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THE QUASICONFORMAL EQUIVALENCE OF RIEMANN SURFACES AND THE UNIVERSAL SCHOTTKY SPACE

HIROSHIGE SHIGA

Abstract. In the theory of Teichmüller space of Riemann surfaces, we consider the set of Riemann surfaces which are quasiconformally equivalent. For topologically finite Riemann surfaces, it is quite easy to examine if they are quasiconformally equivalent or not. On the other hand, for Riemann surfaces of topologically infinite-type, the situation is rather complicated.

In this paper, after constructing an example which shows the complexity of the problem, we give some geometric conditions for Riemann surfaces to be quasiconformally equivalent.

Our argument enables us to obtain a universal property of the deformation spaces of Schottky regions, which is analogous to the fact that the universal Teichmüller space contains all Teichmüller spaces.

1. Introduction

In the theory of Teichmüller space of Riemann surfaces, we consider the set of Riemann surfaces which are quasiconformally equivalent. Here, we say that two Riemann surfaces are quasiconformally equivalent if there is a quasiconformal homeomorphism between them. Hence, at the first stage of the theory, we have to know a condition for Riemann surfaces to be quasiconformally equivalent.

The condition is quite obvious if the Riemann surfaces are topologically finite. Indeed, the genus, the number of punctures, and the number of borders of surfaces completely determine the quasiconformal equivalence. On the other hand, for Riemann surfaces of topologically infinite-type, the situation is rather difficult. For example, viewing Royden algebras of open Riemann surfaces, Nakai ([10]; see also [11]) obtains an algebraic criterion for the equivalence. He shows that two Riemann surfaces are quasiconformally equivalent if and only if the Royden algebras of those Riemann surfaces are isomorphic. However, it is hard to examine the condition in general since the Royden algebras are huge function spaces. In this paper, we consider geometric conditions for the quasiconformal equivalence of open Riemann surfaces.

First, we give examples of Riemann surfaces in order to show the difficulty of the problem. We say that two homeomorphic Riemann surfaces R_1 and R_2 are *quasiconformally equivalent near the ideal boundary* if they are quasiconformally equivalent outside of compact subsets of those surfaces. At first glance, it seems to be true that if two Riemann surfaces are quasiconformally equivalent near the

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ideal boundary, then they are quasiconformally equivalent. However, it is not true. We may construct a counterexample in §3. Namely, we construct two homeomorphic Riemann surfaces R_1, R_2 and compact subsets K_i of R_i $(i = 1, 2)$ such that $R_1 \setminus K_1$ and $R_2 \setminus K_2$ are conformally equivalent but R_1 and R_2 are *not* quasiconformally equivalent. This example shows that the quasiconformal equivalence is not a boundary property. In the second example, we show that domains given by Schottky groups are not quasiconformally equivalent to domains given by boundary groups of Schottky spaces.

To give conditions for open Riemann surfaces to be quasiconfomally equivalent, we show a gluing lemma for quasiconformal mappings on Riemann surfaces (Lemma 4.1). By using the gluing lemma, we shall give a condition under which Riemann surfaces are quasiconformally equivalent. MacManus [9] obtains similar results from a different point of view, that is, a viewpoint of uniform domains, while we are considering the problems from the theory of Riemann surfaces of infinite-type.

In §6, we will discuss a universality of Schottky regions which are complements of the limit sets of Schottky groups. In fact, we show that Schottky regions are quasiconformally equivalent to each other (Theorem 6.2). The result makes a striking contrast to the second example in §3.

At the end, we present *the universal Schottky space* which includes all Schottky spaces.

2. Preliminaries

In this section, we give definitions, terminology, and known facts used in the later sections.

Let R be an open Riemann surface. A sequence $\{W_n\}_{n=1}^{\infty}$ of subdomains of R is called a *regular exhaustion* of R if it satisfies the following conditions:

(1) Each W_n is a relatively compact domain in R bounded by a finite number of mutually disjoint smooth simple closed curves in R.
(2) Every connected component of the complement of W_n $(n \in \mathbb{N})$ is not compact in R.
(3) $W_1 \subset W_2 \subset \cdots \subset W_n \subset W_{n+1} \subset \dots$ and $R = \bigcup_{n=1}^{\infty} W_n$.

It is known that any open Riemann surface has a regular exhaustion (cf. [2]).

A Riemann surface which is homeomorphic to a triply connected planar domain is called *a pair of pants*. If a Riemann surface is decomposed into pairs of pants $\{P_n\}$, then we say that the Riemann surface admits a pants decomposition $\{P_n\}$.

The Douady-Earle extension.

Let ϕ be an orientation preserving homeomorphism from $\mathbb{R}$ to itself. The mapping ϕ is called *quasisymmetric* if there exists a constant $M > 0$ such that

$$M^{-1} \leq \frac{\phi(x) - \phi(x-t)}{\phi(x+t) - \phi(x)} \leq M$$

holds for any $x \in \mathbb{R}$ and $t > 0$.

It is known that (cf. [1]) if $\phi : \mathbb{R} \to \mathbb{R}$ is quasisymmetric, then it has a quasiconformal extension to the upper halfplane $\mathbb{H}$. Namely, there exists a quasiconformal mapping $f : \mathbb{H} \to \mathbb{H}$ whose boundary value on $\mathbb{R}$ is ϕ.

In the famous paper by Douady and Earle [5], they show that every homeomorphism from $\mathbb{R}$ to itself admits a so-called *conformal natural extension* to $\mathbb{H}$, which is called the Douady-Earle extension. We denote the Douady-Earle extension of ϕ by

$E(\phi)$. The Douady-Earle extension $E(\phi)$ is a homeomorphism on $\mathbb{H}$ with boundary value ϕ and it is conformal natural, that is, for any $\gamma_1, \gamma_2 \in \mathrm{PSL}(2,\mathbb{R})$,

$$\gamma_1 \circ E(\phi) \circ \gamma_2 = E(\gamma_1 \circ \phi \circ \gamma_2)$$

holds. Moreover, $E(\phi)$ is real analytic in $\mathbb{H}$ and if ϕ is quasisymmetric, then $E(\phi)$ is quasiconformal in $\mathbb{H}$.

Teichmüller space and Schottky space.

Let R be a hyperbolic Riemann surface and let Γ_R be a Fuchsian group acting on $\mathbb{H}$ which represents R. A quasiconformal mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called a *quasiconformal deformation* of Γ_R if it is conformal on the lower halfplane $\mathbb{L}$ and $f \circ \Gamma_R \circ f^{-1} \subset \mathrm{PSL}(2,\mathbb{C})$. We say that two quasiconformal deformations f, g of Γ_R are equivalent if there exists a Möbius transformation A such that $g = A \circ f$. The Teichmüller space $\mathscr{T}(\Gamma_R)$ of the Fuchsian group Γ_R is the set of equivalence classes of quasiconformal deformations of Γ_R.

Let $\mathrm{Belt}(\Gamma_R; \mathbb{H})$ be the set of bounded measurable functions μ on $\mathbb{C}$ with $\|\mu\|_\infty < 1$ satisfying

$$\mu(\gamma(z))\overline{\gamma'(z)}\gamma'(z)^{-1} = \mu(z) \quad \text{(a.e. in } \mathbb{H})$$

for any $\gamma \in \Gamma_R$ and $\mu(z) = 0$ for any $z \in \mathbb{L}$. $\mathrm{Belt}(\Gamma_R; \mathbb{H})$ is a complex Banach space by the usual way.

