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ROTATIONAL PROPERTIES OF HOMEOMORPHISMS WITH INTEGRABLE DISTORTION

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Abstract. We establish a modulus inequality, with weak assumptions on the Sobolev regularity, for homeomorphisms with integrable distortion. As an application, we find upper bounds for the pointwise rotation of planar homeomorphisms with p-integrable distortion. When the mapping is entire we bound the local pointwise rotation and when the mapping is restricted to a bounded convex domain $\Omega \subset \mathbb{C}$ we concentrate on the rotation along the boundary. Furthermore, we show that these bounds are sharp in a very strong sense. Our examples will also prove that the modulus of continuity result, due to Koskela and Takkinen, for the homeomorphisms with p-integrable distortion is sharp in this strong sense.

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping. Then the classical modulus inequalities state that

$$\frac{M(\Gamma)}{K} \leq M(f(\Gamma)) \leq KM(\Gamma), \tag{1.1}$$

where Γ is an arbitrary path family. Inequalities (1.1) are of fundamental importance for the quasiconformal mappings, and similar modulus inequalities for more general classes of mappings have been studied in, for example, [9] and [11].

In this paper, we concentrate on homeomorphisms of finite distortion and generalize the modulus inequality from [11], by weakening the assumption on the Sobolev regularity, to obtain the following.

Theorem 1.1. *Let Ω be a domain and let $f : \Omega \to \mathbb{C}$ be a homeomorphism of finite distortion for which $K_f(z) \in L^1_{loc}(\Omega)$. Then, given a family Γ of paths $\gamma \subset \Omega$ we have*

$$M(f(\Gamma)) \leq M_{K_f}(\Gamma). \tag{1.2}$$

The only additional assumption, that $K_f(z) \in L^1_{\mathrm{loc}}(\Omega)$, made in Theorem 1.1 is necessary for the modulus $M_{K_f}(\Gamma)$ to make sense. Thus Theorem 1.1 is optimal, in the sense that we cannot relax our assumptions any further, and we believe that it has many applications in the study of mappings with integrable distortion.

We will use the modulus inequality (1.2) to study the rotational properties of homeomorphisms with p-integrable distortion both in the entire plane and in bounded convex domains. To this end, let $\Omega \subset \mathbb{C}$ be a bounded convex C^1-regular

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domain, let $\Omega' \subset \mathbb{C}$ be a bounded C^1-regular Jordan domain, and let $f_0 : \partial\Omega \to \partial\Omega'$ be a homeomorphism. The convergence of the double integral

$$\int_{\partial\Omega\times\partial\Omega} \big| \log |f_0(z) - f_0(x)| \big| \, |dz| \, |dx| \tag{1.3}$$

is a necessary and sufficient condition for f_0 to have a $W^{1,1}(\Omega)$-regular homeomorphic extension $f : \overline{\Omega} \to \overline{\Omega}'$ with integrable distortion, that is,

$$\int_\Omega K_f(z)\, dz < \infty;$$

see, for example, [1] and [2].

However, the geometric properties of the homeomorphisms f_0 for which the integral (1.3) converges are much less understood. Our first application of Theorem 1.1 is to bound the spiraling of the homeomorphisms f_0 for which the double integral (1.3) converges, which gives a necessary geometric condition for the existence of the desired extension. Moreover, in this setting the C^1-regularity of the boundaries is not needed. Therefore, we formulate our first rotational result as follows.

Theorem 1.2. *Let Ω be a bounded convex domain, normalized by $0, 1 \in \partial\Omega$, and let Ω' be a bounded Jordan domain. Suppose that $f : \overline{\Omega} \to \overline{\Omega'}$ is a $W^{1,1}(\Omega)$-regular homeomorphism with integrable distortion over Ω, fix the branch of the argument by $\arg(f(1) - f(0)) \in [0, 2\pi)$ and for any point $z \in \partial\Omega$ denote $|z|_f = \min_{z_0\in\partial\Omega,\, |z_0|\ge|z|} |f(z_0) - f(0)|$. Then*

$$|\arg(f(z) - f(0))| \le C_{\Omega,f} \frac{\left(\log\left(\frac{1}{|z|_f}\right)\right)^{\frac{1}{2}}}{|z|}, \tag{1.4}$$

whenever $z \in \partial\Omega$ and $|z| > 0$ is small enough.

Furthermore, we will present examples verifying that the bound (1.4) is essentially sharp.

Our second application of the modulus inequality (1.2) is to study global pointwise rotational properties of homeomorphisms with p-integrable distortion. Research in this direction was initiated in [3], where Astala, Iwaniec, Prause, and Saksman proved that given a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$, normalized by $f(0) = 0, f(1) = 1$, its pointwise rotation at the origin is sharply bounded by

$$|\arg(f(r))| \le \frac{1}{2}\left(K - \frac{1}{K}\right)\log\left(\frac{1}{r}\right) + c_K \qquad \text{for all } 0 < r < 1. \tag{1.5}$$

Later on in [8] the study of pointwise rotation was extended to homeomorphisms of finite distortion with p-exponentially integrable distortion, that is,

$$e^{pK_f(z)} \in L^1_{\mathrm{loc}}(\mathbb{C}) \qquad \text{for some } p > 0.$$

There it was shown that given a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ with p-exponentially integrable distortion, normalized by $f(0) = 0, f(1) = 1$, we can bound the pointwise rotation at the origin by

$$|\arg(f(z))| \le \frac{c}{p}\log^2\left(\frac{1}{|z|}\right) \qquad \text{for all small } |z|, \tag{1.6}$$

and that (1.6) is sharp up to the exact value of the constant c.

It is natural to ask how much more rotation can we have if we relax our assumption on the distortion to p-integrable, that is,

$$K_f(z) \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{C}) \qquad \text{for some } p \geq 1,$$

instead of being exponentially integrable. To answer this question we use some ideas from [8] together with Theorem 1.1 and obtain the following.

Theorem 1.3. *Fix an arbitrary $p > 1$ and let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism of finite distortion, normalized by $f(0) = 0$ and $f(1) = 1$, for which $K_f(z) \in L^p_{loc}(\mathbb{C})$. Then*

$$|\arg(f(z))| \leq c_{f,p} \frac{1}{|z|^{\frac{2}{p}}} \tag{1.7}$$

when $|z| \to 0$, where $c_{f,p}$ is a constant that does not depend on z.

In the case $p = 1$ we can slightly improve the bound (1.7) and obtain the following.

Theorem 1.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism of finite distortion, normalized by $f(0) = 0$ and $f(1) = 1$, for which $K_f(z) \in L^1_{loc}(\mathbb{C})$. Then*

$$|\arg(f(z))| \leq \frac{c_f(|z|)}{|z|^2} \tag{1.8}$$

when $|z| \to 0$, where $c_f(|z|) \to 0$ as $|z| \to 0$.

Moreover, the above theorems are sharp in the following strong sense.

Theorem 1.5. *Given any $p \geq 1$ and any function $h(r) : (0, 1] \to (0, \infty)$, for which $h(r) \to 0$ when $r \to 0$, we can find a homeomorphism of p-integrable distortion $f_0 : \mathbb{C} \to \mathbb{C}$, normalized by $f(0) = 0$, $f(1) = 1$, and a sequence r_n of positive radii converging to zero such that*

$$|\arg(f_0(r_n))| \geq \frac{h(r_n)}{r_n^{\frac{2}{p}}} \tag{1.9}$$

for every r_n.

We fix the branch of the argument in (1.7), (1.8), and (1.9) by $\arg(1) = 0$.

Furthermore, we will see that the maximal rotations (1.7) and (1.8) depend on the stretching of the mapping f, in the same way as the maximal rotation for homeomorphisms with exponentially integrable distortion; see [8].

We would like to note that, while the jump from quasiconformal mappings to mappings with exponentially integrable distortion does not change the maximal rotation significantly (both (1.5) and (1.6) are logarithmic), the assumption that the distortion is merely integrable increases the growth order of maximal rotations (1.7) and (1.8) to polynomial.

