^1$, which induces a map $\psi : \mathbb{P}^1 \to Y_{d,n} = \overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1,d))$.

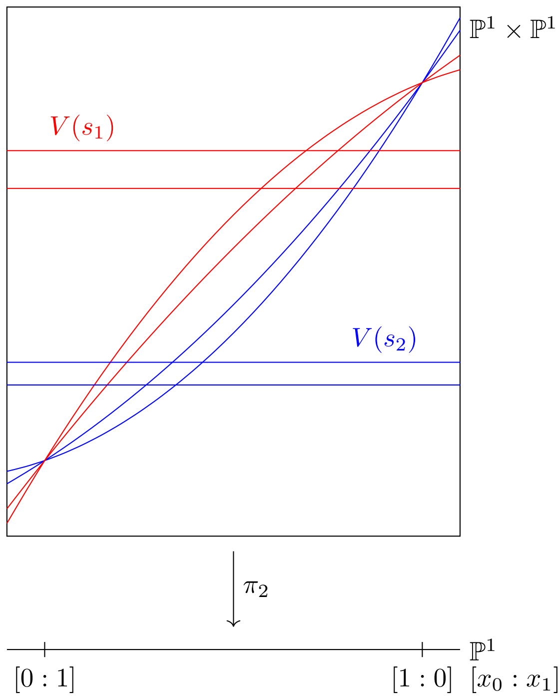

FIGURE 2. An illustration of $V(s_1), V(s_2) \subset \mathbb{P}^1 \times \mathbb{P}^1$ for $d = 4, k = 2$.

Moreover, for a sufficiently general choice of the parameters a_i, b_i, c_i, d_i, the map ψ intersects the boundary of $Y_{d,n}$ transversally and a point $x = [x_0 : x_1]$ maps to the boundary if and only if there is a special point $p \in S$ with $x = \pi_2(p)$. Let

$$B = \{i : \sigma_i \text{ passes through } p\} \subset \{1, \ldots, n\}$$

be the set of indices of sections through p and let $m = 0$ if p is not a base point, $m = 1$ if p is a base point different from $([1:0],[1:0])$ and $([0:1],[0,1])$ and $m = k$ otherwise. Then at x the map ψ intersects exactly the boundary divisor $D_{B,m}$.

Proof. We first show that the blow-up of the special points resolves the indeterminacies of our rational map. Note that all base points except $([1:0],[1:0])$ (which is easily seen to behave similar to $([0:1],[0:1])$) lie in $\mathbb{A}^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$. Thus we will set $w = x_1 = 1$ and use affine coordinates z, x_0.

Around the point $p = (c_i, c_i/b_j)$ we can identify

$$\mathrm{Bl}_p\mathbb{A}^2 = \{((z,w),[T:S]) : (z - c_i)S - (x_0 - c_i/b_j)T\} \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Then the second component of the map the map μ extends around p by sending $((z, x_0), [T : S])$ to

$$\frac{(z-a_1)\ldots(z-b_{j-1}x_0)(T-b_jS)(z-b_{j+1}x_0)\ldots(z-b_kx_0)}{(z-c_1)\ldots(z-c_{i-1})T(z-c_{i+1})\ldots(z-d_kx_0)}.$$

Similarly, for $q = (0,0)$, we have

$$\mathrm{Bl}_q\mathbb{A}^2 = \{((z,w),[T:S]) : zS - x_0T\} \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Here, the second component of μ can be extended by sending $((z, x_0), [T : S])$ to
$$\frac{(z - a_1) \dots (z - a_{d-k})(T - b_1 S) \dots (T - b_k S)}{(z - b_1) \dots (z - b_{d-k})(T - d_1 S) \dots (T - d_k S)}.$$
It is also clear that the sections σ_i factor through $\overline{S}$. As their tangent directions in every special point are distinct, they map to distinct points on the exceptional divisors. As $\pi_2 : \overline{S} \to \mathbb{P}^1$ is dominant, it is flat and as blowups are projective, it is projective. The fibres of π_2 are isomorphic to $\mathbb{P}^1$ except for the fibres over projections of special points, which are nodal genus 0 curves with two branches. The exceptional divisors are one of the branches and as can be seen above, they map to $\mathbb{P}^1 \times \mathbb{P}^1$ with degree $(0, 1)$ or $(0, k)$ at the base points and are contracted for the intersection points which are not base points, and thus map with degree $(0, 0)$. The fact that the induced map $\mathbb{P}^1 \to Y_{d,n}$ meets the boundary transverally follows because the total space $\overline{S}$ of the family is smooth (see [Vak99, §4.4]). $\square$

We will find that for a given subset $B \subset \{1, \dots, n\}$, a particular choice of the sections σ_i will be very useful.

Definition 4.6. For $\alpha \in \mathbb{C}^*$ we define
$$S_\alpha : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1, [x_0 : x_1] \mapsto ([\alpha x_0 : x_1], [x_0 : x_1]).$$
For $B \subset \{1, \dots, n\}$ and $0 \le k \le d$ we set

- $\sigma_j = S_{\alpha_j}$ for general $\alpha_j \in \mathbb{C} \setminus \{0, 1\}$ for $j \in B$,
- $\sigma_j = (p_j, \mathrm{id})$ with general $p_j = [p_j : 1] \in \mathbb{C} \subset \mathbb{P}^1$ for $j \notin B$.

We will specify the required generality for the points α_i, p_j in the proof of Proposition 4.7 below. We denote by $\psi_{B,k} : \mathbb{P}^1 \to Y_{d,n}$ the corresponding curve from Lemma 4.5 and by $C_{B,k} = (\psi_{B,k})_*([\mathbb{P}^1])$ its image cycle in $Y_{d,n}$.

For later use we compute the evaluation of the i-th point along the map $\psi_{B,k}$. Here, for brevity, we identify points $[\lambda : 1] \in \mathbb{P}^1$ with $\lambda \in \mathbb{C}$:
$$(\mathrm{ev}_i \circ \psi_{B,k})([x_0 : x_1])$$
$$= \begin{cases} (p_j, \frac{(p_j - a_1)\dots(p_j - a_{d-k})(x_1 p_j - b_1 x_0)\dots(x_1 p_j - b_k x_0)}{(p_j - b_1)\dots(p_j - b_{d-k})(x_1 p_j - d_1 x_0)\dots(x_1 p_j - d_k x_0)}) \text{ for } i \notin B, \\ ([\alpha_i x_0 : x_1], \frac{(\alpha_i x_0 - a_1 x_1)\dots(\alpha_i x_0 - a_{d-k} x_1)(x_1 \alpha_i x_0 - b_1 x_0 x_1)\dots(x_1 \alpha_i x_0 - b_k x_0 x_1)}{(\alpha_i x_0 - c_1 x_1)\dots(\alpha_i x_0 - c_{d-k} x_1)(x_1 \alpha_i x_0 - d_1 x_0 x_1)\dots(x_1 \alpha_i x_0 - d_k x_0 x_1)}) \\ = ([\alpha_i x_0 : x_1], \frac{(\alpha_i x_0 - a_1 x_1)\dots(\alpha_i x_0 - a_{d-k} x_1)(\alpha_i - b_1)\dots(\alpha_i - b_k)}{(\alpha_i x_0 - c_1 x_1)\dots(\alpha_i x_0 - c_{d-k} x_1)(\alpha_i - d_1)\dots(\alpha_i - d_k)}) \text{ for } i \in B. \end{cases} \tag{11}$$
We will see now that the curves $C_{B,k}$ above were constructed to meet very specific divisors in $Y_{d,n}$.

Proposition 4.7. *The nonzero intersection numbers of the curves $C_{B,k}$ with the generators of the rational Picard group of $Y_{d,n}$ are exactly:*

- $(C_{B,k}, \mathcal{H}) = 1$ *if* $1 \in B$ *and* 0 *otherwise,*
- $(C_{B,k}, D_{\{a,b\},0}) = 1$ *for* $a \in A = \{1, \dots, n\} \setminus B, b \in B$,
- $(C_{B,k}, D_{B,k}) = 2$,
- $(C_{B,k}, D_{\emptyset,1}) = 2k(d - k)$.

Note that for $B = \emptyset, k = 1$ we have $(C_{B,k}, D_{\emptyset,1}) = 2d = 2 + 2(d - 1)$, so in this case the two different numbers from above are added.

Proof. All the base points in the construction of the test curves of Lemma 4.5 lie on the union of the images of the maps S_{b_j}, S_{d_j} for $j = 1, \dots, k$. But the images of S_α, S_β for $\alpha \neq \beta$ intersect exactly at $(0, 0)$, (∞, ∞) and there they have distinct tangent directions. We choose the points p_i such that the sections (p_i, id) miss all those base points and additionally their intersection points with the sections σ_j, $j \in B$ do not have the same second coordinate as one of the base points. Then it is ensured that the conditions of Lemma 4.5 are satisfied.

