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UNIFORMIZATION OF CANTOR SETS WITH BOUNDED GEOMETRY

VYRON VELLIS

ABSTRACT. In this note we provide a quasisymmetric taming of uniformly perfect and uniformly disconnected sets that generalizes a result of MacManus [Rev. Mat. Iberoamericana 15 (1999), pp. 267–277] from 2 to higher dimensions. In particular, we show that a compact subset of $\mathbb{R}^n$ is uniformly perfect and uniformly disconnected if and only if it is ambiently quasiconformal to the standard Cantor set $\mathcal{C}$ in $\mathbb{R}^{n+1}$.

1. INTRODUCTION

The *(quasisymmetric) uniformization problem* asks for necessary and sufficient conditions under which a metric space X is quasisymmetrically homeomorphic to a "standard" space X_0. Roughly speaking, quasisymmetric homeomorphisms are a generalization of conformal maps which preserve relative distances; see §2 for precise definitions. The uniformization problem has been extensively studied in literature for a variety of "standard" spaces: the unit circle $\mathbb{S}^1$ [20], the unit sphere $\mathbb{S}^2$ [3], and geodesic trees [4]. See also [2] for a general overview.

Some of the most simple metric spaces, from a topological point of view, are Cantor sets, i.e., homeomorphic images of the standard ternary Cantor set $\mathcal{C}$. Brouwer's topological characterization of Cantor sets [13, Theorem 7.4] states that a metric space is a Cantor set if and only if it is compact, perfect, and totally disconnected.

In [6], David and Semmes solved the uniformization problem in the case that the standard space X_0 is $\mathcal{C}$. Contrary to Brouwer's uniformization, in the quasisymmetric case one has to assume some quantitative versions of perfectness and total disconnectedness.

A closed nondegenerate metric space X is called *uniformly perfect* if there exists a constant $c \geq 1$ such that for all $x \in X$ and all $r \in (0, \operatorname{diam} X)$, there exists a point in $B(x,r) \setminus B(x, r/c)$. Every uniformly perfect space is perfect; on the other hand, the planar set $[0,1] \times \bigcup_{n=1}^{\infty}\{n!\}$ is perfect but not uniformly perfect.

A metric space X is *uniformly disconnected* if there is a constant $c \geq 1$ such that for all $x \in X$ and all positive $r < \frac{1}{4}\operatorname{diam} X$, there exists $E \subset X$ containing x such that $\operatorname{diam} E \leq r$ and $\operatorname{dist}(E, X \setminus E) \geq r/c$. All uniformly disconnected spaces are totally disconnected; on the other hand, $\mathbb{Z}$ is totally disconnected but not uniformly disconnected.

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Based on these scale-invariant notions, the David and Semmes uniformization is as follows.

Theorem 1.1 ([6, Proposition 15.11])**.** *A metric space is quasisymmetrically homeomorphic to $\mathcal{C}$ if and only if it is compact, doubling, uniformly disconnected and uniformly perfect.*

Recall that a metric space X is doubling if there exists a constant $C > 1$ such that for all $x \in X$ and all $r > 0$, the ball $B(x, r)$ can be covered by at most C many balls of radius $r/2$. Since all Euclidean spaces $\mathbb{R}^n$ are doubling, the doubling condition in Theorem 1.1 can be dropped if $X \subset \mathbb{R}^n$.

Later, MacManus [15] proved a stronger uniformization result for Cantor sets contained in $\mathbb{R}^2$. The improvement here is that the quasisymmetric homeomorphism can be in fact assummed to be defined on the ambient space $\mathbb{R}^2$ and not just $\mathcal{C}$.

Theorem 1.2 ([15, Theorem 3])**.** *For a compact set $X \subset \mathbb{R}^2$ there exists a quasisymmetric mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ with $F(\mathcal{C}) = X$ if and only if X is uniformly perfect and uniformly disconnected.*

Note that Theorem 1.2 is false in $\mathbb{R}^3$ due to the existence of a self-similar wild Cantor set in $\mathbb{R}^3$ called *Antoine necklace*; see [5, pp. 70–75]. By self-similarity, this set is both uniformly perfect and uniformly disconnected, but there exists no homeomorphism of $\mathbb{R}^3$ (let alone a quasisymmetric homeomorphism) that maps this set onto $\mathcal{C}$. See also [16, Appendix A] for recent examples in $\mathbb{R}^4$.

Additionally, the wildness of X is not the only obstruction in the generalization of MacManus' result. In particular, there exists a compact uniformly perfect and uniformly disconnected set $X \subset \mathbb{R}^3$ which is "topologically tame and quasisymmetrically wild", that is, it is ambiently homeomorphic to $\mathcal{C}$ but not ambiently quasisymmetric to $\mathcal{C}$; see [9, Proposition 1.4].

In our main result, we provide a quasisymmetric taming of Cantor sets with bounded geometry. In particular, we show that by increasing the dimension by 1, MacManus' result generalizes to all dimensions $n \geq 3$.

Theorem A. *Let $n \in \mathbb{N}$. For a compact set $X \subset \mathbb{R}^n$ there exists a quasisymmetric map $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $F(\mathcal{C}) = X$ if and only if X is uniformly perfect and uniformly disconnected.*

Here and for the rest, given $n \in \mathbb{N}$, we identify $\mathbb{R}^n$ with the plane $\mathbb{R}^n \times \{0\} = \{(x_1, \dots, x_n, 0) : x_i \in \mathbb{R}\} \subset \mathbb{R}^{n+1}$.

One application of Theorem A is in conformal dynamics. A *uniformly quasiregular map* (abbv. *UQR* map) $f : \mathbb{R}^n \to \mathbb{R}^n$ is a map for which there exists $K \geq 1$ such that for any $m \in \mathbb{N}$, the m-th iterate $f_m = f \circ \cdots \circ f$ is in the Sobolev space $W^{1,n}_{\text{loc}}(\mathbb{R}^n)$ and satisfies

$$|f'_m(x)| \leq K J_{f_m}, \qquad \text{for a.e. } x \in \mathbb{R}^n.$$

Due to the rigidity of conformal maps in dimensions $n \geq 3$, UQR maps play the role of holomorphic maps in the study of conformal dynamics in higher dimensions. A well known problem in conformal dynamics is the characterization of closed sets in $\mathbb{R}^n$ that arise as Julia sets of UQR maps. Iwaniecz and Martin [12] showed that the Cantor set $\mathcal{C}$ is a Julia set of a UQR map of $\mathbb{R}^2$, and Fletcher and Wu [10] showed that the Antoine necklace is a Julia set of a UQR map of $\mathbb{R}^3$. Note that both $\mathcal{C}$ and the Antoine necklace are uniformly perfect and uniformly disconnected because they are self-similar. In general, it is unknown whether self-similar Cantor sets in dimensions $n \geq 3$ are always Julia sets of UQR maps. In [9] we apply Theorem A to show that every uniformly perfect and uniformly disconnected subset of $\mathbb{R}^n$, $n \geq 3$, is the Julia set of a UQR map of $\mathbb{R}^{n+1}$.

Moreover, Theorem 1.2 and Theorem A yield the following quasiconformal embedding result for uniformly disconnected sets.

