

DEGREE GROWTH OF RATIONAL MAPS INDUCED FROM ALGEBRAIC STRUCTURES

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ABSTRACT. For a finite dimensional vector space equipped with a \mathbb{C} -algebra structure, one can define rational maps using the algebraic structure. In this paper, we describe the growth of the degree sequences for this type of rational maps.

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1. INTRODUCTION

In higher dimensional complex dynamics, understanding the degree growth of a rational map under iteration is a fundamental and important issue. Given any dominant rational map $f : \mathbb{P}^d \dashrightarrow \mathbb{P}^d$, one can define its p -th degree $\deg_p(f) := \deg(f^{-1}L_p)$, where L_p denotes a generic linear subspace of \mathbb{P}^d of codimension p . The problem is to describe the behavior of the sequence $\{\deg_p(f^n)\}_{n=1}^\infty$, especially as $n \rightarrow \infty$. It is not difficult to check that this sequence is submultiplicative. Following [RS] one can therefore introduce a numerical invariant which measures the exponential growth of the degree

$$\lambda_p(f) := \lim_{n \rightarrow \infty} \deg_p(f^n)^{1/n} \geq 1.$$

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These are invariants of birational conjugacy that are usually referred to as the *dynamical degrees*.

Understanding the degrees of iterates or computing the dynamical degrees of a rational selfmap are not easy tasks. Most results have been focused on the case $p = 1$; see [DF, FJ, AAdBM, AdMV, B, BK1, BK2, BT, BHM, N] and the references therein. On the other hand, the case $2 \leq p \leq d - 2$ is substantially harder since it is delicate to compute $\text{deg}_p(f)$ even in concrete examples, and there are only a few references in the literature. For monomial maps, the degree growth is obtained in [FW, L]. The degrees of special maps on the moduli space of marked points on the Riemann sphere have been investigated in [KR, R1, R2]. The dynamical degrees of maps preserving fibrations were computed in [DN]. We also refer to [O] for some results on dynamical degrees of automorphisms on hyperkähler manifolds.

In this paper, we study the behavior and the degree growth of rational maps which come from an algebra structure. One motivation to study such maps comes from the recent classification of birational maps of type $(2, 2)$ on \mathbb{P}^d in terms of involutions on certain Jordan algebra by L. Pirio and F. Russo [PR1, PR2, PR3]. There are also some sporadic studies in the literature about the dynamics of this type of maps; for example, see [CD, U] which study the dynamics of rational maps on the matrix algebras. In this paper we study the algebraic structure of these maps more systematically and describe the growth of their degrees.

Our starting point is a finite dimensional complex vector space V equipped with a \mathbb{C} -algebra structure. We shall be mainly interested in the dynamics of two classes of maps.

The first family of maps we deal with are the maps induced from single variable rational map. Here we need to assume that the algebra is *power associative* and has a *multiplicative unit* (see Section 2.1 for definitions). This implies that the power map $x \mapsto x^n$, $n \geq 0$ is well-defined, where x^n is the multiplication in V of x with itself for n times in any order. As a consequence, for a single variable polynomial $P(T) \in \mathbb{C}[T]$, P also induces a selfmap on V . Furthermore, one can show that a generic element of V is invertible. Therefore, any rational function $\varphi = \frac{Q}{P} \in \mathbb{C}(T)$, φ induces a rational selfmap $f_\varphi : V \dashrightarrow V$. Compactifying V , we obtain a rational map $f_\varphi : \mathbb{P}^d \dashrightarrow \mathbb{P}^d$, where $d = \dim_{\mathbb{C}}(V)$.

Theorem A. *Suppose V is a power associative \mathbb{C} -algebra with a multiplicative unit. Then there exists an integer $k \geq 1$ such that for any rational map $\varphi(T) = \frac{Q(T)}{P(T)} \in \mathbb{C}(T)$ and for any integer $p \geq 1$, we have*

$$\text{deg}_p(f_\varphi^n) \asymp \left(\max\{\text{deg}(Q), \text{deg}(P)\}^{\min\{p, k\}} \right)^n.$$

The second type of maps we study are the (generalized) monomial maps. In order for the iterates to be still monomial maps, we need to require that the algebra structure is *abelian* (i.e., commutative, associative and unitary). Now we let $k = \dim_{\mathbb{C}}(V)$, given a $d \times d$ integer matrix $A = (a_{ij}) \in M_d(\mathbb{Z})$ with $\det(A) \neq 0$, we can define the monomial map $F_A : V^d \dashrightarrow V^d$ as

$$F_A(x_1, \dots, x_d) = \left(\prod_j x_j^{a_{1,j}}, \dots, \prod_j x_j^{a_{d,j}} \right),$$

where $x_j \in V$ and the product is given by the multiplication of V . Let the number m be given by $m = \dim_{\mathbb{C}}(\text{red}(V))$, where $\text{red}(V) = V/N(V)$ and $N(V)$ is the nilradical of V , i.e., the ideal of all nilpotent elements.

Theorem B. *Let $\text{diag}(A; m)$ be the block-diagonal matrix with m blocks of the matrix A on the diagonal positions. Then the degree growth of the (generalized) monomial map F_A is given by*

$$\text{deg}_p(F_A^n) \asymp \max_{p-d(k-m) \leq i \leq p} \|\wedge^i \text{diag}(A; m)^n\|.$$

Observe that m only depends on the structure of V and not on the map F_A . Also, the norm $\|\wedge^i \text{diag}(A, m)\|$ can be computed solely in terms of $\|\wedge^j A^n\|$, $j \leq i$, and m . Indeed, one has

$$\|\wedge^i \text{diag}(A, m)^n\| \asymp \max_{i_1 + \dots + i_m = i} \|\wedge^{i_1} A^n\| \cdots \|\wedge^{i_m} A^n\|.$$

There are many other possibilities to associate a rational map to a structure of algebra. We point out here some generalizations of our setting and natural questions that may arise.

Question 1. Let V be an abelian algebra, and pick any rational map

$$f = [f_0 : \dots : f_d] : \mathbb{P}_{\mathbb{C}}^d \dashrightarrow \mathbb{P}_{\mathbb{C}}^d,$$

where each f_j is a homogeneous polynomial of the same degree. Then f_j also induces a map $V^{d+1} \rightarrow V$ which is homogeneous of the same degree. Thus f induces a map $F : \mathbb{P}(V^{d+1}) \dashrightarrow \mathbb{P}(V^{d+1})$. Can we describe the relation between the degree growth of f^N and the degree growth of F^N ?

Question 2. Under the same notation as the above question, one can show that F is birational when f is birational. In the particular case where $d = 2$, it is tempting to explore if similar results as in [DF] can be obtained about the degree growth of the iterates of F ?

Question 3. If V is a power associative algebra with a multiplicative unit, and $P(T) \in \mathbb{C}[T]$ is a polynomial, then we can define the generalized Hénon map $H_P : V^2 \rightarrow V^2$ by

$$H_P(x, y) = (y, P(x) - cy),$$

where $c \in \mathbb{C} \setminus \{0\}$. What is the degree growth of the iterates of H_P ?