Using the projection formula we obtain

$$\begin{aligned}(C_{B,k}.\mathcal{H}) &= \deg((\psi_{B,k})_*[\mathbb{P}^1].(\pi_1 \circ \mathrm{ev}_1)^*\mathcal{O}_{\mathbb{P}^1}(1)) \\ &= \deg((\pi_1 \circ \mathrm{ev}_1 \circ \psi_{B,k})^*\mathcal{O}_{\mathbb{P}^1}(1)).\end{aligned}$$

From equation (11) we see that $\pi_1 \circ \mathrm{ev}_1 \circ \psi$ is an isomorphism for $1 \in B$ and constant for $1 \notin B$. The sections σ_i, $i \in A$, do not meet among each other. For $j \in B$, they meet the section σ_j once and all those sections meet exactly in the points $(0, 0), (\infty, \infty)$, which are base points of degree k. This explains the remaining intersection numbers. $\square$

We need one other test curve in case $d = 0$.

Proposition 4.8. *For $d = 0$ consider the identity map $id : S = \mathbb{P}^1 \times \mathbb{P}^1 \to S$ and constant sections $\sigma_i : \mathbb{P}^1 \to S, q \mapsto (p_i, q)$. Then $\pi_2 : S \to \mathbb{P}^1$ defines a family of stable maps over $\mathbb{P}^1$ where the map over $[x_0 : x_1]$ corresponds to the inclusion $\mathbb{P}^1 \to \mathbb{P}^1 \times \{[x_0 : x_1]\}$ with the horizontal position of the marked points held fixed. This gives a curve $C_{\mathcal{G}}$ in $Y_{0,n}$ which of all generators in $\mathfrak{G}_{d,n}$ intersects exactly the divisor $\mathcal{G}$ with multiplicity 1.*

For formulating results about the relations among the generators of the rational Picard group of $Y_{d,n}$, we group those generators into convenient subsets. For $0 \leq j \leq n$ we define

$$\Delta_j = \{D_{B,k} : |B| = j, 0 \leq k \leq d \text{ and } 1 \leq k \text{ for } j = 0, 1\}.$$

In terms of our interpretation as pointed graphs of rational maps, these are exactly those graphs where the vertical section carries j marked points. Additionally, we let

$$\Delta = \bigcup_{j=0}^{n} \Delta_j$$

be the set of all boundary divisor generators and for $n \geq 4$ we define

$$\Delta' = \Delta \setminus (\Delta_0 \cup \Delta_1 \cup \Delta_{n-1} \cup \Delta_n),$$

where we set $\Delta' = \emptyset$ for $n \leq 4$.

Theorem 4.9. *Let $d \geq 0, n \geq 0$, then in $Pic(Y_{d,n}) \otimes \mathbb{Q}$:*

- *the divisor $\mathcal{G}$ is linearly independent of all other generators for $d = 0$,*
- *the divisor $\mathcal{H}$ is not contained in the span of Δ for $n = 1, 2$,*
- *the set Δ_0 is linearly independent for $n = 0$,*
- *the set $\Delta_0 \cup \Delta_1 \cup \Delta_{n-1} \cup \Delta_n$ is linearly independent modulo the span of Δ' for $n \geq 1$,*
- *the relations among the divisor classes in Δ' are exactly the pullback of relations between the boundary divisors in $\overline{M}_{0,n}$ for $n \geq 4$. These are generated by the relations*

$$\sum_{\substack{i,j \in A \\ k,l \in B}} D(A; B) = \sum_{\substack{i,k \in A \\ j,l \in B}} D(A; B)$$

for distinct $i, j, k, l \in \{1, \ldots, n\}$.

Proof. For $d = 0$, as $(C_{\mathcal{G}}.\mathcal{G}) = 1$ and all intersections with other generators are zero, $\mathcal{G}$ is linearly independent of those.

For $n = 1$ we see that $C_{\{1\},0}$ intersects exactly the divisor $\mathcal{H}$ with multiplicity 1. Hence $\mathcal{H}$ is linearly independent from the span of Δ.

For $n = 2$ we consider the curves $C_{\{i\},0}$ for $i = 1, 2$. They intersect $D_{\{1,2\},0}$ with multiplicity 1 and $C_{\{1\},0}$ also intersects $\mathcal{H}$. Hence the 1-cycle $C_{\{1\},0} - C_{\{2\},0}$ intersects only $\mathcal{H}$ nontrivially, which is therefore linearly independent.

Now let $n \geq 0$ and consider the curves $C_{\emptyset,k}$, which only intersect $D_{\emptyset,k}$ and $D_{\emptyset,1}$ nontrivially. The case $k = 1$ shows that $D_{\emptyset,1}$ is linearly independent from all other boundary divisors. Using this in the case $2 \leq k \leq d$, we also obtain linear independence of the other divisors $D_{\emptyset,k}$ in Δ_0.

Now let $n \geq 1$. For $1 \leq i \leq n$ and $1 \leq k \leq d$ choose $\sigma_j = (p_j, \mathrm{id})$ for $j \neq i$ missing the base points as in Definition 4.6, but $\sigma_i = (0, \mathrm{id})$. Then the only nonzero intersections with boundary divisors are

$$(C.D_{\emptyset,1}) = 2k(d-k), (C.D_{\emptyset,k}) = 1, (C.D_{\{i\},k}) = 1.$$

As the first two types of divisors are in Δ_0, which is already seen to be linearly independent, we obtain the independence of all divisors in Δ_1.

For $n \geq 2$, $0 \leq k \leq d$, we first consider $C_{\{1,\ldots,n\},k}$, which only intersects $D_{\{1,\ldots,n\},k}$ and $D_{\emptyset,1}$. Again we already know that $D_{\emptyset,1}$ is independent of the other boundary divisors and thus we obtain the independence of all elements in Δ_n.

Now if for some $i \in \{1, \ldots, n\}$ we modify the family giving us $C_{\{1,\ldots,n\},k}$ by setting $\sigma_i = (0, \mathrm{id})$, this section no longer meets (∞, ∞) and we obtain a new curve C in $Y_{d,n}$. Note now that

$$(C.D_{\emptyset,1}) = 2k(d-k), (C.D_{\{1,\ldots,n\},k}) = 1, (C.D_{\{1,\ldots,n\}\setminus\{i\},k}) = 1.$$

As we already know the independence of Δ_n, this also gives us the independence of Δ_{n-1}.

Now for $n \geq 4$ we come to the relations among the divisors in Δ'. Remember that there is a morphism

$$F : Y_{d,n} = \overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1, d)) \to \overline{M}_{0,n}$$

by forgetting the map and only remembering the stabilization of the domain curve. The map F is flat by Theorem B.1. For $A, B \subset \{1, \ldots, n\}$ disjoint with $|A|, |B| \geq 2$ and $A \cup B = \{1, \ldots, n\}$, the boundary divisor $D(A; B)$ in $\overline{M}_{0,n}$ pulls back under F to the multiplicity free sum

$$D_{A\cup B} = \sum_{m=0}^{d} D_{A,m} + \sum_{m=0}^{d} D_{B,m} = F^*(D(A; B)).$$

We claim that given a relation

$$\sum_{D\in\Delta'} c_D D = 0 \tag{12}$$

in $\mathrm{Pic}(Y_{d,n}) \otimes \mathbb{Q}$, it is the pullback of a relation in $\overline{M}_{0,n}$. In a first step we will show, that the coefficient $c_{D_{B,k}}$ of $D_{B,k}$ in (12) only depends on B and, moreover, the coefficients for B and $A = \{1, \ldots, n\} \setminus B$ coincide. We will denote them by $c_{A\cup B}$.

Indeed, for $k \geq 1$, we can take the intersection of the relation (12) with the 1-cycle $C_{B,k} - C_{B,0}$. We see that all the intersections with the divisors $D_{\{i,j\},0}$ for $i \in A, j \in B$ cancel and what remains is

$$0 = (C_{B,k} - C_{B,0}, \sum_{D \in \Delta'} c_D D) = 2c_{D_{B,k}} - 2c_{D_{B,0}},$$

that is, $c_{D_{B,k}} = c_{D_{B,0}}$ does not depend on k.

