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FIXED POINTS OF KOCH MAPS

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Abstract. We study endomorphisms constructed by Sarah Koch in [Teichmüller theory and critically finite endomorphisms, *Advances in Mathematics* **248** (2013), 573–617] and we focus on the eigenvalues of the differential of such maps at its fixed points. To each post-critically finite unicritical polynomial, Koch associated a post-critically algebraic endomorphism of $\mathbb{CP}^k$. Koch showed that the eigenvalues of the differentials of such maps along periodic cycles outside the post-critical sets have modulus strictly greater than 1. In this article, we show that the eigenvalues of the differentials at fixed points are either 0 or have modulus strictly greater than 1. This confirms a conjecture proposed by the author in his thesis. We also provide a concrete description of such values in terms of the multiplier of a unicritical polynomial.

1. Introduction

Let M be either $\mathbb{C}^n$ or $\mathbb{CP}^n$ and $f\colon M \to M$ be a holomorphic endomorphism. Denote by $f^{\circ m} = f \circ f \circ \ldots \circ f$ the m-th composition of f. A point $z \in M$ is called a *preperiodic point* of *preperiod* k and of *period* m if $f^{\circ(k+m)}(z) = f^{\circ k}(z)$ and k, m are the smallest integers satisfying such a property. A preperiodic point of preperiod 0 is called a *periodic point*. A periodic point of period 1 is called a *fixed point*. Given a periodic point z of period m, a value $\lambda \in \mathbb{C}$ is called *an eigenvalue of f along the orbit of z* (or *at the fixed point z*) if λ is an eigenvalue of the differential $D_z f^{\circ m}\colon T_z M \to T_z M$.

A point $z \in M$ is called *a critical point* if the differential $D_z f\colon T_z M \to T_{f(z)} M$ is not invertible. The set $C(f)$ containing all critical points of f is called the *critical set* of f. The set

$$PC(f) := \bigcup_{j \geq 1} f^{\circ j}(C(f))$$

is called the *post-critical set* of f. The endomorphism f is called *post-critically algebraic* if $PC(f)$ is an algebraic set of codimension one in M. When $\dim M = 1$, post-critically algebraic rational maps are called *post-critically finite* rational maps.

The family of post-critically finite rational maps is one of the most important families of maps in the theory of one dimensional complex dynamics. In higher dimension, post-critically algebraic endomorphisms are interesting family of maps since many results, which are well-known for post-critically finite rational maps, remain unknown. We refer to [14],[1],[6],[5], [7] [11] for some recent studies about

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post-critically algebraic endomorphisms. In this article, we focus on Conjecture 1 proposed by the author in his thesis [10].

Conjecture 1. *Let f be a post-critically algebraic endomorphism of $\mathbb{CP}^n, n \geq 2$ of degree $d \geq 2$ and λ be an eigenvalue of f along a periodic cycle. Then either $\lambda = 0$ or $|\lambda| > 1$.*

The conjecture has been verified by the author in the case $n = 2$ and in the case in any dimension with the periodic cycles outside the post-critical set. In this article, we shall verify the conjecture for the family of post-critically algebraic endomorphisms associated to unicritical polynomials constructed by Sarah Koch in [8], or *Koch maps* for short.

We shall now describe the family of Koch maps we want to study and we refer to [8] for the original construction. Throughout this article, we fix

$$d \in \mathbb{N}, d \geq 2 \text{ and } \beta^d = 1, \beta \neq 1.$$

The maps $\{G_{k,m} \colon \mathbb{C}^{k+m-1} \to \mathbb{C}^{k+m-1} \mid (k,m) \in \mathbb{N} \times \mathbb{N}^*\}$ constructed by Sarah Koch are of the following forms (see [8, Proposition 6.1 - 6.2]),

- if $k = 0$,

$$G_{0,m} \colon \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \end{pmatrix} \mapsto \begin{pmatrix} -x_{m-1}^d \\ x_1^d - x_{m-1}^d \\ \vdots \\ x_{m-2}^d - x_{m-1}^d \end{pmatrix},$$

- if $k \neq 0$,

$$G_{k,m} \colon \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+m-1} \end{pmatrix} \mapsto \begin{pmatrix} \left(-\frac{\beta x_{k+m-1} - x_{k-1}}{\beta - 1}\right)^d \\ \left(x_1 - \frac{\beta x_{k+m-1} - x_{k-1}}{\beta - 1}\right)^d \\ \vdots \\ \left(x_{k+m-2} - \frac{\beta x_{k+m-1} - x_{k-1}}{\beta - 1}\right)^d \end{pmatrix}.$$

