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ON HYPERBOLIC COBORDISMS AND HURWITZ CLASSES OF HOLOMORPHIC COVERINGS

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Abstract. In this article we show that for every collection $\mathcal{C}$ of an even number of polynomials, all of the same degree $d > 2$ and in general position, there exist two hyperbolic 3-orbifolds M_1 and M_2 with a Möbius morphism $\alpha : M_1 \to M_2$ such that the restriction of α to the boundaries ∂M_1 and ∂M_2 forms a collection of maps Q in the same conformal Hurwitz class of the initial collection $\mathcal{C}$. Also, we discuss the relationship between conformal Hurwitz classes of rational maps and classes of continuous isomorphisms of sandwich products on the set of rational maps.

1. Introduction

Cobordism theory has been studied widely since it was introduced by H. Poincaré in the context of homology theory. Also R. Thom studied cobordism of embeddings. Since then there has been an interest in cobordism of functions, for instance functions with stable singularities. Cobordims can be endowed with geometric structures such as symplectic structures, flat connections, or complex structures.

For example, start with a pair of Kleinian groups Γ_1 and Γ_2 such that Γ_1 is a subgroup of finite index in Γ_2. The inclusion map generates a Möbius morphism $\alpha : M(\Gamma_1) \to M(\Gamma_2)$ which is a finite degree orbifold covering. Since $\partial M(\Gamma_1)$ may be disconnected, the restriction $f := \alpha|_{\partial M(\Gamma_1)}$ forms a collection of finite degree holomorphic coverings from the components of $\partial M(\Gamma_1)$ to the components of $\partial M(\Gamma_2)$. In this situation, it is natural to say that the collection f forms a hyperbolic cobordism.

With this point of view we avoid the homological language and will be interested in the following inverse problem.

Given a collection Q of holomorphic finitely degree (orbifold) coverings, does there exists a pair of Kleinian groups and a Möbius morphism α which is conformally equivalent to the collection Q?

Another motivation to this question is the relational dictionary between rational maps and Kleinian groups. For this reason, the collection of maps will be often taken as a collection of rational endomorphisms of the Riemann sphere. The main results of this article are Theorems 1 and 3 below; these are proven in Sections 3, 4, and 5.

In the last section, we characterize algebraically when two rational maps define the same conformal Hurwitz class. Also we briefly remark that the Hurwitz class of a rational map R can be presented as a space of quasiconformal deformations of a semigroup of holomorphic correspondences and discuss the related questions.

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From now on our surfaces are compact surfaces with finitely many punctures which admit a hyperbolic (orbifold) structure of finite-type. However, some of our results can be extended to the case of infinite-type.

We start with the following definition.

Definition. A branched covering of finite degree d is a triplet (R, S, S') where S and S' are finite collections of Riemann surfaces and $R : S \to S'$ is a continuous surjective mapping so that

(1) If $Y \subset S$ is a component, then $Z = R(Y) \subset S'$ is also a component and the restriction $R : Y \to Z$ is a degree $d_Y \leq d$ branched covering map.
(2) There is a component $\tilde{Y}$ such that $d_{\tilde{Y}} = d$.

We say that a branched covering R is **simple** whenever the number of components of S coincides with the number of components of S'. If R is simple and S is connected, then we say that R is a **single** branched covering.

Branched coverings between Riemann surfaces have been studied widely in the literature. We are interested in the following basic examples.

(1) Rational maps, these are single branched self-coverings of the sphere.
(2) Let Γ be a Kleinian group and let $G < \Gamma$ be a subgroup with $\Omega(G) = \Omega(\Gamma)$; then the natural projection $R : S(G) \to S(\Gamma)$ is a holomorphic branched covering map. When $\Omega(G) \neq \Omega(\Gamma)$, in general, the inclusion map does not induce a holomorphic covering. Here $\Omega(\Gamma)$ denotes the discontinuity set of Γ.

    It is known that the equality $\Omega(G) = \Omega(\Gamma)$ holds when either G is a subgroup of finite index or is a non-elementary normal subgroup. However, there are examples of non-elementary groups G with $\Omega(G) = \Omega(\Gamma)$ but such that G is not normal and has infinite index in Γ. If G and Γ are finitely generated and the limit set of Γ is not a subset of a round circle, then by the Ahlfors finiteness theorem G necessarily has finite index in Γ.

    Even more, as pointed out by the referee, again by the Ahlfors finiteness theorem $G < \Gamma$ has finite index whenever Γ has an invariant component in $\Omega(\Gamma) = \Omega(G)$ for finitely generated groups G and Γ.

Any branched covering can be regarded as a collection of single coverings; we call each of them a *single component* of the branched covering. We say that a branched covering R is a holomorphic covering whenever every single component is a holomorphic (orbifold) unbranched covering between hyperbolic surfaces (orbifolds).

Given a holomorphic covering (R, S, S'), we can improve (R, S, S') into a simple covering in the following way: if X and Y are components of S so that $R(X) = R(Y) = Z \subset S'$, then consider a conformal copy Z' of Z. Let $R' : X \to Z'$ be the respective holomorphic covering; now the holomorphic covering $X \sqcup Y \to R'(X) \sqcup R(Y)$ is simple. By induction on the number of components we construct a simple holomorphic covering (Q, T, T') such that for every single component $R : X \to Z$ of (R, S, S') there exists a single component $Q : X' \to Z'$ of (Q, T, T') and two conformal homeomorphisms $\phi : Z \to Z'$ and $\psi : X \to X'$ so that $\phi \circ R = Q \circ \psi$. The previous discussion also motivates the following definition of Hurwitz classes for non-connected branched coverings.

Definition. Let $f : S \to S'$ be a branched covering, let the **Hurwitz class** $H(f)$ of f consist of the triples (g, N, N') so that $g : N \to N'$ is a branched covering, and let there exist orientation preserving homeomorphisms $\varphi : S \to N$ and $\psi : S' \to N'$ such that $\psi \circ f = g \circ \varphi$.

If $f : S \to S'$ is a single branched covering, then $H(f)$ coincides with the classical Hurwitz space of f. If f is a simple branched covering, then

$$H(f) = \bigotimes_{Y} H(f, Y, f(Y)),$$

where the product is taken over the connected components Y of S.

Given a holomorphic map f, the set

$$CH(f) = \{(g, N, N') \in H(f), \phi, \psi \text{ conformal}\}$$

is called the *conformal Hurwitz class* of the holomorphic covering f.

For example, if $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a finite degree branched covering, then $H(f)$ contains a rational map of the same degree. When f is a rational map the set $H(f) \cap \mathrm{Rat}(\mathbb{C}) = S(f)$ is known as the *Speisser class* of f and was introduced into holomorphic dynamics by A. Erëmenko and M. Lyubich in [4]. By Teichmüller's theorem, if f is holomorphic of finite degree and if $g \in H(f)$ is holomorphic, then the associated homeomorphisms ϕ and ψ can be taken quasiconformal. For general holomorphic maps, this is not true. It is not even clear whether ϕ and ψ can be taken to be local quasiconformal maps for an infinite degree holomorphic map f. A plausible counterexample is an entire map f such that the set of singularities of f^{-1} has positive Lebesgue measure.

The *hyperbolic cobordism* between two holomorphic coverings (R_1, S_1, S_1') and (R_2, S_2, S_2') is given by the triple $(\Re, M, M')$ satisfying the following conditions:

(1) There are geometrically finite Kleinian groups $\Gamma, \Gamma' < PSL(2, \mathbb{C})$ such that

$$M = M(\Gamma) = (B \bigcup \Omega(\Gamma))/\Gamma$$

and

$$M' = M(\Gamma') = (B \bigcup \Omega(\Gamma'))/\Gamma'.$$

Hence, M and M' are oriented hyperbolic 3-orbifolds with natural projections $\pi : B \bigcup \Omega(\Gamma) \to M$, and $\pi' : B \bigcup \Omega(\Gamma') \to M'$. The map $\Re : M \to M'$ is a *surjective Möbius morphism*, that is, there exists an orientation preserving Möbius map α such that the following diagram commutes:

$$\begin{array}{ccc} B \bigcup \Omega(\Gamma) & \xrightarrow{\alpha} & B \bigcup \Omega(\Gamma') \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{\Re} & M'. \end{array} \qquad (1)$$

(2) The boundary ∂M is conformally equivalent to $\bigsqcup S_i$ and $\partial M'$ conformally equivalent to $\bigsqcup S_i'$, so that

$$(\Re|_{\partial M}, \partial M, \partial M') \in \bigotimes_{i=1}^{2} CH(R_i, S_i, S_i').$$

Hence $\Re$ is a local isometry between the respective hyperbolic metrics on $M(\Gamma)$ and $M(\Gamma')$ induced by the Kleinian groups Γ and Γ', respectively.

Given two holomorphic coverings R_1 and R_2, if there exists a hyperbolic cobordism between R_1 and R_2, we will say that $R_1 \sqcup R_2$ forms a *cobordant family of holomorphic coverings*, or that R_1 is *hyperbolically cobordant* to R_2.

Given a finite degree holomorphic branched covering $R : M \to N$, between Riemann surfaces M and N, there are many ways to transform R into a holomorphic covering between hyperbolic orbifold structures supported on M and N.

We consider the simplest construction depending on the ramification data of R and a finite subset $A \subset N$ as follows: first restrict R to $R : \{S = M \setminus R^{-1}(A)\} \to \{S' = N \setminus A\}$. Second, using the ramification data of R produce orbifold structures on S and S' so that R is a holomorphic (orbifold) covering between hyperbolic orbifold structures supported on S and S', respectively.

In particular, if $A = \emptyset$, then the canonical orbifold structure on M and N defined by the ramification data of R must be hyperbolic. For instance, in the case where $R(z) = z^n$, the set A must be non-empty and $card(A \setminus \{0, \infty\}) \geq 1$ for $n \geq 3$.

If $A = V(R)$ is the set of critical values of R, and the surfaces S and S' are hyperbolic, then the triple (R, S, S') is called the *canonical holomorphic representative* of the holomorphic branched covering $R : M \to N$.

Examples.

(1) The *null cobordism* where S and S' are connected is related to the extension of a single holomorphic covering to the respective 3-hyperbolic spaces. This situation has been studied in [3] with applications to holomorphic dynamical systems. In particular, in [3] the authors gave the construction of a geometric extension for generic rational maps. The present article develops the geometrical part of [3] in the case of a collection of holomorphic coverings.