For the claim that the coefficient also remains the same if we switch the roles of A and B, we take the intersection of (12) with $C_{B,k} - C_{A,k}$. Again we see a cancellation and obtain

$$0 = (C_{B,k} - C_{A,k}, \sum_{D \in \Delta'} c_D D) = 2c_{D_{B,k}} - 2c_{D_{A,k}},$$

which concludes the proof that c_D only depends on the partition of the marked points in D.

Thus we know that the relation (12) is of the form

$$0 = \sum_{A \cup B = \{1,\ldots,n\}} c_{A\cup B} D_{A \cup B} = F^*(\sum_{A \cup B = \{1,\ldots,n\}} c_{A \cup B} D(A;B)).$$

But as F is surjective, proper and flat, by [GJRW96, Corollary 2.3] the kernel of the map $F^* : \mathrm{Pic}(\overline{M}_{0,n}) \to \mathrm{Pic}(Y_{d,n})$ is torsion, so the induced map of rational Picard groups is injective. Hence as claimed, the relation (12) is the pullback under F of the relation $\sum_{A \cup B = \{1,\ldots,n\}} c_{A\cup B} D(A;B) = 0$ in $\mathrm{Pic}(\overline{M}_{0,n}) \otimes \mathbb{Q}$.

The form of the relations of boundary divisors in $\overline{M}_{0,n}$ was proved in [Kee92, Theorem 1]. □

Corollary 4.10. *For $d \geq 0, n \geq 0$, the rank of the Picard group of $Y_{d,n}$ is $2^n(d+1) - \binom{n}{2} - 1 + \delta_{n,1} + \delta_{n,2} + \delta_{d,0}$.*

Proof. Using Theorem 4.4, we first count generators. For the boundary divisors $D_{B,k}$, we see that there are 2^n choices for B and $d+1$ choices for k. In the case $k = 0$ we have to substract the $n+1$ choices for a set B with at most 1 element. We also have one additional generator $\mathcal{H}$ for $n = 1, 2$ and $\mathcal{G}$ for $d = 0$.

For the relations, we see as in [Pan99] (cf. the discussion above Lemma 1.2.3) that there are $\binom{n-1}{2} - 1$ independent relations among the boundary components in $\overline{M}_{0,n}$.

Thus the total dimension of $\mathrm{Pic}(Y_{d,n}) \otimes \mathbb{Q}$ is

$$\begin{aligned} &2^n(d+1) - (n+1) + \delta_{n,1} + \delta_{n,2} + \delta_{d,0} - \left(\binom{n-1}{2} - 1 \right) \\ &= 2^n(d+1) - \binom{n}{2} - 1 + \delta_{n,1} + \delta_{n,2} + \delta_{d,0}. \end{aligned}$$

□

For the sake of completeness, we also want to explicitly name one subset of the generators that forms a basis.

Corollary 4.11. *Consider the set $\mathfrak{G}_{d,n}$ of generators of $W = Pic(Y_{d,n}) \otimes \mathbb{Q}$ from Theorem 4.4. Then the set*

$$\mathfrak{B}_{d,n} = \mathfrak{G}_{d,n} \setminus \{D_{B,0}; B \subset \{2,\ldots,n\}, |B| = 2 \text{ and } B \neq \{n-1,n\}\} \tag{13}$$

forms a basis of W. Note that $\mathfrak{B}_{d,n} = \mathfrak{G}_{d,n}$ for $n \leq 3$.

Proof. One sees easily from Corollary 4.10 that for all d, n, the set $\mathfrak{B}_{d,n}$ has $\dim(W)$ elements. For $n \leq 3$ this finishes the proof, so we may assume that $n \geq 4$. Let

$$V = \bigoplus_{D \in \mathfrak{G}_{d,n}} \mathbb{Q}D$$

be the vector space with formal basis $\mathfrak{G}_{d,n}$ together with the natural surjective map $\mathrm{Cl} : V \to W$ by taking the class in the rational Picard group. Let $U \subset V$ denote the kernel of this map. We have seen in the proof of Corollary 4.10 that $\dim(U) = \binom{n-1}{2} - 1$. Moreover, by Theorem 4.9, it is spanned by relations among the generators in Δ' obtained as pullback from the relations

$$\sum_{\substack{i,j \in A \\ k,l \in B}} D(A;B) = \sum_{\substack{i,k \in A \\ j,l \in B}} D(A;B) \tag{14}$$

among boundary divisors $D(A;B)$ in $\overline{M}_{0,n}$. Set $D_B = \sum_k D_{B,k}$, where the sum is over $k = 0, \ldots, d$ if $|B| \geq 2$ and $k = 1, \ldots, d$ otherwise. Then the relation (14) pulls back to

$$\sum_{\substack{i,j \in A \\ k,l \in B}} D_B + \sum_{\substack{i,j \in B \\ k,l \in A}} D_B = \sum_{\substack{i,k \in A \\ j,l \in B}} D_B + \sum_{\substack{i,k \in B \\ j,l \in A}} D_B. \tag{15}$$

Now consider the canonical projection $V \to V'$ on the subspace $V' \subset V$ generated by the divisors $D_{B,0}$ with $|B| = 2$. From (15) we see that under this projection, the space U maps to the space U' generated by elements of the form

$$D_{\{k,l\},0} + D_{\{i,j\},0} - D_{\{j,l\},0} - D_{\{i,k\},0}. \tag{16}$$

We have another projection $V' \to V''$ where V'' is the span of

$$\{D_{B,0}; B \subset \{2, \ldots, n\}, |B| = 2, \text{ and } B \neq \{n-1, n\}\}.$$

One sees immediately that $\dim(V'') = \binom{n-1}{2} - 1 = \dim(U)$ and we claim that the image U'' of the induced projection $U' \to V''$ is all of V''. This in turn would imply, that the map $U \to V''$ is an isomorphism and from this one immediately concludes that $\mathfrak{B}_{d,n}$ forms a basis of W.

First note that under $V' \to V''$ the elements

$$D_{\{1,2\},0}, D_{\{1,3\},0}, \ldots, D_{\{1,n\},0}, D_{\{n-1,n\},0}$$

map to 0 by definition. Taking $i = 1, k = n, l = n-1$ in (16) we see that $-D_{\{j,n-1\},0} \in U''$. Switching k and l we also have $-D_{\{j,n\},0} \in U''$. But now take $2 \leq i, j \leq n-2$ arbitrarily distinct and $k = n, l = n-1$, then we have $D_{\{i,j\},0} - D_{\{i,n\},0} - D_{\{j,n-1\},0} \in U''$, so also $D_{\{i,j\},0} \in U''$. But this finishes the proof. ☐

4.1.3. *Identification of divisors.* Now that we have a basis of the rational Picard group of $Y_{d,n}$, we can find an algorithm to explicitly represent a given divisor D as a linear combination in this basis. We will see that all the information that is needed are the intersection numbers of D with the test curves $C_{B,k}$ from Proposition 4.7 together with the intersection $(C_{\mathcal{G}}.D)$ for $d = 0$. We now give explicit formulas for the coefficients of the basis elements.

Proposition 4.12. *Let D be a rational divisor class on $Y_{d,n}$ and let $N_{B,k} = (C_{B,k}.D)$. Then D has a unique representation*

$$D = \sum_{D_{B,k} \in \mathfrak{B}_{d,n}} c_{D_{B,k}} D_{B,k} \underbrace{+c_{\mathcal{H}}\mathcal{H}}_{for\ n=1,2} \underbrace{+c_{\mathcal{G}}\mathcal{G}}_{for\ d=0} . \tag{17}$$

The coefficients are determined as follows:

- $c_{\mathcal{G}} = (C_{\mathcal{G}}.D)$ *for* $d = 0$,
- $c_{\mathcal{H}} = N_{\{1\},0}$ *for* $n = 1$, $c_{\mathcal{H}} = N_{\{1\},0} - N_{\{2\},0}$ *for* $n = 2$ *and* 0 *otherwise,*
- $c_{\{1,2\},0} = N_{\{2\},0}$ *for* $n = 2$,
- $c_{\{1,j\},0} = N_{\{j\},0}$ *for* $2 \leq j \leq n-2$, $n \geq 3$,
- $c_{\{1,n-1\},0} = \frac{1}{2}(N_{\{1\},0} - N_{\{2\},0} - \cdots - N_{\{n-2\},0} + N_{\{n-1\},0} - N_{\{n\},0})$ *for* $n \geq 3$,
- $c_{\{1,n\},0} = \frac{1}{2}(N_{\{1\},0} - N_{\{2\},0} - \cdots - N_{\{n-2\},0} - N_{\{n-1\},0} + N_{\{n\},0})$ *for* $n \geq 3$,
- $c_{\{n-1,n\},0} = \frac{1}{2}(-N_{\{1\},0} + N_{\{2\},0} + \cdots + N_{\{n-2\},0} + N_{\{n-1\},0} + N_{\{n\},0})$ *for* $n \geq 3$,
- $c_{\{k,l\},0} = 0$ *for* $k, l > 1$ *if* $\{k,l\} \neq \{n-1,n\}$ *for* $n \geq 3$,
- $c_{B,k} = \frac{1}{2}(N_{B,k} - \frac{k(d-k)}{d} N_{\emptyset,1} - \chi_B(1)c_{\mathcal{H}} - \sum_{a \notin B, b \in B} c_{\{a,b\},0})$ *for* $B \subset \{1,\ldots,n\}$, $k \geq 0$ *and* $(|B|,k) \neq (2,0)$.