The map $G_{k,m}$ induces a holomorphic endomorphism of $\mathbb{CP}^{k+m-2}$ which is closely related to maps on moduli spaces used in Thurston's topological characterization of rational maps. We refer to [8],[9],[3] for further discussion. In [8], Koch showed that $G_{k,m}$ is post-critically algebraic. It is natural to ask whether Conjecture 1 is true for $G_{k,m}$. The eigenvalues of Koch maps along a periodic cycle outside the post-critical set are well understood. It is a consequence of its construction that those values have modulus strictly bigger than 1.

Theorem 1 ([8, Corollary 7.2])**.** *Let μ be an eigenvalue of $G_{k,m}$ along a periodic cycle outside the post-critical set. Then $|\mu| > 1$.*

We refer also to [2] for a further discussion about the arithmetics of such values. In [8], Koch asked whether we have the same conclusion for eigenvalues along a cycle inside the post-critical set. In this article, we answer this question in the positive (and hence verify Conjecture 1) for the case when the cycle is a fixed point.

Theorem A. *Let μ be an eigenvalue of a map $G_{k,m}$ at a fixed point. Then, either $\mu = 0$ or $|\mu| > 1$.*

In fact, we can have even better understanding about the values of such eigenvalues. Thanks to Theorem 1, we only need to study the eigenvalues at a fixed point inside the post-critical set of $G_{k,m}$. However, the original construction does not provide much information about fixed points inside post-critical set. In order to explain our result, let us take a closer look at $G_{k,m}$.

Let $(k,m) \in \mathbb{N} \times \mathbb{N}^*$. To a point $z \in \mathbb{C}^{k+m-1}$, we can associate a polynomial P_z of the following form

$$P_z(t) = \begin{cases} t^d - z_{m-1}^d \text{ if } k = 0 \\ \left(t - \frac{\beta z_{k+m-1} - z_{k-1}}{\beta - 1}\right)^d \text{ if } k \neq 0. \end{cases}$$

If z is a fixed point of $G_{k,m}$, then for all $1 \leq i \leq k+m-1$, $z_i = P_z^{\circ i}(0)$. Moreover, P_z is post-critically finite. Indeed,

- if $k = 0$ then $P_z(z_{m-1}) = 0$,
- if $k \neq 0$ then

$$\begin{aligned} P_z(z_{k+m-1}) &= \left(z_{k+m-1} - \frac{\beta z_{k+m-1} - z_{k-1}}{\beta - 1}\right)^d = \left(\frac{-z_{k+m-1} + z_{k-1}}{\beta - 1}\right)^d \\ &= \left(\frac{\beta z_{k-1} - \beta z_{k+m-1}}{\beta - 1}\right)^d = P_z(z_{k-1}). \end{aligned}$$

We shall call P_z the *polynomial associated to z* since P_z plays an important role in the study of the eigenvalues of $G_{k,m}$ at a fixed point z. More precisely, our main result, which completes the description of eigenvalues of Koch maps at fixed points, is the following.

Theorem A'. *Let μ be an eigenvalue of a map $G_{k,m}$ at a fixed point $z = (z_1, \ldots, z_{k+m-1})$. Let P_z be the polynomial associated to z and z_1 is preperiodic of preperiod k' and of period m' to a cycle of multiplier λ under P_z. Then only one of the following cases happens:*

1. $\mu = 0$.
2. *μ is an eigenvalue of a map $G_{k',m'}$ at a fixed point outside the post-critical set $PC(G_{k',m'})$.*
3. *We have*
$$\mu^m = \lambda^{\frac{m}{m'}}, \mu^{m'} \neq \lambda.$$

We can see that Theorem A is a direct consequence of Theorem A'. Indeed, if Case 1 or Case 2 happens, Theorem A follows from Theorem 1. If Case 3 happens, since the polynomial P_z is a post-critically finite polynomial, Theorem A follows from the equation $\mu^m = \lambda^{\frac{m}{m'}}$ and the fact that a non-vanishing multiplier of a post-critically finite polynomial has modulus strictly bigger than 1 (see [12, Corollary 14.5]).