(2) *The trivial cobordisms.* Consider the identity maps $\mathrm{Id}_i : S_i \to S_i$, where S_i is a Riemann surface for $i = 1, 2$. Then the existence of a cobordism between Id_1 and Id_2 reduces to the existence of a hyperbolic manifold with boundary conformally equivalent to $S_1 \sqcup S_2$. Then we have:
   - If S_1 is quasiconformally equivalent to S_2, then by the Bers simultaneous uniformization theorem there exists a quasifuchsian group uniformizing the surfaces S_1 and S_2, so that $S_1 \sqcup S_2$ is conformally equivalent to the boundary of a hyperbolic 3-manifold. The quasifuchsian group can be chosen a fuchsian group if there is an anticonformal isomorphism between S_1 and S_2.
   - For any surface S_1 consisting of a finite number of hyperbolic components, there exists a connected hyperbolic surface S_2 such that $S_1 \sqcup S_2$ can be uniformized by a geometrically finite function group. This uniformization is given by the Klein-Maskit combination theorems in such a way that $S_1 \sqcup S_2$ bounds an oriented hyperbolic 3-orbifold. This observation will be needed in the proof of Theorem 1.
   - Whenever S is a compact hyperbolic closed connected Riemann surface with an even number of cusps, there exists a Schottky-type group uniformizing S so that S is conformally equivalent to the boundary of a 3-hyperbolic manifold.

Connected transitivity. Given three single holomorphic coverings R_1, R_2, and R_3 such that the pairs (R_1, R_2) and (R_2, R_3) are each hyperbolically cobordant by manifolds M_1 and M_2, and assuming that the canonical homorphisms $\pi_1(S_i) \to \pi_1(M_i)$ and $\pi_1(S'_i) \to \pi_1(M'_i)$ are injective, then R_1 is hyperbolically cobordant to R_3. In fact, this follows from Thurston's hyperbolization theorem.

In general, without the single and injective assumptions, it is not clear that the manifold, resulting by gluing M_1 and M_2 along the boundary components associated to R_2, is hyperbolic. This is because the result of gluing hyperbolic manifolds along the boundary may not be hyperbolic. For instance, consider a geometrically finite hyperbolic 3-manifold M which has an essentially embedded annulus A. Let S_1 and S_2 be not necessarily different components of ∂M containing the boundary of A. Take a copy of M, say M', and make $V = M \sqcup_{S_1 \sqcup S_2} M'$ by gluing M and M' along $S_1 \sqcup S_2$. Then V does not accept a hyperbolic metric since V contains a torus which is not homotopic to the ideal boundary of V. This indicates that there might be obstacles to the existence of a hyperbolic cobordism between multiple coverings.

Now we formulate our first main theorem.

Theorem 1. *Given a simple holomorphic covering F_1, there exists another holomorphic covering F_2 such that $F_1 \sqcup F_2$ forms a family of cobordant holomorphic coverings.*

Moreover, if F_1 has a single component R_0 with degree $\deg(R_0) > 1$, then the covering F_2 contains only one single component, say Q_0, with degree larger than 1 and $\deg(Q_0) = \sum_{i=0}^{n} \deg(R_i) - n$ where R_i are single components of F_1 and $n+1$ is the number of these components.

We need the following definition.

Definition. A holomorphic covering $Q : S \to S'$ is called an anticonformal copy of the holomorphic covering map $R : T \to W$ if there are anticonformal homeomorphisms $\alpha : S \to T$ and $\beta : S' \to W$ so that $\beta \circ Q = R \circ \alpha$. Given a holomorphic covering R, we call the Hurwitz class $H(R)$ **symmetric** if and only if $H(R)$ contains an anticonformal copy of an element $g \in H(R)$. Finally, we say that a holomorphic covering is symmetric if its Hurwitz class is symmetric.

Let us note that if $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a branched covering, then $H(f)$ is symmetric whenever $H(f)$ contains a real rational map, that is, all coefficients are real. In particular, if B is a Blaschke endomorphism, then $H(B)$ is symmetric. Moreover, if $\bigsqcup_B H(B)$ is the union of all Hurwitz classes of Blaschke endomorphisms and $g \in \bigsqcup_B H(B)$ then $H(g)$ is symmetric. Also for every natural number d, by the Theorem 3.4 in [1], the set $\bigsqcup_B H(B) \cap \mathrm{Rat}_d(\mathbb{C})$ is connected and contains an open and everywhere dense subset of the space of $\mathrm{Rat}_d(\mathbb{C})$. Here $\mathrm{Rat}_d(\mathbb{C})$ denotes the set of rational maps of degree d. A Blaschke endomorphism is a rational map B with $B^{-1}(\mathbb{D}) = \mathbb{D}$ where $\mathbb{D}$ is the open unit disk in $\mathbb{C}$.

In general it is not clear that the Hurwitz class of any finite degree branched covering between closed Riemann surfaces is symmetric. But we believe that is true for Hurwitz classes of rational maps.

Definition. Two cobordant holomorphic coverings R_1 and R_2 are called **simply cobordant** if and only if M and M' are homeomorphic to $S_1 \times [0,1]$ and $S'_1 \times [0,1]$, respectively.

Theorem 2. *Two symmetric single holomorphic coverings R_1 and R_2 belong to the same Hurwitz class if and only if R_1 and R_2 are simply cobordant.*

Theorem 1 shows that any finite family of coverings can be included in a family of cobordant coverings which is non-simple and includes single univalent components on the boundary. Theorem 2 gives a condition for when a pair of holomorphic coverings of the Riemann sphere is cobordant.

With Theorem 2 at hand we improve Theorem 1 into Theorem 3. First recall that a holomorphic polynomial map $P : \mathbb{C} \to \mathbb{C}$ of degree $d > 1$ is in general position if there are $d - 1$ different finite critical values $V(P)$. It is known that two polynomial maps in general position belong to the same Hurwitz class if and only if these polynomials have the same degree. Also, every polynomial in general position is symmetric.

Theorem 3. *The canonical holomorphic representatives of every collection of an even number of polynomials in general position of the same degree $d > 2$ form a hyperbolic cobordism.*

2. Some background on Kleinian groups

For the convenience of the reader here we collect some facts from Kleinian group theory which will be used in this article. We follow the books of M. Kapovich [5] and A. Marden [6] which give a modern introduction to Kleinian groups.

Denote by B the Poincaré model of the hyperbolic 3-space, that is, the unit ball in $\mathbb{R}^3$ equipped with the Poincaré metric. Given a group Γ of automorphisms of the Riemann sphere, we denote by $\Omega(\Gamma)$ the discontinuity set of Γ on $\overline{\mathbb{C}}$. The isometry group of B acts on the Riemann sphere $\overline{\mathbb{C}} = \partial B$ as the whole group of Möbius transformations $\mathrm{Mob}(\mathbb{C})$ including anticonformal automorphisms. A discrete subgroup Γ of $\mathrm{Isom}(B)$ is a Kleinian group if $\Omega(\Gamma) \neq \emptyset$. Historically, a Kleinian group is defined as a subgroup of orientation preserving isometries of B, but we need the extended definition in order to apply Brook's deformation theorem. Also we follow the definition in [5] where it is shown that many classical theorems for orientation preserving Kleinian groups extend to the general case without many difficulties.

Define $S(\Gamma) = \Omega(\Gamma)/\Gamma$ and $M(\Gamma) = (B \sqcup \Omega(\Gamma))/\Gamma$ and note that $S(\Gamma) = \partial M(\Gamma)$.

Both spaces $S(\Gamma)$ and $M(\Gamma)$ can be endowed with a hyperbolic orbifold structure. For an orbifold O, let $|O|$ be the underlying space of O. When Γ contains orientation reversing elements, one has to be cautious with the fact that $|S(\Gamma)|$ is a proper subset of $\partial|M(\Gamma)|$. The points in $\partial|M(\Gamma)| \setminus |S(\Gamma)|$ are interior points in the orbifold structure contained in the singular locus. In other words, neighborhoods of these points are modeled by the quotient of a ball by the action of a finite group of isometries of B. The simplest example to have in mind is the space X which is the quotient of $\mathbb{C}$ by the map $z \mapsto \overline{z}$. Then X admits the structure of a manifold with boundary homeomorphic to the closure of the upper half-space. Alternatively, X also possesses the structure of an orbifold without boundary where the real line is the singular locus of the orbifold.

Definition. A 3-manifold M is called geometrically finite if there exists a compact submanifold with boundary M_0 such that $M \setminus M_0$ is a disjoint union of finitely many pieces V_i such that either

- V_i is homeomorphic to $\mathbb{S}^1 \times (\mathbb{D} \setminus \{0\})$, where $\mathbb{D}$ is the open unit disk in $\mathbb{C}$ and $\mathbb{S}^1$ is the unit circle in $\mathbb{C}$; or
- V_i is homeomorphic to $[0,1] \times (\mathbb{D} \setminus \{0\})$ so that the punctured disks $\{0\} \times (\mathbb{D} \setminus \{0\})$ and $\{1\} \times (\mathbb{D} \setminus \{0\})$ belong to ∂M.

A Kleinian group Γ is called geometrically finite if and only if it contains a finite index subgroup Γ_0 such that $M(\Gamma_0)$ is geometrically finite.

The pieces V_i in the definition are usually known as solid cusp torii and solid pairing tubes, respectively. There are many equivalent definitions of geometrically finite Kleinian groups; see for example [5] and [6].

2.1. **Pinching.** In what follows we describe the pinching procedure for a finite family of disjoint simple closed geodesics on a Riemann surface S contained in $S(\Gamma)$ for a geometrically finite Fuchsian group $\Gamma < PSL(2,\mathbb{C})$, according to theorems of B. Maskit [7] and K. Ohshika [10]; see also the pinching theorem of Section 5.15 in [6].

For $r > 0$ let $A_r = \{z : \frac{1}{r} < |z| < r\}$ be the round symmetric annulus and consider the homeomorphism of the plane $F(z) = z|z|$; note that $F(z)$ is quasiconformal. Take the sequence μ_n of Beltrami differentials on A_r defined by $\mu_n = \frac{\overline{\partial} F^n}{\partial F^n}|_{A_r}$ where F^n is the nth iterated of F and the partial derivatives are taken in the sense of distributions. Then $\|\mu_n\| \to 1$ as $n \to \infty$.

Let l_i be a finite collection of disjoint simple closed geodesics in S; then by the collar lemma there exists r_0 and a family of conformal embeddings $h_i : A_{r_0} \to S(\Gamma)$ with $h_i(\mathbb{S}^1) = l_i$ and the closed sets $\overline{h(A_{r_0})}$ are mutually disjoint. By taking the simultaneous push-forward of μ_n by the maps h_i, we obtain a sequence $\tilde{\nu}_n$ of Beltrami differentials on S supported on the union of the annular neighborhoods $h_i(A_{r_0})$. Now lift the sequence $\tilde{\nu}_n$ over $\Omega(\Gamma)$ by the natural projection $\Omega(\Gamma) \to S(\Gamma)$ to get a sequence ν_n of Beltrami differentials in $\Omega(\Gamma)$ with $\|\nu_n\| \to 1$. If f_n is a solution of the Beltrami equation with coefficient ν_n, then the group $\Gamma_n = f_n \circ \Gamma \circ f_n^{-1}$ in $PSL(2,\mathbb{C})$ is quasifucshian. In case that Γ acts on $\mathbb{D}$ and $S = \mathbb{D}/\Gamma$ then all the maps f_n are holomorphic outside $\overline{\mathbb{D}}$.