For each $\mu \in \mathrm{Belt}(\Gamma_R; \mathbb{H})$, there exists a quasiconformal deformation $w_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of Γ_R with

$$\frac{\partial w_\mu(z)}{\partial \overline{z}} = \mu(z) \frac{\partial w_\mu(z)}{\partial z}, \text{ a.e.}$$

Hence, we have a projection $\pi_T : \mathrm{Belt}(\Gamma_R; \mathbb{H}) \to \mathscr{T}(\Gamma_R)$ by sending $\mu \in \mathrm{Belt}(\Gamma_R; \mathbb{H})$ to the equivalence class of w_μ. It is known that the Teichmüller space $\mathscr{T}(\Gamma_R)$ admits a complex structure so that the projection π_T is holomorphic. It is also known that the complex structures of $\mathscr{T}(\Gamma_R)$ and $\mathscr{T}(\Gamma_{R'})$ are the same if R and R' are quasiconformally equivalent.

If the Riemann surface R is the upper halfplane $\mathbb{H}$, then the group Γ_R is the trivial group $\{id\}$. We denote by $\mathscr{T}$ the Teichmüller space $\mathscr{T}(\{id\})$ and we call it *the universal Teichmüller space*. For any hyperbolic Riemann surface R, there exists a natural holomorphic embedding

$$\iota_R : \mathscr{T}(\Gamma_R) \hookrightarrow \mathscr{T}. \tag{2.1}$$

For more details on Teichmüller spaces, see [6] and [7].

Schottky space is defined in a similar way to Teichmüller space. Let G_g be a Schottky group of genus $g > 1$. A quasiconformal mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called a *quasiconformal deformation* of G_g if $f \circ G_g \circ f^{-1} \subset \mathrm{PSL}(2,\mathbb{C})$. We say that two quasiconformal deformations f, g of G_g are equivalent if there exists a Möbius transformation A such that g is homotopic to $A \circ f$ rel $\Lambda(G_g)$. The Schottky space $\mathscr{S}_g$ of genus g is the set of equivalence classes of quasiconformal deformations of G_g.

Let $\mathrm{Belt}(G_g; \mathbb{C})$ be the set of bounded measurable functions μ on $\mathbb{C}$ with $\|\mu\|_\infty < 1$ satisfying

$$\mu(\gamma(z))\overline{\gamma'(z)}\gamma'(z)^{-1} = \mu(z), \text{ a.e.}$$

for any $\gamma \in G_g$. By the same way as in Teichmüller spaces, we have a projection $\pi_S : \mathrm{Belt}(S_g; \mathbb{C}) \to \mathscr{S}_g$ and the Schottky space $\mathscr{S}_g$ admits a complex structure so

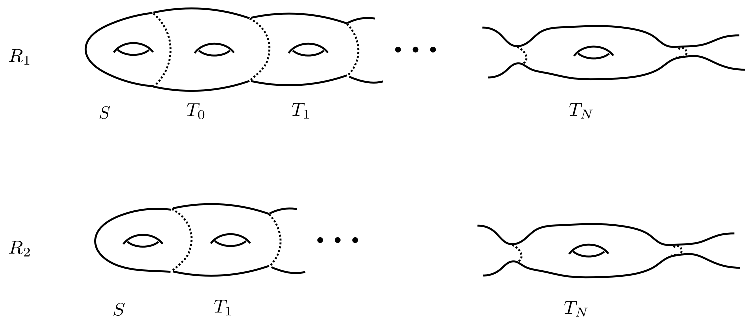

FIGURE 1

that the projection π_S is holomorphic. It is known that the complex structure of $\mathscr{S}_g$ depends only on the genus g.

Remark 2.1. The Schottky space defined above is called the strong deformation space of G_g in [8], in which the complex structure of the space is discussed.

Teichmüller space of a closed set.

Let E be a closed set in $\widehat{\mathbb{C}}$. We denote by $\mathrm{Belt}(\mathbb{C}) = \mathrm{Belt}(\{id\};\mathbb{C})$ the set of bounded measurable functions μ on $\mathbb{C}$ with $\|\mu\|_\infty < 1$. Two functions μ_1, μ_2 are said to be equivalent if there exists a Möbius transformation A such that $A \circ w_{\mu_1}$ is homotopic to w_{μ_2} rel E. We define Teichmüller space of E, which is denoted by $\mathscr{T}(E)$, by the set of equivalence classes.

3. EXAMPLES OF RIEMANN SURFACES ON QUASICONFORMAL NON-EQUIVALENCE

In this section, we construct two examples of pairs of Riemann surfaces which are not quasiconformally equivalent. In the first example, we construct two Riemann surfaces R_1 and R_2 which are quasiconformally equivalent near the ideal boundary but not quasiconformally equivalent. The second one is an example of Riemann surfaces defined by Cantor sets. The example has its own interest and is also related to the result in Theorem 6.2 in §6.

Example 3.1. Put $a_n = (n!)^{-1}$ and take pairs of pants P_n bounded by three hyperbolic closed geodesics whose lengths are $1, 1$ and a_n $(n = 0, 1, 2, \dots)$. We glue P_n and P_{n+1} along two boundary curves with length 1 to make a Riemann surface T_n of genus 1 with two boundary curves of lengths a_n and a_{n+1}. Since T_n and T_{n+1} have a boundary curve of the length a_{n+1}, we may glue them along the boundary curves. By repeating this operation for $n = 0, 1, 2, \dots$, we get a Riemann surface $R_1' = \bigcup_{n=0}^{\infty} \overline{T_n}$ which is a Riemann surface of infinite genus with a geodesic boundary curve of length 1. We take a Riemann surface S of genus 1 with a geodesic boundary curve of length 1 by gluing two boundary curves of P_0. Gluing R_1' and S along the boundary curves, we have an open Riemann surface R_1 of infinite genus.

Next, we make a Riemann surface R_2' in the same way as R_1' but we do it from $n = 1$ instead of $n = 0$ for R_1'. Then, R_2' is still a Riemann surface of infinite genus with a geodesic boundary of length 1. Hence, since we can glue R_2' and S along the boundary curves, we have an open Riemann surface R_2 of infinite genus (Figure 1).

Obviously, both R_1 and R_2 are homeomorphic and they have the same subsurface $\bigcup_{n=1}^{\infty} \overline{T_n}$. Hence, $R_1 \setminus K_1$ and $R_2 \setminus K_2$ are conformally equivalent for $K_1 = \overline{S \cup T_0}$ and $K_2 = \overline{S}$. In particular, they are quasiconformally equivalent near the ideal boundary. However, we may show that there are no quasiconformal mappings between R_1 and R_2.