The local rotational properties of homeomorphisms of finite distortion go hand in hand with the local stretching properties. Hence, we must have a good understanding of the local stretch of homeomorphisms with p-integrable distortion in order to understand the local rotation of these mappings. For studying the local stretch our starting point is the modulus of continuity result of Koskela and Takkinen; see Theorem 3 in [10]. They proved that if $f : \mathbb{C} \to \mathbb{C}$ is a homeomorphism of p-integrable distortion, normalized by $f(0) = 0$, and $|z|$ is sufficiently small, then

$$|f(z)| \geq e^{-c_{f,p}|z|^{-\frac{2}{p}}}, \tag{1.10}$$

where $c_{f,p}$ is a constant and the exponent $\frac{2}{p}$ is sharp in the sense that the claim does not hold for any smaller exponent. We aim to improve this result in the following way.

Theorem 1.6. *Let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism of finite distortion, normalized by $f(0) = 0$, such that $K_f(z) \in L^1_{loc}(\mathbb{C})$. Then for any $z \in B\left(0, \frac{1}{2}\right)$ it holds that*

$$|f(z)| \geq e^{-\frac{c_f(|z|)}{|z|^2}}, \tag{1.11}$$

where $c_f(|z|) \to 0$ when $|z| \to 0$.

Moreover, given any $p \geq 1$ and any function $h(r) : (0, 1] \to (0, \infty)$, for which $h(r) \to 0$ when $r \to 0$, we can find a homeomorphism $f_0 : \mathbb{C} \to \mathbb{C}$ with p-integrable distortion and a sequence r_n of positive radii converging to zero such that

$$|f_0(r_n) - f_0(0)| \leq e^{-h(r_n) r_n^{-\frac{2}{p}}} \tag{1.12}$$

for every r_n.

Here the improvement compared with the result of Koskela and Takkinen is the convergence of $c_f(|z|)$ in (1.11) and, more importantly, the stronger sharpness of (1.10) portrayed by examples (1.12).

The paper is organized in the following way. Section 2 contains the necessary prerequisites and the framework for the pointwise rotation. In section 3 we consider the bounded case and prove Theorems 1.1 and 1.2. Finally, the global situation is considered in section 4, where we prove Theorems 1.3, 1.4, 1.5, and 1.6 while also presenting examples verifying optimality of our theorems.

2. Prerequisites

Let $\Omega \subset \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be a sense-preserving homeomorphism. We say that f has finite distortion if the following conditions hold:

- $f \in W^{1,1}_{\text{loc}}(\Omega)$,
- $J_f(z) \in L^1_{\text{loc}}(\Omega)$,
- $|Df(z)|^2 \leq J_f(z)K(z)$ almost everywhere in Ω,

for a measurable function $K(z) \geq 1$, which is finite almost everywhere. The smallest such function is denoted by $K_f(z)$ and called the distortion of f. Here $Df(z)$ denotes the differential matrix of f at the point z and the norm $|Df(z)|$ is defined by

$$|Df(z)| = \max\{|Df(z)e| : e \in \mathbb{C}, |e| = 1\},$$

whereas $J_f(z)$ is the Jacobian of the mapping f at the point z. Such a mapping is said to have a p-integrable distortion if

$$K_f(z) \in L^p_{\text{loc}}(\Omega),$$

which we might shorten to integrable distortion if $p = 1$. The distortion $K_f(z)$ has the disadvantage of being insufficiently regular to deal with variational equations and for this reason we will use the outer distortion

$$\mathbb{K}_f(z) = \frac{||D_f(z)||^2}{J_f(z)}, \tag{2.1}$$

where $||A||^2 = \frac{1}{2}\operatorname{Tr}(A^T A)$ is the mean-Hilbert-Schmidt norm, when motivating Theorem 1.2 with connections to Dirichlet's energy. These definitions for the distortion are related through the equation

$$\mathbb{K}_f(z) = \frac{1}{2}\left(K_f(z) + \frac{1}{K_f(z)}\right),$$

and thus

$$\frac{1}{2}K_f(z) < \mathbb{K}_f(z) < 2K_f(z),$$

from which we see that both distortions are p-integrable simultaneously.

For a detailed exposition of mappings of finite distortion see, for example, [1] or [7].

Given a bounded domain Ω' and a mapping $h \in W^{1,2}(\Omega')$ we say that Dirichlet's energy of h is

$$\int_{\Omega'} ||D_h(z)||^2 \, dz. \tag{2.2}$$

Given any mapping $h_0 \in W^{1,2}(\Omega')$ the minimizer of the Dirichlet's energy (2.2) over the family $h \in W_0^{1,2}(\Omega') + h_0$, where $W_0^{1,2}(\Omega')$ denotes the family of Sobolev functions with compact support, exists and is unique; see, for example, [2]. Furthermore, for any homeomorphism $f : \Omega \to \Omega'$ of finite distortion with $K_f(z) \in L^1_{\text{loc}}(\Omega)$ it holds that the inverse $f^{-1} \in W^{1,2}_{\text{loc}}(\Omega')$ is a mapping of finite distortion and

$$\int_\Omega \mathbb{K}_f(z) \, dz = \int_{\Omega'} ||D_{f^{-1}}(x)||^2 \, dx; \tag{2.3}$$

see, for example, Theorem 2.1 in [6] and Theorem 5.9 in [7].

Moreover, let Ω be a bounded convex domain, let Ω' be a bounded Jordan domain, and let $f_0 : \overline{\Omega} \to \overline{\Omega'}$ be a $W^{1,1}(\Omega)$-regular homeomorphism with integrable distortion over Ω, as in Theorem 1.2. Then it was proved by Hencl, Koskela, and Onninen (see [6]) that the minimization problem

$$\min_{f \in \mathbb{F}} \int_\Omega \mathbb{K}_f(z) \, dz, \quad f = f_0 \text{ on } \partial\Omega, \tag{2.4}$$

where $\mathbb{F}$ is the family of all mappings satisfying the above conditions, has a unique diffeomorphic solution whose inverse is harmonic in Ω'. So, if f minimizes (2.4), then f^{-1} is the solution of the Dirichlet's problem

$$\min_{h \in W_0^{1,2}(\Omega') + f^{-1}} \int_{\Omega'} ||D_h(z)||^2 \, dz. \tag{2.5}$$

The result of Hencl, Koskela, and Onninen highlights the need to conclude when the family $\mathbb{F}$ is non-empty, for which Theorem 1.2 gives a necessary geometric condition.

Let us then move on to define what we mean by the pointwise rotation. We will first consider the global situation $f : \mathbb{C} \to \mathbb{C}$ and then indicate the necessary adjustments for the bounded case $f : \Omega \to \Omega'$.

Let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism of finite distortion. When we study the pointwise rotation of the mapping f at a point $z_0 \in \mathbb{C}$ we are interested in the change of the argument of $f(z_0 + te^{i\theta}) - f(z_0)$ as the parameter t goes from 1 to $r > 0$, which we can write as

$$|\arg(f(z_0 + re^{i\theta}) - f(z_0)) - \arg(f(z_0 + e^{i\theta}) - f(z_0))|.$$

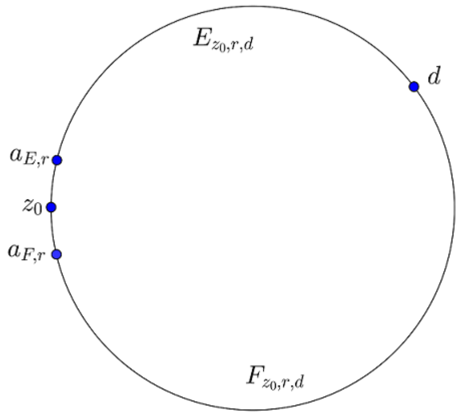

FIGURE 1. Measuring pointwise rotation along the boundary.

This can also be understood as the winding of the set $f\left([z_0 + re^{i\theta}, z_0 + e^{i\theta}]\right)$ around the point $f(z_0)$. As we are interested in the maximal change of the argument, over an arbitrary direction θ, we study the supremum

$$\sup_{\theta\in[0,2\pi)} |\arg(f(z_0 + re^{i\theta}) - f(z_0)) - \arg(f(z_0 + e^{i\theta}) - f(z_0))|. \tag{2.6}$$

Finally, we study the growth of (2.6) at the limit $r \to 0$ and say that a function $g : (0, 1] \to [0, \infty)$ bounds the pointwise rotation of f at the point z_0 if

$$\limsup_{r\to 0} \frac{\sup_{\theta\in[0,2\pi)} |\arg(f(z_0 + re^{i\theta}) - f(z_0)) - \arg(f(z_0 + e^{i\theta}) - f(z_0))|}{g(r)} \leq c \tag{2.7}$$

for some constant $c < \infty$.