Then the following theorem is true.

Theorem 4. *Let Γ_n be a family of quasifuchsian groups as constructed above. After taking a suitable subsequence there exists a geometrically finite Kleinian group $\Gamma_\infty = \lim \Gamma_{n_k}$ in the topology of convergence on generators so that*

- $\Gamma_\infty \simeq \Gamma$.
- *The interior of $M(\Gamma_\infty)$ is homeomorphic to the interior of $M(\Gamma)$.*
- *The surface $S(\Gamma_\infty)$ is homeomorphic to $S(\Gamma) \setminus (\bigcup l_i)$. Even more, the homeomorphism can be chosen to be holomorphic outside $\bigcup h_i(A_{r_0})$ and each l_i determines a pair of punctures in $S(\Gamma_\infty)$,*

The previous theorem is proved by Maskit for function groups, with the condition that the closed simple curves $l \in S(\Gamma)$ to be pinched must have loxodromic representatives in the group, which represent different conjugacy classes. The last condition always holds for simple closed geodesics which belong to the same connected component of $S(\Gamma)$ for a given quasifuchsian group Γ. In [10], Ohshika extends the theorem of Maskit to all geometrically finite groups. Our version follows the exposition of Marden in the pinching theorem of Section 5.15 of [6].

Klein-Maskit combination theorem. We will need the following theorem in the proof of Theorem 1. See [7].

Theorem 5 (Klein-Maskit's combination Theorem I [8].)**.** *For $i = 1, 2$, let Γ_i be a Kleinian group with region of discontinuity Ω_i and a fundamental region F_i. Assume that there is a simple closed loop γ contained in the interior of $F_1 \cap F_2$, bounding two complemented disks D_1 and D_2 with $\overline{D}_i \subset F_i$. Then $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is a Kleinian group, such that:*

(1) *The group Γ is isomorphic to the free product $\Gamma_1 * \Gamma_2$. If Γ_1 and Γ_2 are geometrically finite, then Γ is so.*
(2) *Let $S_i = K_i / \mathrm{Stab}(K_i)$ be surfaces where $K_i \subset \Omega(\Gamma_i)$ are the components containing γ and $\mathrm{Stab}(K_i) < \Gamma_i$ are their respective stabilizers. Then $S(\Gamma)$ is homeomorphic to*

$$(S(\Gamma_1) \setminus S_1) \sqcup (S(\Gamma_2) \setminus S_2) \sqcup (S_1 \# S_2),$$

where $S_1 \# S_2$ is the connected sum of the surfaces S_1 and S_2 along the respective projections of D_1 and D_2.

Even more, this homeomorphism can be chosen holomorphic on $S(\Gamma) \setminus S_1 \# S_2$.
(3) *The manifold $M(\Gamma)$ is homeomorphic to the disk sum $M(\Gamma_1)$ with $M(\Gamma_2)$ induced by the disks determining the connected sum $S_1 \# S_2$.*

Disk patterns and Brook's deformation theorem. The following construction is needed in the proof of Theorem 3.

Definition. Let Γ be a geometrically finite torsion-free Kleinian group. Then a collection K of closed sets $K_i \subset S(\Gamma)$ is called a **round disk collection** if and only if the set K consists of finitely many elements and every element K_i is either a homeomorphic projection of a compact round disk $D \subset \Omega(\Gamma)$ to $S(\Gamma)$ or is a closed punctured disk in $S(\Gamma)$ where the puncture corresponds to a cusp of $S(\Gamma)$ and K_i is covered by a round disk $D \setminus \{p\} \subset \Omega(\Gamma)$, where D is precisely invariant under its parabolic stabilizer $\gamma \in \Gamma$ and p is the fixed point of γ with $p \in \partial D$.

Definition. A finite round disk collection $K \subset S(\Gamma)$ is called a **pattern of round disks** if and only if the following holds:

(1) No point in $S(\Gamma)$ is covered by the interior of more than two disks in K.
(2) Given two different disks K_i and K_j, then either the interiors of K_i and K_j are disjoint or their boundaries are orthogonal.

Each disk $K_i \subset K$ is covered by a round disk $D_i \in \overline{\mathbb{C}}$. If $C(D_i)$ is the convex hull of ∂D_i in B with respect to the Poincaré metric, then $C(D_i)$ is invariant under the stabilizer of D_i in Γ. Let $V(D_i)$ be the component of $\overline{B} \setminus C(D_i)$ containing D_i. Let $Y(K_i) \subset M(\Gamma)$ be the projection of $V(D_i)$ in $M(\Gamma)$ and

$$M_K = M(\Gamma) \setminus \bigcup_{K_i \in K} Y(K_i);$$

then on the manifold M_K there exists a natural polyhedral geometric structure $\mathcal{G}$ which on the interior of M_K coincides with the hyperbolic structure of $\mathrm{int}(M(\Gamma))$, on $M(\Gamma) \cap (\bigcup \partial Y(K_i))$ the structure $\mathcal{G}$ is a polyhedral piecewise geodesic structure, and on the remaining boundary components of M_K the structure coincides with the Möbius structure inherited from $M(\Gamma)$.

Let $\tilde{K}$ be the collection of all round disks in $\Omega(\Gamma)$ which cover all $K_i \subset K$. Let $\Gamma_K < \mathrm{Isom}(B)$ be the group generated by Γ and the reflections with respect to the circles ∂D for $D \in \tilde{K}$. Then Theorem 13.1 in [5] states as follows.

Theorem 6. *The group Γ_K is geometrically finite and $M(\Gamma_K)$ is an orbifold diffeomorphic to M_K equipped with the structure $\mathcal{G}$.*

Remarks. (1) In particular, if a component $\Omega_0 \subset \Omega(\Gamma)$ does not intersect $\tilde{K}$, then the stabilizer of Ω_0 in Γ_K coincides with the stabilizer of Ω_0 in Γ. Therefore, if a disk pattern K completely covers exactly one component $S \subset \partial M(\Gamma)$, then $\partial(M(\Gamma_K))$ is a 2-dimensional orbifold which is conformally equivalent to $\partial M(\Gamma) \setminus S$.

(2) The charts around points on $\partial(Y(K_i))$ are modeled with the quotient of the unit 3-dimensional ball by a finite group, this group is generated by reflections on planes passing through the origin. In particular, in this structure the points in $\partial Y(K_i) \setminus K_i$ are interior points of M_K equipped with the structure $\mathcal{G}$.

The following simple example shows how this procedure works. Let S be any Riemann surface and let Γ be a Fuchsian group uniformizing S. Then $M(\Gamma)$ is a hyperbolic manifold homeomorphic to $S \times [0, 1]$, the boundary of $M(\Gamma)$ consists of S and an anticonformal copy of S. Let τ be the reflection with respect to the unit circle, then τ commutes with Γ. Let $G = \langle \Gamma, \tau \rangle$. Thus G is a Kleinian group and $M(G)$ is a non-orientable hyperbolic orbifold, so that $\partial M(G)$ is conformally equivalent to S. The underlying space of the orbifold $M(G)$ is a manifold which still is homeomorphic to $S \times [0, 1]$ but now, one of the components consists of interior points of the orbifold structure on $M(G)$.

The following theorem justifies the existence of a pattern of disks for a quasiconformal deformation of a given geometrically finite group. This theorem is part of the proof of the Brooks orbifold deformation theorem. More precisely, see the steps 1 to 4 in the proof presented in Section 13.5 of [5]. The statement is as follows.

Theorem 7. *For any torsion-free geometrically finite Kleinian group Γ there exists a quasiconformal homeomorphism h such that the group $\Gamma_h = h \circ \Gamma \circ h^{-1}$ is so that $S(\Gamma_h)$ admits a pattern K which completely covers $S(\Gamma_h)$.*

With this background we can proceed to prove our theorems.

3. Proof of Theorem 1

3.1. Connected sums of single coverings.

Now we reproduce a topological operation between branched coverings which is a sort of "connected sum" of coverings. This operation consists of taking the connected sum of the target surfaces and a pull-back with respect to the branched coverings. Note that there are different ways to make a pull-back. We choose the simplest as follows.

Start with two single finite degree branched coverings $R_1 : S \to S'$ and $R_2 : T \to T'$. We construct $R_0 = R_1 \# R_2$, the connected sum of branched covering maps R_1 and R_2, as a branched covering between two surfaces U and W such that $\deg(R_0) = \deg(R_1) + \deg(R_2) - 1$ and $W = T' \# S'$ is the connected sum with respect to topological disks $D_S \subset S'$ and $D_T \subset T'$ not containing the branched points (critical values) of R_1 and R_2, respectively.

Topological construction of U. Set $\deg(R_1) = n$ and $\deg(R_2) = m$. Let $S'' = S' \setminus D_S$ and $T'' = T' \setminus D_T$ and $h : \partial D_S \to \partial D_T$ be a gluing homeomorphism. Let $S_0 = R_1^{-1}(S'') \subset S$ and $T_0 = R_2^{-1}(T'') \subset T$. Let us fix the following system of curves and homeomorphisms.

(1) Take two components of the boundaries, α_0 a component of ∂S_0 and β_0 a component of ∂T_0. Let $\phi_0 : \alpha_0 \to \beta_0$ be a homeomorphism such that $R_2 \circ \phi_0 = h \circ R_1$.

(2) If γ is another component of either ∂S_0 or ∂T_0, and different from α_0 and β_0, then fix a homeomorphism ϕ_γ which is either $h \circ R_1|_\gamma$ or $h^{-1} \circ R_2|_\gamma$, depending on the case. For $i = 1, \ldots, m-1$, let $\{S_i\}$ be $m-1$ copies of S'' and for $j = 1, \ldots, n-1$ let $\{T_i\}$ be a family of $n-1$ copies of T''. Then

$$U = S_0 \sqcup T_0 \sqcup \{\sqcup S_i\} \sqcup \{\sqcup T_i\} / \sim,$$

where the quotient is taken according to the system of homeomorphisms. More precisely, the homeomorphism ϕ_0 identifies α_0 with β_0, and the map ϕ_γ identifies the component γ, which is either in ∂S_0 or in ∂T_0, with the respective copies of T'' or S''. The identification is taken in such a way that U is a connected surface and there exists a branched covering $R_1 \# R_2 : U \to W$ so that $R_1 \# R_2|_{S_0} = R_1 : S_0 \to S''$ and $R_1 \# R_2|_{T_0} = R_2 : T_0 \to T''$. The restriction of $R_1 \# R_2$ on each one of the remaining copies, of either T'' or S'', is univalent.

Then, we have that

$$\operatorname{genus}(U) = \operatorname{genus}(S) + (m-1)\operatorname{genus}(S') + \operatorname{genus}(T) + (n-1)\operatorname{genus}(T').$$

Moreover, punctures and holes satisfy the same equation as the genus.