Let us explain briefly our approach. Instead of using the original construction of $G_{k,m}$, we introduce

- a partial order $\preceq$ on $\mathbb{N} \times \mathbb{N}^*$,
- a dynamically equivalent family of maps, that we denote by

$$\{F_{k,m} \colon \mathcal{M}_{k,m} \to \mathcal{M}_{k,m}, (k,m) \in \mathbb{N} \times \mathbb{N}^*\},$$

where $\mathcal{M}_{k,m}$ is a subspace of the vector spaces of complex sequences $\mathbb{C}^{\mathbb{N}^*}$.

More precisely, with the convention $x_0 := 0$, the space $\mathcal{M}_{k,m}$ is defined as

$$\mathcal{M}_{k,m} = \{\boldsymbol{x} = (x_i)_{i\geq 1} \mid \forall\, i \geq k+1, x_i = x_{i+m} \text{ and } \beta x_{k+m} - x_k = 0\}$$

and the map $F_{k,m}\colon \mathcal{M}_{k,m} \to \mathcal{M}_{k,m}$ is defined as

$$F_{k,m}(\boldsymbol{x}) = \boldsymbol{y} \Leftrightarrow \begin{cases} \beta y_{k+m} - y_k = 0 \\ y_i = x_{i-1}^d + y_1 \quad \text{for all } i \geq 2. \end{cases}$$

The construction shall be presented in details in Section 2. For each $(k,m) \in \mathbb{N}\times\mathbb{N}^*$, the maps $F_{k,m}$ and $G_{k,m}$ are conjugate. Thus, to prove Theorem A', we need to study an eigenvalue μ of $F_{k,m}$ at a fixed point $\boldsymbol{z} \in \mathcal{M}_{k,m}$. The full statement of what we can prove is Theorem B'. Briefly, to each fixed point $\boldsymbol{z}$ in $\mathcal{M}_{k,m}$, we associate a pair of integers (k',m') and a post-critically finite polynomial P_z whose critical value is preperiodic of preperiod k' to a cycle of period m'. The associated polynomial P_z for a $\boldsymbol{z} = (z_i)_{i\geq 1} \in \mathcal{M}_{k,m}$ is simply $P_z(t) = t^d + z_1$. The partial order characterizes the following property of the family $\{F_{k,m}\}$:

$$(k',m') \preceq (k,m) \Leftrightarrow \begin{cases} \mathcal{M}_{k',m'} \subseteq \mathcal{M}_{k,m}, \\ F_{k,m}|_{\mathcal{M}_{k',m'}} = F_{k',m'}. \end{cases}$$

Moreover, we shall show that $\boldsymbol{z} \notin PC(F_{k',m'})$. Thus, if $(k',m') = (k,m)$, our fixed point $\boldsymbol{z}$ is outside the post-critical set $PC(F_{k,m})$ and we are in Case 2.

If $(k',m') \neq (k,m)$, then $\mathcal{M}_{k',m'}$ is a proper subset of $\mathcal{M}_{k,m}$ which is invariant under $F_{k,m}$ and the restriction of $F_{k,m}$ to $\mathcal{M}_{k',m'}$ is exactly $F_{k',m'}$. If μ has associated eigenvectors tangent to $\mathcal{M}_{k',m'}$, since $\boldsymbol{z} \notin PC(F_{k',m'})$, we are again in the Case 2. Otherwise, μ is the eigenvalue of the transpose $D_{\boldsymbol{z}}F_{k,m}^*$ of the derivative $D_{\boldsymbol{z}}F_{k,m}$ acting on the annihilator $\mathcal{M}_{k',m'}^0 = \{\omega \in \mathcal{M}_{k,m}^* \mid \omega|_{\mathcal{M}_{k',m'}} = 0\}$. In such a case, either $\mu = 0$, i.e. Case 1, or we will show that $\mathcal{M}_{k',m'}^0$ has a set of generators on which $D_{\boldsymbol{z}}F_{k,m}^*$ acts cyclically (Lemma 4.14 and Lemma 4.17) and we obtain Case 3 by solving a linear algebra problem.