In this light Theorems 1.3 and 1.4 state that for any homeomorphism of finite distortion f, such that $K_f(z) \in L^p_{\mathrm{loc}}(\mathbb{C})$, $f(0) = 0$, and $f(1) = 1$, the function

$$g(r) = \frac{1}{r^{\frac{2}{p}}}$$

bounds its pointwise rotation at the origin. Moreover, examples (1.9) verifying optimality of these theorems show that the functions

$$g(r) = \frac{h(r)}{r^{\frac{2}{p}}}$$

do not bound the rotation of such mappings for any function h such that $h(r) \to 0$ when $r \to 0$.

We then make the necessary adjustments for the situation $f : \overline{\Omega} \to \overline{\Omega'}$, where we are interested in measuring the pointwise rotation along the boundary of Ω.

Fix any two points $z_0, d \in \partial\Omega$ and choose $0 < r < |d - z_0|$. Since Ω is convex and $z_0 \in \partial\Omega$ we see that for any small enough $r > 0$ there exists only two points on $\partial\Omega$, which we denote by $a_{E,r}$ and $a_{F,r}$, satisfying $|z_0 - a_{E,r}| = |z_0 - a_{F,r}| = r$. So, for all small enough $r > 0$ the points $a_{E,r}, a_{F,r}$ and d divide $\partial\Omega$ into three parts, of which we choose those that do not contain the point z_0 and denote them by $E_{z_0,r,d}$ and $F_{z_0,r,d}$; see Figure 1.

We then measure the winding of the sets $f(E_{z_0,r,d})$ and $f(F_{z_0,r,d})$ around the point $f(z_0)$, which we can write as

$$\max_{i\in\{E,F\}} |\arg(f(a_{i,r}) - f(z_0)) - \arg(f(d) - f(z_0))|. \tag{2.8}$$

We are again interested in the growth of (2.8) as $r \to 0$, and say that the pointwise rotation of the mapping f at the point z_0 is bounded by a function $g : (0, 1] \to [0, \infty)$ if

$$\limsup_{r\to 0} \frac{\max_{i\in\{E,F\}} |\arg(f(a_{i,r}) - f(z_0)) - \arg(f(d) - f(z_0))|}{g\left(\frac{r}{|d-z_0|}\right)} \le c \tag{2.9}$$

for some constant $c < \infty$. Here we stress that the function g does not depend on the choice of the point d, which just serves as a starting point for measuring the rotation.

Theorem 1.2 then says that for any mapping f satisfying its assumptions the function

$$g(r) = \frac{\left(\log\left(\frac{1}{r_f}\right)\right)^{\frac{1}{2}}}{r},$$

where $r_f = \min_{z\in\partial\Omega,\, |z|\ge r} |f(z) - f(0)|$, bounds the pointwise rotation of f at the origin. Note that here we use the normalization $d = 1$ from (1.4).

Next we note that we can normalize general pointwise rotation in terms of Theorems 1.2, 1.3, and 1.4. Let us start with the global case.

Corollary 2.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism with p-integrable distortion and let $z_0 \in \mathbb{C}$ be arbitrary. Then there exists a homeomorphism $f_0 : \mathbb{C} \to \mathbb{C}$ with p-integrable distortion, normalized by $f_0(0) = 0$ and $f_0(1) = 1$, such that the pointwise rotation of the mapping f at the point z_0 is the same as the pointwise rotation of the mapping f_0 at the origin.*

Proof. Define $f_0(z) = h[f(z_0 + z) - f(z_0)]$, where h is chosen such that $f_0(1) = 1$. Clearly f_0 is a homeomorphism of p-integrable distortion, since the mapping f is, and furthermore it is easy to see that f_0 satisfies the desired normalization. Moreover, the pointwise rotation of f_0 at the origin is the same as the pointwise rotation of f at the point z_0, since the constant h plays no role in (2.6).

Then we normalize the general case for a bounded convex domain Ω.

Corollary 2.2. *Let Ω be a bounded convex domain, let Ω' be a bounded Jordan domain, and let $f : \overline{\Omega} \to \overline{\Omega}'$ be a $W^{1,1}(\Omega)$-regular homeomorphism with integrable distortion. Choose any point $z_0 \in \partial\Omega$. Then there exists a homeomorphism f_0, normalized by $f_0(0) = 0$ and $f_0(1) = 1$, and domains Ω_0, Ω_0' satisfying the assumptions of Theorem 1.2, such that the pointwise rotation of f_0 at the origin, with the proper normalization, is the same as the pointwise rotation of f at the point z_0.*

Proof. Choose an arbitrary point $d \in \partial\Omega \setminus \{z_0\}$ as the starting point for measuring the pointwise rotation of f at the point z_0. Fix a domain $\Omega_0 = \left\{\frac{z-z_0}{d-z_0} : z \in \Omega\right\}$ and note that it is bounded and convex due to assumptions on Ω. Then define $f_0(z) = h[f((d - z_0)z + z_0) - f(z_0)]$, where the constant h is chosen such that $f_0(1) = 1$. It is easy to check that f_0 satisfies the assumptions of Theorem 1.2, and that Ω_0' is a bounded Jordan domain since it is a stretched and translated image of Ω'. Moreover, the mapping f_0 clearly has the desired normalization. So, the only thing left is to make sure that the pointwise rotations are the same.

To this end, define the sets $E_{z_0,r,d}$ and $F_{z_0,r,d}$, for all small enough r, as in the definition for the pointwise rotation; see Figure 1. From the definition of the domain Ω_0 we see that the sets $E_{z_0,r,d}$ and $F_{z_0,r,d}$ have translated and stretched copies $E_{0,\frac{r}{|d-z_0|},1}$ and $F_{0,\frac{r}{|d-z_0|},1}$, that lie at the boundary of Ω_0. From the definition of f_0 we see that the winding of $f_0\left(E_{0,\frac{r}{|d-z_0|},1}\right)$ or $f_0\left(F_{0,\frac{r}{|d-z_0|},1}\right)$ around the origin is the same as the winding of $f\left(E_{z_0,r,d}\right)$ or $f\left(F_{z_0,r,d}\right)$ around the point $f(z_0)$. Hence, we see that any function $g:(0,1]\to[0,\infty)$ satisfies (2.9) either for both f and f_0 or for neither of them.

Thus when studying the pointwise rotation for general entire homeomorphisms of p-integrable distortion we can restrict ourselves to the situation of Theorems 1.3 and 1.4, and if the mapping is defined on a bounded convex domain we can restrict ourselves to the situation of Theorem 1.2, with the additional normalization for the mapping f.

Then let us briefly define the modulus of path families. For a closer look on the topic we recommend, for example, [13]. We call a continuous function $\gamma : I \to \mathbb{C}$, where $I \subset \mathbb{R}$ is an interval, a path and denote both the function and its image by γ. Let Γ be a family of paths. We say that a Borel-measurable function $\rho : \mathbb{C} \to [0,\infty)$ is admissible with respect to Γ if

$$\int_\gamma \rho(z)\,|dz| \geq 1 \tag{2.10}$$

for any locally rectifiable path $\gamma \in \Gamma$. We denote the modulus of a path family Γ by $M(\Gamma)$ and define it by

$$M(\Gamma) = \inf_{\rho \text{ admissible}} \int_{\mathbb{C}} \rho^2(z)\,dz. \tag{2.11}$$

We will also need a weighted version of (2.11), where the weight function $\omega : \mathbb{C} \to [0,\infty)$ is measurable and locally integrable, which we define by

$$M_\omega(\Gamma) = \inf_{\rho \text{ admissible}} \int_{\mathbb{C}} \rho^2(z)\omega(z)\,dz. \tag{2.12}$$

We say that a homeomorphism $f : \Omega \to \mathbb{C}$ satisfies the Lusin (N) condition if for each $E \subset \Omega$ holds

$$|E| = 0 \qquad \Rightarrow \qquad |f(E)| = 0,$$

where $|A|$ denotes the Lebesgue measure of the set A.