2. ALGEBRA STRUCTURE AND QUADRATIC MAPS

2.1. The algebra structure. Let V be a finite dimensional complex vector space equipped with a \mathbb{C} -algebra structure, that is, a \mathbb{C} -linear map $\mu : V \otimes V \rightarrow V$. We shall always denote multiplicatively this law, i.e., we will denote $\mu(x \otimes y)$ by xy .

Recall the following list of classical definitions.

- V is *unitary* if it has a unit 1 (i.e., $1 \cdot x = x \cdot 1 = x$ for all x);
- V is *associative* if $x(yz) = (xy)z$ for all x, y, z ;
- V is *commutative* if $xy = yx$ for all x, y ;
- V is *alternative* if $x(xy) = x^2y$ and $(yx)x = yx^2$ for all x, y ;
- V is *power-associative* if the algebra generated by any element is associative;
- V is *abelian* if the multiplication law is commutative, associative and unitary (this is the typical setting for commutative algebra, e.g., [AM]);
- V is a *Jordan algebra* if it is commutative and alternative.

Set $d = \dim_{\mathbb{C}} V$. Observe that the space \mathcal{A} of all algebra structures on V is $\text{Hom}_{\mathbb{C}}(V \otimes V, V)$, which is an affine space of dimension d^3 .

The space of abelian (resp. associative, alternative, power-associative, Jordan) algebras is a Zariski closed subset of \mathcal{A} .

When V is power associative, we define the set of nilpotent elements as $N(V) := \{x \in V \mid x^N = 0 \text{ for some integer } N\}$. When V is abelian, then $N(V)$ is an ideal.

2.2. Quadratic maps. There is a natural identification between d -dimensional complex commutative (but not necessarily associative) algebras and homogeneous quadratic polynomial maps from \mathbb{C}^d to itself, as follows.

Choose a basis of V as a complex vector space $V = \mathbb{C}e_1 \oplus \dots \oplus \mathbb{C}e_d$, and write $e_i \cdot e_j = \sum_k a_{ij}^k e_k$ with $a_{ij}^k \in \mathbb{C}$. Since V is commutative, we have $a_{ij}^k = a_{ji}^k$. Then $f_V(x) = x^2$ is a polynomial map as above, and we have

$$f_V(z_1, \dots, z_d) = \left(\sum_{1 \leq i, j \leq d} a_{ij}^1 z_i z_j, \dots, \sum_{1 \leq i, j \leq d} a_{ij}^d z_i z_j \right).$$

Conversely, given a homogeneous quadratic polynomial map $f : \mathbb{C}^d \rightarrow \mathbb{C}^d$, one can define the algebra structure as

$$x \cdot y = \frac{1}{2} (f(x + y) - f(x) - f(y)).$$

The two operations are inverse to each other. Therefore, if we use f_V to denote again the induced quadratic map $\mathbb{P}^{d-1} \dashrightarrow \mathbb{P}^{d-1}$, then we can see that the space of all quadratic rational selfmaps on \mathbb{P}^{d-1} can be identified with the space of f_V over all algebra structures on \mathbb{C}^d . Moreover, the following results show that the dynamics of f_V plays a special role in the structure of V .

Proposition 2.1. *The algebra is unitary iff f_V admits a fixed point x such that $df_V(x) = 2 \text{ id}$.*

Proof. In one direction this is obvious since $(1 + tx)^2 = 1 + 2tx + O(t^2)$. In the other direction, pick a fixed point x and let y be any other point. Then $(x + ty)^2 = x^2 + 2t(x \cdot y) + O(t^2) = x^2 + df_V(x)(ty) + O(t^2)$, whence $x \cdot y = y$ as required. \square

Proposition 2.2. *Suppose V and V' are dimension d commutative algebras. Then V is isomorphic to V' iff $f_V : \mathbb{P}^{d-1} \dashrightarrow \mathbb{P}^{d-1}$ is conjugated in $\text{PGL}(d, \mathbb{C})$ to $f_{V'}$.*

Proof. Identify both V and V' to \mathbb{C}^d . Being isomorphic then means that there exists $A \in \text{GL}(d, \mathbb{C})$ such that $A(x \cdot_V y) = Ax \cdot_{V'} Ay$. This implies the conjugacy in \mathbb{C}^d .

Conversely, being conjugated in \mathbb{P}^{d-1} implies the existence of a matrix in $\text{GL}(d, \mathbb{C})$ and a function $\lambda(x)$ such that $A(x \cdot_V x) = \lambda(x) Ax \cdot_{V'} Ax$. For degree reasons, λ is a constant. Changing A by μA replaces λ by $\lambda\mu$ hence we can assume $\lambda = 1$. \square

Remark. Any two-dimensional unitary commutative algebra is abelian, hence isomorphic to either $\mathbb{C}[x]/(x^2)$ or to $\mathbb{C} \oplus \mathbb{C}$.

Indeed, the algebra structure is determined by $x^2 = a + bx$. When $a + b^2/4 = 0$, then there exists an element $y := \mu x + \lambda$ such that $y^2 = 0$ and we are in the former case. When $a + b^2/4 \neq 0$, then there exists an element $y := \mu x + \lambda$ such that $y^2 = y$ and we are in the latter case.

3. SOME LEMMAS ABOUT DEGREE GROWTH

In this section, we will prove several lemmas for the degrees and degree growth of rational maps. The maps we study in this paper all preserve some fibration. The dynamical degrees for maps preserving a fibration were computed by Dinh, Nguyễn and Truong [DN, DNT] (see [T] and [D] for a purely algebraic approach to these computations). Their results are very general and their computations are quite involved. For the convenience of the reader we have preferred to reprove some of their (elementary) results that we shall need in the latter section.

First, let us define higher degrees. For a rational selfmap $f : X \dashrightarrow X$ on a projective variety X of dimension d , and given an ample divisor D , we define the p -th degree of f with respect to D as $\text{deg}_{D,p}(f) = f^*[D]^p \cdot [D]^{d-p}$, where $[D]$ is the class of D in $H^{1,1}(X)$. When $X = \mathbb{P}^d$ and $D = H$ is a generic hyperplane, this coincides with the usual definition of degrees. In this case, we can compute the degree by $\text{deg}_p(f) = f^{-1}L_p \cdot L_{d-p}$, where L_p and L_{d-p} are generic linear subspaces of \mathbb{P}^d of codimensions p and $d - p$, respectively, and $f^{-1}L_p$ is the proper transform of L_p under f .