Corollary A. *Let $n \geq 2$ be an integer and let $X \subset \mathbb{R}^n$ be a bounded uniformly disconnected set. There exists a quasisymmentric homeomorphism of $\mathbb{R}^N$ that maps X into $\mathcal{C}$, where $N = 2$ if $n = 2$, and $N = n + 1$ if $n \geq 3$.*

Corollary A has an application in hyperbolic geometry. If $X \subset \mathbb{S}^2$ is a Cantor set, then by the Uniformization Theorem, $S := \mathbb{S}^2 \setminus X$ is necessarily a hyperbolic Riemann surface. Hence, S has a pants decomposition, that is, $S = \bigcup_{i=1}^{\infty} P_i$, where each P_i is a topological sphere with three disks removed. The collection of boundary curves of the pairs of pants, called the *cuffs of the decomposition*, may be enumerated by $(\alpha_j)_{j=1}^{\infty}$. Each α_j is a simple closed curve on S and generates a class $[\alpha_j]$ of simple closed curves that are freely homotopic to α_j. Denote by $\ell_S[\alpha_j]$ the infimum of hyperbolic lengths of curves in $[\alpha_j]$. Pommerenke [17] proved that the Cantor set X is uniformly perfect if and only if $\inf \ell[\alpha_j] > 0$. In a recent paper with Fletcher [8], we apply Corollary A to show that a similar statement holds for uniformly disconnected sets: a Cantor set $X \subset \mathbb{R}^2$ is uniformly disconnected if and only if there exists a pair of pants decomposition such that the associated cuffs satisfy $\sup \ell[\alpha_j] < \infty$.

By properties of quasisymmetric homeomorphisms, one direction of Theorem A is clear. Namely, if there exists a quasisymmetric map $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $F(\mathcal{C}) = X$, then X is compact, uniformly perfect and uniformly disconnected. For the converse, which is the content of this paper, we use the existence of a quasisymmetric homeomorphism $f : \mathcal{C} \to X$, and we extend this mapping quasisymmetrically to $\mathbb{R}^{n+1}$. In §2 we give some basic definitions, in §3 we discuss some bi-Lipschitz Schoenflies theorems in higher dimensions, and in §4 we prove Theorem A and Corollary A.

2. Preliminaries

For $n \in \mathbb{N}$, a point $x \in \mathbb{R}^n$, and $r > 0$ we denote by $B^n(x, r)$ and $\overline{B^n}(x, r)$ the open and closed, respectively, ball centered at x and with radius r.

A homeomorphism $f\colon D \to D'$ between two domains in $\mathbb{R}^n$ is called K-*quasiconformal* for some $K \geq 1$ if, for all $x \in D$, f satisfies the distortion inequality

$$\limsup_{r \to 0} \frac{\sup_{y \in \partial B^n(x,r)} |f(x) - f(y)|}{\inf_{y \in \partial B^n(x,r)} |f(x) - f(y)|} \leq K.$$

A homeomorphism $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is said to be η-*quasisymmetric* if there exists a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ such that for all $x, a, b \in X$ with $x \neq b$

$$\frac{d_Y(f(x), f(a))}{d_Y(f(x), f(b))} \leq \eta\left(\frac{d_X(x, a)}{d_X(x, b)}\right).$$

A quasisymmetric mapping between two domains in $\mathbb{R}^n$ is quasiconformal. The converse holds true for the smaller class of uniform domains which contains $\mathbb{R}^n$. For a systematic treatment of quasiconformal mappings see [23].

It follows easily from the definitions above that quasisymmetric maps preserve the notions of uniform perfectness and uniform disconnectedness quantitatively.

Lemma 2.1 ([14, Theorem 1.3.4])**.** *If $f : X \to Y$ is η-quasisymmetric and X is c-uniformly perfect (resp. c-uniformly disconnected), then Y is c'-uniformly perfect (resp. c'-uniformly disconnected) with c' depending only on η and c.*

A map $f\colon X \to Y$ between metric spaces is *L-bi-Lipschitz* for some $L \geq 1$ if

$$L^{-1}d_X(x,y) \leq d_Y(f(x),f(y)) \leq Ld_X(x,y)$$

for all $x, y \in X$. Note that an L-bi-Lipschitz mapping is L^2t-quasisymmetric.

A weaker notion of bi-Lipschitz mappings is that of *bounded length distortion* (*BLD*) mappings. A mapping $f : (X,d_X) \to (Y,d_Y)$ is L-BLD if there exists $L \geq 1$ such that

$$L^{-1}\ell(\gamma) \leq \ell(f(\gamma)) \leq L\ell(\gamma)$$

for all paths $\gamma : [0,1] \to X$. Here and for the rest, ℓ denotes the length of a path. Clearly, L-bi-Lipschitz mappings are L-BLD mappings but BLD mappings need not be bi-Lipschitz even if they are homeomorphisms. However, BLD homeomorphisms between geodesic spaces are bi-Lipschitz.

Lemma 2.2. *Let $f : X \to Y$ be an L-BLD homeomorphism between two geodesic metric spaces. Then f is L-bi-Lipschitz.*

An embedding $f : (X,d_X) \to (Y,d_Y)$ is a (λ,L)-*quasisimilarity* for some $\lambda > 0$ and $L \geq 1$ if

$$L^{-1}\lambda d_X(x,y) \leq d_Y(f(x),f(y)) \leq L\lambda d_X(x,y) \qquad \text{for all} \quad x,y \in X.$$

Note that $(\lambda,1)$-quasisimilarities are similarities with scaling factor λ, while $(1,L)$-quasisimilarities are L-bi-Lipschitz, and $(1,1)$-quasisimilarites are isometries.

While similarities preserve relative distances between nondegenerate sets, quasisymmetric maps quasi-preserve relative distances between nondegenerate sets. Specifically, if $f : X \to Y$ is η-quasisymmetric and $E, E' \subset X$ are nondegenerate closed sets, then

$$\frac{1}{2}\phi\left(\frac{\operatorname{dist}(E,E')}{\operatorname{diam} E}\right) \leq \frac{\operatorname{dist}(f(E),f(E'))}{\operatorname{diam} f(E)} \leq \eta\left(2\frac{\operatorname{dist}(E,E')}{\operatorname{diam} E}\right), \tag{2.1}$$

where $\phi(t) = (\eta(t^{-1}))^{-1}$; see for example [22, p. 532]. Moreover, if $f : X \to Y$ is η-quasisymmetric and $A \subset B \subset X$ are such that $0 < \operatorname{diam} A \leq \operatorname{diam} B < \infty$, then $\operatorname{diam} f(B)$ is finite and

$$\left(2\eta\left(\frac{\operatorname{diam} B}{\operatorname{diam} A}\right)\right)^{-1} \leq \frac{\operatorname{diam} f(A)}{\operatorname{diam} f(B)} \leq \eta\left(2\frac{\operatorname{diam} A}{\operatorname{diam} B}\right). \tag{2.2}$$

For the proof of (2.2) see [11, Proposition 10.8].

3. Bi-Lipschitz Schoenflies theorems for multiply connected domains

The classical Schönflies theorem states that every embedding of $\mathbb{S}^1$ in $\mathbb{R}^2$ extends to a homeomorphism of $\mathbb{R}^2$. Tukia [19] proved a bi-Lipschitz version of Schönflies theorem.

Theorem 3.1 ([19])**.** *If $f : \mathbb{S}^1 \to \mathbb{R}^2$ is L-bi-Lipschitz, then f extends in an L'-bi-Lipschitz way to $\mathbb{R}^2$ with L' depending only on L.*

It is well known that in higher dimensions Theorem 3.1 fails even under strong topological assumptions. In particular, Tukia [19, §15] constructed a bi-Lipschitz embedding of $\mathbb{S}^2$ into $\mathbb{R}^3$ that can be extended as a homeomorphism of $\mathbb{R}^3$ but not as a quasisymmetric (let alone bi-Lipschitz) homeomorphism of $\mathbb{R}^3$.