Given any homeomorphism $f : \Omega \to \mathbb{C}$ we introduce a Borel-measurable function $L_f : \Omega \to [0,\infty]$ defined by

$$L_f(z) = \limsup_{h\to 0} \frac{|f(z+h) - f(z)|}{|h|},$$

and note that if f is differentiable at a point z, then $|Df(z)| = L_f(z)$.

We use c as a generic constant whose value might change even in the middle of inequalities, and we use c_a if the constant depends on a parameter a. The boundary of a given set A is denoted by ∂A, the unit disc by $\mathbb{D}$, the radius of a given ball B by $r(B)$ and for any ball B and constant c we denote $cB(a,r) = B(a,cr)$. We define the distance between two disjoint compact sets A and B, as usually, by $\operatorname{dist}(A,B) = \min_{x\in A, y\in B} |x-y|$.

3. Bounded case

We start by proving Theorem 1.1 on the modulus inequality. To this end, we have to show that if Ω is a domain and $f : \Omega \to \mathbb{C}$ is a homeomorphism with integrable distortion, then, given a path family Γ of paths $\gamma \subset \Omega$, we have

$$M(f(\Gamma)) \leq M_{K_f}(\Gamma). \tag{3.1}$$

Proof of Theorem 1.1. Since f is a homeomorphism with integrable distortion the inverse f^{-1} is a mapping of finite distortion and $f^{-1} \in W^{1,2}_{\text{loc}}(f(\Omega))$; see Theorem 5.9 in [7]. Therefore, by Fuglede's theorem (see [13, p. 95]), if $\bar{\Gamma}$ is the family of all paths $\gamma \in f(\Gamma)$ for which f^{-1} is absolutely continuous on every closed subpath of γ, then $M(\bar{\Gamma}) = M(f(\Gamma))$. Given any ρ, which is admissible with respect to Γ, we define $\bar{\rho}(z) = \rho(f^{-1}(z))L_{f^{-1}}(z)$ for $z \in f(\Omega)$ and $\bar{\rho}(z) = 0$ otherwise. Due to Theorem 5.3 in [13] we obtain for any locally rectifiable $\bar{\gamma} \in \bar{\Gamma}$ that

$$\int_{\bar{\gamma}} \bar{\rho}(z)\, |dz| \geq \int_{f^{-1}\circ\bar{\gamma}} \rho(z)\, |dz| \geq 1,$$

which shows that $\bar{\rho}$ is admissible with respect to $\bar{\Gamma}$.

Since $f^{-1} \in W^{1,2}_{\text{loc}}(f(\Omega))$ is a mapping of finite distortion it satisfies the Lusin N-condition; see Theorem 4.5 in [7]. Moreover, the Gehring-Lehto theorem asserts that both f and f^{-1} are differentiable almost everywhere, and we know from [4] that $K_{f^{-1}}(z) = K_f\left(f^{-1}(z)\right)$ for every $z \in f(\Omega)$, even if the mapping f would not satisfy the Lusin (N) condition. As the mapping f^{-1} is additionally a homeomorphism in $W^{1,2}_{\text{loc}}(f(\Omega))$ we can use the change of variables formula, see Theorem 2 in [5], to obtain

$$\begin{aligned} M(f(\Gamma)) = M(\bar{\Gamma}) \leq \int_{f(\Omega)} \bar{\rho}^2(z)\, dz &= \int_{f(\Omega)} \rho^2(f^{-1}(z))L^2_{f^{-1}}(z)\, dz \\ &= \int_{f(\Omega)} \rho^2(f^{-1}(z))|Df^{-1}(z)|^2\, dz \\ &\leq \int_{f(\Omega)} \rho^2(f^{-1}(z))K_{f^{-1}}(z)J_{f^{-1}}(z)\, dz \\ &= \int_{f(\Omega)} \rho^2(f^{-1}(z))K_f(f^{-1}(z))J_{f^{-1}}(z)\, dz \\ &= \int_{\Omega} K_f(x)\rho^2(x)\, dx. \end{aligned}$$

Since ρ was an arbitrary admissible function with respect to Γ this proves inequality (3.1).

Our proof of Theorem 1.1 closely follows the proof given in [11], the difference being that we only assume $f \in W^{1,1}_{\text{loc}}(\Omega)$ instead of $f \in W^{1,2}_{\text{loc}}(\Omega)$. Note that any mapping f satisfying the assumptions of Theorems 1.2, 1.3, 1.4, or 1.6 also satisfies assumptions of Theorem 1.1, and hence the mapping f must satisfy the modulus inequality (3.1) for any path family Γ.

Then let us move on to prove Theorem 1.2. Let Ω be a bounded convex domain such that $0, 1 \in \partial\Omega$. Fix an arbitrary $r > 0$ and let E be the part of $\partial\Omega$ which does not contain the origin and has the endpoints 1 and a, where $|a| = r$. We remind that given small enough r there are only two possibilities to choose a from. Then let F be the part of $\partial\Omega$ that is disjoint from E and has the endpoints 0 and b, where

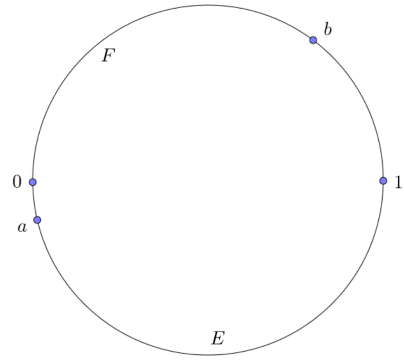

FIGURE 2. The sets E and F.

$|b| > \frac{1}{2}$, and let Γ be the family of all paths $\gamma \in \Omega$ connecting these two sets; for illustration see Figure 2. Inequality (3.1) applied to a mapping f, which satisfies the assumptions of Theorem 1.2 and hence also of Theorem 1.1, and to the path family Γ will give the desired bound for the pointwise rotation at the origin. To use this approach we first have to estimate the moduli $M_{K_f}(\Gamma)$ and $M(f(\Gamma))$.

Let us start with the modulus $M_{K_f}(\Gamma)$. To this end we must estimate the distance between the sets E and F. First note that when r is small the distance between these sets is clearly shortest near the origin. Since Ω is convex we can find a cone starting from the origin whose interior lies inside the domain Ω; see Figure 3 for illustration. Let us denote the sides of this cone by E_c and F_c, and check that for an arbitrary point $d \in E_c$

$$\operatorname{dist}(d, F_c) = c_\Omega |d|,$$

where c_Ω depends on the angle α of the cone. Then note, that for an arbitrary point $z_0 \in E$ for which $r \le |z_0| \le 2r$, where r is small, we have

$$\operatorname{dist}(z_0, F) \ge \operatorname{dist}(z_0, F_c) \ge \operatorname{dist}(l, F_c),$$

where $l \in E_c$ and $|l| = r$. Thus we obtain that the distance between the sets E and F satisfies

$$\operatorname{dist}(E, F) \ge c_\Omega r \tag{3.2}$$

for small r. Here the constant c_Ω can be as small as we wish, as can be seen by choosing Ω to be a sector of the unit disc with an arbitrary small angle.