Let us elaborate on the motivation of this approach. The main objective is to characterize when a fixed point belongs to the post-critical set. More precisely, in [8], Koch used the Thurston's pull-back map induced by a topological polynomial f on the oriented topological 2-sphere. When f is unicritical, the inverse of Thurston's pull-back map induces a map on moduli spaces which extends holomorphically to an endomorphism G_f of $\mathbb{CP}^n$ (the form in homogeneous coordinates of G_f is $G_{k,m}$ in this article). The properties of G_f which are most important to us are that it is non-trivial (see [8, Proposition 5.11]) and it is post-critically algebraic (see [8, Theorem 5.18]). However, this construction provides little information regarding eigenvalues at fixed points on the post-critical set. A major difficulty when tackling Conjecture 1 for this family is to give a simple characterization for when a fixed point belongs to the post-critical set, which is in fact the boundary of moduli spaces in the compactification $\mathbb{CP}^n$ (see also [9, Conjecture 8.2.1]). Thus, we seek for another construction. Since there always exists a unicritical polynomial realizing the action of f on its post-critical set ([4, Theorem 1]), we choose a normal form of a degree $d \geq 2$ unicritical polynomial by setting the critical points to be $\{0,\infty\}$, i.e. $P(t) = t^d + c, c \in \mathbb{C}$ and we want to obtain a post-critically finite polynomial. Instead of finding only the parameter c, we look for the sequence defined by the orbit of c, i.e. the sequence $\boldsymbol{z} = (z_i) = (P^{\circ i}(0))_{i\geq 1}$. Such a sequence needs to satisfy two conditions:

(1) $z_{i+1} = z_i^d + c$.
(2) For some $(k,m) \in \mathbb{N} \times \mathbb{N}^*$, $z_{k+1} = z_{k+m+1}$.

The integers k, m are defined by the action of f on $PC(f)$. Then, the parameter c can be computed in terms of $\boldsymbol{z}, k, m$ (See Lemma 2.3). Moreover, by substituting (1) in (2), we deduce that

$$z_k^d = z_{k+m}^d \Leftrightarrow \exists \beta^d = 1, \beta \neq 1, z_k = \beta z_{k+m}.$$

Among $d-1$ choices of β, there exists a root β encoding the combinatorial information of dynamics of f. It is uniquely determined by the so-called Hurwitz class of f (see [8, Example 2.13 and Proposition 5.3]). Thus, for each d-th root of unity $\beta \neq 1$ and a pair $(k,m) \in \mathbb{N} \times \mathbb{N}^*$, we can construct a map $F_{k,m}$ on a space of sequences $\mathcal{M}_{k,m}$, whose fixed points are the orbits of critical values of unicritical polynomials. Then, as explained in the previous paragraph, we can characterize the relative position between fixed points of $F_{k,m}$ and the post-critical set $PC(F_{k,m})$ in terms of some discrete information and overcome the aforementioned difficulty in the original construction.

2. An alternative construction of Koch maps

2.1. Construction of $F_{k,m}$ and $\mathcal{M}_{k,m}$. Recall that in this article, we fix

$$d \in \mathbb{N}, d \geq 2 \text{ and } \beta^d = 1, \beta \neq 1.$$

Let $(k,m) \in \mathbb{N} \times \mathbb{N}^*$ and denote by $\mathcal{E} = \mathbb{C}^{\mathbb{N}^*}$ the vector space of complex sequences $\boldsymbol{x} = (x_i)_{i \geq 1}$. Set $\mathbf{0} := (0, 0, \ldots)$. Let $\mathcal{L} \subset \mathcal{E}$ be the one-dimensional subspace consisting of constant sequences,

$$\mathcal{L} = \{\boldsymbol{x} \in \mathcal{E} \mid \forall i, j \geq 1, \quad x_i = x_j\},$$

and $\mathcal{H}_{k,m} \subset \mathcal{E}$ be the hyperspace of $\mathcal{E}$ defined by

$$\mathcal{H}_{k,m} := \{\boldsymbol{x} \in \mathcal{E} \mid \beta x_{k+m} - x_k = 0\},$$

with the convention $x_0 := 0$. In particular, when $k = 0$,

$$\mathcal{H}_{0,m} = \{\boldsymbol{x} \in \mathcal{E} \mid x_m = 0\}.$$

Lemma 2.1. *Given $(k,m) \in \mathbb{N} \times \mathbb{N}^*$, we have $\mathcal{E} = \mathcal{H}_{k,m} \oplus \mathcal{L}$.*

Proof. On the one hand, given $\boldsymbol{x} \in \mathcal{E}$, define $\boldsymbol{y} \in \mathcal{E}$ by

$$y_i := x_i - \kappa \text{ with } \kappa = \begin{cases} x_m & \text{if } k = 0 \\ \frac{\beta x_{k+m} - x_k}{\beta - 1} & \text{if } k \geq 1 \end{cases}.$$