Theorem 3.1 was generalized for annuli in $\mathbb{R}^2$ and annuli in higher dimensions by Sullivan [18]. Given $L > 1$ and integer $n \geq 2$, we say that a domain $D \subset \mathbb{R}^n$ is an *L-bi-Lipschitz ball*, if it is the image of the unit ball $\mathbb{B}^n$ under a (λ, L)-quasisimilarity homeomorphism of $\mathbb{R}^n$. For the proof of the following theorem see Theorem 3.17 and §5.9 in [21].

Theorem 3.2. *Given $n \in \{2, 3, \dots\}$, $L_0, L > 1$, and $\delta \in (0, 1)$ there exists $L' > 1$ with the following property. Let $D_1 \subset D_2 \subset \mathbb{R}^n$ be two L_0-bi-Lipschitz balls such that*

$$\operatorname{dist}(D_1, \partial D_2) \geq \delta \operatorname{diam} D_2 \qquad \text{and} \qquad \operatorname{diam} D_1 \geq \delta \operatorname{diam} D_2.$$

If $f : \partial D_1 \cup \partial D_2 \to \mathbb{R}^n$ is an L-bi-Lipschitz embedding that can be extended homeomorphically to $\overline{D_2} \setminus D_1$ then f extends in an L'-bi-Lipschitz way to $\overline{D_2} \setminus D_1$.

In this section, we generalize Theorem 3.2 to multiply connected domains with controlled topology and geometry.

For $d > 1$, $\lambda > 1$, and $n \in \{2, 3, \dots\}$ denote by $\mathscr{U}_n(\lambda, d)$ the collection of bounded domains $U \subset \mathbb{R}^n$ whose boundary components are boundaries of λ-bi-Lipschitz balls of diameters and mutual distances bounded below by $d^{-1} \operatorname{diam} U$.

Proposition 3.3 is the main result of this section.

Proposition 3.3. *Let $U \in \mathscr{U}_n(\lambda, d)$ and $f : \partial U \to \mathbb{R}^n$ be an L-bi-Lipschitz map that extends homeomorphically to $\overline{U}$. Then f extends in an L'-bi-Lipschitz way to $\overline{U}$ with L' depending only on L, λ, d, and n.*

We start with the simple observation that every domain in $\mathscr{U}_n(\lambda, d)$ has a finite number of boundary components.

Lemma 3.4. *Every domain $U \in \mathscr{U}_n(\lambda, d)$ contains at most N boundary components with N depending only on n, d.*

Proof. Let $D_1, \dots, D_m$ be some bounded components of $\mathbb{R}^n \setminus \overline{U}$. For each $i \in \{1, \dots, m\}$, choose $x_i \in D_i$. Then, for each distinct i, j, we have $x_i \in \overline{B^n}(x_1, \operatorname{diam} U)$ and $|x_i - x_j| \geq d^{-1} \operatorname{diam} U$. By the doubling property of $\mathbb{R}^n$, we have that $m \leq N_0 d^{-n}$ for some universal N_0. □

Below, for a domain $U \in \mathscr{U}_n(\lambda, d)$, we write $U = U(D_0; D_1, \dots, D_m)$ if $D_0, \dots, D_m$ are λ-bi-Lipschitz balls with

1. $U = D_0 \setminus \bigcup_{i=1}^m \overline{D_i}$;
2. $\overline{D_1}, \dots, \overline{D_m}$ are contained in D_0 and are mutually disjoint;
3. for all $i \in \{1, \dots, m\}$ we have $\operatorname{diam} D_i \geq d^{-1} \operatorname{diam} D_0$;
4. for all distinct $i, j \in \{0, \dots, m\}$ we have $\operatorname{dist}(\partial D_i, \partial D_j) \geq d^{-1} \operatorname{diam} D_0$.

For the rest of §3 we denote by $\mathcal{S}_0$ the open n-cube $(-1, 1)^n$ in $\mathbb{R}^n$. For each $m \in \mathbb{N}$ and $k \in \{1, \dots, m\}$ define the open cubes

$$\mathcal{S}_{m,k} := \left(\frac{4k - 2m - 3}{2m + 1}, \frac{4k - 2m - 1}{2m + 1} \right) \times \left(\frac{-1}{2m + 1}, \frac{1}{2m + 1} \right)^{n-1} \subset \mathbb{R}^n$$

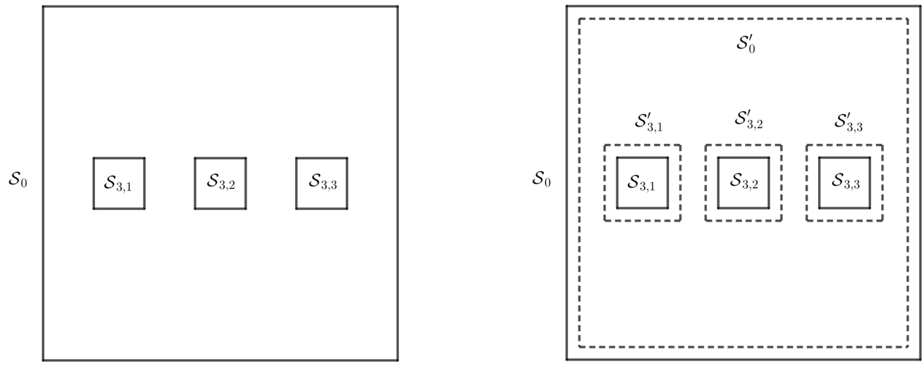

FIGURE 1. The profile of cubes $\mathcal{S}_0$, $\mathcal{S}_{m,k}$, $\mathcal{S}'_0$, and $\mathcal{S}'_{m,k}$ in the first two dimensions in the case $m = 3$

and the domains

$$\mathcal{U}_0 := \mathcal{S}_0, \qquad \mathcal{U}_m := \mathcal{S}_0 \setminus \bigcup_{k=1}^{m} \overline{\mathcal{S}_{m,k}}, \quad \text{for } m \in \mathbb{N}.$$

In Lemma 3.5, we show that every domain $U \in \mathscr{U}_n(\lambda, d)$ is quasisimilar to $\mathcal{U}_m$ for some $m \in \mathbb{N}$. This allows us to reduce the proof of Proposition 3.3 to the case $U = \mathcal{U}_m$.

Lemma 3.5. *For $n \in \{2, 3, \dots\}$, $\lambda > 1$, and $d > 0$ there exists $L \geq 1$ with the following property. If $U = U(D_0; D_1, \dots, D_m)$ is in $\mathscr{U}_n(\lambda, d)$ and has diameter equal to 1, then there exists an L-bi-Lipschitz homeomorphism $f : \overline{U} \to \overline{\mathcal{U}_m}$ with $f(\partial D_0) = \partial \mathcal{S}_0$ and $f(\partial D_i) = \partial \mathcal{S}_{m,i}$ for $i = 1, \dots, m$.*

Note that the order of domains D_i, $i \in \{1, \dots, m\}$ is arbitrary. Therefore, Lemma 3.5 implies that we can bi-Lipschitz map U onto $\mathcal{U}_m$ and match the cubes $\partial \mathcal{S}_{m,i}$ with the inner boundary components of of U, in any order.

For the proof of Lemma 3.5, given $k \in \mathbb{N}$ we say that a point $x \in \mathbb{R}^n$ is a *k-dyadic point* if there exists $(i_1, \dots, i_n) \in \mathbb{Z}^n$ such that $x = (i_1 2^{-k}, \dots, i_n 2^{-k})$.

Remark 3.6. For any $k \in \mathbb{N}$ and any $x \in \mathbb{R}^n$, there exists a k-dyadic point $x' \in \mathbb{R}^n$ such that $|x - x'| < 2^{-k}\sqrt{n}$.

To prove Lemma 3.5 we require the following result.