Fix some small $r > 0$. From (3.2) it follows that the function

$$\rho_0(z) = \begin{cases} \frac{1}{c_\Omega r} & \text{if } z \in \Omega \text{ and } \operatorname{dist}(z, E) < c_\Omega r, \\ 0 & \text{otherwise,} \end{cases}$$

is admissible with respect to Γ. Hence we get an upper bound

(3.3)

$$M_{K_f}(\Gamma) \le \int_\Omega \rho_0^2(z) K_f(z)\, dz \le \frac{1}{c_\Omega r^2} \int_{\{z \in \Omega:\ \operatorname{dist}(z;E) < c_\Omega r\}} K_f(z)\, dz = \frac{c_{\Omega,f}(r)}{r^2}$$

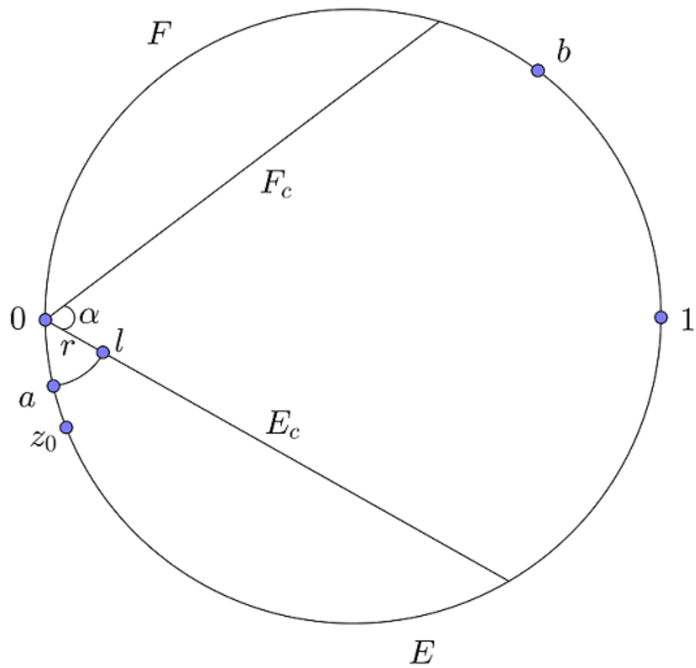

FIGURE 3. The cone between the sets E and F.

for the modulus $M_{K_f}(\Gamma)$. Since $K_f(z) \in L^1_{\text{loc}}(\Omega)$ and the measure of the set $\{z \in \Omega : \operatorname{dist}(z, E) < c_\Omega r\}$ converges to zero, we obtain convergence $c_{\Omega,f}(r) \to 0$ when $r \to 0$.

Next we will estimate the modulus $M(f(\Gamma))$ with a similar method as in [8]. First we note that we can assume without loss of generality that $f(0) = 0$. Then we write the modulus $M(f(\Gamma))$ in the polar form

$$M(f(\Gamma)) = \inf_{\rho \text{ admissible}} \int_{\mathbb{C}} \rho^2(z)\, dz = \inf_{\rho \text{ admissible}} \int_0^{2\pi} \int_0^{\infty} \rho^2(r,\theta) r\, dr d\theta,$$

and seek a lower bound for

$$\int_0^{\infty} \rho^2(r,\theta) r\, dr \tag{3.4}$$

that holds for an arbitrary direction $\theta \in [0, 2\pi)$ and an arbitrary admissible ρ. The main idea is to note that the images $f(E)$ and $f(F)$ must cycle around the origin alternately; see Figure 4 for illustration.

To see this, fix an arbitrary direction $\theta \in [0, 2\pi)$ and denote by L_θ the half-line starting from the origin to the direction θ. Assume that the image $f(E)$ winds once around the origin when z moves from a point t_0 to a point t_2 along the set E and $f(t_0) \in L_\theta$. Furthermore, suppose that there exists a point $z_0 \in F$ such that $f(z_0) \in L_\theta$ and $|f(z_0)| > |f(t_0)|$. Then, as the image $f(F)$ contains the origin and the point $f(z_0)$ and the mapping $f : \overline{\Omega} \to \overline{\Omega}'$ is a homeomorphism, the image

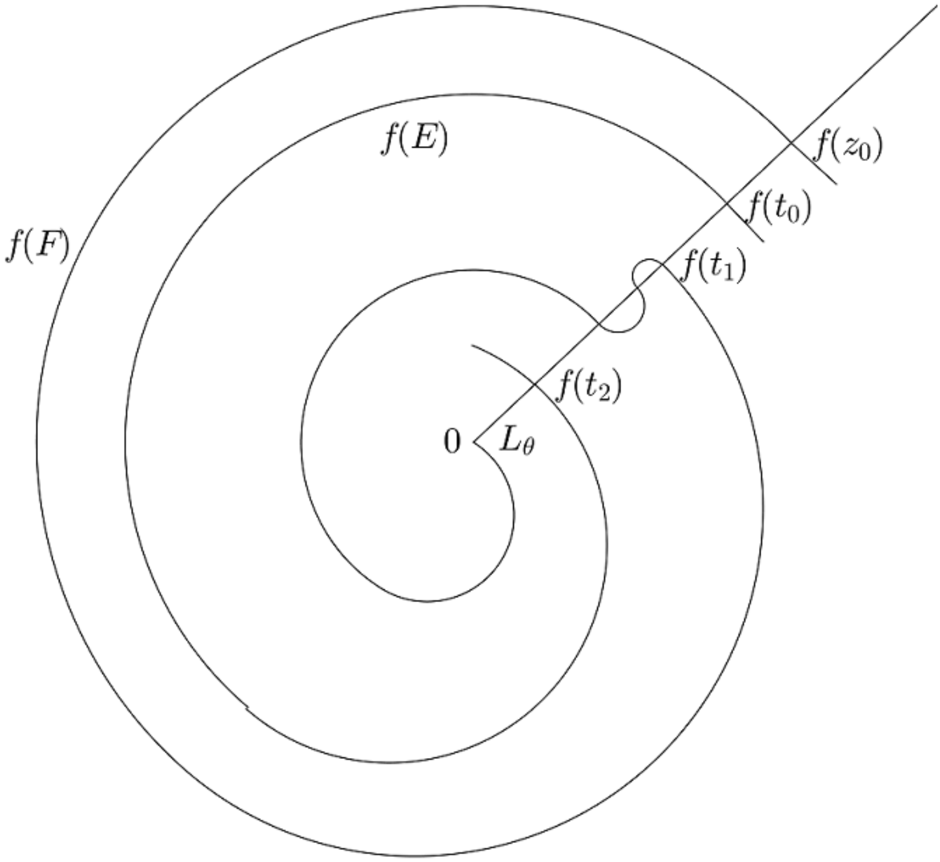

FIGURE 4. The images $f(E)$ and $f(F)$ cycling alternately around the origin.

$f(F)$ must intersect the line segment $(f(t_2), f(t_0))$ at least once, say at a point $f(t_1)$, where $t_1 \in F$. We can choose the point t_1 such that either the line segment $(f(t_1), f(t_0))$ or the line segment $(f(t_2), f(t_1))$ belongs to the path family $f(\Gamma)$. Hence, as the image $f(E)$ cycles around the origin $\left\lfloor \frac{|\arg(f(a))-\arg(f(1))|}{2\pi} \right\rfloor$ times, we can find at least

$$n_1(r) = \left\lfloor \frac{|\arg(f(a)) - \arg(f(1))|}{2\pi} \right\rfloor - 1 \tag{3.5}$$

disjoint line segments belonging to the path family $f(\Gamma)$, if we assume that there exists a point $z_0 \in F$ such that $f(z_0) \in L_\theta$ and $|f(z_0)| > |f(t_0)|$ for every such line segment. We note that this condition is always satisfied when the point t_0 is sufficiently close to the origin, as the mapping f is a homeomorphism, and so in a general case we can find at least

$$n(r) = \left\lfloor \frac{|\arg(f(a)) - \arg(f(1))|}{2\pi} \right\rfloor - 1 - c_f \tag{3.6}$$

disjoint line segments that belong to the path family $f(\Gamma)$. As we are interested in the extremal rotation, and thus assume that the winding of the image $f(E)$ around the origin approaches infinity as $r \to 0$, both (3.5) and (3.6) are positive when r is small. Moreover, since we assume that $|\arg(f(a))| \to \infty$ when $r \to 0$ we obtain a trivial estimate

$$n(r) \geq \frac{1}{2} \left\lfloor \frac{|\arg(f(a)) - \arg(f(1))|}{2\pi} \right\rfloor, \tag{3.7}$$

which holds for all small r. Here we would like to note that (3.7) does not depend on the direction θ, but gives a lower bound for the number of the desired line segments from an arbitrary direction θ.