Then $\beta y_{k+m} - y_k = 0$ hence $\boldsymbol{y} \in \mathcal{H}_{k,m}$. Note that $\boldsymbol{x} - \boldsymbol{y} \in \mathcal{L}$ hence

$$\mathcal{E} = \mathcal{H}_{k,m} + \mathcal{L}.$$

On the other hand, assume $\boldsymbol{x} \in \mathcal{H}_{k,m} \cap \mathcal{L}$. Then $\beta x_{k+m} - x_k = 0$ and $x_{k+m} = x_k$. Since $\beta \neq 1$, we have $x_k = x_{k+m} = 0$. Moreover, $\boldsymbol{x}$ is a constant sequence. Thus, $\boldsymbol{x}$ vanishes identically; that is

$$\mathcal{H}_{k,m} \cap \mathcal{L} = \{\mathbf{0}\}.$$

□

Denote by

$$\pi_{k,m}\colon \mathcal{E} \to \mathcal{E}$$

the projection to $\mathcal{H}_{k,m}$ parallel to $\mathcal{L}$. In particular, $\pi_{k,m}(\mathcal{E}) = \mathcal{H}_{k,m}$. Consider the map $\mathcal{Q} : \mathcal{E} \to \mathcal{E}$ defined by

$$\mathcal{Q}(\boldsymbol{x}) = \boldsymbol{y} \quad \text{with} \quad y_1 := 0 \text{ and } y_i = x_{i-1}^d, i \geq 1.$$

Definition 2.2. Given $(k, m) \in \mathbb{N} \times \mathbb{N}^*$, the map $F_{k,m} := \mathcal{E} \to \mathcal{E}$ is defined as

$$F_{k,m} := \pi_{k,m} \circ \mathcal{Q}.$$

We shall now study some important properties of $F_{k,m}$.

2.1.1. *Properties of $F_{k,m}$.*

Lemma 2.3. *Given $(k, m) \in \mathbb{N} \times \mathbb{N}^*$, for every $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}$, we have*

$$F_{k,m}(\boldsymbol{x}) = \boldsymbol{y} \Leftrightarrow \begin{cases} \beta y_{k+m} - y_k = 0 \\ y_i = x_{i-1}^d + y_1 \quad \textit{for all } i \geq 2. \end{cases}$$

In particular, with the convention $x_0 := 0$, we have

$$y_1 = \begin{cases} -x_{m-1}^d & \textit{if } k = 0, \\ -\frac{\beta x_{k+m-1}^d - x_{k-1}^d}{\beta - 1} & \textit{if } k \geq 1. \end{cases}$$

Proof. Assume $\boldsymbol{x} \in \mathcal{E}$ and $\boldsymbol{y} = \pi_{k,m}(\mathcal{Q}(\boldsymbol{x})) = F_{k,m}(\boldsymbol{x})$. On the one hand, $\boldsymbol{y} \in \pi_{k,m}(\mathcal{E}) = \mathcal{H}_{k,m}$, i.e.

$$\beta y_{k+m} - y_k = 0.$$

On the other hand, set $\boldsymbol{z} = \mathcal{Q}(\boldsymbol{x})$, i.e. $z_1 = 0$ and $z_i = x_{i-1}^d$ for all $i \geq 1$. Then since $\boldsymbol{y} = \pi_{k,m}(\boldsymbol{z})$, for all $i \geq 1$,

$$y_i = z_i - \kappa \text{ with } \kappa := \begin{cases} z_m & \text{if } k = 0 \\ \frac{\beta z_{k+m} - z_k}{\beta - 1} & \text{if } k \geq 1 \end{cases}.$$

In particular,

$$y_1 = z_1 - \kappa = -\kappa = \begin{cases} -x_{m-1}^d & \text{if } k = 0, \\ -\frac{\beta x_{k+m-1}^d - x_{k-1}^d}{\beta - 1} & \text{if } k \geq 1. \end{cases}$$

whence for all $i \geq 2$, $y_i = z_i + y_1$, i.e.