If R_1 and R_2 are holomorphic branched coverings between Riemann surfaces, then by taking a conformal gluing in the construction of U we can assume that $R_1 \# R_2 : U \to W$ is a holomorphic branched covering. In other words, the Hurwitz class of a topological connected sum between surfaces contains a holomorphic branched covering.

Hence any branched covering of the Riemann sphere, in general position and degree d, can be presented as the connected sum of $d-1$ copies of z^2.

If $R_1(z) = z^2$ and $R_2(z) = z^3$, then $R_1 \# R_2$ is a degree 4 branched self-covering of the topological sphere.

Figure 1 illustrates yet another example; in this case, it shows the connected sum of rational maps in general position of degree 3 and 4.

3.2. **Pinching.** We can recover the factors of the connected sum $R_1 \# R_2$ by a pinching procedure as follows.

Let us note that if $R_1 \# R_2 : S_1 \to S_2$ is a connected sum of coverings with $\deg(R_1), \deg(R_2) \geq 2$, then $R_1 \# R_2 : U \to W$ is a finite degree covering between hyperbolic surfaces where

$$W = S_2 \setminus \operatorname{CritVal}(R_1 \# R_2) \quad \text{and} \quad U = S_1 \setminus (R_1 \# R_2)^{-1}(\operatorname{CritVal}(R_1 \# R_2)).$$

In other words, $R_1 \# R_2$ always has a canonical representative whenever

$$\deg(R_1), \deg(R_2) \geq 2.$$

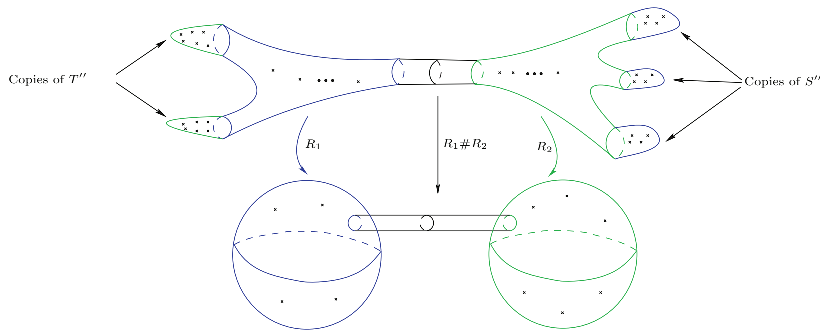

FIGURE 1. The connected sum of rational maps R_1 and R_2 of degree 3 and 4, respectively. The dots depict the respective critical values below and their preimages above.

Let G, Γ_2 be the Fuchsian groups uniformizing the surfaces U and W in the unit disk, respectively. Let $\pi_U, \pi_W : \mathbb{D} \to U, W$ be the respective uniformizing projections; then there exists α a Möbius automorphism of $\mathbb{D}$ satisfying $R_1\#R_2 \circ \pi_U = \pi_W \circ \alpha$ and such that the subgroup $\Gamma_1 = \alpha G \alpha^{-1} < \Gamma_2$ has finite index.

We will say that the fixed pair of groups $\Gamma_1 < \Gamma_2$ and the inclusion map uniformizes $R_1\#R_2$.

On the other hand, the pair $\Gamma_1 < \Gamma_2$ acts on $\mathbb{D}^*$ where $\mathbb{D}^* = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and defines an orbifold covering map $Q : U^* \to W^*$, where $U^* = \mathbb{D}^*/\Gamma_1$ and $W^* = \mathbb{D}^*/\Gamma_2$ are the anticonformal copies of U and W, respectively, and such that Q is an anticonformal copy of $R_1\#R_2 : U \to W$.

Let C be the unique simple closed geodesic in the isotopy class of a simple loop on W providing the connected sum. According to Theorem 4:

- There exists a sequence f_k of quasiconformal automorphisms of the Riemann sphere holomorphic outside the unit disk such that the groups $\Gamma_{2,k} = f_k \circ \Gamma_2 \circ f_k^{-1}$ are quasifuchsian groups converging to a Kleinian group $\Gamma_{2,\infty}$ with an invariant component of $\Omega(\Gamma_{2,\infty})$.
- The surface $S(\Gamma_{2,\infty})$ is homeomorphic to $(W \setminus C) \sqcup W^*$ and C determines a pair of punctures on $S(\Gamma_2, \infty)$. Moreover, the homeomorphism can be chosen conformal outside a tubular neighborhood of C.

In other words $S(\Gamma_{2,\infty})$ is conformally equivalent to

$$(S' \setminus \{x\}) \sqcup (T' \setminus \{y\}) \sqcup W^*,$$

where x and y are the additional cusps determined by C. Indeed the perforations x and y are known as *accidental cusps* (accidental parabolics) which appear in pinching processes. Let $\Gamma_{1,\infty}$ be the respective limit of the groups $f_k \circ \Gamma_1 \circ f_k^{-1}$; then $\Gamma_{1,\infty} \cong \Gamma_1$. Since $(R_1\#R_2)^{-1}(C)$ consists of finitely many simple closed geodesics on U, then according to Theorem 4 the surface $S(\Gamma_{1,\infty})$ is topologically equivalent to $(U \setminus (R_1\#R_2)^{-1}(C)) \sqcup U^*$. These equivalences can be chosen conformal outside tubular neigborhoods of the curves in $(R_1\#R_2)^{-1}(C)$. In conclusion, $\Gamma_{1,\infty} < \Gamma_{2,\infty}$ induces a covering map $H : S(\Gamma_{1,\infty}) \to S(\Gamma_{2,\infty})$ so that the restriction of H to the component associated to U^* is in the conformal Hurwitz class of Q. Among the restrictions to the other components of $S(\Gamma_{1,\infty})$ which are coverings there are only two which have degree larger than one, the other restrictions are univalent.

The pair of non-univalent coverings belongs to the conformal Hurwitz class of the coverings R_1 and R_2.

The result of pinching the manifolds of the example depicted in Figure 1 is shown in Figure 2.

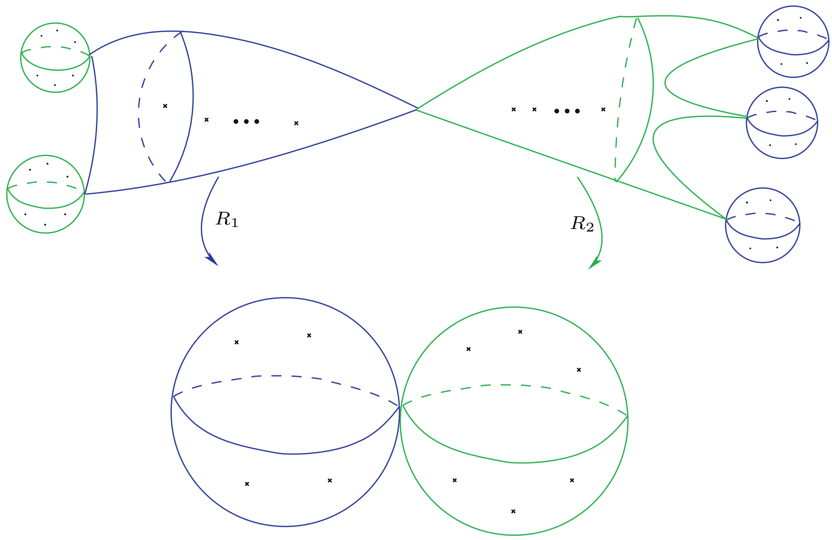

FIGURE 2. In this picture we applied pinching to the example in Figure 1.

3.3. **Klein-Maskit combinations and the connected sums of coverings.** Now we are ready to prove Theorem 1.

Proof of Theorem 1. First, begin with two single holomorphic covering maps $R_1 : S \to S'$ and $R_2 : T \to T'$ of degree n and m, respectively, denote by Γ and Γ' the respective Fuchsian uniformizing groups of S and S', and groups G and G' for the surfaces T and T' acting on the unit disk $\mathbb{D}$ so that $\Gamma < \Gamma'$ and $G < G'$ and the inclusion maps induce the covering R_1 and R_2, respectively. If $\mathbb{D}^* = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, then inclusion maps $(\Gamma, \mathbb{D}^*) \to (\Gamma', \mathbb{D}^*)$ and $(G, \mathbb{D}^*) \to (G', \mathbb{D}^*)$ define anticonformal copies of R_1 and R_2, respectively. Denote by $Q_1 : S^* \to (S')^*$ and $Q_2 : T^* \to (T')^*$ these anticonformal copies. We have that $[\Gamma' : \Gamma] = n$ and $[G' : G] = m$; following the construction of the previous section we construct a covering map of degree $n + m - 1$. Define $R_0 := Q_1 \# Q_2$; thus R_0 maps U to $W = (S')^* \#(T')^*$.

Group construction of U**.** We follow the topological construction above with the Klein-Maskit combination theorem. Let $\tau(z) = 1/\overline{z}$ be the reflection and fix suitable round closed disks $D \subset \tau(F(\Gamma))$ and $\widetilde{D} \subset \tau(F(G))$, where $F(\Gamma)$ and $F(G)$ are fundamental regions for the actions of Γ and G in $\mathbb{D}$, respectively. Now let $h \in PSL(2, \mathbb{C})$ be an element so that $h(\partial D) = \partial\widetilde{D}$ and h maps the interior of D onto the exterior of $\widetilde{D}$. Then the pair of groups Γ' and $h^{-1} \circ G' \circ h$ and the disks $D_1 = D$ and $D_2 = \overline{\mathbb{C}} \setminus D_1$ satisfy the conditions of the Klein-Maskit combination theorem. Since $h^{-1} \circ G' \circ h$ is a Möbius copy of G' we can assume, by taking suitable Möbius copies, that the groups Γ' and G' already satisfy the conditions of

Theorem 5. By Theorem 5, the orbit space $S(\langle\Gamma', G'\rangle)$ is conformally equivalent to $S' \sqcup T' \sqcup ((S')^*\#(T')^*)$.

Consider elements $\{e, \sigma_2, \ldots, \sigma_n\} \subset \Gamma'$ and $\{e, g_2, \ldots, g_m\} \subset G'$ such that $\Gamma' = \Gamma \cup \cdots \cup \sigma_n\Gamma$ and $G' = G \cup \cdots \cup g_m G$. Let $\Gamma_i = g_i \circ \Gamma' \circ g_i^{-1}$ be $m-1$ Möbius copies of Γ' and let $G_j = \sigma_j \circ G' \circ \sigma_j^{-1}$ be $n-1$ Möbius copies of G'. Then by an inductive application of the Klein-Maskit combination theorem the group $H = \langle \Gamma, G, \Gamma_2, \ldots, \Gamma_m, G_2, \ldots, G_n\rangle$ is isomorphic to

$$\Gamma * G * \prod_{i=2}^{m} \Gamma_i * \prod_{j=2}^{n} G_j.$$

Hence, the manifold $M(H)$ is a disk sum of the manifolds $M(\Gamma), M(G), M(\Gamma_i)$, and $M(G_j)$, where the latter are $n-1$ Möbius copies of $M(\Gamma')$ and $m-1$ Möbius copies of $M(G')$, respectively.