We can write the $n(r)$ disjoint line segments in the form $(x_j e^{i\theta}, y_j e^{i\theta}) \subset L_\theta$, where the coefficients x_j, y_j satisfy

$$0 < x_1 < y_1 < \cdots < x_{n(r)} < y_{n(r)} \leq c_f$$

and the constant $c_f = \sup_{z \in \partial\Omega} |f(z)|$ does not depend on θ or r. Hence we can estimate

$$\int_0^\infty \rho^2(r,\theta) r \, dr \geq \sum_{j=1}^{n(r)} \int_{x_j}^{y_j} \rho^2(r,\theta) r \, dr. \tag{3.8}$$

Then note, that since the line segments $(x_j e^{i\theta}, y_j e^{i\theta})$ belong to the path family $f(\Gamma)$ for an arbitrary j and ρ is admissible with respect to $f(\Gamma)$ we can use the reverse Hölder inequality to estimate

$$\int_{x_j}^{y_j} \rho^2(r,\theta) r \, dr \geq \left(\int_{x_j}^{y_j} \rho(r,\theta) \, dr \right)^2 \left(\int_{x_j}^{y_j} \frac{1}{r} \, dr \right)^{-1} \geq \frac{1}{\log\left(\frac{y_j}{x_j}\right)}$$

over any line segment $(x_j e^{i\theta}, y_j e^{i\theta})$. Combining this with (3.8) we obtain

$$\int_0^\infty \rho^2(r,\theta) r \, dr \geq \sum_{j=1}^{n(r)} \frac{1}{\log\left(\frac{y_j}{x_j}\right)}. \tag{3.9}$$

To estimate this further we use the arithmetic-harmonic means inequality

$$\sum_{j=1}^{n} a_j \geq \frac{n^2}{\sum_{j=1}^{n} \frac{1}{a_j}}, \tag{3.10}$$

which holds whenever every a_j is positive.

To this end we first continue the estimate (3.9) by

$$\sum_{j=1}^{n(r)} \frac{1}{\log\left(\frac{y_j}{x_j}\right)} \geq \sum_{j=1}^{n(r)-1} \frac{1}{\log\left(\frac{x_{j+1}}{x_j}\right)} + \frac{1}{\log\left(\frac{c_f}{x_{n(r)}}\right)}. \tag{3.11}$$

Then we use (3.10) with the choices $a_j = \frac{1}{\log\left(\frac{x_{j+1}}{x_j}\right)}$, when $j \in \{1, 2, ..., n(r)-1\}$, and $a_{n(r)} = \frac{1}{\log\left(\frac{c_f}{x_{n(r)}}\right)}$, and obtain from (3.11) that

$$\sum_{j=1}^{n(r)} \frac{1}{\log\left(\frac{y_j}{x_j}\right)} \geq \frac{n^2(r)}{\log\left(\frac{c_f}{x_1}\right)} \geq \frac{n^2(r)}{\log\left(\frac{c_f}{r_f}\right)}, \tag{3.12}$$

where $r_f = \min_{z\in E} |f(z)|$. This bound holds for an arbitrary direction θ, an arbitrary admissible ρ and for all small r. By combining (3.12) with (3.8) we obtain

$$\int_0^\infty \rho^2(r,\theta) r\, dr \geq \frac{n^2(r)}{\log\left(\frac{c_f}{r_f}\right)},$$

which then gives

$$M(f(\Gamma)) \geq \frac{cn^2(r)}{\log\left(\frac{c_f}{r_f}\right)}. \tag{3.13}$$

We now have the bounds (3.3) and (3.13) for the moduli $M_{K_f}(\Gamma)$ and $M(f(\Gamma))$, when r is small. These estimates together with the modulus inequality (1.2) show that

$$\frac{cn^2(r)}{\log\left(\frac{c_f}{r_f}\right)} \leq \frac{c_{\Omega,f}(r)}{r^2},$$

which simplifies to

$$n(r) \leq c_{\Omega,f}(r) \frac{\left(\log\left(\frac{1}{r_f}\right)\right)^{\frac{1}{2}}}{r}. \tag{3.14}$$

Inequality (3.14) together with the estimate (3.7) proves Theorem 1.2. In fact we even prove a slightly stronger result, due to the convergence $c_{\Omega,f}(r) \to 0$ when $r \to 0$.

We will show examples regarding optimality of Theorem 1.2 at the end of the next section.

4. Global case

As we mentioned in the introduction, the modulus inequality (1.2) can also be used for bounding the local stretching and rotational properties of homeomorphisms with p-integrable distortion. We demonstrate this at the beginning of this section by establishing the bounds (1.7), (1.8), and (1.11) using Theorem 1.1. Then at the end of this section we present examples verifying sharpness of these bounds in the form (1.9) and (1.12).

Let us start with the pointwise stretching and begin the proof of Theorem 1.6. To this end, we show that given any homeomorphism $f : \mathbb{C} \to \mathbb{C}$, normalized by $f(0) = 0$, with integrable distortion and any point $z_0 \in B\left(0, \frac{1}{2}\right)$ the inequality

$$|f(z_0)| \geq e^{-\frac{c_f(|z_0|)}{|z_0|^2}} \tag{4.1}$$

holds for some positive function c_f such that $c_f(|z_0|) \to 0$ when $|z_0| \to 0$.

Given any $z_0 \in B\left(0, \frac{1}{2}\right) \setminus \{0\}$ we can assume, using rotation if necessary, that $z_0 > 0$. Fix sets $\bar{F} = [z_0, \infty)$ and $\bar{E} = [x_0, 0]$, where $x_0 < 0$ and $|f(x_0)| = 1$, and let $\bar{\Gamma}$ be the family of paths which connect these two sets. Then define function

$$\rho_0(z) = \begin{cases} \frac{1}{z_0} & \text{if } \operatorname{dist}(z, \bar{E}) < z_0, \\ 0 & \text{otherwise,} \end{cases}$$

which is clearly admissible with respect to $\bar{\Gamma}$, and estimate

$$M_{K_f}(\bar{\Gamma}) \leq \int_{\mathbb{C}} K_f(z)\rho_0^2(z)\, dz = \frac{1}{z_0^2} \int_{\{z:\, \operatorname{dist}(z,\bar{E}) < z_0\}} K_f(z)\, dz = \frac{c_f(|z_0|)}{z_0^2}. \tag{4.2}$$

FIGURE 5. The sets E and F.

Since $K_f(z)$ is locally integrable and the measure of the set $\{z : \operatorname{dist}(z, \bar{E}) < z_0\}$ converges to zero, we deduce that $c_f(|z_0|) \to 0$ as $|z_0| \to 0$.

To estimate the modulus $M(f(\bar{\Gamma}))$ we first note that the set $f(\bar{E})$ contains the origin and the point $f(x_0)$ with the modulus 1, and that the set $f(\bar{F})$ is unbounded. Then it is well known (see [13, chapter 11] and the references therein) that the smallest possible modulus for the path family $f(\bar{\Gamma})$ occurs when $f(\bar{E}) = [0, 1]$ and $f(\bar{F}) = (-\infty, -|f(z_0)|]$. Hence we can estimate (see [12, Theorem 7.26]) that

$$M(f(\bar{\Gamma})) \geq c \log\left(\frac{1}{|f(z_0)|}\right). \tag{4.3}$$

By combining (4.2) and (4.3) with the modulus inequality (1.2), which holds for the mapping f, we get

$$c \log\left(\frac{1}{|f(z_0)|}\right) \leq \frac{c_f(|z_0|)}{z_0^2},$$

which yields

$$|f(z_0)| \geq e^{-\frac{c_f(|z_0|)}{z_0^2}},$$

where $c_f(|z_0|) \to 0$ when $|z_0| \to 0$, and proves inequality (4.1).

Then we move on to bound the pointwise rotation from above and begin the proofs of Theorems 1.3 and 1.4. Fix an arbitrary $p \geq 1$ and let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism with p-integrable distortion, which hence satisfies the assumptions of Theorem 1.1, normalized by $f(0) = 0$.

Choose an arbitrary point $z_0 \in \mathbb{C} \setminus \{0\}$, for which $|z_0| < 1$. By a possible rotation we can assume that z_0 lies on a positive part of the real axis. Fix the line segments $E = [z_0, 1]$ and $F = (-\infty, 0]$ (see Figure 5), and let Γ be the family of paths connecting them.