$$y_i = x_{i-1}^d + y_1.$$

Conversely, assume $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}$ such that $\beta y_{k+m} - y_k = 0$ and for all $i \geq 2$, $y_i = x_{i-1}^d + y_1$. In particular, $\boldsymbol{y} \in \mathcal{H}_{k,m}$. Set $\boldsymbol{z} = \boldsymbol{y} - \mathcal{Q}(\boldsymbol{x})$. Then for all $i \geq 2$,

$$z_i = y_i - x_{i-1}^d = y_1.$$

Moreover, $z_1 = x_0^d + y_1 = y_1$. Hence $\boldsymbol{z} \in \mathcal{L}$. In other words,

$$\boldsymbol{y} = \pi_{k,m}(\mathcal{Q}(\boldsymbol{x})) = F_{k,m}(\boldsymbol{x}).$$

□

Although $F_{k,m}$ is defined on a vector space of infinite dimension, we will now see that the dynamics of $F_{k,m}$ is captured entirely by some finite dimensional vector space. Given a sequence $\boldsymbol{x} \in \mathcal{E}$, $\boldsymbol{x}$ is preperiodic of preperiod k to a cycle of period m if for all $i \geq k+1, x_i = x_{i+m}$ and k, m are the smallest integers satisfying such a property. Given $(k, m) \in \mathbb{N} \times \mathbb{N}^*$, let $\mathcal{P}_{k,m} \subset \mathcal{E}$ be the subspace of preperiodic sequences of preperiod at most k to a cycle of period dividing m, i.e.

$$\mathcal{P}_{k,m} := \{\boldsymbol{x} \in \mathcal{E} \mid x_{i+m} = x_i \text{ for } i \geq k+1\}.$$

Since sequences in $\mathcal{P}_{k,m}$ are uniquely determined by the first $k+m$ entries, the vector space $\mathcal{P}_{k,m}$ has finite dimension $k+m$.

Definition 2.4. Given $(k,m) \in \mathbb{N}\times\mathbb{N}^*$, define

$$\mathcal{M}_{k,m} := \mathcal{P}_{k,m} \cap \mathcal{H}_{k,m}.$$

Note that the constant sequence $(1,1,\ldots)$ is in $\mathcal{P}_{k,m}\setminus\mathcal{H}_{k,m}$ and that $\mathcal{H}_{k,m}$ has codimension one in $\mathcal{E}$. Hence, $\mathcal{M}_{k,m}$ is a vector space of dimension $k+m-1$. Lemma 2.5 and Lemma 2.6 show the importance of $\mathcal{M}_{k,m}$ to the dynamics of $F_{k,m}$.

Lemma 2.5. *We have $F_{k,m}(\mathcal{M}_{k,m}) = \mathcal{M}_{k,m}$ and $F_{k,m}\colon \mathcal{M}_{k,m}\to\mathcal{M}_{k,m}$ is a nondegenerate homogeneous map of degree d.*

Proof. Let us first prove that $\mathcal{Q}(\mathcal{M}_{k,m}) \subseteq \mathcal{P}_{k,m}$. Assume $\boldsymbol{x}\in\mathcal{M}_{k,m}$. Since $\boldsymbol{x}\in\mathcal{P}_{k,m}\cap\mathcal{H}_{k,m}$, $\boldsymbol{x}$ has preperiod k and period dividing m and $\beta x_{k+m} - x_m = 0$. In particular, since $\beta^d = 1$, we have $x_{k+m}^d = x_m^d$. Consequently, setting $\boldsymbol{y} := \mathcal{Q}(\boldsymbol{x})$,

$$\begin{cases} y_{k+1} = x_k^d + y_1 = x_{k+m}^d + y_1 = y_{k+m+1} & \\ y_i = x_{i-1}^d + y_1 = x_{m+i-1}^d + y_1 = y_{i+m} & \text{for all } i \geq k+2. \end{cases}$$

Thus $\mathcal{Q}(\boldsymbol{x}) \in \mathcal{P}_{k,m}$.

Since $\pi_{k,m}(\mathcal{P}_{k,m}) \subset \mathcal{P}_{k,m}$, $F_{k,m}(\mathcal{M}_{k,m}) \subset \mathcal{P}_{k,m}$. Since $F_{k,m}(\mathcal{E}) \subset \mathcal{H}_{k,m}$,

$$F_{k,m}(\mathcal{M}_{k,m}) \subset \mathcal{P}_{k,m}\cap\mathcal{H}_{k,m} = \mathcal{M}_{k,m}.$$

Clearly, the map $\mathcal{Q}$ is homogeneous of degree d and the map $\pi_{k,m}$ is homogeneous of degree 1, thus $F_{k,m}$ is homogeneous of degree d.