Suppose that there exists a K-quasiconformal mapping $F : R_1 \to R_2$ for some $K \geq 1$. We take a sufficiently large $N \in \mathbb{N}$ with $N > K$. We consider the closed geodesic α_N of $\partial T_N \subset R_1$ with length a_N and the geodesic $[F(\alpha_N)]$ homotopic to $F(\alpha_N)$ in R_2. It follows from Wolpert's formula ([14],[15]) that the hyperbolic length $\ell([F(\alpha_N)])$ of $[F(\alpha_N)]$ in R_2 satisfies an inequality,

$$K^{-1} a_N \leq \ell([F(\alpha_N)]) \leq K a_N.$$

Hence, we have

$$a_{N+1} = \frac{1}{(N+1)!} < N^{-1} a_N \leq \ell([F(\alpha_N)]) \leq N a_N < \frac{N}{N!} = a_{N-1}. \tag{3.1}$$

If the geodesic $[F(\alpha_N)]$ transversely intersects with some α_i in R_2, then it follows from the collar theorem (cf. [3]) that the length $\ell([F(\alpha_N)])$ is large enough. If $[F(\alpha_N)] \cap \alpha_i = \emptyset$ for any $i \in \mathbb{N}$, from the geometry of S and T_n ($n \in \mathbb{N}$) we see that $\ell([F(\alpha_N)])$ is larger than a_N for a sufficiently large N.

Hence, we conclude that only the closed geodesic of $\overline{T_N} \cap \overline{T_{N+1}}$ of length a_n in R_2 has the length satisfying (3.1). Therefore, the subsurface $\overline{S} \cup \bigcup_{n=0}^{N-1} \overline{T_n}$ of R_1 which is of genus $N+1$ has to be mapped a subsurface of R_2 of genus N. It is absurd because F is a homeomorphism. Thus, we have a contradiction.

Example 3.2. Let G be a Schottky group of genus $g > 1$. The group is constructed from $2g$ (topological) closed disks $D_1, D_2, \dots, D_{2g}$ with $D_i \cap D_j = \emptyset$ ($i \neq j$) and $\gamma_i \in \mathrm{PSL}(2, \mathbb{C})$ ($i = 1, 2, \dots, g$) which map the outside of D_{2i-1} onto the inside of D_{2i}. The group G is a Kleinian group generated by $\gamma_1, \gamma_2 \dots, \gamma_g$ and it is a purely loxodromic free group of rank g. The region of discontinuity $\Omega(G)$ of G is a connected domain in $\widehat{\mathbb{C}}$ and the complement $\Lambda(G)$, the limit set of G, is a Cantor set. Thus, $\Omega(G)$ is an open Riemann surface of infinite-type.

Now, we consider a Kleinian group G' of *Schottky-type* with cusps. We construct the group G' as follows.

Take $2g$ closed disks $D'_1, D'_2, \dots, D'_{2g}$ such as $D'_i \cap D'_j = \emptyset$ for $1 \leq i < j \leq 2g-1$, $D'_i \cap D'_{2g} = \emptyset$ for $1 \leq i \leq 2g-2$ but D'_{2g} is tangential to D'_{2g-1} at one point z_0. We also take $\delta_i \in \mathrm{PSL}(2, \mathbb{C})$ ($i = 1, 2, \dots, g-1$) which map the outside of D'_{2i-1} onto the inside of D'_{2i}, and $\delta_g \in \mathrm{PSL}(2, \mathbb{C})$ which maps the outside of D'_{2g-1} onto the inside of D'_{2g} fixing z_0. Hence, δ_g is a parabolic transformation with the fixed point z_0. The group G' is generated by $\delta_1, \delta_2, \dots, \delta_g$. The group G' is still a Kleinian group and a free group of rank g, but it contains parabolic elements δ_g.

We may take a sequence $\{G_n\}_{n=1}^{\infty}$ of Schottky groups of genus g such that it converges to G'. Hence, the group G' is regarded as a group on the boundary of Schottky space.

The limit set $\Lambda(G')$ of G' is also a Cantor set and the region of discontinuity $\Omega(G')$ is an open Riemann surface of infinite-type.

Thus, we have two open Riemann surfaces $\Omega(G)$ and $\Omega(G')$ of infinite-type both of which are complements of some Cantor sets. Then, we insist on the following.

Claim. $\Omega(G)$ and $\Omega(G')$ are not quasiconformally equivalent.

Since both G and G' are quasiconformal deformations of Fuchsian groups, we may assume that G and G' are Fuchsian groups, so that $\Lambda(G), \Lambda(G') \subset \mathbb{R}$. Suppose that there exists a quasiconformal mapping f from $\Omega(G)$ onto $\Omega(G')$. Then we have the following ([13, Theorem 1. 2 and Corollary 1. 3]):

(1) The mapping f is extended to a quasiconformal mapping from $\widehat{\mathbb{C}}$ onto itself. We use the same letter f for the extended mapping.

(2) The mapping f is extended to a homeomorphism of the Martin compactifications. We denote the extended homeomorphism by f^* (for the Martin compactification, see [4]).

Let $p \in \Lambda(G')$ be a parabolic fixed point. From (1) above, there exists a point $q \in \Lambda(G)$ such that $f(q) = p$. Moreover, it follows from (2) that there exists a unique limit of $f^*(z)$ as $z \to q$ in the Martin compactification of $\Omega(G)$. On the other hand, in the Martin compactification of $\Omega(G')$, there are more than two points over a parabolic fixed point ([13, Theorem 1. 1 (A)]; see also [12]). Therefore, we may find a non-convergent sequence $\{f^*(z_n)\}_{n=1}^{\infty}$ as $z_n \to q$. Thus, we have a contradiction.

4. A gluing lemma

In this section, we shall prove the following lemma.

Lemma 4.1. *Let X, Y be Riemann surfaces. We consider simple closed curves $\alpha \subset X$ and $\beta \subset Y$ with $X \setminus \alpha = X_1 \sqcup X_2$ and $Y \setminus \beta = Y_1 \sqcup Y_2$, respectively. Suppose that there exist quasiconformal mappings $f_i : X_i \to Y_i$ $(i = 1, 2)$ such that $f_1(\alpha) = f_2(\alpha) = \beta$. Then, there exists a quasiconformal mapping $f : X \to Y$. Moreover, the maximal dilatation of f depends only on those of f_1, f_2 and the local behavior of those mappings near α.*

Remark 4.1. Since α is a simple closed curve, the quasiconformal mappings f_1 and f_2 are extended homeomorphically to α. We use the fact in the statement of the above lemma.

Remark 4.2. If we suppose that α is piecewise smooth and f_1, f_2 agree on α, then the conclusion is easy. But we do not assume them in this lemma.

Proof. We take simple closed curves $\alpha_i \subset X_i$ $(i = 1, 2)$ near α so that α and α_i bound annuli $A_i \subset X_j$. We put $B_i = f_i(A_i)$ and $\beta_i = f_i(\alpha_i)$. Then, B_i are also annuli, which are bounded by β and β_i $(i = 1, 2)$. First of all, we show that f_1 and f_2 can be real analytic on α_1 and α_2, respectively.

There exist $r_i, k_i > 1$ such that each A_i is conformally equivalent to a circular annulus

$$\mathcal{A}_i := \{z \in \mathbb{C} \mid 1 < |z| < r_i\} \simeq \mathbb{H}/\langle z \mapsto k_i z\rangle$$

via a conformal mapping $\varphi_i : \mathcal{A}_i \to A_i$ $(i = 1, 2)$. We also take $\rho_i, \kappa_i > 1$ so that each B_i is conformally equivalent to a circular annulus

$$\mathcal{B}_i = \{z \in \mathbb{C} \mid 1 < |z| < \rho_i\} \simeq \mathbb{H}/\langle z \mapsto \kappa_i z\rangle\}$$

via $\psi_i : \mathcal{B}_i \to B_i$.