3.1. Product map. Product maps are the simplest maps preserving a fibration. Let $f : X \dashrightarrow X$ and $g : X' \dashrightarrow X'$ be two rational maps, and $h = f \times g$. Pick any two ample divisors D, D' on X, X' , respectively, and write $d = \dim X, d' = \dim X'$. By abuse of notation we shall again denote by D and D' their pull-back to $X \times X'$ so that $D + D'$ is again ample on $X \times X'$. The degrees of (f, g) can be then computed by a direct calculation as

$$\text{deg}_{D+D',p}(h) = \sum_{i+j=p} \binom{p}{i} \binom{d+d'-p}{d-i} \text{deg}_{D,i}(f) \text{deg}_{D',j}(g).$$

In dynamics, we are interested in the growth of degrees under iteration. For this reason, we introduce the following notation for *asymptotic equivalence*. For two sequences $\{a_j\}, \{b_j\}$ of positive numbers, we say the two sequences are *asymptotic equivalent*, denoted by $a_j \asymp b_j$, if there is a constant $C > 0$ such that $C^{-1}a_j \leq b_j \leq Ca_j$ for all large enough j .

For a rational map f , the asymptotic behavior of the degree sequence $\{\text{deg}_{D,p}(f^n)\}_{n=1}^\infty$ is independent of the ample divisor D , and is invariant under birational conjugation (see [DS] using analytic arguments or [T, D] using algebraic ones). An important class of invariants to measure the asymptotic growth of the degree sequence is the *dynamical degrees*. The p -th dynamical degree of a rational map f , denoted by $\lambda_p(f)$, is defined as

$$\lambda_p(f) = \lim_{n \rightarrow \infty} \text{deg}_{D,p}(f^n)^{1/n}.$$

The existence of the limit follows here from the sub-multiplicativity of the sequence $\{C \text{deg}_{D,p}(f^n)\}_{n=1}^\infty$ for some constant $C > 0$; see [DS].

With the above notation, we can first conclude that for product map $h = f \times g$, we have

$$(3.1) \quad \text{deg}_{D+D',p}(h^n) \asymp \max_{i+j=p} \{ \text{deg}_{D,i}(f^n) \text{deg}_{D',j}(g^n) \}.$$

A consequence of this is a formula for the dynamical degrees of h :

$$\lambda_p(h) = \max_{i+j=p} \{ \lambda_i(f) \lambda_j(g) \}.$$

3.2. Skew product with fibers \mathbb{P}^1 . Another type of rational maps preserving a fibration that we will encounter is a map $F : \mathbb{P}^d \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^d \times \mathbb{P}^1$ which preserves the fibration $\pi : \mathbb{P}^d \times \mathbb{P}^1 \rightarrow \mathbb{P}^d$. Denote by H_d the pull-back of $\mathcal{O}(1)_{\mathbb{P}^d}$ on $\mathbb{P}^d \times \mathbb{P}^1$, and by H_1 the pull-back of $\mathcal{O}(1)_{\mathbb{P}^1}$ on the same variety.

Proposition 3.1. *Suppose that the rational map $F : \mathbb{P}^d \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^d \times \mathbb{P}^1$ is of the form $F(z, t) = (g(z), h(z, t))$, where $z \in \mathbb{P}^d$, $t \in \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$. Suppose $F^*H_1 = \delta_d H_d + \delta_1 H_1$.*

The p -th degree of F with respect to the ample divisor $D := H_d + H_1$ can be computed as

$$\deg_{D,p}(F) = (d + 1 - p) \deg_p(g) + p(\delta_1 + (d + 1 - p)\delta_d) \cdot \deg_{p-1}(g).$$

Observe that δ_1 is the degree in t of the rational function $h(z, t)$ whereas δ_d is the degree in z of $h(z, t)$.

Proof of Proposition 3.1. One has

$$\begin{aligned} F^*H_d^p &= \deg_p(g)H_d^p, \\ F^*(H_d^{p-1} \cdot H_1) &= \deg_{p-1}(g)H_d^{p-1} \cdot (\delta_d H_d + \delta_1 H_1). \end{aligned}$$

The first equality is by definition. For the second equality, we first claim the following.

Claim. for any subvariety Z of $\mathbb{P}^d \times \mathbb{P}^1$, we can find a linear subspace $L \subset \mathbb{P}^d$ of codimension $p - 1$ and a point $q \in \mathbb{P}^1$ such that $L \times \{q\}$, $L \times \mathbb{P}^1$ and $\mathbb{P}^d \times \{q\}$ all intersect Z properly.

Recall that two pure dimensional subvarieties W and W' intersect properly when the codimension of any irreducible component of $W \cap W'$ is equal to $\text{codim}(W) + \text{codim}(W')$.

Here we show the claim for $L \times \{q\}$, and leave the other two cases to the reader. Since Z is irreducible, the projection of Z on \mathbb{P}^1 is either a single point $\{q'\}$ or \mathbb{P}^1 . In the first case, we can pick any point $q \neq q'$ in \mathbb{P}^1 and any linear subvariety L of codimension $p - 1$. For the latter case, pick a generic $q \in \mathbb{P}^1$ such that $Z' = (\mathbb{P}^d \times \{q\}) \cap Z$ is a variety of pure dimension $\dim(Z) - 1$. Next, pick a generic linear subspace $L \in \mathbb{P}^d$ which intersects Z' properly. Then $L \times \{q\}$ will intersect Z properly.

Now let $\Gamma \subset (\mathbb{P}^d \times \mathbb{P}^1) \times (\mathbb{P}^d \times \mathbb{P}^1)$ be the graph of F . Denote by π_1 and π_2 the projections from Γ onto the first and second components, and set

$$Z = \{q \in \mathbb{P}^d \times \mathbb{P}^1 \mid \dim(\pi_2^{-1}(q)) > 0\}.$$

The class $\pi_2^*(H_d^{p-1} \cdot H_1)$ is represented by $\pi_2^{-1}(L \times \{p\})$ (see e.g. [T, Lemma 3.1]), which is equal to $\pi_2^{-1}(L \times \mathbb{P}^1) \cap \pi_2^{-1}(\mathbb{P}^d \times \{q\})$. On the other hand, this intersection represents the class $\pi_2^*H_d^{p-1} \cdot \pi_2^*H_1$. Thus we have

$$\pi_2^*(H_d^{p-1} \cdot H_1) = \pi_2^*H_d^{p-1} \cdot \pi_2^*H_1.$$

Notice that $\pi_2^* H_d^{p-1} = \pi_1^*(\deg_{p-1}(g)H_d^{p-1}) + E$ for some class E such that $\pi_{1*}(E) = 0$, therefore

$$\begin{aligned} F^*(H_d^{p-1} \cdot H_1) &= \pi_{1*}\pi_2^*(H_d^{p-1} \cdot H_1) = \pi_{1*}(\pi_2^*H_d^{p-1} \cdot \pi_2^*H_1) \\ &= \pi_{1*}(\pi_1^*(\deg_{p-1}(g)H_d^{p-1}) \cdot \pi_2^*H_1) + \pi_{1*}(E \cdot \pi_2^*H_1) \\ &= \deg_{p-1}(g)H_d^{p-1} \cdot (\pi_{1*}\pi_2^*H_1) + 0 \\ &= \deg_{p-1}(g)H_d^{p-1} \cdot F^*H_1 = \deg_{p-1}(g)H_d^{p-1} \cdot (\delta_d H_d + \delta_1 H_1). \end{aligned}$$