Lemma 3.7. *Let $\{p_1, \dots, p_m\}$ and $\{q_1, \dots, q_m\}$ be two families of distinct k-dyadic points in $\mathcal{S}_0$. Given $r = \frac{1}{10} 2^{-k}$, there exists $L > 1$ depending only on n and k, and there exists an L-bi-Lipschitz homeomorphism*

$$f : \overline{\mathcal{S}_0} \setminus \bigcup_{i=1}^{m} B^n(p_i, r) \to \overline{\mathcal{S}_0} \setminus \bigcup_{i=1}^{m} B^n(q_i, r)$$

such that for all i, f maps $\partial B^n(p_i, r)$ onto $\partial B^n(q_i, r)$.

Proof. Let l_1 denote the line segment $[p_1, q_1]$. Then $\operatorname{dist}(l_1, \partial \mathcal{S}_0) \geq 2^{-k}$. We modify l_1 on its intersections with balls $B^n(p_i, 5r)$ for $i \neq 1$ replacing $l_1 \cap B^n(p_i, 5r)$ (if nonempty) by an arc on $\partial B^n(p_i, 5r)$ of minimal length. We denote the new curve by γ_1. Since we have chosen r small enough, it follows that the set

$$D_1 = \bigcup_{x\in\gamma_1} B^n(x, 2r)$$

is a λ_1-bi-Lipschitz ball. In fact, since there are a finite number of arrangement for the points $\{p_1, \dots, p_m, q_1\}$ in $\mathcal{S}_0$ (depending only on k and n), we have that λ_1 depends only on n and k. Moreover, D_1 contains $B^n(p_1, r)$ and $B^n(q_1, r)$ and does not intersect any of the balls $B^n(p_i, r)$ for $i \neq 1$. Applying Theorem 3.2 on $\overline{D_1} \setminus B^n(p_1, r)$, there exists $L_1 > 1$ depending only on k, n and an L_1-bi-Lipschitz map

$$f_1 : \overline{\mathcal{S}_0} \setminus \bigcup_{i=1}^{m} B^n(p_i, r) \to \overline{\mathcal{S}_0} \setminus \left(B^n(q_1, r) \cup \bigcup_{i=2}^{m} B^n(p_i, r) \right)$$

such that f_1 is identity on $\overline{\mathcal{S}_0} \setminus D_1$ and maps $\partial B(p_1, r)$ onto $\partial B(q_1, r)$.

Working as above, for each $j \in \{2, \dots, m\}$, there exists an L_j-bi-Lipschitz map

$$f_j : \overline{\mathcal{S}_0} \setminus \left(\bigcup_{i=j}^{m} B^n(p_i, r) \cup \bigcup_{i=1}^{j-1} B^n(q_i, r) \right) \to \overline{\mathcal{S}_0} \setminus \left(\bigcup_{i=1}^{j} B^n(q_i, r) \cup \bigcup_{i=j+1}^{m} B^n(p_i, r) \right)$$

such that L_j depends only on n, k, the map f_j is identity on

$$\bigcup_{i=j+1}^{m} B^n(p_i, r) \cup \bigcup_{i=1}^{j-1} B^n(q_i, r),$$

and f_j maps $\partial B(p_j, r)$ onto $\partial B(q_j, r)$. Here we use the convention $\bigcup_{i=j+1}^{m} B^n(p_i, r) = \emptyset$ if $j = m$. Define now

$$f = f_m \circ \cdots \circ f_1 : \overline{\mathcal{S}_0} \setminus \bigcup_{i=1}^{m} B^n(q_i, r) \to \overline{\mathcal{S}_0} \setminus \bigcup_{i=1}^{m} B^n(p_i, r).$$

Note that f maps each $\partial B^n(p_i, r)$ onto $\partial B^n(q_i, r)$. Since $m \leq (2^{k+1} + 1)^n$, the bi-Lipschitz constant of f depends only on n, k. □

We can now show Lemma 3.5.

Proof of Lemma 3.5. By Lemma 3.4, $m \leq N$ for some N depending only on d and n. Applying a λ_0-bi-Lipschitz homeomorphism of $\mathbb{R}^n$ for some $\lambda_0 > 1$ depending only on λ and n, we may assume that $D_0 = \mathcal{S}_0$.

Let us outline the proof. In the first step we shrink the domains D_i isotopically to small balls with dyadic centers. In the second step we apply Lemma 3.7 to move these balls to some arranged positions and in the third step we move the balls from these positions to balls inside the cubes $\mathcal{S}_{m,k}$. In the fourth and final step we inflate these balls isotopically to cubes $\mathcal{S}_{m,i}$.

To this end, we define three collections $\{x_1, \dots, x_m\}$, $\{y_1, \dots, y_m\}$, $\{z_1, \dots, z_m\}$ of dyadic points. Let $k \in \mathbb{N}$ be the smallest integer such that

$$2^{-k} \leq (2\lambda^2 \sqrt{n}(2m + 1)d)^{-1}.$$

First, there exists $x_0 \in U$ such that

$$\operatorname{dist}(x_0, \partial U) \geq (2d)^{-1} \operatorname{diam} U = (2d)^{-1} \operatorname{diam} \mathcal{S}_0 = d^{-1}\sqrt{n}.$$

By Remark 3.6, there exists a k-dyadic point x_0' such that $|x_0' - x_0| < 2^{-k}\sqrt{n}$. By the choice of k, the k-dyadic points

$$x_i = x_0' + 2^{-k}(i-1)(1,0,\dots,0), \qquad i = 1,\dots,m$$

satisfy

$$\begin{aligned} \mathrm{dist}(x_i, \partial U) &\geq \mathrm{dist}(x_0, \partial U) - |x_0 - x_0'| - |x_0' - x_i| \\ &\geq d^{-1}\sqrt{n} - 2^{-k}\sqrt{n} - (m-1)2^{-k} \\ &\geq \tfrac{1}{2}\sqrt{n}d^{-1}. \end{aligned}$$

Second, since each domain D_i is a λ-bi-Lipschitz ball of diameter at least $d^{-1}\sqrt{n}$, there exist for each i a point $y_i' \in D_i$ such that $\mathrm{dist}(y_i', \partial D_i) \geq \lambda^{-2}d^{-1}\sqrt{n}$. By Remark 3.6, there exists a k-dyadic point y_i such that $|y_i - y_i'| < 2^{-k}\sqrt{n}$. By the choice of k, we have that $y_i \in D_i$ and

$$\mathrm{dist}(y_i, \partial D_i) \geq \mathrm{dist}(y_i', \partial D_i) - |y_i - y_i'| \geq \tfrac{1}{2}\lambda^{-2}d^{-1}\sqrt{n}.$$

Third, for each i let z_i' be the center of the open cube $\mathcal{S}_{m,i}$. By Remark 3.6, there exists a k-dyadic point z_i such that $|z_i - z_i'| < 2^{-k}\sqrt{n}$. By the choice of k,

$$\mathrm{dist}(z_i, \partial \mathcal{S}_{m,i}) \geq (2m+1)^{-1} - 2^{-k}\sqrt{n} \geq \tfrac{1}{2}(2m+1)^{-1}.$$

We now construct four bi-Lipschitz homeomorphisms. Let

$$r = \tfrac{1}{10}\min\left\{2^{-k}, \tfrac{1}{2}(2m+1)^{-1}, \tfrac{1}{2}\lambda^{-2}d^{-1}\sqrt{n}\right\} = \tfrac{1}{10}2^{-k}.$$