Then, $\phi_i := \psi_i^{-1} \circ f_i|_{A_i} \circ \varphi_i$ from $\mathcal{A}_i$ onto $\mathcal{B}_i$ are lifted to quasiconformal mappings $\widehat{\phi}_i : \mathbb{H} \to \mathbb{H}$ with

$$\widehat{\phi}_i(k_i z) = \kappa_i \widehat{\phi}_i(z)$$

for any $z \in \overline{\mathbb{H}}$. In particular,

$$\widehat{\phi}_i(k_i x) = \kappa_i \widehat{\phi}_i(x) \tag{4.1}$$

holds for any $x \in \mathbb{R}$.

We take the Douady-Earle extension $\widehat{\Phi}_i$ of $\widehat{\phi}_i|_{\mathbb{R}}$. Since $\widehat{\phi}_i$ satisfy (4.1) on $\mathbb{R}$, $\widehat{\Phi}_i$ also satisfy the equations on $\overline{\mathbb{H}}$. Moreover, they are real analytic in $\mathbb{H}$. Therefore, the quasiconformal mappings $\widehat{\Phi}_i : \mathbb{H} \to \mathbb{H}$ are projected quasiconformal mappings $\Phi_i : \mathcal{A}_i \to \mathcal{B}_i$. Hence, $F_i := \psi_i \circ \Phi_i \circ \varphi_i^{-1} : A_i \to B_i$ are real analytic quasiconformal mappings with the same boundary values as $f_i|_{A_i}$ $(i = 1, 2)$.

We define quasiconformal mappings $\tilde{f}_i$ from X_i onto Y_i by f_i on $X_i \setminus A_i$ and F_i on $A_i \cup \{\alpha_i\}$. They are real analytic in A_i. Let $\tilde{\alpha}_i$ be non-trivial smooth Jordan curves in A_i. Then, $\tilde{f}_i$ are real analytic on $\tilde{\alpha}_i$. Thus, by considering $\tilde{f}_i$ and $\tilde{\alpha}_i$ instead of f_i and α_i, respectively, we may assume that f_i are real analytic on α_i.

Now, we consider an annulus A in X bounded by α_1 and α_2. We also consider an annulus B in Y bounded by $\beta_1 := f_1(\alpha)$ and $\beta_2 := f_2(\alpha)$. We take $r, k > 1$ so that A is conformally equivalent to the circular annulus

$$\mathcal{A} := \{z \in \mathbb{C} \mid 1 < |z| < r\} \simeq \mathbb{H}/\langle z \mapsto kz\rangle$$

via a conformal mapping $h_A : \mathcal{A} \to A$. We also take $\rho, \kappa > 1$ so that B is conformally equivalent to the circular annulus

$$\mathcal{B} := \{z \in \mathbb{C} \mid 1 < |z| < \rho\} \simeq \mathbb{H}/\langle z \mapsto kz\rangle$$

via a conformal mapping $h_B : \mathcal{B} \to B$.

We denote by $\pi_A : \overline{\mathbb{H}} \setminus \{0, \infty\} \to \overline{A} \simeq \overline{\mathbb{H}} \setminus \{0, \infty\}/\langle z \mapsto kz\rangle$ and $\pi_B : \overline{\mathbb{H}} \setminus \{0, \infty\} \to \overline{B} \simeq \overline{\mathbb{H}} \setminus \{0, \infty\}/\langle z \mapsto \kappa z\rangle$, the quotient mappings for $\overline{A}$ and $\overline{B}$, respectively. We may assume that $\pi_A(\mathbb{R}_{<0}) = \alpha_1$, $\pi_A(\mathbb{R}_{>0}) = \alpha_2$, $\pi_B(\mathbb{R}_{<0}) = \beta_1$ and $\pi_B(\mathbb{R}_{>0}) = \beta_2$. Then, the smooth homeomorphism $f_1|_{\alpha_1} : \alpha_1 \to \beta_1$ is lifted to a smooth homeomorphism from $\mathbb{R}_{<0}$ to itself and $f_2|_{\alpha_2} : \alpha_2 \to \beta_2$ is also lifted to a smooth homeomorphism from $\mathbb{R}_{>0}$ to itself. Thus, we have a strictly increasing homeomorphism Ψ on $\mathbb{R}$ onto itself which is smooth in $\mathbb{R} \setminus \{0\}$ with $\Psi(0) = 0$. The mapping Ψ satisfies

$$\Psi(kx) = \kappa\Psi(x) \tag{4.2}$$

for any $x \in \mathbb{R}$.

We may normalize the function as $\Psi(1) = 1$ and $\Psi(-1) = -1$. We show that Ψ is quasisymmetric on $\mathbb{R}$.

We put

$$M = \sup_{x>0,t>0} \frac{\Psi(x) - \Psi(x-t)}{\Psi(x+t) - \Psi(x)}$$

and

$$m = \inf_{x>0.t>0} \frac{\Psi(x) - \Psi(x-t)}{\Psi(x+t) - \Psi(x)}.$$

We show that $0 < m \leq M < \infty$ in several steps.

If $x = ky$ and $t = ks$ $(s > 1)$, then we have from (4.2)

$$\begin{aligned}
\Psi(x) - \Psi(x-t) &= \Psi(ky) - \Psi(k(y-s)) = \kappa(\Psi(y) - \Psi(y-s)),\\
\Psi(x+t) - \Psi(x) &= \kappa(\Psi(y+s) - \Psi(y)).
\end{aligned}$$

Thus, we see

$$M = \sup_{x\in\{0\}\cup[1,k],t>0} \frac{\Psi(x)-\Psi(x-t)}{\Psi(x+t)-\Psi(x)} \tag{4.3}$$

and

$$m = \inf_{x\in\{0\}\cup[1,k],t>0} \frac{\Psi(x)-\Psi(x-t)}{\Psi(x+t)-\Psi(x)}. \tag{4.4}$$

(i) If $x \in [1,k]$ and $0 < t \le \frac{1}{2}$, then we have

$$\Psi(x)-\Psi(x-t) = \Psi'(x-\theta t)t$$

and

$$\Psi(x+t)-\Psi(x) = \Psi'(x+\theta' t)t$$

for some $\theta, \theta' \in (0,1)$. Thus,

$$\frac{\Psi(x)-\Psi(x-t)}{\Psi(x+t)-\Psi(x)} = \frac{\Psi'(x-\theta t)}{\Psi'(x+\theta' t)}.$$

Since $x \in [1,k]$,

$$\frac{1}{2} \le x-\theta t < x+\theta' t \le k+\frac{1}{2}.$$

We conclude that there exist $0 < m_1 < M_1 < \infty$ such that

$$m_1 \le \frac{\Psi(x)-\Psi(x-t)}{\Psi(x+t)-\Psi(x)} \le M_1 \tag{4.5}$$

for any $x \in [1,k]$ and $t \in (0, \frac{1}{2}]$.