We first estimate the moduli $M_{K_f}(\Gamma)$ from above when $p > 1$. To this end, fix balls $B_j = B(2^j z_0, 2^j z_0)$, where j goes through the integers from 0 to n and n is the smallest integer for which $2^n z_0 \geq 1$. Then define

$$\rho_0(z) = \begin{cases} \frac{2}{z_0} & \text{if } z \in B_0, \\ \frac{2}{2z_0} & \text{if } z \in B_1 \setminus B_0, \\ \vdots & \vdots \\ \frac{2}{2^n z_0} & \text{if } z \in B_n \setminus B_{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

To see that the function ρ_0 is admissible with respect to Γ note that every point $z \in E$ belongs to some ball $\frac{1}{2}B_j$, we have $\rho_0(z) \geq \frac{2}{r(B_j)}$ when $z \in B_j$ and $B_j \cap F = \emptyset$

for every j. Using the function ρ_0 and the Hölder inequality we estimate

$$M_{K_f}(\Gamma) \le \int_{\mathbb{C}} K_f(z)\rho_0^2(z)\,dz \le \left(\int_{B(0,4)} K_f^p(z)\,dz\right)^{\frac{1}{p}} \left(\int_{B(0,4)} \rho_0^{\frac{2p}{p-1}}(z)\,dz\right)^{\frac{p-1}{p}} = c_{f,p}\left(\int_{B(0,4)} \rho_0^{\frac{2p}{p-1}}(z)\,dz\right)^{\frac{p-1}{p}}. \tag{4.4}$$

In order to estimate this further we use the definition of ρ_0 to obtain

$$\begin{aligned}\int_{B(0,4)} \rho_0^{\frac{2p}{p-1}}(z)\,dz &\le \sum_{j=0}^{n}\int_{B_j}\left(\frac{2}{r(B_j)}\right)^{\frac{2p}{p-1}} dz\\ &= c_p\sum_{j=0}^{n}\frac{(r(B_j))^2}{(r(B_j))^{\frac{2p}{p-1}}}\\ &= c_p\sum_{j=0}^{n}\frac{1}{z_0^{\frac{2}{p-1}}}\frac{1}{2^{\frac{2j}{p-1}}}\\ &= c_p z_0^{-\frac{2}{p-1}}\sum_{j=0}^{n}\frac{1}{2^{\frac{2j}{p-1}}}.\end{aligned} \tag{4.5}$$

For any fixed $p > 1$ the series

$$\sum_{j=0}^{\infty}\frac{1}{2^{\frac{2j}{p-1}}}$$

converges to some constant c_p, and by combining (4.5) with (4.4) we get the upper bound

$$M_{K_f}(\Gamma) \le c_{f,p} z_0^{-\frac{2}{p}}. \tag{4.6}$$

When $p = 1$ we proceed as in the proof of Theorem 1.6 and use the function

$$\rho_0(z) = \begin{cases} \frac{1}{z_0} & \text{if } \operatorname{dist}(z,E) < z_0,\\ 0 & \text{otherwise,}\end{cases}$$

which is clearly admissible, and estimate

$$M_{K_f}(\Gamma) \le \int_{\mathbb{C}} K_f(z)\rho_0^2(z)\,dz = \frac{1}{z_0^2}\int_{\{z:\ \operatorname{dist}(z,E)\,<\,z_0\}} K_f(z)\,dz = \frac{c_f(z_0)}{z_0^2}. \tag{4.7}$$

As in the proof of Theorem 1.6 we see that $c_f(z_0) \to 0$ when $z_0 \to 0$.

Then we must estimate the modulus $M(f(\Gamma))$ for $p \ge 1$. We do this in a similar manner as in the proof of Theorem 1.2. This is possible since f is again a homeomorphism and the set $f(F)$ contains the origin and points with an arbitrary big modulus, and hence the images $f(E)$ and $f(F)$ must cycle around the origin alternately; see Figure 4 for the illustration.

So, let us write the modulus in the polar form

$$M(f(\Gamma)) = \inf_{\rho\ \text{admissible}}\int_{\mathbb{C}} \rho^2(z)\,dz = \inf_{\rho\ \text{admissible}}\int_0^{2\pi}\int_0^{\infty}\rho^2(r,\theta)r\,drd\theta, \tag{4.8}$$

and define

$$n(z_0) = \left\lfloor \frac{|\arg(f(z_0)) - \arg(f(1))|}{2\pi} \right\rfloor - 1. \tag{4.9}$$

With a similar reasoning as in the proof of Theorem 1.2 we see that given an arbitrary direction θ we can find $n(z_0)$ separate line segments $(x_i e^{i\theta}, y_i e^{i\theta})$ that belong to the path family $f(\Gamma)$ and whose coefficients satisfy

$$r_f \leq x_1 < y_1 < \cdots < x_{n(r)} < y_{n(r)} \leq c_f,$$

where $c_f = \sup_{z\in E} |f(z)|$ and $r_f = \min_{z\in E} |f(z)|$. Using these line segments we calculate as before (see inequalities (3.8), (3.9), and (3.12)) and get the estimate

$$\int_0^\infty \rho^2(r,\theta) r\, dr \geq \frac{n^2(z_0)}{\log\left(\frac{c_f}{r_f}\right)}, \tag{4.10}$$

which holds for an arbitrary admissible ρ and an arbitrary direction θ. The constant c_f is finite and does not depend on θ or z_0, at least for small z_0, and hence is irrelevant at the limit $z_0 \to 0$; whereas the constant r_f must be estimated using the modulus of continuity results (1.10) and (1.11).

Let us first cover the case $p > 1$. Combine the modulus of continuity result (1.10), when z_0 is small, with (4.10) to estimate

$$\int_0^\infty \rho^2(r,\theta) r\, dr \geq \frac{z_0^{\frac{2}{p}} n^2(z_0)}{c_{f,p}}.$$

This together with (4.8) yields

$$M(f(\Gamma)) \geq c_{f,p} z_0^{\frac{2}{p}} n^2(z_0). \tag{4.11}$$

So, when z_0 is small the bounds (4.6) and (4.11) for the moduli $M_{K_f}(\Gamma)$ and $M(f(\Gamma))$ together with the modulus inequality (1.2) imply

$$c_{f,p} z_0^{\frac{2}{p}} n^2(z_0) \leq c_{f,p} \left(\frac{1}{z_0}\right)^{\frac{2}{p}},$$

which simplifies to

$$n(z_0) \leq c_{f,p} \left(\frac{1}{z_0}\right)^{\frac{2}{p}}.$$

This together with the definition (4.9) proves the bound (1.7), and thus also Theorem 1.3.

We approach the case $p = 1$ similarly and use (1.11), when z_0 is small, to continue from (4.10) by estimating

$$\int_0^\infty \rho^2(r,\theta) r\, dr \geq \frac{z_0^2 n^2(z_0)}{q_f(z_0)},$$

where the function q_f satisfies $q_f(z_0) \to 0$ when $z_0 \to 0$. From this and (4.8) we obtain

$$M(f(\Gamma)) \geq \frac{z_0^2 n^2(z_0)}{q_f(z_0)}. \tag{4.12}$$

And again, the bounds (4.7) and (4.12), which hold when z_0 is small, for the moduli coupled with the modulus inequality (1.2) give, after simplification, that

$$n(z_0) \leq \frac{\sqrt{c_f(z_0)q_f(z_0)}}{z_0^2}. \tag{4.13}$$

This together with the definition (4.9) proves the bound (1.8), since $\sqrt{c_f(z_0)q_f(z_0)} \to 0$ when $z_0 \to 0$, and hence also Theorem 1.4.

Finally, to finish the proof of Theorem 1.6, to prove Theorem 1.5, and to discuss the sharpness of Theorem 1.2 we show examples verifying optimality of the bounds established earlier in this paper.