Finally, we can compute

$$\begin{aligned} F^*(D^p) &= F^*\left(H_d^p + p H_d^{p-1} \cdot H_1\right) \\ &= (\deg_p(g) + p \deg_{p-1}(g)\delta_d) H_d^p + p \deg_{p-1}(g)\delta_1 H_d^{p-1} \cdot H_1 \end{aligned}$$

and the result follows by intersecting with the class

$$D^{d+1-p} = H_d^{d+1-p} + (d+1-p)H_d^{d-p} \cdot H_1. \quad \square$$

4. DEGREE GROWTH OF THE SQUARING MAP

In this section we study the degree growth of the squaring map $f_V : \mathbb{P}^{d-1} \dashrightarrow \mathbb{P}^{d-1}$ introduced in the previous section. The methods we use in this section will be generalized in later sections to prove our main Theorems A and B. For squaring maps, these methods are more intuitive and serve as an illustration for the more complicated cases.

4.1. The abelian case. We assume V is an abelian algebra. Since V is finite dimensional as a complex vector space, the \mathbb{C} -algebra V is Artinian ([AM, §8, Exercise 2]). By the structure theorem for Artin rings ([AM, Theorem 8.7]), we can decompose V as a finite direct product of Artinian local rings $V \simeq \prod_{i=1}^k V_i$.

For the Artin local ring V_i , the maximal ideal \mathfrak{m}_i is nilpotent (i.e., $\mathfrak{m}_i^l = 0$ for some $l \geq 1$) and $V_i/\mathfrak{m}_i \simeq \mathbb{C}$. As a \mathbb{C} -vector space, we can write V_i as the direct sum $V_i \simeq \mathbb{C} \oplus \mathfrak{m}_i$. For V , we introduce the map

$$\Phi : V \simeq \mathbb{C}^k \times \prod_{i=1}^k \mathfrak{m}_i \longrightarrow V$$

sending $((a_1, \dots, a_k), (h_1, \dots, h_k))$ to $\sum_{i=1}^k a_i \exp(h_i)$, where

$$\exp(h) = \sum_{j=0}^{\infty} h^j / j!.$$

In the source, we write the argument in the form $((a_1, \dots, a_k), (h_1, \dots, h_k))$ as an element of $\mathbb{C}^k \times \prod_{i=1}^k \mathfrak{m}_i$. This represents the element $\sum_{i=1}^k (a_i 1_i + h_i) \in V$, where 1_i means the image of the identity element of V_i in $V \simeq \prod_{i=1}^k V_i$. Whereas in the target, we write the image $\sum_{i=1}^k a_i \exp(h_i)$ as an element of V , which, under the isomorphism $V \simeq \mathbb{C}^k \times \prod_{i=1}^k \mathfrak{m}_i$, is represented by

$$\left((a_1, \dots, a_k), \left(a_1 \sum_{j=1}^{\infty} \frac{h_1^j}{j!}, \dots, a_k \sum_{j=1}^{\infty} \frac{h_k^j}{j!} \right) \right).$$

Notice that $\exp(h_i)$ is well defined since $h_i \in \mathfrak{m}_i$ is nilpotent, so the sum is indeed finite.

Moreover, we claim that Φ is a birational map. Indeed, using the vector space isomorphism $V \simeq \mathbb{C}^k \times \prod_{i=1}^k \mathfrak{m}_i$, we can describe the birational inverse of Φ concretely as

$$\Phi^{-1}((a_1, \dots, a_k), (h_1, \dots, h_k)) = \sum_{i=1}^k (a_i 1_i + \log(1 + a_i^{-1} h_i)) ,$$

where $\log(1 + x) = \sum_{j=1}^{\infty} (-1)^{j-1} x^j / j$. Again, we write the argument of Φ^{-1} as an element of $\mathbb{C}^k \times \prod_{i=1}^k \mathfrak{m}_i$ and the image as an element of V .

The usual rules for the exponential and logarithm functions hold. Hence the fact that Φ and Φ^{-1} are inverse to each other is a reflection of the fact that $\log(\exp(h)) = h$. However, we emphasize here that Φ and Φ^{-1} are *not* ring homomorphisms. They are inverse to each other only as rational maps.

Example 4.1. Suppose that $V = \mathbb{C}[x]/(x^3)$, then $V \simeq \mathbb{C} \cdot 1 \oplus \mathbb{C}x \oplus \mathbb{C}x^2$ and $\mathfrak{m} = xV \simeq \mathbb{C}x \oplus \mathbb{C}x^2$ as \mathbb{C} -vector spaces. The element $a + bx + cx^2 \in V$ is represented as $(a, bx + cx^2)$ as an element of $\mathbb{C} \oplus \mathfrak{m}$. By definition, we have $\Phi(a + bx + cx^2) = a \exp(bx + cx^2) = a + abx + (ac + \frac{ab^2}{2})x^2$ and $\Phi^{-1}(a + bx + cx^2) = a + \log[1 + a^{-1}(bx + cx^2)] = a + \frac{b}{a}x + \frac{2ac - b^2}{2a^2}x^2$.

Define

$$F[(a_1, \dots, a_k), (h_1, \dots, h_k)] = [(a_1^2, \dots, a_k^2), (2h_1, \dots, 2h_k)]$$

then

$$\begin{aligned} & (\Phi \circ F)[(a_1, \dots, a_k), (h_1, \dots, h_k)] \\ &= \sum_{i=1}^k a_i^2 \exp(2h_i) \\ &= \sum_{i=1}^k (a_i \exp(h_i))^2 \\ &= \left(\sum_{i=1}^k a_i \exp(h_i) \right)^2 \quad \text{by the definition of the product ring} \\ &= (f_V \circ \Phi)[(a_1, \dots, a_k), (h_1, \dots, h_k)] . \end{aligned}$$

Thus, f_V is birationally conjugate to a product of power maps and linear maps.

Recall that the reduced algebra associated to V is by definition the quotient of V by its nilradical $N(V)$. Since for each V_i , we have $\text{red}(V_i) \cong \mathbb{C}$, thus $\text{red}(V) \cong \mathbb{C}^k$. From the product structure of f_V , we obtain the following.

Theorem 4.1. *Suppose V is abelian, and write $k := \dim_{\mathbb{C}} \text{red}(V)$. Then for any p , we have*

$$\text{deg}_p(f_V^n) \asymp \left(2^{\min\{p, k\}} \right)^n \quad \square$$

Example 4.2. Suppose that V is power associative and generated by one element. Then it is automatically abelian, and there exists a polynomial $P(x) \in \mathbb{C}[x]$ such that

$$V \simeq \mathbb{C}[x]/(P(x)) \simeq \mathbb{C}[x]/\prod_{i=1}^k (x - z_i)^{k_i} \simeq \bigoplus_{i=1}^k \mathbb{C}[x_i]/((x_i - z_i)^{k_i}) .$$

In this case, k is the number of different (complex) roots of the defining polynomial P .