(ii) If $\frac{1}{2} < t < x$, we have

$$\Psi(x)-\Psi(x-t) \le \Psi(x) \le \Psi(k) = \kappa$$

and

$$\Psi(x+t)-\Psi(x) \ge \Psi\left(x+\frac{1}{2}\right) - \Psi(x).$$

For $\widetilde{m}_2 = \inf_{1\le x\le k} \left\{\Psi\left(x+\frac{1}{2}\right) - \Psi(x)\right\} > 0$, we get

$$\frac{\Psi(x)-\Psi(x-t)}{\Psi(x+t)-\Psi(x)} \le \frac{\kappa}{\widetilde{m}_2} < \infty.$$

Also, we have

$$\Psi(x)-\Psi(x-t) \ge \Psi(x) - \Psi\left(x-\frac{1}{2}\right)$$

and

$$\Psi(x+t)-\Psi(x) \le \Psi(x+t) \le \Psi(2x) \le \Psi(2k) = \kappa\Psi(2).$$

For $\widehat{m}_2 = \inf_{1\le x\le k} \left\{\Psi(x) - \Psi\left(x-\frac{1}{2}\right)\right\} > 0$, we get

$$\frac{\Psi(x)-\Psi(x-t)}{\Psi(x+t)-\Psi(x)} \ge \frac{\widehat{m}_2}{\kappa\Psi(2)} > 0.$$

(iii) If $x \in [1,k]$ and $t \ge \frac{1}{2}$, then we put

$$M_3 = \sup_{1\le x\le k, t\ge \frac{1}{2}} \frac{\Psi(x)-\Psi(x-t)}{\Psi(x+t)-\Psi(x)}$$

and

$$m_3 = \inf_{1 \le x \le k, t \ge \frac{1}{2}} \frac{\Psi(x) - \Psi(x-t)}{\Psi(x+t) - \Psi(x)}.$$

We take sequences $\{x_n\}, \{t_n\}$ so that $x_n \in [1, k], t_n \ge \frac{1}{2}$ and

$$\lim_{n \to \infty} \frac{\Psi(x_n) - \Psi(x_n - t_n)}{\Psi(x_n + t_n) - \Psi(x_n)} = M_3.$$

If $\{t_n\}$ is bounded, it is obvious that $M_3 < \infty$. We suppose that $\{t_n\}$ is unbounded. Since $x_n \in [1, k]$, we have

$$x_n - t_n \in [1 - t_n, k - t_n], \quad x_n + t_n \in [1 + t_n, k + t_n].$$

Hence, we have

$$\Psi(x_n) - \Psi(x_n - t_n) \le \Psi(k) - \Psi(1 - t_n) = \kappa - \Psi(1 - t_n)$$

and

$$\Psi(x_n + t_n) - \Psi(x_n) \ge \Psi(1 + t_n) - \Psi(k) = \Psi(1 + t_n) - \kappa.$$

We take $m(n) \in \mathbb{N}$ such that

$$k^{m(n)} \le t_n \le k^{m(n)+1}.$$

Note that $m(n) \to \infty$ as $n \to \infty$. Then

$$\Psi(1 - t_n) \ge \Psi(-t_n) \ge \Psi(-k^{m(n)+1}) = \kappa^{m(n)+1}\Psi(-1) = -\kappa^{m(n)+1}$$

and

$$\Psi(1 + t_n) \ge \Psi(k^{m(n)}) = \kappa^{m(n)}.$$

Thus, we have

$$\frac{\Psi(x_n) - \Psi(x_n - t_n)}{\Psi(x_n + t_n) - \Psi(x_n)} \le \frac{1 + \kappa^{m(n)}}{\kappa^{m(n)-1} - 1}$$

and we get

$$M_3 \le \kappa$$

as $m(n) \to \infty$. A similar argument shows that $m_3 > 0$.

Thus, we conclude that $0 < m < M < \infty$. By using the same argument as above, we can show that

$$0 < \inf_{x<0, 0<t} \frac{\Psi(x) - \Psi(x-t)}{\Psi(x+t) - \Psi(x)} \le \sup_{x<0, 0<t} \frac{\Psi(x) - \Psi(x-t)}{\Psi(x+t) - \Psi(x)} < \infty.$$

(iv) If $x = 0$, we have

$$\begin{aligned} \kappa^{n-1} \quad &\le \Psi(t) \quad \le \kappa^n \\ -\kappa^n = \kappa^n \Psi(-1) \quad &\le \Psi(-t) \quad \le \kappa^{n-1}\Psi(-1) = -\kappa^{n-1} \end{aligned}$$

if $k^{n-1} \le t \le k^n$ $(n \in \mathbb{N})$. Hence,

$$\kappa^{-1} \le \frac{\Psi(0) - \Psi(-t)}{\Psi(t) - \Psi(0)} \le \kappa.$$

The same argument gives us the same estimate for $\kappa^{-n} \le t \le \kappa^{-n+1}$ $(n \in \mathbb{N})$.

It follows from (i) – (iv) that Ψ is quasisymmetric on $\mathbb{R}$.

Now, we take the Douday-Earle extension $E(\Psi)$ of Ψ. It is a quasiconformal self-mapping of $\mathbb{H}$ because of the quasisymmetricity of Ψ. Since Ψ satisfies (4.2), the equation

$$E(\Psi)(kz) = \kappa E(\Psi)(z)$$

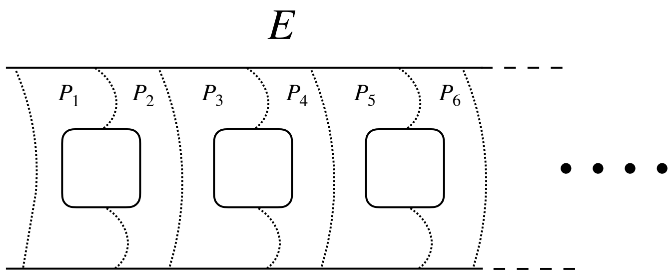

FIGURE 2

also holds for any $z \in \mathbb{H}$. Therefore, $E(\Psi)$ is projected to a quasiconformal mapping ψ from A to B. Moreover, we have $\psi|_{\alpha_1} = f_1|_{\alpha_1}$, $\psi|_{\alpha_2} = f_2|_{\alpha_2}$. We define a map $f : X \to Y$ by

$$f(p) = \begin{cases} f_i(p), & (p \in X_i \setminus A; i = 1, 2), \\ \psi(p), & (p \in \overline{A}). \end{cases}$$

The map f is a homeomorphism and quasiconformal except on $\alpha_1 \cup \alpha_2$. It follows from the removability for quasiconformal mapping that f is quasiconformal on X. Moreover, from the construction we see that the maximal dilatation of f depends only on those of f_i and the local behavior of them near α. □

5. Conditions for the quasiconformal equivalence of Riemann surfaces

Let R_1, R_2 be open Riemann surfaces which are homeomorphic to each other. Suppose that R_1 and R_2 are quasiconformally equivalent near the ideal boundaries, namely, there exist compact subsets K_j of R_j $(j = 1, 2)$ and a quasiconformal mapping such that $f(R_1 \setminus K_1) = R_2 \setminus K_2$. As we have seen in the previous section, the quasiconformal equivalence near the ideal boundaries does not imply the quasiconformal equivalence of the surfaces in general. In this section, we will give sufficient conditions for two open Riemann surfaces which are quasiconformally equivalent near the ideal boundaries to be quasiconformally equivalent.