The inclusion of H in $\Gamma' * G'$ induces a holomorphic (orbifold) covering

$$\hat{\imath} : M(H) \to M(\langle\Gamma', G'\rangle)$$

which has finite degree, thus H has finite index in $\langle\Gamma', G'\rangle$. Then the restriction $\hat{\imath} : S(H) \to S(\langle\Gamma', G'\rangle)$ is so that there exist three surfaces S_1, S_2, and S_3 in $S(H)$ where $\deg(\hat{\imath}|_{S_j}) > 1$, the space $S_1 \sqcup S_2 \sqcup S_3$ is conformally equivalent to $S \sqcup T \sqcup U$, and $\hat{i}(S_1 \sqcup S_2 \sqcup S_3)$ is conformally equivalent to $S' \sqcup T' \sqcup W$. Moreover, the map $\hat{\imath}$ belongs to $CH(R_1, R_2, Q_1\#Q_2) = CH(R_1) \sqcup CH(R_2) \sqcup CH(Q_1\#Q_2)$. If $O \subset S(H) \setminus \{S_1 \sqcup S_2 \sqcup S_3\}$, then $\hat{\imath} := O \to \hat{\imath}(O)$ is an univalent holomorphic surjective map, even more $\hat{\imath}(O)$ is either S' or T'. So $\hat{\imath}$ is a non-simple holomorphic covering containing single univalent components.

In conclusion, $\hat{\imath}$ is a Möbius morphism which defines a hyperbolic cobordism between the collections $R_1 : S \to S'$, $R_2 : T \to T'$, $Q_1'\#Q_2 : U \to W$ and single univalent components.

For the general case with three or more coverings $R_i : S_i \to S_i'$, $i = 1, \ldots, k$, we proceed inductively. This finishes the proof. □

Now what can we say about non-simple holomorphic coverings? We start with the following examples of uniformizable non-simple holomorphic coverings.

(1) Let Γ be a geometrically finite Fuchsian group such that $\gamma \circ \Gamma \circ \gamma^{-1} = \Gamma$ for $\gamma(z) = \frac{1}{z}$. Then $G = \langle \Gamma, \gamma\rangle$ is isomorphic to an HNN-extension of Γ and is a geometrically finite Kleinian group, so that $M(G) = (\Omega(G) \cup B)/G$ is a geometrically finite hyperbolic orbifold with connected boundary which is a hyperbolic orbifold conformally equivalent to $\mathbb{D}/\Gamma$. More, the inclusion $\Gamma < G$ induces a degree 2 branched covering $\pi : M(\Gamma) \to M(G)$ so that $\pi(\partial M(\Gamma)) = \partial M(G)$ and for any component $S \in \partial M(\Gamma)$ the restriction $\pi|_S : S \to \partial M(G)$ is a conformal equivalence. Now let $\Gamma_0 < \Gamma$ be a subgroup of index d. Then the inclusion Γ_0 in Γ induces a branched covering map $p : M(\Gamma_0) \to M(\Gamma)$ of degree d, so that for any component $S \subset \partial M(\Gamma_0)$ the restriction $p|_S$ is an orbifold covering map of degree d. Then $\pi \circ p$ is a non-simple holomorphic covering of degree d.

(2) Let Γ be a Kleinian group such that $\partial M(\Gamma)$ is connected and the components of $\Omega(\Gamma)$ are simply connected with stabilizers of infinite index. Such groups are known as web-groups. Since geometrically finite Kleinian groups are also residually finite we can choose a subgroup $H < \Gamma$ of finite index

such that $\partial M(H)$ is disconnected. The map $\pi : \partial M(H) \to \partial M(\Gamma)$ given by the canonical holomorphic orbifold covering induced by the inclusion $H \subset \Gamma$ is non-simple with at least two components S_1 and $S_2 \subset \partial M(H)$ such that $\deg(\pi_{S_1}) > \deg(\pi|_{S_2})$.

We have not found in the literature whether for any connected hyperbolic Riemann surface S there exists a web-group G with $\partial M(G)$ conformally equivalent to S. However, M. Kapovich kindly pointed out that this construction can be done using the Brooks deformation theorem (see [5]).

Given a holomorphic covering (R, S, S'), if S' is connected, then we say that R is primitive. We call a primitive holomorphic covering uniformizable if there exists a pair of web-groups $H < \Gamma$, with H of finite index, so that the canonical holomorphic covering $\pi : M(H) \to M(\Gamma)$ belongs to $CH(R, S, S')$. So far we have no examples of non-simple non-uniformazible holomorphic coverings.

Connected sum of non-simple coverings. Given two primitive holomorphic coverings (R_1, S_1, S_1') and (R_2, S_2, S_2'), we construct a connected sum $R_1\#R_2$, in a similar but slightly different way as in Subsection 3.1 as follows: for $i = 1, 2$ fix two open topological disks $t_i \subset S_i'$, not containing branching points of R_i, respectively, together with a homeomorphism $\phi : \partial t_1 \to \partial t_2$. Let g_1 be a single component of R_1; then the map $\phi \circ g_1$ defines a family of homeomorphisms from the components of $g_1^{-1}(\partial t_1)$ to ∂t_2. Let $S_2'' = S_2' \setminus t_2$ and we glue the copies of S_2'' to the surface $g_1^{-1}(S_1' \setminus t_1)$ along the family of homeomorphisms $\phi \circ g_1$ on $g_1^{-1}(\partial t_1)$. Using induction with respect to all single components of R_1 we construct a non-simple holomorphic covering $(\hat{R}_1, T, T')$ with $T' = S_1'\#S_2'$. Now repeat the process for a single component g_2 of R_2 to get a non-simple holomorphic covering $(\hat{R}_2, W, W')$ with $W' = S_1'\#S_2'$ and finally we put $R_1\#R_2 = (\hat{R}_1, T, T') \sqcup (\hat{R}_2, W, W')$ which is a primitive non-simple holomorphic covering.

Let us note that the case of holomorphic coverings over a surface of genus zero is special in the following sense: let R_1 and R_2 be non-simple holomorphic coverings onto the Riemann sphere with finitely many points removed. If g is a single component of R_1, then the induced single component f of $R_1\#R_2$ belongs to $H(g)$ up to forgetting additional perforations on the source and the target surfaces. We call such a primitive covering a *primitive covering of genus zero*. This observation leads to the following lemma.

Lemma 8. *Let $R_1 : S_1 \to S_1'$ and $R_2 : S_2 \to S_2'$ be holomorphic coverings over S_1' and S_2' which are Riemann spheres with finitely many points removed. Then we can choose for $i = 1, 2$ disks $t_i \subset S_i'$ and a gluing map $\phi : \partial t_1 \to \partial t_2$ such that every single component of $R_1\#R_2$ belongs to either $CH(R_1)$ or $CH(R_2)$ up to forgetting additional perforations.*

Proof. Let $t \subset \overline{\mathbb{C}}$ be a round open disk such that $t \subset S_1'$ and not containing branching points of R_1. Let $X \subset \overline{\mathbb{C}}$ be the set $\overline{\mathbb{C}} \setminus (S_2' \cup V(R_2))$, where $V(R_2)$ is the set of branching points of R_2, and $\gamma \in PSL(2, \mathbb{C})$ is so that $\gamma(X) \subset t$. Then for the coverings R_1 and $\gamma \circ R_2$ we choose $t_1 = t$ and $t_2 = \overline{\mathbb{C}} \setminus t$. Taking $\phi = \mathrm{Id}$ on ∂t finishes the proof of the lemma. □

Another application of the Klein-Maskit combination theorem and the arguments of the proofs of Theorem 1 and Lemma 8 allow us to improve Theorem 1 as follows.

Theorem 9. *Let R_1 and R_2 be two primitive uniformizable genus zero holomorphic coverings. Then there exists a primitive uniformizible genus zero holomorphic covering Q so that for every single component q of Q there exists r, which is either a single component of R_1 or a single component of R_2, so that $q \in CH(r)$ after, perhaps, forgetting additional perforations.*

Proof. Let Γ_1 and Γ_2 be web-groups uniformizing R_1 and R_2. As in Theorem 1 we apply the Klein-Maskit combination theorem to Γ_1 and Γ_2 and construct a finite index subgroup of $\Gamma_1 * \Gamma_2$ compatible to the connected sum $R_1 \# R_2$. □

Remark. The Klein-Maskit theorems are generalized to Kleinian groups in higher dimensions, so our Theorem 1 generalizes in that setting as well.

4. Proof of Theorem 2

Theorem 2 is a direct application of the Bers simultaneous uniformization theorem.

Proof of Theorem 2. Assume that R_1 and R_2 are two symmetric holomorphic coverings forming a simple hyperbolic cobordism. Then the respective geometrically finite Kleinian groups Γ_1 and Γ_2 are quasifuchsian. We can assume that $\Gamma_1 < \Gamma_2$. Since every quasifuchsian group is quasiconformally equivalent to a Fuchsian group, Γ_2 admits an orientation reversing quasiconformal involution $\tau : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ commuting with Γ_2, interchanging components of $\Omega(\Gamma_2)$ and which is the identity on the limit set $\Lambda(\Gamma_2)$. Hence, τ commutes with Γ_1. Since R_1 and R_2 are symmetric, then R_1 and R_2 belong to the same Hurwitz class.

Assume that two symmetric holomorphic coverings R_1 and R_2 belong to the same Hurwitz class. Then we can construct cobordisms between R_1 and an anticonformal copy of R_1, together with a cobordism between R_2 and an anticonformal copy of R_2. The homeomorphisms ϕ and ψ associated to R_1 and R_2 allow us to glue the given cobordisms along the anticonformal copies to get a cobordism between R_1 and R_2. □

What follows is an example of a topological cobordism between the simplest rational maps.