Let us start with Theorem 1.6. We must show that the bounds (1.10) and (1.11) are sharp by providing examples satisfying (1.12) for any given $p \geq 1$. To this end, fix an arbitrary $p \geq 1$ and choose an arbitrary function $h(r) : (0, 1] \to (0, \infty)$ such that $h(r) > r^{\frac{1}{p}}$ and $h(r) \to 0$ when $r \to 0$, where the former is just an auxiliary technical assumption that can be made without loss of regularity. Then let us construct a homeomorphism of p-integrable distortion $f : \mathbb{C} \to \mathbb{C}$, normalized by $f(0) = 0$, and find a sequence r_n of positive radii converging to zero such that

$$|f(r_n)| \leq e^{-ch(r_n)r_n^{-\frac{2}{p}}}.$$

We will fix the radii r_n later, but assume that $r_{n+1} < \frac{r_n}{2}$ and $r_1 < \frac{1}{2}$, and fix $x_n = 2r_n$. These choices guarantee that $r_n < x_n < r_{n-1}$, so we can construct disjoint annuli $A_n = B(0, x_n) \setminus B(0, r_n)$. Given an arbitrary annulus $A = B(0, a) \setminus B(0, b)$ we define the corresponding radial stretching map by

$$\psi_A(z) = \begin{cases} z & \text{if } z \notin B(0, a), \\ a\frac{z}{|z|}\left|\frac{z}{a}\right|^{K_A} & \text{if } z \in B(0, a) \setminus B(0, b), \\ \left(\frac{b}{a}\right)^{K_A - 1} z & \text{if } z \in B(0, b). \end{cases} \tag{4.14}$$

Note that ψ_A is K_A-quasiconformal mapping which is conformal outside of the annulus A. We construct the mapping f iteratively starting from the mapping

$$f_1(z) = \psi_{A_1}(z) = \begin{cases} z & \text{if } z \notin B(0, x_1), \\ x_1\frac{z}{|z|}\left|\frac{z}{x_1}\right|^{K_{A_1}} & \text{if } z \in B(0, x_1) \setminus B(0, r_1), \\ \left(\frac{1}{2}\right)^{K_{A_1} - 1} z & \text{if } z \in B(0, r_1), \end{cases}$$

and defining f_n, for every $n > 1$, by

$$f_n(z) = \psi_{f_{n-1}(A_n)} \circ f_{n-1}(z). \tag{4.15}$$

From the definition (4.14) it follows that f_n is conformal outside of the annuli $A_1, A_2, ..., A_n$, and K_{A_n}-quasiconformal inside the annuli A_n. Let us fix

$$K_{A_n} = h(r_n)\left(\frac{1}{r_n}\right)^{\frac{2}{p}}, \tag{4.16}$$

where we choose r_n so small that $h(r_n) < \frac{1}{n^2}$, and calculate

$$\sum_{n=1}^{\infty} |A_n| K_{A_n}^p \leq c \sum_{n=1}^{\infty} (h(r_n))^p < \infty. \tag{4.17}$$

From the definition (4.15) we see that the sequence f_n is uniformly Cauchy so there exists the limit

$$f = \lim_{n\to\infty} f_n, \tag{4.18}$$

which is clearly a homeomorphism. Since f_n is quasiconformal for every n and it differs from the mapping f_{n-1} only inside the ball $B(0, x_n)$, where we recall that $x_n \to 0$ when $n \to \infty$, the limit f is absolutely continuous on almost every line parallel to the coordinate axes and differentiable almost everywhere. Furthermore, we can estimate using (4.14) and (4.15) that

$$|f_z(z)| \leq cK_{A_n}$$

and

$$|f_{\bar{z}}(z)| \leq cK_{A_n},$$

when $z \in A_n$, while noting that $|D_f(z)| \leq 1$ elsewhere. When paired with the estimate (4.17) this shows that $D_f(z) \in L^1_{\text{loc}}(\mathbb{C})$, which together with the absolute continuity guarantees $f \in W^{1,1}_{\text{loc}}(\mathbb{C})$. As f is a homeomorphism this also shows $J_f(z) \in L^1_{\text{loc}}(\mathbb{C})$. Hence f is a finite distortion homeomorphism with the distortion

$$K_f(z) = \begin{cases} K_{A_n} & \text{if } z \in A_n, \\ 1 & \text{otherwise,} \end{cases}$$

that is, p-integrable due to (4.17). Thus the mapping f satisfies the assumptions regarding (1.12). Hence, all that is left is to estimate

$$\begin{aligned} |f(r_n)| = |f_n(r_n)| &= \left(\frac{1}{2}\right)^{K_{A_1}+\cdots+K_{A_{n-1}}-(n-1)} \left(\frac{1}{2}\right)^{K_{A_n}-1} r_n \\ &\leq \left(\frac{1}{2}\right)^{K_{A_n}-1} = 2^{-h(r_n) r_n^{-\frac{2}{p}}+1} = e^{-\log(2) h(r_n) r_n^{-\frac{2}{p}}+\log(2)} \end{aligned}$$

which proves the condition (1.12), since we can choose the function h such that the convergence $h(r) \to 0$ is as slow as we wish, and thus finishes the proof of Theorem 1.6.

In order to prove Theorem 1.5 we note that we can add rotation to the construction by changing the building block (4.14) to the form

$$\phi_A(z) = \begin{cases} z & \text{if } z \notin B(0,a), \\ a\frac{z}{|z|}\left|\frac{z}{a}\right|^{(1+i)K_A} & \text{if } z \in B(0,a) \setminus B(0,b), \\ \left(\frac{b}{a}\right)^{K_A-1} e^{iK_A \log\left(\frac{b}{a}\right)} z & \text{if } z \in B(0,b). \end{cases} \tag{4.19}$$

Then continuing as above, defining $f_{s,1} = \phi_{A_1}$ and $f_{s,n}(z) = \phi_{f_{s,n-1}(A_n)} \circ f_{s,n-1}(z)$, we obtain the limit map

$$f_s = \lim_{n\to\infty} f_{s,n} \tag{4.20}$$

which is a homeomorphism. The estimate (4.17) will still hold since $K_{f_s} = cK_f$ inside the annuli A_n, for some fixed constant c, and moreover the mapping f_s is conformal outside the annuli A_n. Thus we can verify, similarly as above, that f_s is a mapping of p-integrable distortion. From the definitions (4.19) and (4.20) we see that $f(0) = 0$, $f(1) = 1$, and

$$|\arg(f_s(r_n))| \geq \left|\arg\left(\frac{1}{2}\right)^{(1+i)K_{A_n}}\right| = \frac{\log(2)h(r_n)}{r_n^{\frac{2}{p}}} \tag{4.21}$$

for every r_n, which proves (1.9) since we can choose the function h such that the convergence is as slow as we wish. This finishes the proof of Theorem 1.5.

For studying the sharpness of Theorem 1.2 we use the above construction with the additional assumptions that $p = 1$ and that the distortion (4.16) satisfies $K_{A_n} > K_{A_1} + \cdots + K_{A_{n-1}}$ for every n, which is possible by choosing the radii r_n to be sufficiently small. We restrict such mappings f_s to the set $\overline{\Omega} = \{z : |z| \leq 1 \text{ and } \Im(z) \geq 0\}$ and denote these restrictions, which satisfy the assumptions of Theorem 1.2, by $f_{s,\Omega}$. Then choose any $r_n \in \partial\Omega$ and use (4.21) to estimate

$$|\arg(f_{s,\Omega}(r_n))| \geq \frac{ch(r_n)}{r_n^2}.$$

On the other hand, Theorem 1.2 gives the bound

$$\begin{aligned} |\arg(f_{s,\Omega}(r_n))| &\leq \frac{\left(\log\left(\frac{2^{K_{A_1}+\cdots+K_{A_n}-n}}{r_n}\right)\right)^{\frac{1}{2}}}{r_n} \\ &= \frac{\left(\log(2)(K_{A_1}+\cdots+K_{A_n}-n)+\log\left(\frac{1}{r_n}\right)\right)^{\frac{1}{2}}}{r_n} \\ &\leq \frac{\left(2\log(2)K_{A_n}+\log\left(\frac{1}{r_n}\right)\right)^{\frac{1}{2}}}{r_n} \\ &\leq \frac{c\sqrt{h(r_n)}}{r_n^2}, \end{aligned}$$

where we first use the bound (1.4), then the assumption $K_{A_n} > K_{A_1} + \cdots + K_{A_{n-1}}$, and finally the definition (4.16) for the distortion. Thus we see that Theorem 1.2 is essentially sharp as we can choose the function $h(r)$ such that the convergence $h(r) \to 0$ is as slow as we wish.

To finish, we comment on the remarks made earlier on the relation between the pointwise rotation and the pointwise stretch in the situation of Theorems 1.3 and 1.4. If we would instead of using the modulus of continuity results just assume that

$$|f(z)| \geq e^{-c_{f,p}|z|^{-\frac{2-\alpha p}{p}}} \tag{4.22}$$

for some $\alpha \in \left(0, \frac{2}{p}\right)$, when estimating (4.10) and continue as in the proof, we would bound the rotation by

$$|\arg(f(z))| \leq c_{f,p}|z|^{-\frac{4-\alpha p}{2p}}. \tag{4.23}$$

This shows that the maximal pointwise rotation indeed depends on the pointwise stretching.

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