Here, χ_B is the characteristic function of the set B, so $\chi_B(m) = 1$ if $m \in B$ and $\chi_B(m) = 0$ otherwise.

Proof. From Corollary 4.11 it is clear that a unique representation of D in the form (17) must exist. To arrive at the formulas above one takes the intersection of equation (17) with the test curves $C_{\mathcal{G}}$, $C_{B,k}$ and checks that the resulting linear system uniquely determines the coefficients to be the numbers above. Here one should proceed in the order suggested above, except that one needs to determine $c_{\emptyset,1}$ using $N_{\emptyset,1}$ before calculating the other numbers $c_{B,k}$. ☐

4.1.4. *Geometric divisors.* We now want to define several divisors on $Y_{d,n}$ and use Proposition 4.12 to identify them in terms of our basis.

First of all, for every $i \in \{1,\ldots,n\}$ we have the evaluation map $\mathrm{ev}_i : Y_{d,n} \to \mathbb{P}^1 \times \mathbb{P}^1$. This gives us divisors

$$\mathcal{H}_{i,1} = (\pi_1 \circ \mathrm{ev}_i)^* \mathcal{O}_{\mathbb{P}^1}(1),$$
$$\mathcal{H}_{i,2} = (\pi_2 \circ \mathrm{ev}_i)^* \mathcal{O}_{\mathbb{P}^1}(1).$$

For $n = 1, 2$, the divisor $\mathcal{H}_{1,1} = \mathcal{H}$ is an element of or basis $\mathfrak{B}_{d,n}$ of the rational Picard group. While it is possible to describe the elements $\mathcal{H}_{i,1}$ in other cases as well using the formulas in Proposition 4.12, the resulting representation is not very illuminating. However, we will see that by substracting suitable multiples of the divisors $\mathcal{H}_{i,j}$ from other geometric divisors below, the representation of these divisors becomes much nicer. As the divisors $\mathcal{H}_{i,1}$ are easy to handle in any case, we will use them in our expression of other geometric divisors.

As a first step we apply this to the divisors $\mathcal{H}_{i,2}$.

Proposition 4.13. *The divisor class of $\mathcal{H}'_{i,2} = \mathcal{H}_{i,2} - d\mathcal{H}_{i,1}$ has the form*

$$\mathcal{H}'_{i,2} = \sum_{k=1}^{d} \left(\sum_{B \not\ni i} \frac{k^2}{2d} D_{B,k} + \sum_{B \ni i} (\frac{k^2}{2d} - k) D_{B,k} \right) \underbrace{+\mathcal{G}}_{for\ d=0} .$$

Proof. To compute the intersection numbers of the divisors $\mathcal{H}_{i,j}$ with $C_{B,k}$ consider again equation (11). Then it is clear that

$$(C_{B,k}.\mathcal{H}_{i,j}) = \deg(\pi_j \circ \mathrm{ev}_i \circ \psi_{B,k}).$$

Hence we conclude

$$\begin{aligned}(C_{B,k}.\mathcal{H}_{i,1}) &= \chi_B(i),\\ (C_{B,k}.\mathcal{H}_{i,2}) &= d\chi_B(i) + k(1-2\chi_B(i)).\end{aligned}$$

Thus,

$$(C_{B,k}.\mathcal{H}'_{i,2}) = k(1-2\chi_B(i)).$$

Going through the recipe of Proposition 4.12 we find the desired formula. □

The reason for substracting $d\mathcal{H}_{i,2}$ was that it eliminates intersections with all the test curves $C_{B,0}$, which would make the formulas much more complicated.

For the next definitions, we will use that the evaluation maps, which were already considered above, are flat.

Lemma 4.14. *For $i = 1, \ldots, n$, the evaluation maps $ev_i : Y_{d,n} \to \mathbb{P}^1 \times \mathbb{P}^1$ are flat and surjective.*

Proof. All that we will use, is that there exists a transitive action of an algebraic group H on the target $X = \mathbb{P}^1 \times \mathbb{P}^1$, which leaves the curve class $\beta = (1, d)$ invariant. In our case, we can take the natural action of $H = \mathrm{PGL}_2 \times \mathrm{PGL}_2$. Using Lemma C.6, we obtain an induced action of H on $Y_{d,n}$ (by postcomposition) making ev_i equivariant. Then surjectivity is immediate, as H acted transitively on X. Moreover, by generic flatness, the map ev_i is flat over some open subset $U \subset X$. But then it is flat over $g.U$ for all $g \in H$ and using again the transitivity of the action, it is flat everywhere. □

One divisor that will be important later is the subset of $Y_{d,n}$ where the i-th marked point is a fixed point of the self-map. When looking at the graph Γ of a map from $\mathbb{P}^1$ to itself, the set of fixed points is exactly (the projection of) the intersection of Γ with the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$. Thus we make the following definition.

Definition 4.15. For $d, n \geq 0$ we call

$$D_{i=\mathrm{fix}} = \mathrm{ev}_i^{-1}(\Delta) \subset \overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1, d))$$

the i-th fixed point divisor.

By Lemma 4.14, this is an effective Cartier divisor (i.e., of pure codimension 1). This implies that the generic points of its irreducible components are supported in the open part $Y^o_{d,n} \subset Y_{d,n}$, since otherwise an entire boundary divisor $D_{B,k}$ would have to map to $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ via ev_i. This can be excluded in various ways, for instance, observing that the componentwise $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-action on $\mathbb{P}^1 \times \mathbb{P}^1$ induces a $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-action on $Y_{d,n}$ as in Corollary 2.3 making ev_i an equivariant map. But $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ will leave $D_{B,k}$ invariant, while not leaving $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ invariant.

As $\mathcal{O}(\Delta) = \pi_1^*(\mathcal{O}(1)) \otimes \pi_2^*(\mathcal{O}(1))$ we have

$$D_{i=\mathrm{fix}} = \mathcal{H}_{i,1} + \mathcal{H}_{i,2}$$

as divisor classes in $\mathrm{Pic}(Y_{d,n})$.

Next, for a point $p \in \mathbb{P}^1 \times \mathbb{P}^1$ we want to consider the locus $D_p \subset Y_{d,n}$ of stable maps $f : C \to \mathbb{P}^1 \times \mathbb{P}^1$, where p lies on the graph $f(C)$. For this, consider the forgetful map $F : Y_{d,n+1} \to Y_{d,n}$ of the last marking $n+1$, which we interpret as the universal curve over $Y_{d,n}$. Here, we have the evaluation map $\mathrm{ev}_{n+1} : Y_{d,n+1} \to \mathbb{P}^1 \times \mathbb{P}^1$, which is flat. Hence, we can pull back the cycle $\{p\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ and then push it forward to $Y_{d,n}$ via F. Indeed, we define

$$D_p = F_* \mathrm{ev}_{n+1}^*[\{p\}].$$

As $\{p\}$ is codimension 2 and as F has fibres of dimension 1, this should indeed be a divisor in $Y_{d,n}$. We now compute its class in the rational Picard group.