First, for each i, let $B^n(a_i, R_i)$ be a ball and let g_i be a λ-bi-Lipschitz homeomorphism of $\mathbb{R}^n$ mapping $B^n(a_i, R_i)$ onto D_i. Note that $\frac{2\sqrt{n}}{\lambda d} \leq R_i \leq 2\lambda\sqrt{n}$. Let $D_i' = g_i(B(a_i, (1 + (2\lambda^2 d)^{-1})))$. Then, D_i' is a λ-bi-Lipschitz ball containing D_i such that the Hausdorff distance

$$\tfrac{1}{2}(\lambda d)^{-2}\sqrt{n} \leq \mathrm{dist}_H(\partial D_i, \partial D_i') \leq \tfrac{1}{2}d^{-1}\sqrt{n}.$$

Therefore, for each i we have that $\partial D_i' \subset U$ and for all distinct i, j we have $\mathrm{dist}(\partial D_i', \partial D_j') \geq d^{-1}\sqrt{n}$. Applying Theorem 3.2 on each annulus $\overline{D_i} \setminus D_i'$ we obtain an L_1-bi-Lipschitz map

$$F_1 : \overline{U} \to \overline{\mathcal{S}_0} \setminus \bigcup_{i=1}^{m} B^n(y_i, r)$$

so that F_1 is the identity on $\overline{\mathcal{S}_0} \setminus \bigcup_{i=1}^{m} D_i'$ and maps each ∂D_i onto $\partial B^n(y_i, \frac{3}{2}r)$, with L_1 depending only on n, λ, d.

Second, for each $i = 1,\dots,m$ following the same ideas as above, we can construct an L_2-bi-Lipschitz map

$$F_2 : \overline{\mathcal{U}_m} \to \overline{\mathcal{S}_0} \setminus \bigcup_{i=1}^{m} B^n(z_i, r)$$

with $F_2(\partial \mathcal{S}_{m,i}) = \partial B(z_i, r)$, and L_2 depending only on n and d.

Third, by Lemma 3.7, there exists an L_3-bi-Lipschitz homeomorphism

$$F_3 : \overline{\mathcal{S}_0} \setminus \bigcup_{i=1}^{m} B^n(y_i, r) \to \overline{\mathcal{S}_0} \setminus \bigcup_{i=1}^{m} B^n(x_i, r)$$

with $F_3(\partial B^n(y_i, r)) = \partial B^n(x_i, r)$, and with $L_3 > 1$ depending only on n, k, hence only on n, λ, d.

Fourth, by Lemma 3.7, there exists an L_4-bi-Lipschitz homeomorphism

$$F_4 : \overline{\mathcal{S}_0} \setminus \bigcup_{i=1}^{m} B^n(z_i, r) \to \overline{\mathcal{S}_0} \setminus \bigcup_{i=1}^{m} B^n(x_i, r)$$

with $F_4(\partial B^n(z_i, r)) = \partial B^n(x_i, r)$ and L_4 depending only on n, λ, d.

The map $F_2^{-1} \circ F_4^{-1} \circ F_3 \circ F_1 : \overline{U} \to \overline{\mathcal{U}_m}$ is an L-bi-Lipschitz homeomorphism such that $F(\partial D_i) = \partial \mathcal{S}_{m,i}$ for all $i \in \{1, \dots, m\}$, and with L depending only on n, λ, and d. □

We are now ready to prove Proposition 3.3.

Proof of Proposition 3.3. Proposition 3.3 is trivial if ∂U has only one component. For the rest, we assume that ∂U has at least two components.

Since the embedding f can be extended homeomorphically to $\overline{U}$, there exists a domain $U' \subset \mathbb{R}^n$ such that $\partial U' = f(\partial U)$ and f can be extended to a homeomorphism of $\overline{U}$ onto $\overline{U'}$. Applying two similarities with comparable scaling factors, we may assume that $\operatorname{diam} U = \operatorname{diam} U' = 1$. Since f is L-bi-Lipschitz, we have that $U' \in \mathscr{U}_n(\lambda', d')$ for some $\lambda', d' > 1$ depending only on n, λ, d. By Lemma 3.5, applying two L_0-bi-Lipschitz maps, with L_0 depending only on n, λ, d, we may assume that $U = U' = \mathcal{U}_m$, that f maps $\partial \mathcal{S}_0$ onto $\partial \mathcal{S}_0$, and for each $k \in \{1, \dots, m\}$, f maps $\partial \mathcal{S}_{m,k}$ onto $\partial \mathcal{S}_{m,k}$.

Define the open cube

$$\mathcal{S}_0' := \left(\frac{1/2}{2m+1} - 1, 1 - \frac{1/2}{2m+1} \right)^n$$

and for each $k = 1, \dots, m$ define the open cube

$$\mathcal{S}_{m,k}' := \left(\frac{4k - 2m - 7/2}{2m+1}, \frac{4k - 2m - 1/2}{2m+1} \right) \times \left(-\frac{3/2}{2m+1}, \frac{3/2}{2m+1} \right)^{n-1}$$

so that $\mathcal{S}_{m,k} \subset \mathcal{S}_{m,k}' \subset \mathcal{S}_0' \subset \mathcal{S}_0$ for each $k = 1, \dots, m$; see Figure 1. Extend f to $\partial \mathcal{S}_0'$ and to each $\mathcal{S}_{m,k}'$ with identity and note that the new embedding, which we still denote by f, is L_1-bi-Lipschitz with L_1 depending only on L and m, hence only on L, n, and d.

Applying Theorem 3.2 on each $\overline{\mathcal{S}_{k,m}'} \setminus \mathcal{S}_{k,m}$ we obtain L_1'-bi-Lipschitz extensions g_k on $\overline{\mathcal{S}_{k,m}'} \setminus \mathcal{S}_{k,m}$ with L_1' depending only on n, d and L. Similarly, we obtain an L_1''-bi-Lipschitz extension g_0 on $\mathcal{S}_0 \setminus \mathcal{S}_0'$. The map

$$f : \overline{\mathcal{U}_m} \to \overline{\mathcal{U}_m} \qquad \text{with} \quad f(x) = \begin{cases} g_0(x), & \text{if } x \in \overline{\mathcal{S}_0} \setminus \mathcal{S}_0' \\ x, & \text{if } x \in \overline{\mathcal{S}_0'} \setminus \bigcup_k \mathcal{S}_{m,k}' \\ g_k(x), & \text{if } x \in \overline{\mathcal{S}_{m,k}'} \setminus \mathcal{S}_{m,k} \end{cases}$$

is L'-BLD for some L' depending only on n, L and d. By Lemma 2.2, f is L'-bi-Lipschitz. □

4. Proof of Theorem A and Corollary A

Here we prove the following quantitative version of Theorem A. The proof of Corollary A is given in §4.1.

Theorem 4.1. *Let $n \in \mathbb{N}$ and let $X \subset \mathbb{R}^n$ be a compact c-uniformly perfect and c-uniformly disconnected set. There exists an η'-quasisymmetric homeomorphism $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $F(\mathcal{C}) = X$, and with η' depending only on n and c.*

We define $\mathcal{W}$ to be the set of finite words formed from the letters $\{1, 2\}$, including the empty word ε. Define $\mathcal{W}^N$ to be the set of words in $\mathcal{W}$ whose length is exactly N. Given $w \in \mathcal{W}$, we denote by $|w|$ the number of letters that w has, with $|\varepsilon| = 0$.

Let $I_\varepsilon = [0, 1]$ and given $I_w = [a, b]$ let $I_{w1} = [a, a+\frac{1}{3}(b-a)]$, $I_{w2} = [b-\frac{1}{3}(b-a), b]$. For each $w \in \mathcal{W}$, let $\mathcal{C}_w = I_w \cap \mathcal{C}$.