Let us consider the convex combination between z^2 and z^3:

$$f_t(z) = (1-t)z^2 + tz^3$$

for $t \in I = [0, 1]$. Then f_t defines a rational endomorphism F of 3-manifolds $X = \mathbb{C} \times I$ by the formula

$$F(z, t) = (f_t(z), t).$$

Then

i) The map F is not a branched self-covering of X.
ii) Let $X_{t_1,t_2} = \mathbb{C} \times [t_1, t_2]$; then the restriction of F to X_{t_1,t_2} is a branched self-covering of X_{t_1,t_2} for $0 < t_1 < t_2 < 1$.
iii) If $0 < t_1 < t_2 < 1$, then for $t_1 \leq t \leq t_2$ the real polynomial f_t is in general position and the sets of the critical values $\{v_1\}, \{v_2\}$ of F in X_{t_1,t_2} forms two embedded arcs connecting the boundaries of X_{t_1,t_2}. Moreover, for each i the set $F^{-1}(v_i)$ consists of two curves $\alpha_{i,j}$, with $j = 1, 2$, one is mapped homeomorphically onto the image by F, while the other consists of the critical points. Let $M = X_{t_1,t_2} \setminus (v_1 \bigcup v_2)$ and $M = F^{-1}(M') \subset X_{t_1,t_2}$ be two

3-manifolds; then $F : M \to M'$ is a covering. The manifold M is homeomorphic to the product of the five punctured sphere times a closed interval and the manifold M' is homeomorphic to the three punctured sphere times a closed interval. Since f_{t_i} are symmetric maps and $f_{t_2} \in H(f_{t_1})$, then by Theorem 2 the maps f_{t_1} and f_{t_2} are simply cobordant. Then on M and M' there are hyperbolic structures depending on the extremes t_1 and t_2 such that F is a Möbius morphism on M. This means that for $0 < t_1 < t_2 < 1$ we can define two hyperbolic 3-dimensional orbifold structures on which the map F is a Möbius morphism.

iv) The endomorphism $F : X \to X$ is a Hausdorff limit (this is a particular case of Gromov-Hausdorff limit) of Möbius morphisms $F : X_{t_1,t_2} \to X_{t_1,t_2}$ for $t_1 \to 0$ and $t_2 \to 1$.

v) If $t \neq 0, 1$, then we can obtain two functions f_0 and f_1 from the map f_t using a pinching procedure with respect to peripheral curves. For the convenience of the reader we sketch this procedure when $f_1(z) = z^3$. Fix a $t \neq 0, 1$ and a Jordan curve $\gamma \in \mathbb{C}$ so that $\alpha = f_t^{-1}(\gamma)$ is a connected Jordan curve. Then the finite critical values belong to the interior of γ. Let $A(\gamma)$ be an annular neighborhood of γ so that $A(\alpha) = f_t^{-1}(A(\gamma))$ is an annular neighborhood of α and $f_t : A(\alpha) \to A(\gamma)$ is a covering of degree 3. Let ν_j be a sequence of Beltrami differentials supported in $A(\gamma)$ as constructed in Section 2.1 and consider the extension of each ν_j on $\overline{\mathbb{C}}$ by zero outside $A(\gamma)$. Let $\mu_j(z) = v_j(f_t)\frac{\overline{f_t'}}{f_t'}$ be the pull-back of ν_j with respect to f_t. Let ϕ_j and ψ_j be solutions of the Beltrami equation for μ_j and ν_j, respectively, with $\phi_j(0) = \psi_j(0) = 0$, $\phi_j(\infty) = \psi_j(\infty) = \infty$, and $\phi_t'(\infty) = \psi_j'(\infty) = 1$. Then $p_j = \psi_j \circ f_t \circ \phi_j^{-1}$ are polynomials of degree 3. Moreover, ϕ_j forms a family of univalent normalized holomorphic functions defined on a neighborhood of infinity V; then ψ_j also forms a holomorphic family on $U = f_t^{-1}(V)$. After taking a suitable subsequence we can assume that $\phi_j \to \phi_0$ and $\psi_j \to \psi_0$ converge uniformly on compact subsets of U and V, respectively. Moreover, ϕ_0 and ψ_0 are non-constant functions. Hence p_j converges to a degree 3 polynomial p_0 and, even more, $p_0|_{\phi_0(U)} = \psi_0 \circ f_t \circ \phi_0^{-1}|_{\phi_0(U)}$. We claim that $p_0(z) = z^3$, otherwise, p_0 has a critical value $v_0 \neq 0, \infty$. Thus p_j also has a finite critical value v_j converging to v_0. But $v_j = \psi_j(\frac{4}{27}(1-t)^3t^2)$ belongs to the bounded component of $\mathbb{C} \setminus \{\psi_j(A(\gamma))\}$. Since ψ_j converges to ψ_0 and the moduli of the annuli $\psi_j(A(\gamma))$ converges to ∞, we have a contradiction to $v_0 \neq 0$. Thus $p_0(z) = z^3$, as claimed.

In the case where $f_0(z) = z^2$, we consider a curve γ closed to 0 so that $f_t^{-1}(\gamma)$ is the union of two curves α and β. Here f_t is a degree 2 covering on α, and β belongs to the exterior of α. Now we take the normalization of ϕ_j and ψ_j given by $\phi_j(0) = \psi_j(0) = 0$, $\phi_j'(0) = \psi_j'(0) = 1$ and $\phi_j(\infty) = \psi(\infty) = \infty$ and proceed as above.

vi) Now on X_{t_1,t_2} with $0 < t_1 < t_2 < 1$ we put a quasiconformal deformation converging on the components of the boundary $\partial X_{t_1,t_2}$ to z^2 and z^3, respectively, to get a limit. This limit seems to be a sort of double limit converging in Hausdorff topology to a non-uniformizable object. This would give a non-geometric completion of the respective Teichmüller space. We suspect that any limit of this type belongs to $H(F, \mathbb{C} \times [0,1], \mathbb{C} \times [0,1])$.

vii) Let $J(f_t)$ be the Julia set of f_t; then $J(F) = \bigcup_{t\in[0,1]}(J(f_t), t) \subset \mathbb{C}\times[0,1]$, is a closed set completely invariant under F. Computer experiments suggest that $\mathbb{C} \times I \setminus J(F)$ consists of two components. The one containing the set $\{0\} \times [0,1]$ is simply connected and the other one has infinitely generated fundamental group. The experiments also suggest that the set $J(F)$ is a non-locally connected embedding of $\mathbb{S}^1 \times [0,1]$ with infinitely many cusps accumulating to a compact subset of the interior of $\mathbb{S}^1 \times [0,1]$.

5. Proof of Theorem 3

As the previous discussion showed, the construction of cobordisms between rational maps often involves the introduction of single univalent components. In this chapter we look for a construction avoiding these components. We call this type of construction a *pure cobordism.*

Let $\mathbb{D}_\infty$ be the completion of $\mathbb{C}$ by adding the circle at infinity $\infty \cdot \mathbb{S}^1$. Then any monic holomorphic polynomial P of degree d can be extended to the circle at infinity by the formula $\infty \cdot e^{i\theta} \mapsto \infty \cdot e^{di\theta}$ so that P defines a branched self-covering $\hat{P}$ of $\mathbb{D}_\infty$ and $\hat{P}|_{\partial\mathbb{D}_\infty}(z) = z^d$. Now we identify $\mathbb{D}_\infty$ with the unit disk $\mathbb{D} \subset \mathbb{C}$. Let us consider two monic polynomials P_1 and P_2, acting on $\mathbb{C}$, then define a finite degree branched covering $F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ as follows:

$$F(z) = \begin{cases} \hat{P}_1(z) & \text{for } z \in \mathbb{D}, \\ \gamma \circ \hat{P}_2 \circ \gamma(z) & \text{for } z \in \overline{\mathbb{C}} \setminus \mathbb{D}, \gamma(z) = \frac{1}{\bar{z}}. \end{cases}$$

In holomorphic dynamics the map F is known as the *formal mating* of the monic polynomials P_1 and P_2 (see for example [9] or [11]).

Now we need the following theorem.

Theorem 10. *We have two polynomials P_1 and P_2 of the same degree $d > 2$ with canonical holomorphic representatives. Assume one of them, say P_2, is symmetric; then there exists a rational map R such that the following holomorphic coverings form a hyperbolic cobordism:*

- $P_1 : \mathbb{C} \setminus (P_1^{-1}(V(P_1))) \to \mathbb{C} \setminus V(P_1)$,
- $P_2 : \mathbb{C} \setminus (P_2^{-1}(V(P_2))) \to \mathbb{C} \setminus V(P_2)$,
- $R : \mathbb{C} \setminus (R^{-1}(V(R))) \to \mathbb{C} \setminus V(R)$.

Proof. Given two polynomials $P_1$and P_2 as in the statement of the theorem, we can assume that P_1 and P_2 are monic. Let us consider their formal mating F. Let $Q \in H(F, \overline{\mathbb{C}}, \overline{\mathbb{C}}) \cap \mathrm{Rat}$ be a rational map. First note that since $F^{-1}(\partial\mathbb{D}) = \partial\mathbb{D}$, then there is a Jordan curve $\delta \subset \overline{\mathbb{C}}$ such that $Q^{-1}(\delta)$ is a connected Jordan curve. If $V(Q)$ is the set of critical values of Q, then Q is a holomorphic covering of finite degree from $S_1 = \overline{\mathbb{C}} \setminus Q^{-1}(V(Q))$ to $S_2 = \overline{\mathbb{C}} \setminus V(Q)$.

Let $\Gamma_1 < \Gamma_2$ be two finitely generated Fuchsian groups uniformizing $Q : S_1 \to S_2$ in $\mathbb{D}$. Then we can assume that the map $R : \mathbb{D}^*/\Gamma_1 \to \mathbb{D}^*/\Gamma_2$ induced by the inclusion is a rational map satisfying $R(z) = \overline{Q(\overline{z})}$. We claim that the triple (R, P_1, P_2) forms a cobordant family.

Indeed, let us apply a pinching procedure to the covering Q, with respect to the curve δ. Then by Theorem 4, we obtain a Kleinian group $\Gamma_{2,\infty}$ with an isomorphism $\rho : \Gamma_2 \to \Gamma_{2,\infty}$ so that $\Gamma_{2,\infty}$ is a geometrically finite function group with $S(\Gamma_{2,\infty}) = S_2' \sqcup T_2' \sqcup T_2''$ where S_2' is anticonformally equivalent to S_2 and T_2' and T_2'' are finitely punctured Riemannian spheres. Each of them contains only one accidental cusp, that is, determined by δ.