4.2. Power associative algebras. In this section, we will deal with the squaring map f_V when V is a power associative algebra with 1. We start by analyzing the structure of the algebra. For any non-zero $x \in V$ denote by $\mathbb{C}[x]$ the algebra generated by x and $\delta(x) := \dim_{\mathbb{C}} \mathbb{C}[x]$. Since \mathbb{C} is always a subspace of $\mathbb{C}[x]$, $\delta(x) \geq 1$, and since V is power associative, $\mathbb{C}[x]$ is abelian. Moreover, $\mathbb{C}[x]$ is invariant under f_V , i.e., $f_V(\mathbb{C}[x]) \subseteq \mathbb{C}[x]$.

Observe that

$$V_k = \{x \in V \mid (1, x, x^2, \dots, x^k) \text{ are linearly dependent}\}$$

is a Zariski closed subset of V since it is defined by the vanishing of finitely many determinants of matrices of size $(k + 1) \times (k + 1)$. Since $V_k = \{x \mid \delta(x) \leq k\}$ and $V_k \subseteq V_l$ if $k \leq l$, we conclude that the function $x \mapsto \delta(x)$ is lower semicontinuous for the Zariski topology.

Introduce $\delta = \delta_V := \max\{\delta(x) \mid x \in V\}$ and $U' := \{x \in V \mid \delta(x) = \delta_V\}$. The latter is a Zariski dense open subset of V . Let $F := V \setminus U'$ and pick $x, y \in U'$, we have the following observations:

- (1) $y \in \mathbb{C}[x] \iff x \in \mathbb{C}[y] \iff \mathbb{C}[x] = \mathbb{C}[y]$,
- (2) if $y \notin \mathbb{C}[x]$, then $\mathbb{C}[x] \cap \mathbb{C}[y] \cap U' = \emptyset$, i.e., $\mathbb{C}[x] \cap \mathbb{C}[y] \subseteq F$.

Moreover, we claim the following.

Claim. There is a further open dense subset $U \subset U'$ such that for any two $x, y \in U$, we have $\mathbb{C}[x] \cong \mathbb{C}[y]$ as \mathbb{C} -algebras.

Proof of the Claim. For each $x \in U'$, there is a canonical map $\mathbb{C}[T] \rightarrow \mathbb{C}[x]$, where T is a variable, by sending $T \mapsto x$. Thus, $\mathbb{C}[x] \cong \mathbb{C}[T]/(P_x(T))$ for a unique monic polynomial $P_x(T)$ of degree δ . The coefficients of $P_x(T)$ depend algebraically on x . Let $S^\delta \mathbb{C}$ denote the δ -th symmetric product of \mathbb{C} . Then one can parameterize all monic polynomials of degree δ either by \mathbb{C}^δ (using the coefficients, excluding the leading one), or by $S^\delta \mathbb{C}$ (using the δ roots of the polynomial, counting multiplicities); and the two spaces $S^\delta \mathbb{C}$ and \mathbb{C}^δ are isomorphic.

We define the map $U \rightarrow S^\delta \mathbb{C}$ by sending x to the multiset of complex solutions of $P_x(T)$. Each multiset consists of δ elements, hence gives rise to a partition of δ by counting the multiplicities of different elements. Denote Γ as the set of all partitions of δ , then the above process defines a map $S^\delta \mathbb{C} \rightarrow \Gamma$. Use ψ to denote the composition $U \rightarrow S^\delta \mathbb{C} \rightarrow \Gamma$. Define a partial order \preceq on Γ by $\gamma \preceq \gamma'$ if γ' is a refinement of γ . Also, define $U_\gamma = \{x \in U' \mid \psi(x) \preceq \gamma\}$.

Notice that each inequality $\gamma \prec \gamma'$ (this notation means $\gamma \preceq \gamma'$ but $\gamma \neq \gamma'$) can be factored into a sequence of minimal refinements such that each partition in the sequence is obtained by decomposing one number in the previous partition as the sum of two positive numbers. For instance, we may have $\gamma = \{\nu_1, \nu_2\}$ and $\gamma' = \{\nu'_1, \nu''_1, \nu_2\}$, where $\nu'_1 + \nu''_1 = \nu_1$ as an example of a minimal refinement. Then an element $x \in U_{\gamma'}$ will have the corresponding polynomial $P_x(T)$ of the form $P_x(T) = (T - \alpha_0)^{\nu_1} (T - \alpha_1)^{\nu_2}$, where the α_i 's are not necessarily different, and $x \in U_\gamma \subset U_{\gamma'}$ will then be defined by the close condition $\alpha_0 = \alpha_1$ in $U_{\gamma'}$. Generalizing the above instance, one can obtain the conclusion that if $\gamma \prec \gamma'$, then $U_\gamma \subset U_{\gamma'}$ is a closed subset. Therefore, with respect to this partial order, ψ is lower semicontinuous.

Lower semicontinuity implies that we can find $\gamma_1, \dots, \gamma_m \in \Gamma$, no two of them are comparable under “ \preceq ”, such that if we define $U_i = \{x \in U' \mid \psi(x) \preceq \gamma_i\}$, then each U_i is closed in U' , and $U' = \bigcup_{i=1}^m U_i$. However, since U' is an open dense subset of the irreducible space V , U' is itself irreducible. This implies that $m = 1$, and there is a unique partition $\gamma \in \Gamma$, and an open dense subset $U \subset U'$ such that $U = \{x \in U' \mid \psi(x) = \gamma\}$.

Finally, since we have $\mathbb{C}[T]/((T - \alpha)^k) \cong \mathbb{C}[T]/(T^k)$ for all $\alpha \in \mathbb{C}$ (via the isomorphism $T \mapsto T + \alpha$), the isomorphic class of $\mathbb{C}[x]$ for $x \in U$ only depend on the multiplicity of different roots of $P_x(T)$, which is recorded as $\psi(x)$. Since $\psi(x) = \gamma$ for all $x \in U$, we conclude that for all $x \in U$, $\mathbb{C}[x]$ are isomorphic to each other. \square

Fix a monic polynomial $P_0(T)$ of degree δ whose roots gives rise to the partition γ . Then for each $x \in U$, we have the isomorphism

$$\phi_x : V_0 := \mathbb{C}[T]/(P_0(T)) \longrightarrow \mathbb{C}[x] \cong \mathbb{C}[T]/(P_x(T)).$$

Notice that ϕ_x depend algebraically on x .