We say that an open Riemann surface R admits a *bounded pants decomposition* if there exists a pants decomposition $\{P_n\}_{n=1}^{\infty}$ of R such that each P_n is bounded by hyperbolic closed geodesics and the lengths of the geodesics are in $[M^{-1}, M]$, where $M > 0$ is a constant independent of n.

Definition 5.1. Let E be an end of an open Riemann surface R. We say that E is an *infinite ladder end* (ILE) if E is an end of infinite genus having a bounded pants decomposition $\{P_n\}_{n=1}^{\infty}$ given by the dotted lines as in Figure 2.

Theorem 5.1. *Let R_1, R_2 be homeomorphic open Riemann surfaces which are quasiconformally equivalent near the ideal boundaries.*

(1) *If the genus of R_1 is finite, then R_1 and R_2 are quasiconformal equivalent.*
(2) *If R_1 has an ILE, then R_1 and R_2 are quasiconformally equivalent.*

Proof. From the assumption, there exist compact subsets K_i of R_i $(i = 1, 2)$ and a quasiconformal mapping f on $R_1 \setminus K_1$ such that $f(R_1 \setminus K_1) = R_2 \setminus K_2$.

(1) Let $R_1 = \bigcup_{n=1}^{\infty} W_n$ be a regular exhaustion of R_1. Each W_n is a relatively compact subregion of R_1 bounded by a finite number of mutually disjoint simple closed curves, and every connected component of the complement of W_n is not relatively compact in R_1. Hence, there exists $N \in \mathbb{N}$ such that $K_1 \subset W_N$ and the genus of W_N is the same as that of R_1. Also, the number of the connected components of $R_1 \setminus W_N$ is not more than that of the boundary components of W_N. Thus, it has to be finite.

Let $E_1, \dots, E_k$ be the set of connected components of $R_1 \setminus W_N$. Since W_N is of the same genus as R_1, every E_j is a planar and so is $f(E_j)$. Hence, we may take a simple closed curve α_j in E_j which separates the ideal boundary of E_j and the relative boundaries of E_j. We see that there is a unique connected component of $R_2 \setminus \bigcup_{j=1}^{k} f(\alpha_j)$ which is relatively compact in R_2.

Indeed, if we have that there are two relatively compact connected components in $R_2 \setminus \bigcup_{j=1}^{k} f(\alpha_j)$, then each of them together with its connected components of the complement is a subdomain of R_2 with no relative boundaries. It is absurd because of the connectivity of R_2. It has to be unique.

We denote by S_2 the relatively compact connected component of $R_2 \setminus \bigcup_{j=1}^{k} f(\alpha_j)$. It is also seen that there is a unique connected component of $R_1 \setminus \bigcup_{j=1}^{k} \alpha_j$. The component is denoted by S_1. Then, both S_1 and S_2 are open Riemann surfaces of the same genus bounded by the same number of simple closed curves. Hence, they are quasiconformally equivalent as well as their complements. Thus, we see from Lemma 4.1 that R_1 and R_2 are quasiconformally equivalent.

(2) Let $E \subset R_1$ be an ILE of R_1 with a bounded pants decomposition $\{P_n\}_{n=1}^{\infty}$ as Figure 2 shows. Every boundary curve of P_n $(n \in \mathbb{N})$ is the hyperbolic geodesic whose length is in $[M^{-1}, M]$ for some $M > 0$ independent of n.

From the assumption, there exist a compact subset K_i of R_i $(i = 1, 2)$ and a quasiconformal mapping $f : R_1 \setminus K_1 \to R_2 \setminus K_2$. We may assume that K_1 is the closure of a regular region S_1 of R_1 and E is a connected component of $R_1 \setminus S_1$. We put $S_2 = R_2 \setminus \overline{f(R_1 \setminus S_1)}$.

Since $K_1 = \overline{S_1}$, the boundary $\partial K_1 = \partial S_1$ consists of finitely many Jordan curves in R_1. Hence, so is $f(\partial K_1) = \partial S_2$. In particular, the number of boundary components of S_2 are the same as that of S_1. If the genus of S_2 is the same as that of S_1, then S_1 and S_2 are quasiconformally equivalent. Thus, it follows from Lemma 4.1 that R_1 and R_2 are quasiconformally equivalent.

Suppose that the genus of S_2 is greater than the genus of S_1 and let $m \in \mathbb{N}$ be the difference of them. For a bounded pants decomposition $\{P_n\}_{n=1}^{\infty}$ of E as in Figure 2, pairs of pants $P_1, \dots, P_{2m}$ make a regular region W_m of genus m with two boundary components. By gluing S_1 and W_m, we get a regular region S_1' of the same genus as that of S_2. We also see that S_1' is bounded by the same number of closed curves as S_2. Therefore, S_1' and S_2 are quasiconformally equivalent.

Now, we consider an end $E_m := E \setminus \bigcup_{n=1}^{2m} \overline{P_n}$. The end E_m is still an ILE end with a bounded pants decomposition $\{P_n\}_{n \geq m+1}$. On the other hand, the end $E' := f(E)$ is also an ILE and it admits a bounded pants decomposition $\{P_n'\}_{n=1}^{\infty}$ as Figure 2. It follows from Wolpert's formula that the hyperbolic length of any boundary curve of P_n' is in $[K(f)^{-1}M^{-1}, K(f)M]$, where $K(f)$ is the maximal dilatation of f. Therefore, P_i and P_j' are quasiconformally equivalent for any $i \geq m+1$ and for any $j \in \mathbb{N}$. We may also see that the maximal dilatations of quasiconformal mappings from P_i onto P'_j ($i \geq m+1, j \in \mathbb{N}$) can be uniformly bounded. From Lemma 4.1 we see that E_m and E' are quasiconformally equivalent.

From the assumption, $R_1 \setminus (S'_1 \cup E_m)$ and $R_2 \setminus (S_2 \cup E')$ are quasiconformally equivalent. By using Lemma 4.1 again, we conclude that R_1 and R_2 are quasiconformally equivalent.

The same argument works for f^{-1} when the genus of S_1 is greater than the genus of S_2. Thus, we complete the proof of the theorem. □

6. A universality of Schottky regions and the universal Schottky space

Let G_g ($g > 1$) be a Schottky group of genus g. Then, the limit set $\Lambda(G_g)$ of G_g is a Cantor set in $\widehat{\mathbb{C}}$. We call the complement $\Omega(G_g)$ of $\Lambda(G_g)$, which is the region of discontinuity of G_g, a *Schottky region* for genus g.

Let $\Omega(G'_g)$ be another Schottky region for the same genus g. Then the quotient surfaces $X := \Omega(G_g)/G_g$, $X' := \Omega(G'_g)/G'_g$ are compact Riemann surfaces of genus g. We see that there is a quasiconformal mapping from X onto X' and the mapping is lifted to a group equivariant quasiconformal map from $\Omega(G_g)$ onto $\Omega(G'_g)$. Therefore, Schottky regions $\Omega(G_g)$ and $\Omega(G'_g)$ for genus g are quasiconformal equivalent as open Riemann surfaces of infinite-type. In fact, the quasiconformal mapping is extended to a quasiconformal mapping on $\widehat{\mathbb{C}}$.