Lemma 4.2. *Let X be a metric space and let $f : \mathcal{C} \to X$ be an η-quasisymmetric homeomorphism. For each $D > 1$, there exists $k \in \mathbb{N}$ depending only on η and D with the following property. For any integer $m \geq \log k/\log 2$, there exists a partition $\mathscr{E}_1, \dots, \mathscr{E}_k$ of $\mathcal{W}^m$ such that for any $i \in \{1, \dots, k\}$ and any distinct $w, w' \in \mathscr{E}_i$,*

$$\operatorname{dist}(f(\mathcal{C}_w), f(\mathcal{C}_{w'})) \geq D \max\{\operatorname{diam} f(\mathcal{C}_w), \operatorname{diam} f(\mathcal{C}_{w'})\}. \tag{4.1}$$

Proof. Set $d = (\eta^{-1}((2D)^{-1}))^{-1}$. We show that the lemma holds for k being the integer part of $2d^{\log 2/\log 3} + 1$. Fix an integer $m \geq \log k/\log 2$. Let l be the integer part of $\log d/\log 3 + 1$. By definition, $k \geq 2^l$. For distinct $u, u' \in \mathcal{W}^{m-l}$ we have

$$\operatorname{dist}(\mathcal{C}_u, \mathcal{C}_{u'}) \geq 3^{-(m-l)} \geq d3^{-m}.$$

Therefore, for all $w \in \mathcal{W}^m$, there exist at most 2^l (hence at most k) words $w' \in \mathcal{W}^m$ such that $\operatorname{dist}(\mathcal{C}_w, \mathcal{C}_{w'}) \geq d3^{-m}$.

Let now $\{w_1, \dots, w_{2^m}\}$ be an enumeration of $\mathcal{W}^m$ such that for all $1 \leq i < j \leq 2^m$, the set $\mathcal{C}_{w_i}$ lies to the left of the set $\mathcal{C}_{w_j}$. For each $i = 1, \dots, k$ define A_i to be the integers in $\{1, \dots, 2^m\}$ that are of the form $i + rk$ with $r \in \mathbb{N} \cup \{0\}$ and set $\mathscr{E}_i = \{w_j : j \in A_i\}$. It is now straightforward to see that the sets $\mathscr{E}_i$ form a partition of $\mathcal{W}^m$ and that for all $i \in \{1, \dots, k\}$ and all distinct $w, w' \in \mathcal{E}_i$,

$$\operatorname{dist}(\mathcal{C}_w, \mathcal{C}_{w'}) \geq d3^{-m} = d \max\{\operatorname{diam} \mathcal{C}_w, \operatorname{diam} \mathcal{C}_{w'}\}.$$

Now inequality (4.1) follows by quasisymmetry of f and (2.1). □

We are now ready to prove Theorem 4.1. Let us first give an informal outline of the proof. The proof consists of two steps. The first step is the construction of a bi-Lipschitz mapping $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that unlinks X. Roughly speaking, we apply Lemma 4.2 to break X into several pieces which we lift in specific heights with those being close lifted to different heights while those being far lifted to the same height. Then, we repeat the same process to each of these pieces breaking them into further smaller pieces and lifting them to specific heights. The heights will be of the form of Lipschitz functions ϕ_i that we define during the proof.

The subsets and heights from the first step are chosen carefully to make sure that the lifted subsets are contained in disjoint cubes. In the second step, we apply Proposition 3.3 to construct a quasiconformal mapping $G : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that maps the unlinked image $\Phi(X)$ onto $\mathcal{C}$. The composition $G \circ \Phi$ is the desired map F of Theorem A.

Proof of Theorem 4.1. Let X be a compact, c-uniformly perfect and c-uniformly disconnected subset of $\mathbb{R}^n$. By Theorem 1.1, there exists an η-quasisymmetric homeomorphism $f : \mathcal{C} \to X$ with η depending only on n and c. Without loss of generality, we assume that $\operatorname{diam} X = 1$. For the rest of the proof we write

$X_w = f(\mathcal{C}_w)$. Let k be the number obtained by Lemma 4.2 for $D := 1+8\sqrt{n+1}$ and for η. Let also N be the smallest positive integer such that

$$3^{-N} \leq \tfrac{1}{2}\eta^{-1}\left((20k\sqrt{n})^{-1}\right) \qquad \text{and} \qquad N \geq \log k/\log 2.$$

By (2.1) and (2.2), for any two distinct $w, w' \in \mathcal{W}$ with $|w| = |w'|$, and any $u \in \mathcal{W}^N$,

$$(2\eta(3^N))^{-1} \operatorname{diam} X_w \leq \operatorname{diam} X_{wu} \leq \eta(2\cdot 3^{-N}) \operatorname{diam} X_w, \tag{4.2}$$

$$\operatorname{dist}(X_w, X_{w'}) \geq (2\eta(1))^{-1} \max\{\operatorname{diam} X_w, \operatorname{diam} X_{w'}\}. \tag{4.3}$$

Let $\mathscr{E}_1^{\varepsilon}, \dots, \mathscr{E}_k^{\varepsilon}$ be the sets of Lemma 4.2 corresponding to f, D, and $m = N$. Define $\phi_1 : X \to \mathbb{R}$ by

$$\phi_1|_{X_w}(x) = (4k)^{-1}(i-1), \qquad \text{for } w \in \mathscr{E}_i^{\varepsilon} \text{ and } i \in \{1,\dots,k\}.$$

Inductively, suppose that for some $j \in \mathbb{N}$ we have defined $\phi_j : X \to \mathbb{R}$ such that $\phi_j|_{X_w}$ is constant whenever $w \in \mathcal{W}^{jN}$. For each $w \in \mathcal{W}^{jN}$, let $\zeta_w : \mathcal{C} \to \mathcal{C}_w$ be a similarity and note that $f|_{\mathcal{C}_w} \circ \zeta_w : \mathcal{C} \to X_w$ is η-quasisymmetric. Let $\mathscr{E}_1^w, \dots, \mathscr{E}_k^w$ be the sets of $\mathcal{W}^N$ from Lemma 4.2 applied to $f|_{\mathcal{C}_w} \circ \zeta_w$, D, and $m = N$. Define $\phi_{j+1} : X \to \mathbb{R}$ such that

$$\phi_{j+1}|_{X_{wu}}(x) = \phi_j|_{X_w}(x) + (4k)^{-1}(i-1)\operatorname{diam} X_w,$$

where $w \in \mathcal{W}^{jN}$, $u \in \mathscr{E}_i^w$ and $i \in \{1,\dots,k\}$.

By (4.2), for positive integers $j < j'$ we have

$$\begin{aligned}\|\phi_j - \phi_{j'}\|_\infty \leq \sum_{l=j}^{j'-1} \|\phi_l - \phi_{l+1}\|_\infty &\leq \sum_{l=j}^{j'-1} (4k)^{-1}(k-1) \max_{w\in\mathcal{W}^{lN}} \operatorname{diam} X_w \\ &\leq \sum_{l=j}^{j'-1} \tfrac{1}{4}\left(\eta(2\cdot 3^{-N})\right)^l \\ &\leq \frac{\left(\eta(2\cdot 3^{-N})\right)^j}{4 - 4\eta(2\cdot 3^{-N})}\end{aligned}$$

which, by choice of N, goes to 0 as $j \to \infty$. Therefore, the mappings ϕ_j converge uniformly to a mapping $\phi : X \to \mathbb{R}$.