Proposition 4.16. *The divisor class of D_p has the form:*

$$D_p = \sum_{k=1}^{d} \frac{k^2}{2d} \sum_{B \subset \{1,\dots,n\}} D_{B,k} \underbrace{+\mathcal{G}}_{for\ d=0} .$$

Proof. As the class of $\{p\}$ in the Chow group of $\mathbb{P}^1 \times \mathbb{P}^1$ is exactly

$$[\{p\}] = c_1(\mathcal{O}(1,0)) \cap c_1(\mathcal{O}(0,1)) \cap [\mathbb{P}^1 \times \mathbb{P}^1],$$

we know that

$$\begin{aligned} \mathrm{ev}_{n+1}^*[\{p\}] &= c_1(\mathcal{H}_{n+1,2}) \cap c_1(\mathcal{H}_{n+1,1}) \cap [Y_{d,n+1}] \\ &= c_1(\mathcal{H}'_{n+1,2}) \cap c_1(\mathcal{H}_{n+1,1}) \cap [Y_{d,n+1}]. \end{aligned}$$

Here we use that $c_1(\mathcal{H}_{n+1,1})^2 = \mathrm{ev}_{n+1}^* c_1(\mathcal{O}(1,0))^2 = 0$. But now we can apply the formula for $\mathcal{H}'_{n+1,2}$ from Proposition 4.13. Note in the following, that for $n+1 \in B$, $1 \le k \le n$ we have

$$F_* \left(c_1(\mathcal{H}_{n+1,1}) \cap D_{B,k} \right) = 0,$$

because a general point of the cycle on the left has a vertical section with a fixed horizontal position, and this is already a codimension 2 condition. Using this, we compute

$$\begin{aligned} &F_* \mathrm{ev}_{n+1}^*[\{p\}] \\ &= F_* c_1(\mathcal{H}_{n+1,1}) \cap \left(\sum_{k, B \not\ni n+1} \frac{k^2}{2d} D_{B,k} + \sum_{k, B \ni i} (\frac{k^2}{2d} - k) D_{B,k} \underbrace{+\mathcal{G}}_{\text{for } d=0} \right) \\ &= F_* c_1(\mathcal{H}_{n+1,1}) \cap \left(\sum_{k,B} \frac{k^2}{2d} D_{B,k} \underbrace{+\mathcal{G}}_{\text{for } d=0} \right) \\ &= F_* c_1(\mathcal{H}_{n+1,1}) \cap F^* \left(\sum_{k,B} \frac{k^2}{2d} D_{B,k} \underbrace{+\mathcal{G}}_{\text{for } d=0} \right), \end{aligned}$$

where in the last line, the sum of boundary divisors is on $Y_{d,n}$. But then, for using the projection formula to obtain the desired result, we only need that

$$F_* c_1(\mathcal{H}_{n+1,1}) \cap [Y_{d,n+1}] = [Y_{d,n}].$$

But this is clear, since over the locus $Y_{d,n}^o$ with smooth domain curve, any subvariety $\mathrm{ev}_{n+1}^{-1}(\{p_1\} \times \mathbb{P}^1)$ (for $p_1 \in \mathbb{P}^1$) maps birationally onto its image via F. $\square$

4.2. **The Picard group of $M(d|d_1,\dots,d_n)$ finished.**

Corollary 4.17. *Let $\boldsymbol{d} = (d|d_1,\dots,d_n)$ be admissible and set*

$$d_T = d + 1 + \sum_{i=1}^{n} d_i.$$

Then the rational Picard group of $Y_{d,n}^{ss,\mathbf{d}}$ is the quotient of $Pic(Y_{d,n}) \otimes_{\mathbb{Z}} \mathbb{Q}$ by the linear span of the divisors $D_{B,k}$ with $k + \sum_{i\in B} d_i > \frac{d_T}{2}$ and the divisors $D_{i=fix}$ for all i with $d_i > d_T/2 - 1$.

Proof. By restricting to the open set $Y_{d,n}^{ss,\mathbf{d}}$ we divide out the divisor classes of all codimension 1 components of $Y_{d,n} \setminus Y_{d,n}^{ss,\mathbf{d}}$. We will use Lemma 2.9 to identify this locus.

The first case that can make a closed point of $Y_{d,n}$ unstable is when a marked point i with weight d_i becomes a fixed point and $d_i + 1 > d_T/2$. This is exactly accounted for by dividing by $D_{i=\mathrm{fix}}$. We note that this is the only cause of instability away from the boundary.

Now a general point of the boundary divisor $D_{B,k}$ corresponds to a parametrized graph with exactly one vertical section of multiplicity k over a nonfixed point and with the points in B on the vertical section. As we have already taken care of instabilities from marked points being fixed points, this is unstable iff $k+\sum_{i\in B} d_i > \frac{d_T}{2}$. □

Remark 4.18. Note that even in very simple cases, the set $Y_{d,n} \setminus Y_{d,n}^{ss,\mathbf{d}}$ does not have to be of pure codimension 1: take $d = 2, d_1 = \cdots = d_n = 0$, then while for any B the entire boundary divisor $D_{B,2}$ is unstable, a general point on $D_{B,1}$ will be stable (since horizontal and vertical section do not intersect on the diagonal). However, the codimension 1 set in $D_{B,1}$ where this intersection point hits the diagonal is unstable. Still, for computing the Class group we only needed to identify the codimension 1 components of the unstable locus.

Corollary 4.19. *For $\boldsymbol{d} = (d|d_1,\dots,d_n)$ admissible, the map*

$$\phi^* : Pic(M(d|d_1,\dots,d_n)) \otimes_{\mathbb{Z}} \mathbb{Q} \to Pic(Y_{d,n}^{ss,\boldsymbol{d}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism.

Proof. For $(d,n) \neq (2,0),(1,1)$, this is exactly Corollary 4.2. For $d = 2, n = 0$ we have the isomorphism $j : Y_{d,n}^s \to Z_d^s$ from Lemma 3.3 inducing an isomorphism $M(2,0) \cong \mathbb{P}^2$. Then one checks that the generator $D_{\emptyset,1}$ of the rational Picard group corresponds exactly to the line at infinity $\mathbb{P}^2 \setminus \mathbb{A}^2$ and hence forms a basis of the rational Picard group. On the other hand, in the case $d = 1, n = 1$, we have seen that the only nonempty moduli spaces are all isomorphic to $M(1|1)$. But we saw $M(1|1) \cong \mathbb{P}^1$ in Lemma 3.2. On the other hand, Corollary 4.17 gives generators $\mathcal{H}, D_{\emptyset,1}$, but we must divide by the class

$$D_{1=\mathrm{fix}} = 2\mathcal{H} + \frac{1}{2} D_{\emptyset,1}.$$

Hence the Picard group of $Y_{1,1}^{ss,(1|1)}$ is of rank 1 and one checks that the pullback map φ^* induces an isomorphism with $\mathrm{Pic}(\mathbb{P}^1) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$. □

Remark 4.20. The inverse of the map ϕ^* above is determined by giving its values on a generating set of the right-hand side. Such a generating set is given by the set of boundary divisors $D_{B,k}$ with $k + \sum_{i \in B} d_i \leq \frac{d_T}{2}$ together with the class $\mathcal{H}$ for $n = 1, 2$ and $\mathcal{G}$ for $d = 0$. Now the images $\tilde{D}_{B,k} = \phi(D_{B,k} \cap Y_{d,n}^{ss,\mathbf{d}})$ are again divisors and we have $(\phi^*)^{-1}(D_{B,k}) = \tilde{D}_{B,k}$. This follows since $\phi^{-1}\tilde{D}_{B,k} = D_{B,k}$ is reduced as ϕ is a geometric quotient.

For the possible generators $\mathcal{H}, \mathcal{G}$, it is more difficult to interpret their images geometrically. But here we can use the divisors $D_{i=\text{fix}} \subset Y_{d,n}$ computed earlier as alternative generators. From their definition it is obvious that they are stable under the PGL_2-action on $Y_{d,n}$ and so their images under ϕ are again divisors in $M(d|d_1, \ldots, d_n)$, giving the image under $(\phi^*)^{-1}$. To see that the divisors $D_{B,k}$ and $D_{1=\text{fix}}$ generate the rational Picard group of $Y_{d,n}^{ss,\mathbf{d}}$ we observe that by Proposition 4.13 we have the following equality in $\mathrm{Pic}(Y_{d,n}) \otimes \mathbb{Q}$:

$$\begin{aligned} D_{i=\text{fix}} &= \mathcal{H}_{i,1} + \mathcal{H}_{i,2} = (d+1)\mathcal{H}_{i,1} + \mathcal{H}'_{i,2} \\ &= (d+1)\mathcal{H}_{i,1} + \sum_{k=1}^{d} \left(\sum_{B \not\ni i} \frac{k^2}{2d} D_{B,k} + \sum_{B \ni i} (\frac{k^2}{2d} - k) D_{B,k} \right) \underbrace{+\mathcal{G}}_{\text{for } d=0} . \end{aligned}$$

Thus, for ($n = 1, 2$ and $d > 0$) or ($n > 2$ and $d = 0$), the missing generator $\mathcal{H} = \mathcal{H}_{1,1}$ or $\mathcal{G}$ of $\mathrm{Pic}(Y_{d,n}^{ss,\mathbf{d}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ can be expressed in terms of boundary divisors and $D_{1=\text{fix}}$. The only cases left, namely $d = 0$ and $n = 1, 2$ are covered by the analysis in Lemma 3.1, which shows that for $d = 0$ the case $n = 1$ does not occur and $M(0; 1, 1) \cong \{pt\}$ has trivial Picard group, hence also $\mathrm{Pic}(Y_{0,2}^{ss,(1,1)}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong 0$ by Corollary 4.19.