We also see in Example 3.2 that for a Kleinian group G' of Schottky-type with cusps, $\Omega(G_g)$ and $\Omega(G')$ are not quasiconformally equivalent while both are the complements of some Cantor sets.

Now, we consider a Schottky group G_h of genus $h \neq g$. Of course, there are no group equivariant quasiconformal mappings between $\Omega(G_g)$ and $\Omega(G_h)$ since those groups represent topologically different Riemann surfaces. However, it may be possible that $\Omega(G_g)$ and $\Omega(G_h)$ are quasiconformally equivalent as open Riemann surfaces. In fact, it is always possible. We may show the following.

Theorem 6.1. *Schottky regions are quasiconformally equivalent to each other. More precisely, for any Schottky groups G, G' there exists a quasiconformal mapping f on $\widehat{\mathbb{C}}$ such that $f(\Omega(G)) = \Omega(G')$.*

As an immediate consequence, we have the following universality of Teichmüller spaces of Schottky regions.

Corollary 6.1. *For any $g, h > 1$, the Teichmüller space of a Schottky region of genus g and the Teichmüller space of a Schottky region of genus h are the same.*

Proof of Theorem 6.1. Let P be a pair of pants bounded by three hyperbolic geodesics $\alpha_1, \alpha_2, \alpha_3$ of length 1. We make infinite copies $\{P_n\}_{n\in\mathbb{Z}}$ of P and construct a Riemann surface X_∞ as follows (see also Figure 3).

Let $\alpha_{1,n}, \alpha_{2,n}$, and $\alpha_{3,n}$ be boundary curves of P_n corresponding to α_1, α_2, and α_3, respectively. First, we put $X_1 = P_1$, which is the surface of the 1st generation. We glue P_1 and P_2 by identifying $\alpha_{2,1}$ and $\alpha_{1,2}$. We also glue P_1 and P_3 by identifying $\alpha_{3,1}$ and $\alpha_{1,3}$. The resulting surface denoted by X_2 is the surface of the 2nd generation, which is bounded by 5 geodesics, $\alpha_{1,1}, \alpha_{2,2}, \alpha_{3,2}, \alpha_{2,3}$, and $\alpha_{3,3}$. Inductively, we make X_{k+1} from X_k ($k \in \mathbb{N}$) by attaching copies of P along all boundary curves of X_k except $\alpha_{1.1}$. Symmetrically, we make X_{-k} for $k \in \mathbb{N}$ (see Figure 3).

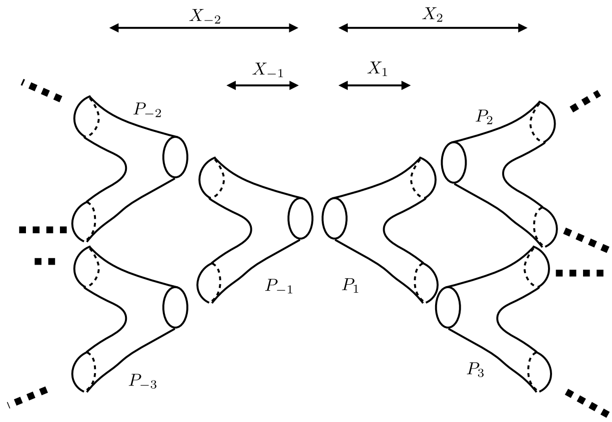

FIGURE 3

We obtain the Riemann surface X_∞ by identifying $\alpha_{1,1} \subset \partial \bigcup_{k\in\mathbb{N}} X_k$ and $\alpha_{1,-1} \subset \partial \bigcup_{k\in\mathbb{N}} X_{-k}$. Then, both X_k and X_{-k} are subsurfaces of X_∞ bounded by $2^k + 1$ geodesics of length 1. X_k is made by $P_1, P_2, \ldots, P_{2^k-1}$ and X_{-k} is by $P_{-1}, \ldots, P_{-2^k+1}$.

Let G be a Schottky group of genus $g > 1$. We show that $\Omega(G)$ is quasiconformally equivalent to X_∞.

From the definition of Schottky groups, there are mutually disjoint $2g$ Jordan curves $C_1, C_2, \ldots, C_{2g}$ in $\widehat{\mathbb{C}}$ such that the outside of them, which is denoted by F_g, is a fundamental domain for G. The group G is a free group of rank g generated by $\gamma_1, \ldots, \gamma_g$ and each γ_j maps the inside of C_{2j-1} onto the outside of C_{2j} ($j = 1, \ldots, g$). Thus, $\Omega(G)$ is constructed from infinite copies of F_g by gluing their boundary curves according to those correspondences (see Figure 4 for $g = 3$). The correspondence gives a regular exhaustion of $\Omega(G)$

$$F_g = W_0 \subset W_1 \subset \cdots \subset W_n \subset W_{n+1} \subset \cdots \subset \Omega(G).$$

The precise construction is the following.

We start at $W_0 := F_g$. It is a region bounded by $2g$ simple closed curves $C_1, C_2, \ldots, C_{2g}$. We put

$$W_1 = \mathrm{Int}\left(\overline{W_0} \cup \bigcup_{j=1}^{g} \gamma_j^{\pm 1}(\overline{F_g})\right).$$

W_1 is a region bounded by $2g(2g-1)$ simple closed curves.

Inductively, we make

$$W_n := \mathrm{Int}\left(\overline{W_{n-1}} \cup \bigcup_{\gamma\in S_n} \gamma(\overline{F_g})\right),$$

where $S_n \subset G$ is the set of $\gamma \in G$ whose word lengths with respect to $\gamma_1^{\pm 1}, \ldots, \gamma_g^{\pm 1}$ are precisely n. For each component c of ∂W_{n-1}, there exist a unique $\gamma \in S_n$ and a unique $C \in \{C_1, \ldots, C_{2g}\}$ such that $c = \gamma(C)$. Thus, W_n is a region bounded by $2g(2g-1)^n$ simple closed curves coming from $C_1, \ldots, C_{2g}$. We also see that the region W_n consists of $N(g) := 1 + \sum_{k=0}^{n-1} 2g(2g-1)^k$ copies of F_g.

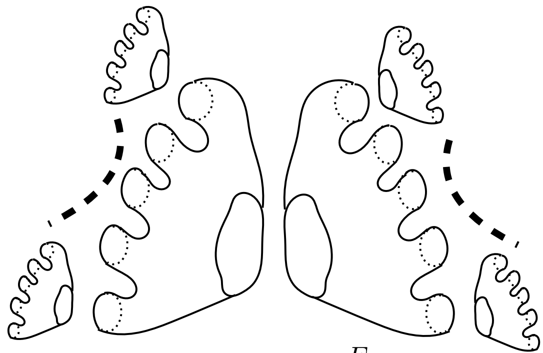

FIGURE 4

Next, we make a regular exhaustion of X_∞ to give a quasiconformal mapping from X_∞ onto $\Omega(G)$.

Let $k \in \mathbb{N} \cup \{0\}$ with $2^k < 2g-1 < 2^{k+1}$. In the above construction of X_∞, we consider a subsurface of X_∞ made by $P_1, \dots, P_{2g-1}$ and denote it by $\mathcal{F}_g$. We see that $X_k \subset \mathcal{F}_g \subset X_{k+1}$ and $\mathcal{F}_g$ is bounded by $2g$ closed geodesics. Since both $\mathcal{F}_g$ and F_g are Riemann surfaces of genus 0 bounded by $2g$ simple closed curves, there exists a quasiconformal mapping F from $\mathcal{F}_g$ onto F_g. The quasiconformal mapping F yields a correspondence between the set of boundary curves of $\mathcal{F}_g$ and that of F_g. We put $\mathcal{C}_j = F^{-1}(C_j)$ $(j = 1, \dots, 2g)$.