We claim that ϕ is Lipschitz with the Lipschitz constant depending only on n and c. To see that, let $x, y \in X$ and let $j \in \mathbb{N}$ be the unique integer such that there exists $w \in \mathcal{W}^{jN}$ and there exist distinct $u, u' \in \mathcal{W}^N$ with $x \in X_{wu}$ and $y \in X_{wu'}$. On the one hand, by (4.2), (4.3),

$$|x-y| \geq \operatorname{dist}(X_{wu}, X_{wu'}) \geq \frac{1}{2\eta(1)} \min_{v\in\mathcal{W}^N} \operatorname{diam} X_{wv} \geq \frac{\operatorname{diam} X_w}{4\eta(1)\eta(3^N)}.$$

On the other hand, by (4.2) and the fact that $\phi_j(x) = \phi_j(y)$,

$$\begin{aligned}|\phi(x)-\phi(y)| &\leq |\phi(x)-\phi_j(x)| + |\phi_j(x)-\phi_j(y)| + |\phi(y)-\phi_j(y)| \\ &\leq 2(4k)^{-1}(k-1)\operatorname{diam} X_w \sum_{l=0}^{\infty} (\eta(2\cdot 3^{-N}))^l \\ &\leq \frac{1}{2-2\eta(2\cdot 3^{-N})} \operatorname{diam} X_w.\end{aligned}$$

Therefore,

$$|\phi(x)-\phi(y)| \le \frac{2\eta(1)\eta(3^N)}{1-\eta(2\cdot 3^{-N})}|x-y|$$

and the claim follows.

Fix $x_0 \in X$, set $B_0 = B^n(x_0, 5\operatorname{diam} X)$ and set $\phi|_{\mathbb{R}^n\setminus B_0} \equiv 0$. Then, the map

$$\phi : (\mathbb{R}^n \setminus B_0) \cup X \to \mathbb{R}$$

is L-Lipschitz for some L depending only on n, c and, by the Kirszbraun Theorem [7, 2.10.43], there exists an L-Lipschitz extension of ϕ to $\mathbb{R}^n$ which we also denote by ϕ. Then, the mapping

$$\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \qquad \text{defined by } \Phi(x,z) = (x, \phi(x)+z)$$

is L'-bi-Lipschitz with $L' = 2(L+2)$ [24, Lemma 5.3.2].

For each $j = 0, 1, \dots$ and each $w \in \mathcal{W}^{jN}$ fix $x_w \in X_w$ and set

$$\mathsf{K}_w := x_w + [-2\operatorname{diam} X_w, 2\operatorname{diam} X_w]^n = \{x_w + (2\operatorname{diam} X_w)y : y \in [-1,1]^n\}.$$

We claim that if $w \in \mathcal{W}^{jN}$ and $u \in \mathcal{W}^N$, then

$$\mathsf{K}_{wu} \subset \mathsf{K}_w \qquad \text{and} \qquad \operatorname{dist}(\mathsf{K}_{wu}, \partial\mathsf{K}_w) \ge \frac{1}{2}\operatorname{diam} X_w. \tag{4.4}$$

Indeed, if $x \in \mathsf{K}_{wu}$, then by (4.2) and the choice of N,

$$\begin{aligned}|x-x_w| \le |x-x_{wu}| + |x_{wu}-x_w| &\le 2\sqrt{n}\operatorname{diam} X_{wu} + \operatorname{diam} X_w \\ &\le (1+2\sqrt{n}\eta(2\cdot 3^{-N}))\operatorname{diam} X_w \\ &< \frac{3}{2}\operatorname{diam} X_w\end{aligned}$$

which proves both claims.

We remark that if $w, w' \in \mathcal{W}^{jN}$ are distinct, then K_w may intersect $\mathsf{K}_{w'}$. This is why we lift different sets X_w to different heights.

For each $j = 0, 1, \dots$ and each $w \in \mathcal{W}^{jN}$ define

$$\mathcal{K}_w := \mathsf{K}_w \times [\phi_j(x_w) - 2\operatorname{diam} X_w, \phi_j(x_w) + 2\operatorname{diam} X_w] \subset \mathbb{R}^{n+1}.$$

We first claim that for all $j \in \mathbb{N}$, for all $w \in \mathcal{W}^{jN}$ and for all $u \in \mathcal{W}^N$,

$$\mathcal{K}_{wu} \subset \mathcal{K}_w \qquad \text{and} \qquad \operatorname{dist}(\mathcal{K}_{wu}, \partial\mathcal{K}_w) \ge (8\sqrt{n+1})^{-1}\operatorname{diam}\mathcal{K}_w. \tag{4.5}$$

Let $z = (z_1, \dots, z_{n+1}) \in \mathcal{K}_{wu}$. By (4.2) and the choice of N,

$$\begin{aligned}|z_{n+1} - \phi_j(x_w)| &\le |z_{n+1} - \phi_{j+1}(x_{wu})| + |\phi_j(x_w) - \phi_{j+1}(x_{wu})| \\ &\le 2\operatorname{diam} X_{wu} + (4k)^{-1}(k-1)\operatorname{diam} X_w \\ &\le 2\eta(2\cdot 3^{-N})\operatorname{diam} X_w + \tfrac{1}{4}\operatorname{diam} X_w \\ &\le \frac{1}{2}\operatorname{diam} X_w.\end{aligned}$$

Therefore, $z \in \mathcal{K}_w$ and

$$\operatorname{dist}(z, \partial\mathcal{K}_w) \ge \max\{\operatorname{dist}(\mathsf{K}_{wu}, \partial\mathsf{K}_w), 2\operatorname{diam} X_w - |z_{n+1} - \phi_j(x_w)|\} \ge \frac{1}{2}\operatorname{diam} X_w$$

and the claim follows.

Second, we claim that for all integers $j \ge 0$, all $w \in \mathcal{W}^{jN}$ and all distinct $u, u' \in \mathcal{W}^N$,

$$\operatorname{dist}(\mathcal{K}_{wu}, \mathcal{K}_{wu'}) \ge (8\eta(3^N)\sqrt{n+1})^{-1}\operatorname{diam}\mathcal{K}_w. \tag{4.6}$$

To prove (4.6), let $x \in \mathcal{K}_{wu}$ and $x' \in \mathcal{K}_{wu'}$. There are two cases to consider.

Case 1. Suppose that $u,u' \in \mathscr{E}_i^w$. Then $\phi_{j+1}(x_{wu}) = \phi_{j+1}(x_{wu'})$ and by the choice of D we have

$$\begin{aligned}|x-x'| &\geq |x_{wu} - x_{wu'}| - \operatorname{diam}\mathcal{K}_{wu} - \operatorname{diam}\mathcal{K}_{wu'}\\ &\geq \operatorname{dist}(X_{wu}, X_{wu'}) - 2\max\{\operatorname{diam}\mathcal{K}_{wu}, \operatorname{diam}\mathcal{K}_{wu'}\}\\ &\geq (D - 8\sqrt{n+1})\max\{\operatorname{diam} X_{wu}, \operatorname{diam} X_{wu'}\}\\ &= \max\{\operatorname{diam} X_{wu}, \operatorname{diam} X_{wu'}\}.\end{aligned}$$

Case 2. Suppose that $u \in \mathscr{E}_i^w$ and $u' \in \mathscr{E}_{i'}^w$ with $i \neq i'$. By the choice of N, we calculate the vertical difference of x, x'

$$\begin{aligned}|x-x'| &\geq |\phi_{j+1}(x_{wu}) - \phi_{j+1}(x_{wu'})| - 2\operatorname{diam} X_{wu} - 2\operatorname{diam} X_{wu}\\ &\geq (4k)^{-1}\operatorname{diam} X_w - 4\max\{\operatorname{diam} X_{wu}, \operatorname{diam} X_{wu'}\}\\ &\geq \left(\frac{1}{4k\eta(2\cdot 3^{-N})} - 4\right)\max\{\operatorname{diam} X_{wu}, \operatorname{diam} X_{wu'}\}\\ &\geq \max\{\operatorname{diam} X_{wu}, \operatorname{diam} X_{wu'}\}.\end{aligned}$$

In either case, (4.6) follows now from (4.2).