Remark 4.21. Combining the description of the basis of $\mathrm{Pic}(Y_{d,n}) \otimes_{\mathbb{Z}} \mathbb{Q}$ from Corollary 4.11 with the additional relations from Corollary 4.17 and using the explicit formula for $D_{i=\text{fix}}$ from Remark 4.20, it is essentially a linear algebra question to determine a basis of $\mathrm{Pic}(Y_{d,n}^{ss,\mathbf{d}}) \otimes_{\mathbb{Z}} \mathbb{Q}$. For simplicity let us do it only in the case where all $d_i = 0$ (and thus $d \geq 2$ is even). Then the unstable boundary divisors are exactly $D_{B,k}$ with $k > d/2$, which are all elements of the basis from Corollary 4.11. Thus we obtain a basis of the quotient space by removing these elements from the old basis. We have 2^n choices for B and $\frac{d}{2}$ choices for k, so we are removing $2^n \frac{d}{2}$ basis elements. Using the formula for the Picard rank of $Y_{d,n}$ from Corollary 4.10 we see that $M(d, n)$ has Picard rank

$$\begin{aligned} &2^n(d+1) - \binom{n}{2} - 1 + \delta_{n,1} + \delta_{n,2} - 2^n \frac{d}{2} \\ &= 2^n \left(\frac{d}{2} + 1 \right) - \binom{n}{2} - 1 + \delta_{n,1} + \delta_{n,2}. \end{aligned}$$

Appendix A. Notations

Rat_d	the space Rat_d of degree d maps $\mathbb{P}^1 \to \mathbb{P}^1$
Z_d	the compactification $Z_d = \mathbb{P}(H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d,1)))$ of Rat_d
$\mathcal{Y}_{d,n}$	the moduli stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1,d))$ of stable maps to $\mathbb{P}^1 \times \mathbb{P}^1$ of degree $(1,d)$ with n marked points
$Y_{d,n}$	the coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1,d))$ of $\mathcal{Y}_{d,n}$
$M(d,n)$	the moduli space of degree d self-maps with n marked points (Corollary 2.7)
$M(d\|d_1,\ldots,d_n)$	the moduli space of degree d self-maps with n marked points and weights $d_1,\ldots,d_n$ (Corollary 2.11)
$\mathcal{M}(d\|d_1,\ldots,d_n)$	the quotient stack $\mathcal{Y}_{d,n}^{ss,\mathbf{d}}/\mathrm{PGL}_2$
$A_{d,n}$	the locus of points in $Y_{d,n}$ with PGL_2-isotropy
$D_{B,k}$	the boundary divisor $D(\{1,\ldots,n\} \setminus B, (1,d-k); B, (0,k))$ inside $Y_{d,n}$ or $M(d\|d_1,\ldots,d_n)$
$\mathcal{H}_{i,j}$	the divisor $(\pi_j \circ \mathrm{ev}_i)^*\mathcal{O}_{\mathbb{P}^1}(1)$ on $Y_{d,n}$
$\mathcal{H}'_{i,2}$	the divisor $\mathcal{H}_{i,2} - d\mathcal{H}_{i,1}$ on $Y_{d,n}$
$D_{i=\mathrm{fix}}$	the divisor $\mathrm{ev}_i^{-1}(\Delta) \subset Y_{d,n}$ for the evaluation map $\mathrm{ev}_i : Y_{d,n} \to \mathbb{P}^1 \times \mathbb{P}^1$
$\mathcal{G}$	the divisor $\pi_{\mathbb{P}^1}^*\mathcal{O}_{\mathbb{P}^1}(1)$ on $Y_{0,n} = \overline{M}_{0,n}(\mathbb{P}^1 \times \mathbb{P}^1, (1,0)) \cong \overline{M}_{0,n}(\mathbb{P}^1, 1) \times \mathbb{P}^1$
$\mathfrak{G}_{d,n}$	the generators of the rational Picard group of $Y_{d,n}$ from Theorem 4.4
$\mathfrak{B}_{d,n}$	the basis of the rational Picard group of $Y_{d,n}$ from Corollary 4.11
$\psi_{B,k}$	the test curve $\mathbb{P}^1 \to Y_{d,n}$ constructed in Definition 4.6
$C_{B,k}$	the image cycle $(\psi_{B,k})_*[\mathbb{P}^1]$ in $Y_{d,n}$

Appendix B. Generalities

Theorem B.1. *Let $X = G/P$ be a homogeneous space, where G is an algebraic group and P is a parabolic subgroup. Let $\beta \in A_1(X)$ be a curve class, $n \geq 3$ and consider the forgetful morphism*

$$F : S = \overline{M}_{0,n}(X,\beta) \to \overline{M}_{0,n}.$$

Then F is flat of relative dimension $\delta = dim(X) + \int_\beta c_1(T_X)$.

Proof. By [BM96, Proposition 7.4] applied to $(V,\tau) = (X,g,n,\beta)$, we know that the morphism of Deligne-Mumford stacks $\overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}$ is flat of dimension δ. In general this will not imply that the corresponding morphism of the coarse moduli spaces is also flat.

However, in the case $g = 0$, we can use the Miracle flatness theorem (see for instance [Mat89, Theorem 23.1]). It tells us that a map from a Cohen-Macaulay variety to a smooth variety with constant fibre dimension is flat. Now, indeed, by [KP01] we have that F is a map between irreducible varieties and $\overline{M}_{0,n}$ is smooth by [Knu83]. By [FP97, Theorem 2] the variety $\overline{M}_{0,n}(X,\beta)$ is locally the quotient V/H of a smooth variety V by a finite group H. Smooth varieties are Cohen-Macaulay and we now use the Hochster-Roberts theorem from [HR74] to show that then V/H is also Cohen-Macaulay. The theorem says that if an affine linearly reductive group (like H) acts rationally on a Noetherian k-algebra, then the ring of invariants is

Cohen-Macaulay. But the map $V \to V/H$ satisfies the conditions of [MFK94, 1.§2, Theorem 1.1] and thus V/H is covered by the spectra of rings of invariants as above, hence it is Cohen-Macaulay.

We are thus left to show that the fibres of F are of dimension δ. But this property is preserved when going from the stacks to the coarse moduli spaces, hence we are done by [BM96, Proposition 7.4]. □

Proposition B.2. *Let $\pi : X \to Y$ be a locally finitely presented, flat, separated morphism of relative dimension n. Let $s : Y \to X$ be a section of π such that $s(p)$ is a smooth point of X_p for all geometric points $p \in Y$. Then s is an effective Cartier divisor in X.*

Proof. As s is a section of a separated morphism, it is a closed embedding (see [Sta14, Tag 024T]). On the complement of the singular locus S of the fibres X_p we have that $\pi : X \setminus S \to Y$ is still locally finitely presented and flat and now also smooth, as the geometric fibres are smooth. Then by [Ful98, B7.3], s is a regular embedding into $X \setminus S$. But as $X \setminus S$ and $X \setminus s(Y)$ define an open cover of X, s is also a regular embedding into X, that is a Cartier divisor. □

Appendix C. Group actions on stacks

The following results use definitions and techniques for group actions on stacks introduced by Romagny in [Rom05].

Lemma C.1. *Let S be a scheme, G a flat, separated group scheme of finite presentation over S. Let $\mathcal{M}, \mathcal{N}$ be G-algebraic stacks over S and let $f : \mathcal{M} \to \mathcal{N}$ be a morphism of G-stacks. Then there exists a canonical commutative diagram*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f} & \mathcal{N} \\ \downarrow & & \downarrow \\ \mathcal{M}/G & \xrightarrow{\tilde{f}} & \mathcal{N}/G, \end{array} \tag{18}$$

and this diagram is a fibre product.

Proof. Our assumptions on G are chosen in such a way, that the quotient stacks $\mathcal{M}/G, \mathcal{N}/G$ are isomorphic to the stacks of G-torsors $(\mathcal{M}/G)^*$, $(\mathcal{N}/G)^*$ by [Rom05, Theorem 4.1].

Recall that for a G-stack $(\mathcal{M}, \mu_{\mathcal{M}} : G \times \mathcal{M} \to \mathcal{M})$, the stack $(\mathcal{M}/G)^*$ has as objects over a scheme of T/S triples

$$t = (p : E \to T, h : E \to \mathcal{M}, \sigma).$$

Here E is an algebraic space with a strict action $\nu : G \times E \to E$, p is G-invariant such that fppf-locally on T it is isomorphic to the projection $G \times T \to T$ and the map $(h, \sigma) : E \to \mathcal{M}$ is a morphism of G-stacks.