Next, choose a generic affine (i.e., a translation of a linear) subspace $L \subset V$ of dimension $d - \delta$, such that $L \cap U$ is open and dense in L . Then for a generic $x \in U$, $\mathbb{C}[x]$ intersects L at a single point. One defines the following birational map,

$$\begin{aligned} \Phi : L \times V_0 &\dashrightarrow V \\ (x, v) &\longmapsto \phi_x(v) \in \mathbb{C}[x] \subset V, \end{aligned}$$

whose inverse is given by

$$\begin{aligned} \Psi : V &\dashrightarrow L \times V_0 \\ y &\longmapsto (x = \mathbb{C}[y] \cap L, \phi_x^{-1}(y)) . \end{aligned}$$

Observe that it also induces a birational map

$$\Phi : \mathbb{P}^{d-\delta} \times \mathbb{P}^\delta \dashrightarrow \mathbb{P}(V) .$$

Since Φ is birational, one can lift f_V to a product map $\tilde{f}_V : \mathbb{P}^{d-\delta} \times \mathbb{P}^\delta \dashrightarrow \mathbb{P}^{d-\delta} \times \mathbb{P}^\delta$ which acts as $\tilde{f}_V = \text{id} \times f_{V_0}$.

Therefore, we conclude the following.

Theorem 4.2. *Suppose V is power associative. Let $k = \dim_{\mathbb{C}}(\text{red } \mathbb{C}[x])$ for any $x \in U$ as described above. Then for any p , we have*

$$\text{deg}_p(f_V^n) \asymp \left(2^{\min\{p, k\}}\right)^n \quad \square$$

Example 4.3. Let $V = M_m(\mathbb{C})$ be the algebra of $m \times m$ complex matrices. For a matrix A with m distinct eigenvalues μ_1, \dots, μ_m , its characteristic polynomial is also the minimal polynomial. Thus, $\mathbb{C}[A] \cong \bigoplus_{i=1}^m \mathbb{C}[x]/(x - \mu_i) \cong \mathbb{C}^m$. Notice that “having m distinct eigenvalues” is a generic property in $M_m(\mathbb{C})$. Therefore, for the matrix algebra $M_m(\mathbb{C})$, the number k in the theorem is equal to m .

5. GENERALIZATION TO RATIONAL MAPS

5.1. Polynomial maps. Suppose V is power associative and pick any polynomial $P \in \mathbb{C}[T]$. Then one can look at the map $f_P(v) = P(v)$ on V and compute the degree growth of f_P on the affine space V .

Theorem 5.1. *Suppose V is power associative, and pick $P \in \mathbb{C}[T]$. Then there exists k such that for any p , we have*

$$\deg_p(f_P^n) \asymp (\deg(P)^{\min\{p,k\}})^n.$$

Proof. By the same trick as in the previous section, we first treat the case that V is abelian and generated by one element. That is, we assume

$$V \cong \mathbb{C}[T]/\prod_{i=1}^l (T - z_i)^{m_i} \cong \prod_{i=1}^l \mathbb{C}[T_i]/(T_i^{m_i}).$$

Since f_P preserves each factor of the product decomposition above, it is sufficient to treat the case $V = \mathbb{C}[x]/(x^m)$. A point in V can then be written as $\sum_{i=0}^{m-1} \lambda_i x^i$ and if $P(T) = \sum_j a_j T^j$ we get

$$\begin{aligned} P\left(\sum_{i=0}^{m-1} \lambda_i x^i\right) &= \sum_j a_j \left(\sum_{i=0}^{m-1} \lambda_i x^i\right)^j \\ &= P(\lambda_0) + x[\lambda_1 P'(\lambda_0)] + x^2 \left[\lambda_2 P'(\lambda_0) + \frac{1}{2} \lambda_1^2 P''(\lambda_0)\right] + \dots \\ &\quad + x^j [\lambda_j P'(\lambda_0) + Q_j(\lambda_0, \lambda_1, \dots, \lambda_{j-1})] + \dots \\ &= P(\lambda_0) + \sum_{j=1}^{m-1} x^j (\lambda_j P'(\lambda_0) + Q_j(\lambda_0, \lambda_1, \dots, \lambda_{j-1})). \end{aligned}$$

In other words, we may write

$$\begin{aligned} f_V(\lambda_0, \lambda_1, \dots, \lambda_{m-1}) &= \left(P(\lambda_0), \lambda_1 P'(\lambda_0), \lambda_2 P'(\lambda_0) + \frac{1}{2} \lambda_1^2 P''(\lambda_0), \right. \\ &\quad \dots, \lambda_j P'(\lambda_0) + Q_j(\lambda_0, \dots, \lambda_{j-1}), \\ &\quad \left. \dots, \lambda_{m-1} P'(\lambda_0) + Q_{m-1}(\lambda_0, \lambda_1, \dots, \lambda_{m-2}) \right). \end{aligned}$$

Observe that the constant term (i.e., the first coordinate of the function) is just $P(\lambda_0)$, and for $1 \leq j \leq m - 1$ the coefficient of x^j (i.e., the j -th coordinate of the function) has the following two properties:

- (1) it is a polynomial function of $\lambda_0, \lambda_1, \dots, \lambda_j$;
- (2) it is an affine function in λ_j .

If for $0 \leq i \leq m - 1$ we let $V^{(i)} = \mathbb{C}^{i+1}$ be the first $i + 1$ coordinates of V , then by observation (1) the map f_P induces a selfmap on each $V^{(i)}$. Moreover, for $1 \leq i \leq m - 1$ let $\pi_i : V^{(i)} \rightarrow V^{(i-1)}$ be the projection, then $f_P|_{V^{(i)}}$ preserve the fibration π_i and the map on a generic fiber is a linear isomorphism of \mathbb{P}^1 by (2) above. Moreover, for each $i \geq 1$, the function on the last coordinate $(f_P)_i : V^{(i)} \rightarrow V^{(i)}/V^{(i-1)} \cong \mathbb{C}$, as a function of $\lambda_0, \dots, \lambda_{i-1}$, is of degree either $\deg(P) - 1$ (for $i = 1$) or $\deg(P)$ (for $i \geq 2$).

When we pass to the n -th iterate, we can see that $f_P^n = f_{P^n}$ and $\deg(P^n) = \deg(P)^n$. We then use Proposition 3.1 repeatedly for $i = 1, \dots, m - 1$. Notice that by (1) above, the number δ_1 in Proposition 3.1 always equals one (for each i); by the previous paragraph, the number δ_d in Proposition 3.1 is asymptotic to $\deg(P)^n$. This implies that the degree growth for each p is indeed

$$\deg_p(f_P^n) \asymp \deg(P)^{pn}.$$

This is for the special case of $\mathbb{C}[x] \cong \mathbb{C}[T]/(T^m)$. If $V = \mathbb{C}[x]$, then the same formula is true by (3.1) since we have $\mathbb{C}[x] \cong \prod_i \mathbb{C}[T_i]/(T_i^{m_i})$.