By using this correspondence between C_j and $\mathcal{C}_j$ together with the configuration of $\{W_n\}_{n=0}^\infty$ by copies of F_g, we construct a regular exhaustion of X_∞,

$$\mathcal{F}_g = \widetilde{W_0} \subset \widetilde{W_1} \cdots \subset \widetilde{W_n} \subset \widetilde{W_{n+1}} \subset \cdots \subset X_\infty.$$

Because of those constructions of the exhaustions, the quasiconformal mapping $F : \mathcal{F}_g \to F_g$ gives a quasiconformal mapping $\widetilde{F}$ from $X_\infty \setminus \bigcup_{n\in\mathbb{N}} \partial\widetilde{W_n}$ onto $\Omega(G) \setminus \bigcup_{n\in\mathbb{N}} \partial W_n$. Noting that there are finitely many boundary behaviors of $\widetilde{F}$ near $\bigcup_{n\in\mathbb{N}} \partial\widetilde{W_n}$, we see from Lemma 4.1 that $\Omega(G)$ and X_∞ are quasiconformally equivalent.

Let G' be another Schottky group. Using the same argument as above for G, we may show that $\Omega(G')$ is quasiconformally equivalent to X_∞. Hence, we conclude that $\Omega(G)$ and $\Omega(G')$ are quasiconformally equivalent. As we have already noted (cf. [13]), every quasiconformal mapping on $\Omega(G)$ is extended to a quasiconformal mapping on $\widehat{\mathbb{C}}$. Thus, we have a quasiconformal mapping f on $\widehat{\mathbb{C}}$ with $f(\Omega(G)) = \Omega(G')$, as desired. □

The universal Schottky space.

Let $\mathcal{C}$ be the standard middle $\frac{1}{3}$-Cantor set for $[-1, 1]$. It is obtained by removing the middle one thirds open intervals from $[-1, 1]$ successively. Let us recall the precise construction.

First, we remove an open interval J_1 of length $2/3$ from $E_0 := I = [-1, 1]$ so that $I \setminus J_1$ consists of two closed intervals I_1^{-1}, I_1^1 of the same length, where $I_1^{-1} \subset \mathbb{R}_{<0}$ and $I_1^1 \subset \mathbb{R}_{>0}$. We put $E_1 = I_1^{-1} \cup I_1^1$. We remove an open interval of length $\frac{1}{3}|I_1^i|$

from each $I_1^{\pm 1}$ so that the remainder E_2 consists of four closed intervals of the same length, where $|J|$ is the length of an interval J. Inductively, we define E_{k+1} from $E_k = \bigcup_{i=-2^{k-1}}^{-1} I_k^i \cup \bigcup_{i=1}^{2^{k-1}} I_k^i$ by removing an open interval of length $\frac{1}{3}|I_k^i|$ from each closed interval I_k^i of E_k so that E_{k+1} consists of 2^{k+1} closed intervals of the same length. The Cantor set $\mathcal{C}$ is defined by

$$\mathcal{C} = \bigcap_{k=1}^{\infty} E_k.$$

We put $\widehat{X} := \widehat{\mathbb{C}} \setminus \mathcal{C}$. We denote the Teichmüller space $\mathscr{T}(\mathcal{C})$ of $\mathcal{C}$ by $\mathscr{S}$. Then, we insist on the following.

Theorem 6.2. *For any $g > 1$, there exists a holomorphic injection*

$$\iota_g : \mathscr{S}_g \hookrightarrow \mathscr{S} \tag{6.1}$$

similar to (2.1).

Proof. We take a pants decomposition $\{\mathcal{P}_n\}_{n\in\mathbb{Z}}$ of the Riemann surface $\widehat{X} := \widehat{\mathbb{C}} \setminus \mathcal{C}$ as follows.

We denote the imaginary axis by C_0^0. For any (k, i) $(k \in \mathbb{Z} \setminus \{0\}; i = \pm 1, \cdots \pm 2^{k-1})$, we take a circle C_k^{i-1} which is a circle centered at the midpoint of I_k^i with radius $\frac{5}{6}|I_k^i|$. We see that all C_k^i's are mutually disjoint curves in $\widehat{X}$ and each C_k^i contains $C_{k+1}^{\varepsilon(i)(2|i|-1)}, C_{k+1}^{2i}$, where $\varepsilon(i) = -1$ if $i < 0$ and $\varepsilon(i) = 1$ if $i > 0$. Hence, they make a pants decomposition of $\widehat{X}$. A pair of pants bounded by C_0^0, C_1^1 (resp., C_1^{-1}) and C_1^2 (resp., C_1^{-2}) is denoted by $\mathcal{P}_1$ (resp., $\mathcal{P}_{-1}$). We also denote by $\mathcal{P}_{\varepsilon(i)(2^k+(i-1))}$ a pair of pants bounded by C_k^i, $C_{k+1}^{\varepsilon(i)(2|i|-1)}$ and C_{k+1}^{2i}. Obviously, for every n with $|n| \geq 2$, $\mathcal{P}_n$ is conformally equivalent to $\mathcal{P}_2$.

Because of the construction of $\{\mathcal{P}_n\}_{n\in\mathbb{Z}}$, the configuration of the pants decomposition $\{\mathcal{P}_n\}_{n\in\mathbb{Z}}$ of $\widehat{X}$ is exactly the same as that of the Riemann surface X_∞ of the proof of Theorem 6.1. It is also seen that each $\mathcal{P}_n$ is quasiconformally equivalent to P_n. From Lemma 4.1, we see that the Riemann surface X_∞ is quasiconformally equivalent to $\widehat{X}$.

Let G_g be a Schottky group of genus g and let $\Omega(G_g)$ be the region of discontinuity of G_g. From Theorem 6.1 and the above argument, we see that there exists a quasiconformal mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $f(\widehat{X}) = \Omega(G_g)$. For each quasiconformal deformation h of G_g, $H_h := h \circ f$ is a quasiconformal deformation of the Riemann surface $\widehat{X}$. It is obvious that h_1 and h_2 are equivalent as quasiconformal deformations of G_g if and only if H_{h_1} and H_{h_2} are equivalent as quasiconformal deformations of $\widehat{X}$. Thus, we have a well-defined map $\iota_g : \mathscr{S}_g \to \mathscr{S}$. The injectivity of the map follows from the definitions of $\mathscr{S}_g$ and $\mathscr{S}$.

The complex structure of $\mathscr{S}_g$ is defined by that of the space of Beltrami differentials. It is the same for the complex structure of $\mathscr{S}$. Hence, the map ι_g is holomorphic. □

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Department of Mathematics, Tokyo Institute of Technology, O-okayama, Meguro-ku Tokyo, Japan

Current address: Department of Mathematics, Kyoto Sangyo University, Motoyama, Kamigamo, Kita-ku, Kyoto, Japan

Email address: shiga@cc.kyoto-su.ac.jp