Third, by (4.2), we have that for all $j \in \mathbb{N}$, for all $w \in \mathcal{W}^{jN}$ and for all $u \in \mathcal{W}^N$,

$$\operatorname{diam}\mathcal{K}_{wu} \geq (2\eta(3^N))^{-1}\operatorname{diam}\mathcal{K}_w. \tag{4.7}$$

Finally, by design of Φ,

$$\mathcal{K}_w \cap \Phi(X) = \Phi(X_w) \quad \text{and} \quad \operatorname{dist}(\Phi(X_w), \partial\mathcal{K}_w) \geq \operatorname{diam} X_w. \tag{4.8}$$

For each $j = 0, 1, \dots$ and $w \in \mathcal{W}^{jN}$, let z_w be the center of I_w, define the cube

$$\mathcal{Q}_w = \left[z_w - \tfrac{5}{6}3^{-jN}, z_w + \tfrac{5}{6}3^{-jN}\right] \times \left[-\tfrac{5}{6}3^{-jN}, \tfrac{5}{6}3^{-jN}\right]^n,$$

and let $g_w : \partial\mathcal{K}_w \to \partial\mathcal{Q}_w$ be an orientation preserving similarity map. By (4.5), (4.6) and (4.7), Proposition 3.3 applies and there exists $\Lambda > 1$ depending only on n, η, (hence only on n, c) such that g_w extends as a $(\frac{\operatorname{diam}\mathcal{Q}_w}{\operatorname{diam}\mathcal{K}_w}, \Lambda)$-quasisimilarity homeomorphism of $\mathcal{K}_w \setminus (\bigcup_{u\in\mathcal{W}^N}\mathcal{K}_{wu})$ onto $\mathcal{Q}_w \setminus (\bigcup_{u\in\mathcal{W}^N}\mathcal{Q}_{wu})$.

Define now $G : \mathbb{R}^{n+1} \setminus \Phi(X) \to \mathbb{R}^{n+1} \setminus \mathcal{C}$ so that

(1) $G : \mathbb{R}^{n+1} \setminus \mathcal{K}_\varepsilon \to \mathbb{R}^{n+1} \setminus \mathcal{Q}_\varepsilon$ is a similarity,
(2) for each $w \in \mathcal{W}^{jN}$, the restriction of G on $\mathcal{K}_w \setminus (\bigcup_{u\in\mathcal{W}^N}\mathcal{K}_{wu})$ is g_w.

Since the diameters of cubes $\mathcal{K}_w$ and $\mathcal{Q}_w$ go to zero as $|w| \to \infty$, the map G extends to a homeomorphism $G : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ mapping $\Phi(X)$ onto $\mathcal{C}$.

Let

$$E = \mathcal{C} \cup \bigcup_{j\geq 0}\bigcup_{w\in\mathcal{W}^{jn}} \partial\mathcal{Q}_w.$$

Then E is a closed set of σ-finite $\mathcal{H}^n$-measure in $\mathbb{R}^{n+1}$ and for some $K \geq 1$ depending only on c and n, G^{-1} is K-quasiconformal on $\mathbb{R}^{n+1} \setminus E$. Therefore, by a theorem of Väisälä on removability of singularities [23, Theorem 35.1], G^{-1} extends to a K-quasiconformal homeomorphism of $\mathbb{R}^{n+1}$, hence G is quasiconformal. Set $F = G \circ \Phi$ and note that F extends f. Therefore, $F(X) = \mathcal{C}$. □

4.1. **Proof of Corollary A.** For the proof of Corollary A, recall that a set $E \subset [0,1]$ is *porous* if there exists $c \geq 1$ such that for any interval $I \subset [0,1]$, there exists an interval $J \subset I \setminus E$ of length $|J| \geq c^{-1}|I|$.

Proof of Corollary A. Let $X \subset \mathbb{R}^n$ be bounded and c-uniformly disconnected. It now suffices to show that X is contained in a compact uniformly perfect and uniformly disconnected set and Theorem 4.1 applies.

Replacing X by $\overline{X}$, we may assume that X is compact. By Theorem 1 in [15] (see also [15, p. 275] for discussion and [1, Theorem 3.8] for a more general statement), there exists an η-quasisymmetric map $f : [0,1] \to \mathbb{R}^n$ such that $X \subset f([0,1])$, $\{f(0), f(1)\} \subset X$, and η depending only on c. Set $E = f^{-1}(X) \subset [0,1]$ which is uniformly disconnected and contains 0 and 1.

For each component I of $[0,1]\setminus E$, let $\phi_I : \mathbb{R} \to \mathbb{R}$ be a similarity that maps $[0,1]$ onto $\overline{I}$. Define now

$$E' := E \cup \bigcup_I \phi_I(\mathcal{C}) = \overline{\bigcup_I \phi_I(\mathcal{C})},$$

where the union is over all components I of $[0,1]\setminus X$. We claim that E' is uniformly perfect and uniformly disconnected. Assuming the latter, $X' = f(E')$ is a compact uniformly perfect, and uniformly disconnected set that contains X. By Theorem A, there exists an η-quasisymmetric map $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $F(X) \subset F(X') = \mathcal{C}$.

To prove the claim, recall that $\mathcal{C}$ is C_0-uniformly perfect and C_0-uniformly disconnected for some $C_0 > 1$.

To show that E' is uniformly perfect, fix $x \in E'$ and $r \in (0,1)$. We claim that there exists universal $C \geq 1$ such that

$$E' \cap ((x-r, x+r) \setminus (x - r/C, x + r/C)) \neq \emptyset.$$

If $(x-r, x+r) \cap E = \emptyset$, then there exists I as above such that $(x-r, x+r) \subset I$ and $(x-r, x+r) \cap E' = (x-r, x+r) \cap \phi_I(\mathcal{C})$ and the claim is true for $C = C_0$. Suppose now that $(x-r, x+r) \cap E \neq \emptyset$. If there exists $z \in (x-r, x+r) \cap E$ with $|z - x| \geq r/2$, then the claim is true for $C = 2$. Suppose now that $(x - r/2, x + r/2) \cap E = (x-r, x+r) \cap E \neq \emptyset$ and let $z \in [x, x + r/2) \cap E$ such that $z - x$ is maximal. That is, z is the left endpoint of a component I as above and by the uniform perfectness of $\mathcal{C}$,

$$\phi_I(\mathcal{C}) \cap (x + (2C_0)^{-1}r, x + r) \supset \phi_I(\mathcal{C}) \cap (z + (2C_0)^{-1}r, z + r/2) \neq \emptyset$$

and the claim holds true for $C = 2C_0$.

To show that E' is uniformly disconnected, by [15, Theorem 1], we need to show that E' is porous. Fix an interval $I \subset [0,1]$. The porosity of E implies that there exists an interval $J' \subset I \setminus E$ such that $|J'| \geq |I|/c_1$ where c_1 depends only on C. Now, the porosity of $J' \cap E'$ (since it is a subset of a copy of $\mathcal{C}$) implies that there exists an interval $J \subset J' \setminus E'$ such that $|J| \geq |J'|/c_0$ where c_0 depends only on C_0. Alltogether, $J \subset I \setminus E'$ and $|J| \geq (c_1 c_0)^{-1}|I|$. $\square$

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Department of Mathematics, The University of Tennessee, Knoxville, Tennessee 37916

Email address: vvellis@utk.edu