Isomorphisms between t, t' are pairs (u, α) with a G-morphism $u : E \to E'$ and a 2-isomorphism $\alpha : (h, \sigma) \implies (h', \sigma') \circ u$.

What will be important for us is that $(\mathcal{M}/G)^*, (\mathcal{N}/G)^*$ are algebraic stacks, i.e., they have surjective, fppf atlases $V \to (\mathcal{M}/G)^*$, $U \to (\mathcal{N}/G)^*$ from schemes U, V. Moreover, the natural maps $\mathcal{M} \to (\mathcal{M}/G)^*$ and $\mathcal{N} \to (\mathcal{N}/G)^*$ are actually the universal G-torsors.

As a first step, we want to construct the map $\tilde{f} : (\mathcal{M}/G)^* \to (\mathcal{N}/G)^*$. On objects over T/S it is given by

$$\left(\begin{tikzcd} E \arrow[r, "(h,\sigma)"] \arrow[d] & \mathcal{M} \\ T & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} E \arrow[r, "f\circ(h,\sigma)"] \arrow[d] & \mathcal{N} \\ T & \end{tikzcd}\right).$$

One checks that the resulting diagram (18) is commutative. Let $\mathcal{Z} = \mathcal{N} \times_{(\mathcal{N}/G)^*} (\mathcal{M}/G)^*$, then there is a natural map $\phi : \mathcal{M} \to \mathcal{Z}$. We want to show that it is an isomorphism by taking suitable base changes with smooth maps from schemes. First take the base change with the atlas $V \to (\mathcal{M}/G)^*$ and we obtain

$$\begin{tikzcd} \mathcal{M}_V \arrow[r] \arrow[d] & \mathcal{Z}_V \arrow[r] \arrow[d] & V \arrow[d] \\ \mathcal{M} \arrow[r] & \mathcal{Z} \arrow[r] \arrow[d] & (\mathcal{M}/G)^* \arrow[d] \\ & \mathcal{N} \arrow[r] & (\mathcal{N}/G)^*. \end{tikzcd}$$

It suffices to show $\mathcal{M}_V \to \mathcal{Z}_V$ is an isomorphisms. Because $\mathcal{M}_V \to V$ is a G-torsor we have $V \cong \mathcal{M}_V/G$ and $\mathcal{M}_V$ is an algebraic space. We have reduced the problem to the case where $\mathcal{M}$ is an algebraic space and $(\mathcal{M}/G)^*$ is a scheme. Similarly, by taking the base change of the entire diagram by the atlas $U \to (\mathcal{N}/G)^*$ we reduce to the case $\mathcal{N}$ an algebraic space. But here we see that the morphism $(\mathcal{M}/G)^* \to (\mathcal{N}/G)^*$ is induced by the G-equivariant map $\mathcal{M} \to \mathcal{N}$ of the G-torsor $\mathcal{M}$ over $(\mathcal{M}/G)^*$. Hence the fact that the corresponding diagram is cartesian is simply the definition of the universal property of the G-torsor $\mathcal{N} \to (\mathcal{N}/G)^*$. $\square$

Remark C.2. As the map $\mathcal{N} \to \mathcal{N}/G$ above is fppf (check on any atlas $U \to \mathcal{N}/G$ pulling back to a locally trivial G-bundle on U) any property of $f : \mathcal{M} \to \mathcal{N}$ that is fppf-local on the target is inherited by the induced map $\tilde{f} : \mathcal{M}/G \to \mathcal{N}/G$. Also, arguing as above, for any cartesian diagram of G-algebraic stacks where all morphisms are G-morphisms the induced diagram of the quotients is also cartesian.

Proposition C.3. *Let $\mathcal{M}$ be an orbifold, i.e., a smooth separated Deligne-Mumford stack with connected coarse moduli space and containing a nonempty open substack which is a scheme. Let the smooth group scheme G act on $\mathcal{M}$ with finite, reduced stabilizers at geometric points. Then the quotient $\mathcal{M}/G$ is again a smooth DM stack.*

Proof. We want to use the frame bundle $\mathcal{F} = \mathrm{Fr}\,(T_\mathcal{M})$ of the tangent bundle $T_\mathcal{M}$ of $\mathcal{M}$. By [BNTGP14, Exercise 1.183], $\mathcal{F}$ is an algebraic space. For $n = \dim(\mathcal{M})$, the group GL_n acts on $\mathcal{F}$ on the right by

$$(v_1, \ldots, v_n).(a_{ij})_{i,j=1}^n = (\sum_{i=1}^n a_{i1}v_i, \ldots, \sum_{i=1}^n a_{in}v_i) \tag{19}$$

and we have that $\mathcal{M} = [\mathcal{F}/\mathrm{GL}_n]$. On the other hand, the action of G on $\mathcal{M}$ induces an action of G on $\mathcal{F}$ by

$$g.(v_1, \ldots, v_n) = (g_*v_1, \ldots, g_*v_n), \tag{20}$$

where g_* denotes the pushforward under the map $p \mapsto gp$ on $\mathcal{M}$.

Note that the actions of G and GL_n commute, because $g_* : T_{\mathcal{M}} \to T_{\mathcal{M}}$ is linear in the fibres. Indeed, we have

$$\begin{aligned} g.((v_1, \ldots, v_n).(a_{ij})) &= g.(\sum_i a_{i1} v_i, \ldots, \sum_i a_{in} v_i) \\ &= (\sum_i a_{i1} g_* v_i, \ldots, \sum_i a_{in} g_* v_i) \\ &= (g.(v_1, \ldots, v_n)).(a_{ij}). \end{aligned}$$

This means that we can combine these two actions to an action of $G \times \mathrm{GL}_n$ on $\mathcal{F}$. As $\mathcal{F}$ is an algebraic space, every action of an algebraic group on $\mathcal{F}$ (in the usual sense) is automatically a strict action in the sense of [Rom05].

Note that a geometric point $\sigma = (v_1, \ldots, v_n) \in \mathcal{F}$ in the fibre of $p \in \mathcal{M}$ has finite, reduced stabilizer in $G \times \mathrm{GL}_n$. Indeed, for a pair (g, A) to stabilize σ, the element $g \in G$ must stabilize p. By assumption, the stabilizer of p in G is finite and reduced. Hence the claim follows, as the action of GL_n on the fibre of $\mathcal{F}$ over p is simply transitive, so the stabilizer of σ is isomorphic to the stabilizer of p in G.

With these preparations done, we simply note that

$$\mathcal{M}/G = [\mathcal{F}/\mathrm{GL}_n]/G = (\mathcal{F}/\mathrm{GL}_n)/G = \mathcal{F}/(\mathrm{GL}_n \times G).$$

The second isomorphism is a consequence of [Rom05, Theorem 4.1], the third isomorphism comes from [Rom05, Remark 2.4], as $\mathrm{GL}_n \subset \mathrm{GL}_n \times G$ is a normal subgroup with quotient group G. But now $\mathcal{F}$ is a smooth algebraic space and the action of $G \times \mathrm{GL}_n$ has finite, reduced stabilizers at geometric points. Then by [BCE', Proposition 5.27], the quotient $\mathcal{F}/(\mathrm{GL}_n \times G)$ is again a Deligne-Mumford stack and as $\mathcal{F}$ is smooth and a locally trivial $\mathrm{GL}_n \times G$-torsor over it, it is also smooth. ☐

Lemma C.4. *Let $\mathcal{M}$ be an Deligne-Mumford stack with a strict action $\mu_{\mathcal{M}}$ of flat, separated group scheme G of finite presentation. Let M be a coarse moduli space carrying the induced action μ_M of G and assume M is locally Noetherian. Let N be an algebraic space and $[M/G] \to N$ a tame moduli space* (*in the sense of* [Alp13]). *Then there are natural maps*

$$\mathcal{M}/G \to [M/G] \to N$$

making N a coarse moduli space for $\mathcal{M}/G$ and $[M/G]$.