Finally, let k be the dimension of $\text{red } \mathbb{C}[x]$ for a generic $x \in V$. By the same argument as in Section 4.2, we know that for generic x, y , we have $\mathbb{C}[x] \cong \mathbb{C}[y]$, and we can further use this fact to make f_P a product map. Therefore, we can conclude again that we have

$$\deg_p(f_P^n) \asymp \left(\deg(P)^{\min\{p,k\}} \right)^n. \quad \square$$

5.2. Rational maps. Next, we claim that for a rational function $\varphi(T) = \frac{Q(T)}{P(T)} \in \mathbb{C}[T]$, we have the same result for degree growth for the induced map $f(v) := P(v)^{-1}Q(v)$ on V . We will show in a moment that for a generic $v \in V$, $P(v)$ is invertible, thus f induces a dominant rational map from V to itself.

First, assume $V = \mathbb{C}[x] \cong \mathbb{C}[T]/(T^m)$, $v = \sum_{i=0}^{m-1} \lambda_i x^i$, and $P(x) = \sum_i a_i x^i$. We get

$$P(v) = P(\lambda_0) + \sum_{j=1}^{m-1} x^j (\lambda_j P'(\lambda_0) + Q_j(\lambda_0, \lambda_1, \dots, \lambda_{j-1}))$$

If $P(\lambda_0) \neq 0$, which is the generic case, then $P(v) - P(\lambda_0) \in xV$ is nilpotent. Thus $P(v)$ is invertible, and its inverse is given by

$$\begin{aligned} P(v)^{-1} &= \left\{ P(\lambda_0) \left(1 + \sum_{j=1}^{m-1} x^j \cdot \frac{\lambda_j P'(\lambda_0) + Q_j(\lambda_0, \lambda_1, \dots, \lambda_{j-1})}{P(\lambda_0)} \right) \right\}^{-1} \\ &= P(\lambda_0)^{-1} \cdot \sum_{i=0}^{m-1} \left(- \sum_{j=1}^{m-1} x^j \cdot \frac{\lambda_j P'(\lambda_0) + Q_j(\lambda_0, \lambda_1, \dots, \lambda_{j-1})}{P(\lambda_0)} \right)^i. \end{aligned}$$

Expanding the last line, we get a polynomial expression in x . In order to find the coefficient for x^j , we observe that in the expansion, x^j can be formed by products of terms coming from $x^{j_1}, \dots, x^{j_\ell}$ with $j_1 + \dots + j_\ell = j$. If, say $j_1 = j$, then all the others satisfy $j_i = 0$. However, since j starts from 1 in the sum, this means we are looking at the term of x^j in the linear term $i = 1$, and the contribution of the coefficient from that product is

$$\lambda_j \frac{-P'(\lambda_0)}{P(\lambda_0)^2} + \frac{-Q_j(\lambda_0, \dots, \lambda_{j-1})}{P(\lambda_0)^2}.$$

If all $j_i < j$, then the contribution of the coefficient from that product is a polynomial in $\lambda_1, \dots, \lambda_{j-1}$ for a generic λ_0 , and is rational in $\lambda_0, \lambda_1, \dots, \lambda_{j-1}$. More precisely, the contribution from that product is of the form

$$\frac{\tilde{Q}_j(\lambda_1, \dots, \lambda_{j-1})}{P(\lambda_0)^\ell},$$

and since $1 \leq \ell \leq m$, the degree of this rational function is equivalent to $\deg(P)$ asymptotically. Therefore, we conclude that the coefficient for x^j is a linear function in λ_j and is a rational function in $\lambda_0, \dots, \lambda_j$ of degree asymptotically equivalent to $\deg(P)$.

Furthermore, if we expand the product $f_\varphi(v) = P(v)^{-1}Q(v)$ and look at the coefficient of x^j , then the argument in the previous paragraph can also be applied. We have that the constant term is $Q(\lambda_0)/P(\lambda_0)$. For $1 \leq j \leq m - 1$, the coefficient

of x^j is a rational function in $\lambda_0, \dots, \lambda_j$ of degree asymptotic equivalent to $\deg(\varphi) := \max\{\deg(P), \deg(Q)\}$, and for generic λ_0 it is a linear function of λ_j . Thus using the same notation as in the polynomial case, f induces a selfmap on each $V^{(i)}$ and for $2 \leq i \leq m$, $f_P|_{V^{(i)}}$ preserve the fibration π_i and the map on a generic fiber is a linear isomorphism of \mathbb{P}^1 . Moreover, we have $f_\varphi^n = f_{\varphi^n}$, where φ^n means we iterate the single variable rational map $\varphi(T)$ for n times. Also, notice that $\deg(\varphi^n) = \deg(\varphi)^n$. Thus we can use Proposition 3.1 on each π_i inductively again. To conclude, we obtain in this case that $\mathbb{C}[x] \cong \mathbb{C}[T]/(T^m)$, and we have $\deg_p(f_\varphi^n) \asymp \max\{\deg(Q), \deg(P)\}^{pn}$.

In general, we can write $\mathbb{C}[x] \cong \prod_i \mathbb{C}[T_i]/(T_i^{m_i})$, and for generic x, y , we have $\mathbb{C}[x] \cong \mathbb{C}[y]$, and f_P is birational to a product map. Therefore, we can conclude again that for $k = \dim_{\mathbb{C}}(\text{red } \mathbb{C}[x])$, where x is generic in V , for any p , we have

$$\deg_p(f_\varphi^n) \asymp \left(\max\{\deg(Q), \deg(P)\}^{\min\{p,k\}} \right)^n.$$

This completes the proof of Theorem A. □

6. MAPS OF SEVERAL VARIABLES

In this section, we will assume that V is an abelian (i.e., commutative, associative and unitary) \mathbb{C} -algebra, and $\dim_{\mathbb{C}}(V) = k$. Take any dominant rational map

$$f = (f_0 : \dots : f_d) : \mathbb{C}^d \dashrightarrow \mathbb{C}^d,$$

where each f_j is a rational function. Interpreting the multiplication and inverse in f_j as multiplication and inverse in V (notice that a generic element in V is invertible), f also induces a rational map $F : V^d \dashrightarrow V^d$.

As we saw in Section 4.1, V is Artinian and can be factored as a product $V = \prod_{i=1}^m V_i$ of local Artinian \mathbb{C} -algebras. Let \mathfrak{m}_i be the maximal ideal of V_i , $\pi_i : V_i \rightarrow V_i/\mathfrak{m}_i \cong \mathbb{C}$ be the quotient map, and $F_i : V_i^d \dashrightarrow V_i^d$ be the component of F on V_i . Then one sees that the fibration defined by $\prod \pi_i : V_i^d \rightarrow \mathbb{C}^d$ is preserved by F_i , and the induced map on the base \mathbb{C}^d is exactly f . That is, the following diagram of maps is commutative:

$$\begin{array}{ccc} V_i^d & \xrightarrow{F_i} & V_i^d \\ \Pi \pi_i \downarrow & & \downarrow \Pi \pi_i \\ \mathbb{C}^d & \xrightarrow{f} & \mathbb{C}^d. \end{array}$$

Coming back to V , we can in fact describe the fibration more succinctly using the quotient map $\pi : V \rightarrow V/N(V) \cong \mathbb{C}^m$. That is, the fibration is given by $\prod_{i=1}^k \pi : V^d \rightarrow (V/N(V))^d \cong \mathbb{C}^{md}$, and F is preserving the fibration in the sense that the following diagram is commutative:

$$\begin{array}{ccc} V^d & \xrightarrow{F} & V^d \\ \Pi \pi \downarrow & & \downarrow \Pi \pi \\ \mathbb{C}^{md} & \xrightarrow{\Pi f} & \mathbb{C}^{md}. \end{array}$$

In particular, if $N(V) \neq (0)$, then $\prod \pi$ is a fibration with positive dimensional fibers. In the following, we show that in the special case of monomial maps, F is, moreover, a product map.