On the other hand, we have a geometrically finite function group $\Gamma_{1,\infty}$ satisfying $\Gamma_{1,\infty} < \Gamma_{2,\infty}$. In particular, since $R^{-1}(\delta)$ is connected, then $\Gamma_{1,\infty}$ also is presented by pinching the group Γ_1 with respect to the curve $R^{-1}(\delta)$ and contains only one conjugacy class of an accidental parabolic element. Hence $S(\Gamma_{1,\infty}) = S_1' \sqcup T_1' \sqcup T_1''$, where S_1' is an anticonformal copy of S_1 and the surfaces T_1' and T_1'' are finitely punctured spheres containing only one accidental cusp determined by $R^{-1}(\delta)$. This implies that the map α induced by inclusion is a holomorphic simple covering from $S(\Gamma_{1,\infty})$ to $S(\Gamma_{2,\infty})$ so that $\alpha|_{S_1'} : S_1' \to S_2'$ belongs to $H(R)$. As the preimage of an accidental cusp is an accidental cusp, then the maps $\alpha|_{T_1'}$ and $\alpha|_{T_1''}$ are in the Hurwitz conformal classes of some polynomials, say Q_1 and Q_2, respectively. By construction $Q_1 \in H(P_1)$ and $Q_2 \in H(P_2)$ with orientation reversing homeomorphisms ϕ_2 and ψ_2. Since P_2 is symmetric, then the homeomorphisms ϕ_2 and ψ_2 also can be chosen to be quasiconformal orientation preserving homeomorphisms. If $\phi_i \circ Q_i = P_i \circ \psi_i$, then let μ be the Beltrami differential on $S_2' \sqcup T_2' \sqcup T_2''$ given in local coordinates by

$$\mu(z) = \begin{cases} \frac{\overline{\partial}\phi_1}{\partial\phi_1}(z) & \text{on } T_2', \\ \frac{\overline{\partial}\phi_2}{\partial\phi_2}(z) & \text{on } T_2'', \\ 0 & \text{on } S_2', \end{cases}$$

and let ν be the pull-back of μ by the orbit projection

$$\pi_2 : \Omega(\Gamma_{2,\infty}) \to \Omega(\Gamma_{2,\infty})/\Gamma_{2,\infty}.$$

If f_ν is a solution of the Beltrami equation with respect to ν, then the groups $G_1 = f_\nu \circ \Gamma_{1,\infty} \circ f_\nu^{-1}$ and $G_2 = f_\nu \circ \Gamma_{2,\infty} \circ f_\nu^{-1}$ satisfy our claim and finish the proof of the theorem. □

Let us note that, by Theorem 4, each of the manifolds $M(G_1)$ and $M(G_2)$, constructed in the proof of Theorem 10, is homeomorphic to a set U which is the complement of two open round 3-dimensional balls B_1 and B_2 in the unit ball B with finitely many embedded arcs, connecting the boundary components of ∂U, removed. See Figure 3.

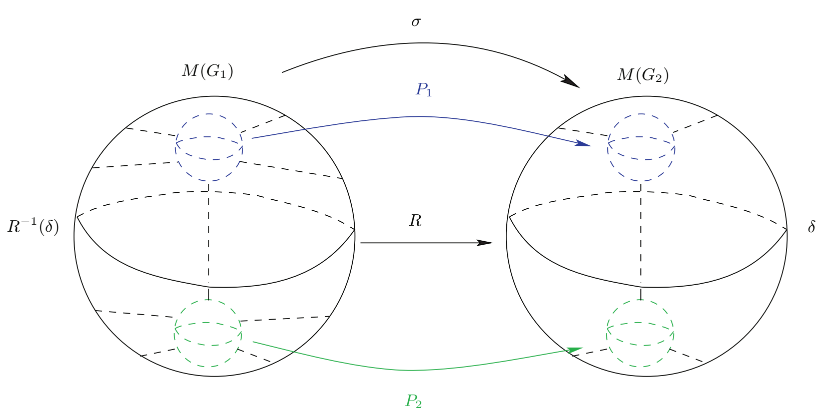

FIGURE 3. A sketch of the manifolds $M(G_1)$ and $M(G_2)$.

The map $\sigma : M(G_1) \to M(G_2)$ induced by the inclusion $G_1 < G_2$ can be extended on U as a finite degree branched covering $\sigma^* : U \to U$ so that in suitable coordinates $\sigma^*|_{\partial U} \in CH(R, \overline{\mathbb{C}}, \overline{\mathbb{C}}) \cup CH(P_1, \overline{\mathbb{C}}, \overline{\mathbb{C}}) \cup CH(P_2, \overline{\mathbb{C}}, \overline{\mathbb{C}})$. In other words, in U there are two Möbius orbifold structures which makes σ^* a Möbius morphism.

Now if τ is the reflection of S^3 with respect to the unit sphere, then on $W = U \cup \tau(U)$ we can extend σ^* to a finite degree self-covering of W by putting

$$\Sigma(z) = \begin{cases} \sigma^*(z) & \text{on } U, \\ \tau \circ \sigma^* \circ \tau(z), & \text{on } \tau(U). \end{cases}$$

Then Σ serves as a topological cobordism between $P_1 \sqcup P_2$ and its anticonformal copies. In what follows, we will show that on W there are two Möbius structures under which Σ is a Möbius morphism. The idea is to apply the arguments of the proof of the Brooks orbifold deformation theorem to put on $M(G_2)$ a non-orientable uniformazible orbifold structure ω so that the component $S_0 \subset \partial M(G_2)$ corresponding to the unit sphere consists of the "interior points" in the orbifold structure ω and other components of $\partial M(G_2)$ equipped with ω are Möbius equivalent to the previous structure.

Then there exists a degree 2 covering $\beta : X \to (M(G_2), \omega)$ such that X is an orientable hyperbolic 3-orbifold with four boundary components which are mapped univalently by β onto $\partial M(G_2) \setminus S_0$.

The following statement is the main lemma of this section, which is an application of arguments of the proof of the Brooks orbifold deformation theorem (see [5]).

Lemma 11. *Let Γ be a geometrically finite Kleinian group without torsion. Let $T \subset S(\Gamma)$ be a proper subcollection of surfaces. Then there exists a geometrically finite Kleinian group G such that $S(G)$ is conformally equivalent to $T \sqcup T^*$ where T^* is an anticonformal copy of T.*

Proof. Let T' be the complement of the collection T in $S(\Gamma)$. Our goal is to destroy the non-empty collection T'. First, assume that there exists a round disk pattern K covering just T'. Let Γ_K be the group generated by Γ and the reflections with respect to all the disks projecting onto the elements of K as in the discussion before Theorem 6. Then by Theorem 6 the hyperbolic orbifold $M(\Gamma_K)$ has as underlying space, the manifold M_K, where M_K is homeomorphic to $M(\Gamma)$. Let $G_K < \Gamma_K$ be the subgroup of orientation preserving elements of Γ_K; then $G_K = \Gamma_K \cap PSL(2, \mathbb{C})$ is a normal order two subgroup of Γ_K containing Γ. By the remark after Theorem 6, if a component $\Omega_0 \in \Omega(\Gamma)$ covers an element of T, then the stabilizer of Ω_0 in G_K coincides with the stabilizer of Ω_0 in Γ. Then $M(G_K)$ admits an anticonformal involution τ and $M(\Gamma_K) = M(G_K)/\tau$. Hence $\partial M(G_K) = T \sqcup T^*$, where T^* is an anticonformal copy of T.

To finish the proof we have to justify the existence of the pattern K. By Theorem 7 there exists a quasiconformal homeomorphism h and a group $\Gamma_h = h \circ \Gamma \circ h^{-1}$ admitting a disk pattern covering the whole surface $S(\Gamma_h)$.

Let us consider a subpattern $K' \subset K$ precisely covering the collection $h(T') \subset S(\Gamma_h)$ and construct a group G_h which uniformizes the surfaces $S(G_h) = h(T) \sqcup (h(T))^*$. Taking a suitable quasiconformal deformation for G_h we obtain a group G, as claimed. □

The lemma above allows to produce a hyperbolic orbifold structure on a double of the manifold $M(\Gamma)$ with respect to a complementary collection T'. In other words, with this lemma one can endow an orbifold structure on the manifold W. The following theorem shows that we can put other orbifold hyperbolic structures on W in such a way that the map Σ becomes a Möbius morphism between these structures.

Theorem 12. *The family of canonical holomorphic representatives of any collection of four polynomials in general position of the same degree $d > 2$ forms a hyperbolic cobordism.*

Proof. Take any pair of polynomials P_1 and P_2 from the given four. By Theorem 10, there exists a rational map R such that $R \sqcup P_1 \sqcup P_2$ forms a cobordant family of coverings. Let $\Gamma_1 < \Gamma_2$ be the Kleinian groups realizing this cobordism, that is, the $\alpha : M_{\Gamma_1} \to M_{\Gamma_2}$ so that α maps $S(\Gamma_1) = V_0 \sqcup V_1 \sqcup V_2$ onto $S(\Gamma_2) = U_0 \sqcup U_1 \sqcup U_2$ and $(\alpha, \partial M(\Gamma_1), \partial M(\Gamma_2))$ belongs to the Hurwitz class $CH(R, P_1, P_2)$. Let H_2 be the geometrically finite group with $S(H_2) = (U_1 \sqcup U_2) \sqcup ((U_1)^* \sqcup (U_2)^*)$ given by Lemma 11; here again $(U_1)^*$ and $(U_2)^*$ are anticonformal copies of U_1 and U_2, respectively.

We claim that there exists a finite index subgroup $H_1 < H_2$ with $S(H_1) = (V_1 \sqcup V_2) \sqcup ((V_1)^* \sqcup (V_2)^*)$ and a projection induced by inclusion of groups

$$\beta : M(H_1) \to M(H_2)$$

is so that $\beta|_{V_i}$ is conformally equivalent to P_i, for $i = 1, 2$, and $\beta|_{(V_i)^*}$ are anticonformal copies of P_i, respectively. Indeed, by the Brooks orbifold deformation theorem as used in Lemma 11 we can assume that the group Γ_2 admits a pattern K which covers only the surface U_0.

Since α is a Möbius morphism, then $K' = \alpha^{-1}(K) \subset V_0$ is also a pattern on $S(\Gamma_1)$ completely covering just the surface V_0. Hence the group G_1, generated by Γ_1 and the reflections with respect to the boundaries of all round disks which project on all elements of K', is a finite index subgroup of the group G_2, where G_2 is generated by Γ_2 and the reflections with respect to the boundaries of all disks projecting on all elements of K. In fact, these families of disks for Γ_1 and Γ_2 coincide.

Therefore the orientation preserving subgroups $H_1 = G_1 \cap PSL(2, \mathbb{C})$ and $H_2 = G_2 \cap PSL(2, \mathbb{C})$ are geometrically finite Kleinian groups such that H_1 has finite index in H_2. Then the groups $H_1 < H_2$ are the desired groups, as claimed.

Let us note that the anticonformal copies of P_1 and P_2 belong to the Hurwitz classes of the polynomials P_3 and P_4. Indeed all polynomials in general position, of the same degree, belong to the same Hurwitz class. Hence, after a suitable quasiconformal deformation of the pair H_1 and H_2 we complete the proof of the theorem. ☐

Choose a surface $S \subset S(G_2)$ together with a pattern precisely covering S. Repeating the construction above, we construct a cobordism between the canonical representatives of six polynomials of the same degree in general position. The iteration of this procedure shows the following statement.

If the canonical holomorphic representatives of a collection of polynomials $P_1, \ldots, P_n$ in general position of the same degree $d > 2$ forms a hyperbolic cobordism, then for every $k < n$, the canonical holomorphic representatives of every collection of $2(n-k)$ polynomials in general position and of degree d forms a hyperbolically cobordant family. An induction argument over Theorems 12 and 2 completes the proof of Theorem 3.

Remark. Unfortunately, we were not able to prove the following desirable statement:

Every finite collection of non-univalent rational (polynomial) maps forms a cobordant family.

This is not clear even in the case of a single rational map R (for more details on this problem see [3]).