Proof. By definition, we have

$$\mathrm{Hom}(\mathcal{M}/G, S) = \mathrm{Hom}_G(\mathcal{M}, S)$$

for any algebraic space S, where the right means homomorphisms of G-stacks with the trivial G action on S. But any morphism $\psi : \mathcal{M} \to S$ must factor uniquely through a morphism $\tilde{\psi} : M \to S$. Because S is an algebraic space, the category $\mathrm{Hom}_G(\mathcal{M}, S)$ is actually a set(oid) and by definition it is given by those morphisms $f : \mathcal{M} \to S$ of stacks such that

$$f \circ \mu_{\mathcal{M}} = f \circ \pi_{\mathcal{M}} : G \times \mathcal{M} \to S.$$

Now as $\mathcal{M}$ is Deligne-Mumford, it is tame and so the formation of its moduli space M commutes with base change. Hence $G \times M$ is a coarse moduli space for $G \times \mathcal{M}$, so the morphisms from these spaces to S agree. Thus the above condition is equivalent to asking for a morphism $\tilde{f} : M \to S$ such that

$$\tilde{f} \circ \mu_M = \tilde{f} \circ \pi_M : G \times M \to S.$$

By now we have shown $\mathrm{Hom}(\mathcal{M}/G, S) = \mathrm{Hom}_G(M, S)$, but as $M \to [M/G]$ is the universal G-torsor over $[M/G]$ this clearly equals $\mathrm{Hom}([M/G], S)$. As M is locally Noetherian, so is $[M/G]$. Then, by [Alp13, Theorem 6.6], the good moduli space $[M/G] \to N$ is universal among maps to algebraic spaces, so $\mathrm{Hom}([M/G], S) = \mathrm{Hom}(N, S)$.

Note that the map $\mathcal{M}/G \to [M/G]$ is obtained from the morphism $\mathcal{M} \to M$ of G-stacks via Lemma C.1, using that $M/G = [M/G]$ by Theorem [Rom05, Theorem 4.1].

Finally, any geometric point $p \in \mathcal{M}/G$ corresponds to a (necessarily) trivial G-torsor $E \to p$ together with a G-equivariant map $E \to \mathcal{M}$. This data is equivalent to specifying an orbit of some geometric point $\widehat{p} \in \mathcal{M}$. But as the geometric points of $\mathcal{M}$ and M agree, this is equivalent to specifying an orbit of a geometric point in M. In turn, this is equivalent to a geometric point of $[M/G]$. By the definition of a tame moduli space, the geometric points of $[M/G]$ and N agree. This shows that geometric points of $\mathcal{M}/G, [M/G]$ and N coincide. □

Remark C.5. A sufficient condition for the morphism $[M/G] \to N$ to be a tame moduli space is to require that G is a smooth, affine, linearly reductive group scheme over a field k and $M = M^{ss} = M^s$ for some G-linearized line bundle $\mathcal{L}$ on M and $N = M /\!\!/ G$. This is the only situation in which we are going to use the above result.

Note that in the lemma above, it does not suffice to ask for instance N to be a geometric quotient of M by G, if M is a scheme. Indeed, in Example 8.6 of [Alp13] we have a geometric quotient $X \to \mathbb{A}^1$ of a scheme X by SL_2 such that $X/G = [X/G]$ is the nonlocally separated affine line, which is an algebraic space. Hence $\mathbb{A}^1$ is not universal for morphisms from X/G to algebraic spaces.

The following type of group action on a stack appears in the study of self-maps.

Lemma C.6. *Let X be a projective, algebraic scheme over $\mathbb{C}$ and $\beta \in A_1 X$. Let G be a group scheme over $S = Spec(\mathbb{C})$ with multiplication $m : G \times G \to G$ and unit $e : S \to G$. Assume that $\sigma : G \times X \to X$ is an algebraic action leaving β invariant.*

Then there is an induced action of G on $\mathcal{M} = \overline{\mathcal{M}}_{0,n}(X, \beta)$, which sends a $\mathbb{C}$-point $(g, (f : C \to X; p_1, \dots, p_n))$ to $(\sigma(g, -) \circ f : C \to X; p_1, \dots, p_n)$.

Proof. By [Rom05, Definition 2.1], an action of G on $\mathcal{M}$ is a morphism of stacks $\mu : G \times \mathcal{M} \to \mathcal{M}$ such that the diagram

$$\begin{array}{ccc} G \times G \times \mathcal{M} & \xrightarrow{m \times \mathrm{id}_{\mathcal{M}}} & G \times \mathcal{M} \\ \Big\downarrow{\scriptstyle \mathrm{id}_G \times \mu} & & \Big\downarrow{\scriptstyle \mu} \\ G \times \mathcal{M} & \xrightarrow{\mu} & \mathcal{M} \end{array} \tag{21}$$

commutes and such that $\mu \circ (e \times \mathrm{id}_{\mathcal{M}}) = \mathrm{id}_{\mathcal{M}}$. Note that we want these equalities of 1-morphisms to hold strictly, that is not up to a choice of 2-morphism between them. We now proceed to construct μ and check the relations above.

As all our fibre products are over the base category $\mathcal{S} = \mathrm{Sch}/S$, an object in $G \times \mathcal{M}$ over a scheme $T \to S$ consists of a tuple

$$\left(\begin{array}{c} T \\ \big\downarrow g \\ G \end{array} , \begin{array}{c} C \xrightarrow{f} X \\ \big\downarrow \pi \quad\quad \\ T \quad\quad \end{array} , p_1, \ldots, p_n : T \to C \right)$$

with p_i sections of π. The functor μ assigns to this the stable family $(\pi : C \to T, g.f : C \to X, p_1, \ldots, p_n)$, where

$$g.f = \sigma \circ ((g \circ \pi) \times f). \tag{22}$$

One checks that this still defines an element of $\mathcal{M}(T)$ as the G-action on X preserves β. Now assume we have a morphism between objects

$$\begin{aligned} P &= (T \xrightarrow{g} G, \quad C \xrightarrow{\pi} T, \quad C \xrightarrow{f} X, \quad p_1, \ldots, p_n : T \to C), \\ P' &= (T' \xrightarrow{g'} G, \quad C' \xrightarrow{\pi'} T', \quad C' \xrightarrow{f'} X, \quad p'_1, \ldots, p'_n : T' \to C'). \end{aligned}$$

This morphism is given by the data $\varphi : T \to T'$, $\overline{\varphi} : C \to C'$ such that $g' \circ \varphi = g$, $f' \circ \overline{\varphi} = f$ and such that

$$\begin{tikzcd} C \arrow[r, "\overline{\varphi}"] \arrow[d, "\pi"] & C' \arrow[d, "\pi'"] \\ T \arrow[r, "\varphi"] & T' \end{tikzcd}$$

becomes a cartesian diagram. Then the induced morphism $\mu(P \to P')$ between $\mu(P)$ and $\mu(P')$ shall be given by the same data $(\varphi, \overline{\varphi})$. To see that this is well-defined, we have to check that $(g'.f') \circ \overline{\varphi} = g.f$. Indeed,

$$\begin{aligned} (g'.f') \circ \overline{\varphi} &= \sigma \circ ((g' \circ \pi') \times f') \circ \overline{\varphi} \\ &= \sigma \circ ((g' \circ \pi' \circ \overline{\varphi}) \times f' \circ \overline{\varphi}) \\ &= \sigma \circ ((g' \circ \varphi \circ \pi) \times f) \\ &= \sigma \circ ((g \circ \pi) \times f) = g.f. \end{aligned}$$

Now that μ is defined, we first check the commutative diagram in (21). First we start with an object:

$$\left(\begin{array}{c} T \\ \big\downarrow g \\ G \end{array} , \begin{array}{c} T \\ \big\downarrow h \\ G \end{array} , \begin{array}{c} C \xrightarrow{f} X \\ \big\downarrow \pi \quad\quad \\ T \quad\quad \end{array} , p_1, \ldots, p_n : T \to C \right) \in G \times G \times \mathcal{M}.$$

Then the images under the upper-right and lower-left corners of the diagram (21) are both stable families $\pi : C \to T$ with identical markings p_i so we only have to show that the maps $C \to X$ coincide. Spelling out the definitions we obtain the equation

$$(m \circ (g \times h)).f = g.(h.f).$$

To see this equality, consider the following diagram:

$$
\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "((g\times h)\circ\pi)\times f"] & C \arrow[d, "((g\times h)\circ\pi)\times f"] \\
G\times G\times X \arrow[r, equal] \arrow[d, "m\times\mathrm{id}"] & G\times G\times X \arrow[d, "\mathrm{id}\times\mu"] \\
G\times X \arrow[d, "\mu"] & G\times X \arrow[d, "\mu"] \\
X \arrow[r, equal] & X.
\end{tikzcd}
$$

The left and right vertical sides are the desired morphisms. Here the lower diagram commutes because σ is an action. The fact that μ acts identically on morphisms is clear, because by definition μ does not change the data $(\varphi, \overline{\varphi})$ of the morphism at all.

The fact that the action μ is compatible with the identity e is also straightforward. □

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