6.1. Generalized monomial maps. Given an $d \times d$ integer matrix $A = (\alpha_{ij}) \in M_d(\mathbb{Z})$ with $\det(A) \neq 0$, we define the (generalized) monomial map $F_A : V^d \dashrightarrow V^d$ as

$$F_A(x_1, \dots, x_d) = \left(\prod_j x_j^{\alpha_{1,j}}, \dots, \prod_j x_j^{\alpha_{d,j}} \right),$$

where $x_j \in V$. We then compactify $V^d \subset \mathbb{P}^{dk}$ and lift the monomial map as $F_A : \mathbb{P}^{dk} \dashrightarrow \mathbb{P}^{dk}$.

The goal of this section is to prove Theorem B. That is, we will compute the degree growth of the generalized monomial map F_A . The method we use is similar to Section 4.1 when we dealt with the squaring map on an abelian algebra.

Now V is finite dimensional as a complex vector space, hence is Artinian as a \mathbb{C} -algebra, hence can be decomposed as a finite direct product of Artinian local rings $V \simeq \prod_{i=1}^m V_i$.

When V is an Artin local ring with maximal ideal \mathfrak{m} , then \mathfrak{m} is nilpotent, i.e., $\mathfrak{m}^l = 0$ for some $l \geq 1$, and $V/\mathfrak{m} \simeq \mathbb{C}$. As a \mathbb{C} -vector space, we can write V as the direct sum $V \simeq \mathbb{C} \oplus \mathfrak{m}$, where \mathfrak{m} is the maximal ideal of V . We introduce the map

$$\Phi : \mathbb{C} \oplus \mathfrak{m} \longrightarrow V$$

sending (a, h) to $a \cdot \exp(h)$, where $\exp(h) = \sum_{i=0}^{\infty} h^i/i!$. The function $\exp(h)$ is well defined since the sum is indeed finite, as we explained in Section 4.1. Moreover, Φ is a birational map with birational inverse

$$\Phi^{-1}(a, h) = a + \log(1 + a^{-1}h).$$

Here $\log(1 + x) = \sum_{i=1}^{\infty} (-1)^{i-1} x^i/i$. Notice that the Φ^{-1} above defines a map $\mathbb{C} \oplus \mathfrak{m} \longrightarrow V$, but using the identification $V \simeq \mathbb{C} \oplus \mathfrak{m}$ (as vector spaces) we can interpret it as a map $V \longrightarrow \mathbb{C} \oplus \mathfrak{m}$. Moreover, one can check that it indeed is the rational inverse of Φ when we interpret it this way.

For d copies of V , we have, as \mathbb{C} -vector spaces, $V^d \simeq \mathbb{C}^d \oplus \mathfrak{m}^{\oplus d}$. We use $\mathfrak{m}^{\oplus d}$ to stress that it is the direct sum of d copies of \mathfrak{m} , not the usual power of ideal \mathfrak{m}^d . Then, the map F_A , after conjugating Φ , becomes a product of a monomial map and a linear map. More precisely, let $f_A : \mathbb{C}^d \dashrightarrow \mathbb{C}^d$ be the usual monomial map on \mathbb{C}^d induced by A , and T_A be the linear map given by A on the vector space $\mathfrak{m}^{\oplus d}$, then for $((a_1, \dots, a_d), (h_1, \dots, h_d)) \in \mathbb{C}^d \oplus \mathfrak{m}^{\oplus d}$, we have

$$\begin{aligned} & F_A \circ \Phi((a_1, \dots, a_d), (h_1, \dots, h_d)) \\ &= F_A(a_1 \exp(h_1), \dots, a_d \exp(h_d)) \\ &= \left(\prod_j (a_j \exp(h_j))^{\alpha_{1,j}}, \dots, \prod_j (a_j \exp(h_j))^{\alpha_{d,j}} \right) \\ &= \left(\left(\prod_j a_j^{\alpha_{1,j}} \right) \cdot \exp\left(\sum_j \alpha_{1,j} h_j\right), \dots, \left(\prod_j a_j^{\alpha_{d,j}} \right) \cdot \exp\left(\sum_j \alpha_{d,j} h_j\right) \right) \\ &= \Phi(f_A(a_1, \dots, a_d), T_A(h_1, \dots, h_d)). \end{aligned}$$

That is, $F_A \circ \Phi = \Phi \circ (f_A, T_A)$. The linear map T_A has degree 1 in codimension $0 \leq j \leq \dim(\mathfrak{m}^{\oplus d}) = d(k-1)$, and degree 0 for $j > d(k-1)$. The degree growth of the usual monomial map f_A is shown as

$$\deg_p(f_A) \asymp \|\wedge^p A^n\|,$$

where $\|\cdot\|$ is any norm (for the proof of this result; see [FW, L]).

Therefore, by (3.1), the degree growth of the map F_A can be described as

$$\deg_p(F_A^n) \asymp \max_{i+j=p} \{ \deg_i(f_A^n) \deg_j(T_A^n) \} \asymp \max_{p+d-k \leq i \leq p} \|\wedge^i A^n\|.$$

Here, we use the convention that negative exterior product is zero.

In the general case, we have $V \simeq \prod_{i=1}^m V_i$ with each V_i being an Artin local ring. Hence $F_A : V^d \dashrightarrow V^d$ is a product of maps $F_{A,i} : V_i^d \dashrightarrow V_i^d$, and each $F_{A,i}$ has a further product structure as explained above. The map F_A is again a product of usual monomial maps and a linear map. The monomial map maps $\mathbb{C}^{md} \rightarrow \mathbb{C}^{md}$ and is associated to the matrix

$$\begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix}.$$

Let $\text{diag}(A; m)$ denote the above block-diagonal matrix with m blocks of the matrix A on the diagonal positions. The degree growth behavior of F_A is therefore as follows:

$$\deg_p(F_A^n) \asymp \max_{p-d(k-m) \leq i \leq p} \|\wedge^i \text{diag}(A; m)^n\|.$$

Notice that the number m is the number of copies of V_i 's in the decomposition for V . Another way to write m is as $m = \dim_{\mathbb{C}}(\text{red}(V))$, where $\text{red}(V) = V/N(V)$ and $N(V)$ is the nilradical of V , i.e., the ideal of all nilpotent elements. This completes the proof of Theorem B.

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