But if we drop the geometrically finiteness condition in the definition of cobordism, then, as it was shown in [3], in the case of a single rational map R the statement above is always true, the corresponding uniformizing group is totally degenerated, and the respective Möbius morphism is Hurwitz equivalent to the radial extension of R in the unit 3-dimensional ball.

6. On conformal Hurwitz classes and sandwich semigroups

According to the discussion above it is interesting to know when two given rational maps belong to the same conformal or anticonformal Hurwitz class or, better, when these maps are conformally or anticonformally conjugated. It turns out that the answer is purely algebraic and does not require any dynamical information. We give a precise answer using sandwich products induced by the given rational maps. Also we suggest another point of view on Hurwitz classes of rational maps as minimal representation spaces of semigroups of holomorphic correspondences associated to the given holomorphic coverings.

The results in this section develop ideas in [3], and the Schreier representation of semigroups as treated in [2]. We start with a brief introduction to Schreier representations.

Definition. Let X be a topological space and let $\mathrm{End}(X)$ be the semigroup of continuous endomorphisms of the space X. Then

(1) $\mathrm{End}(X)$ is a topological semigroup.
(2) X canonically embeds in $\mathrm{End}(X)$ as the ideal I of constant endomorphisms.
(3) I is the unique minimal bi-ideal (left and right) consisting of idempotents.
(4) (Schreier lemma) Let $G < \mathrm{End}(X)$ be a subsemigroup with $G \cap I = A \neq \emptyset$ and let $\rho : G \to \mathrm{End}(X)$ be a homomorphism. Then there exists a map $f : A \to X$ such that $f(g(x)) = \rho(g)(f(x))$ for all $x \in A$ and $g \in G$, here $f(x) := \rho(x)$. In other words, every homomorphism is generated by a map. Even more, the map f is continuous if and only if ρ is continuous, and f is a homeomorphism if and only if ρ is a continuous isomorphism onto its image. We say that ρ is orientation preserving or non-orientable, depending on whether f has the corresponding property.
(5) Let $f : Y \to X$ be a continuous map between topological spaces. Then the set G of all continuous maps $g : X \to Y$ can be transformed into a semigroup with the product:
$$g_1 *_f g_2 = g_1 \circ f \circ g_2.$$
This product is called the sandwich product with respect to f and $G_f = \langle G, *_f \rangle$ is called the sandwich semigroup. If f is not invertible, then G_f does not contain a unit.

The following theorem appears in [2]. For convenience we include the proof.

Theorem 13. *Let $R_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ and $R_2 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be two rational maps, and let G_1 and G_2 be sandwich semigroups of rational maps with respect to R_1 and*

R_2, respectively. If $\rho : G_1 \to G_2$ is an isomorphism, then there exist an element $\gamma \in PSL(2,\mathbb{C})$, and a bijection $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ so that $\rho(R) = \phi \circ R \circ \phi^{-1} \circ \gamma$ for every rational map R.

We say that the homomorphism ρ is orientation preserving or non-orientable depending on whether ϕ has the same property.

Proof. Let $f = \rho|_{\mathbb{C}}$ be the restriction of ρ to the constants; then $f(\overline{\mathbb{C}}) \subset \overline{\mathbb{C}}$. Indeed for a suitable constant $c \in \overline{\mathbb{C}}$ then $c *_{R_1} Q = c$ and hence $\rho(c) *_{R_2} \rho(Q) = \rho(c)$ for every rational map Q. Since ρ is an isomorphism $\rho(Q)$, then $\rho(c)$ is a constant. Also f is a bijection.

Now we show that $\rho(PSL(2,\mathbb{C})) = PSL(2,\mathbb{C})$. Indeed, since for every rational maps R and Q we have $\rho(R *_{R_1} Q) = \rho(R) *_{R_2} \rho(Q)$, taking $Q = c$ a constant then $f(R \circ R_1(c)) = \rho(R) \circ R_2(f(c))$. As f is invertible we have $\deg(R \circ R_1) = \deg(\rho(R) \circ R_2)$. Similarly for ρ^{-1} we have

$$f^{-1}(R \circ R_2(c)) = \rho^{-1}(R) \circ R_1(f^{-1}(c))$$

and

$$\deg(R \circ R_2) = \deg(\rho^{-1}(R) \circ R_1).$$

If $R \in PSL(2,\mathbb{C})$, then $\deg(R_1) = \deg(R \circ R_1) = \deg(\rho(R)) \cdot \deg(R_2) = \deg(\rho(R)) \cdot \deg(\rho^{-1}(R)) \cdot \deg(R_1)$ which implies $\deg(\rho(R)) = 1$.

Let $\gamma = \rho(\mathrm{Id})$ so $\gamma \in PSL(2,\mathbb{C})$. Consider the map $\tau_\gamma : G_2 \to \langle \mathrm{Rat}, *_{\gamma \circ R_2} \rangle$ given by $\tau_\gamma(R) = R \circ \gamma^{-1}$. Then τ_γ is an isomorphism of the sandwich semigroup, which follows from direct computation:

$$\tau_\gamma(R *_{R_2} Q) = \tau_\gamma(R \circ R_2 \circ Q) = R \circ R_2 \circ Q \circ \gamma^{-1} = \tau_\gamma(R) *_{\gamma \circ R_2} \tau_\gamma(Q).$$

Then $\Phi : \tau_\gamma \circ \rho : G_1 \to \langle \mathrm{Rat}, *_{\gamma \circ R_2} \rangle$ is an isomorphism satisfying $\Phi(\mathrm{Id}) = \mathrm{Id}$. If $R = c$ is a constant, then $\Phi(c) = \rho(c) \circ \gamma = \rho(c) = f(c)$. Also $\Phi(R_1) = \Phi(\mathrm{Id} *_{R_1} \mathrm{Id}) = \gamma \circ R_2$, and hence for every $c \in \mathbb{C}$

$$f(R_1(c)) = \Phi(\mathrm{Id} *_{R_1} c) = \gamma \circ R_2(f(c))$$

which implies $R_2 = \gamma^{-1} \circ f \circ R_1 \circ f^{-1}$. We have for every $Q \in \mathrm{Rat}(\mathbb{C})$ and $c \in \mathbb{C}$,

$$f(Q \circ R_1(c)) = \Phi(Q *_{R_1} c) = \Phi(Q) \circ \gamma \circ R_2(f(c)) = \Phi(Q) \circ f \circ R_1(c).$$

Then

$$Q \circ R_1(c) = f^{-1} \circ \Phi(Q) \circ f \circ R_1(c)$$

and so

$$\Phi(Q) = f \circ Q \circ f^{-1}$$

but $\Phi(Q) = \tau_\gamma \circ \rho(Q)$ or $\rho(Q) = \Phi(Q) \circ \gamma$, as we wanted to prove. □

As an immediate corollary we have the following.

Corollary 14. *If $\rho : G_1 \to G_2$ is an isomorphism as in Theorem 12 and $\rho(\mathrm{Id}) = \mathrm{Id}$, then there exists a bijection $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that*

$$\rho(R) = f \circ R \circ f^{-1}.$$

The next theorem is the main result of this section.

Theorem 15. *Let R_1 and R_2 be non-constant rational maps and let G_1 and G_2 be the respective sandwich semigroups on* $\mathrm{Rat}(\mathbb{C})$. *Then*

(1) *The pair R_1, R_2 belongs to the same conformal Hurwitz class if and only if there exists a continuous orientation preserving isomorphism $\rho : G_1 \to G_2$. Moreover, $\rho(\mathrm{Id}) = \mathrm{Id}$ if and only if R_1 is $PSL(2, \mathbb{C})$ conjugated to R_2.*

(2) *A continuous isomorphism ρ reverses orientation if and only if R_1 is an anticonformal copy of R_2. Moreover $\rho(\mathrm{Id}) = \mathrm{Id}$ if and only if R_1 is anticonformally conjugated to R_2.*

Proof. Part 1. Assume that R_1 and R_2 belong to the same conformal Hurwitz class and let $h, g \in PSL(2, \mathbb{C})$ be so that $R_2 = g^{-1} \circ R_1 \circ h$. Then the map $\rho(R) = h^{-1} \circ R \circ g$ defines a continuous orientation preserving isomorphism from G_1 to G_2. Indeed, from direct calculation:

$$\rho(R *_{R_1} Q) = h^{-1} \circ R \circ R_1 \circ Q \circ g$$

$$= \rho(R) *_{R_2} \rho(Q).$$

Now if $\rho : G_1 \to G_2$ is an orientation preserving isomorphism, then by Theorem 12 there exists a $\gamma \in PSL(2, \mathbb{C})$ and a bijection $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ so that $\rho(Q) = f \circ Q \circ f^{-1} \circ \gamma$ for every rational map Q. By conjugation, f defines an automorphism of $\mathrm{Rat}(\mathbb{C})$ with composition as a product. By Proposition 8 in [2], the map f belongs to the group generated by $PSL(2, \mathbb{C})$ and the absolute Galois group. Since ρ is continuous and orientation preserving we have $f \in PSL(2, \mathbb{C})$. Now assume $\rho(\mathrm{Id}) = \mathrm{Id}$; then by Corollary 14, $\rho(R) = f \circ R \circ f^{-1}$ and $f \in PSL(2, \mathbb{C})$ and $\rho(R_1) = f \circ R_1 \circ f^{-1} = \rho(\mathrm{Id} *_{R_1} \mathrm{Id}) = R_2$.

Part 2. If ρ is continuous and orientation reversing, then $\overline{f} \in PSL(2, \mathbb{C})$ and the proof goes as Part 1. □

In conclusion we note the following.

First, it is possible to show that every continuous semigroup product on $\mathrm{Rat}(\mathbb{C})$ which is continuously isomorphic to a sandwich product is a sandwich product itself. So the classes of continuous isomorphisms of sandwich semigroups on $\mathrm{Rat}(C)$ correspond to the conformal Hurwitz classes of rational maps.

Second, that it is not clear at all how to associate the algebraic characterization of the conformal Hurwitz class of symmetric rational maps with the geometric cobordisms point of view.

Finally, similar ideas allow us to consider the Hurwitz space as a representation space of a special class of holomorphic correspondences. This follows using results from [2] with [3]. So we can construct a Teichmüller space of correspondences of the form $R^{-1} \circ R$, called the *deck correspondence associated to R*. From [3] it follows that the Speisser class of R fibers over the moduli space of the deck with fiber equivalent to the conformal Hurwitz class of R. Let $G_R = \langle R^{-1} \circ R, \overline{\mathbb{C}} \rangle$ be the semigroup of holomorphic correspondences generated by the deck correspondence associated to R and constant maps. Consider the space $\mathcal{X}$ of all representations of G_R into the semigroup of holomorphic correspondences on $\overline{\mathbb{C}}$. Then using results from [2], one can consider the Speisser class of a rational map R as a subspace of the connected component of $\mathcal{X}$ containing the identity representation.

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