Let us now prove that $F_{k,m}$ is nondegenerate, i.e. $F_{k,m}^{-1}(\mathbf{0}) = \{\mathbf{0}\}$. Assume $\boldsymbol{x}\in\mathcal{M}_{k,m}$ and

$$\pi_{k,m}\circ\mathcal{Q}(\boldsymbol{x}) = F_{k,m}(\boldsymbol{x}) = 0.$$

Then $\boldsymbol{y} := \mathcal{Q}(\boldsymbol{x}) \in \operatorname{Ker}(\pi_{k,m}) = \mathcal{L}$. By definition of $\mathcal{Q}$, $y_1 = 0$. Since $\boldsymbol{y}\in\mathcal{L}$, $\boldsymbol{y}$ is a constant sequence thus $\boldsymbol{y} = \mathbf{0}$. This implies that $x_i = 0$ for all $i \geq 1$, i.e. $\boldsymbol{x} = \mathbf{0}$.

Since $\mathcal{M}_{k,m}$ has finite dimension and $F_{k,m}\colon\mathcal{M}_{k,m}\to\mathcal{M}_{k,m}$ is homogeneous and nondegenerate, $F_{k,m}$ is surjective, i.e. $F_{k,m}(\mathcal{M}_{k,m}) = \mathcal{M}_{k,m}$. □

Lemma 2.6. *We have that $\bigcap_{n\geq1} F_{k,m}^{\circ n}(\mathcal{E}) = \mathcal{M}_{k,m}$.*

Proof. According to Lemma 2.5, $F_{k,m}(\mathcal{M}_{k,m}) = \mathcal{M}_{k,m}$. Thus,

$$\mathcal{M}_{k,m} \subseteq \bigcap_{n\geq1} F_{k,m}^{\circ n}(\mathcal{E}).$$

Conversely, it is enough to prove that

$$\bigcap_{n\geq2} F_{k,m}^{\circ n}(\mathcal{E}) \subset \mathcal{P}_{k,m}. \tag{2.1}$$

Indeed, since $F_{k,m}(\mathcal{E}) \subset \mathcal{H}_{k,m}$ and $\mathcal{M}_{k,m} = \mathcal{H}_{k,m}\cap\mathcal{P}_{k,m}$, the inclusion (2.1) implies that

$$F_{k,m}(\mathcal{E}) \cap \bigcap_{n\geq2} F_{k,m}^{\circ n}(\mathcal{E}) \subseteq \mathcal{H}_{k,m}\cap\mathcal{P}_{k,m},$$

and hence

$$\bigcap_{n\geq1} F_{k,m}^{\circ n}(\mathcal{E}) \subseteq \mathcal{M}_{k,m} \subseteq \bigcap_{n\geq1} F_{k,m}^{\circ n}(\mathcal{E}).$$

To prove (2.1), we show the following claim: for all $n\geq 2$, if $\boldsymbol{y}\in F_{k,m}^{\circ n}(\mathcal{E})$, then $y_{i+m} = y_i$ for all $i\in\{k+1,\ldots,k+n-1\}$.

Let us prove this claim by induction in n. If $n = 2$, assume $\boldsymbol{y} \in F_{k,m}^{\circ 2}(\mathcal{E})$, i.e. $\boldsymbol{y} = F_{k,m}^{\circ 2}(\boldsymbol{x})$ for some $\boldsymbol{x} \in \mathcal{E}$. Setting $\boldsymbol{z} = F_{k,m}(\boldsymbol{x})$, according to Lemma 2.3, we have $\beta z_{k+m} - z_k = 0$. Thus, since $\boldsymbol{y} = F_{k,m}(\boldsymbol{z})$ and since $\beta^d = 1$, also according to Lemma 2.3, we have

$$y_{k+1+m} = z_{k+m}^d + y_1 = z_k^d + y_1 = y_{k+1},$$

i.e. the claim is true for $n = 2$.

Assume that it holds for some $n > 2$. Assume $\boldsymbol{y} \in F_{k,m}^{\circ(n+1)}(\mathcal{E})$, i.e. $\boldsymbol{y} = F_{k,m}(\boldsymbol{x})$ with $\boldsymbol{x} \in F_{k,m}^{\circ n}(\mathcal{E})$. The induction hypothesis implies that

$$x_{i+m-1} = x_{i-1} \text{ for all } i \in \{k+2, \ldots, k+n\},$$

so that

$$y_{i+m} = y_i \text{ for all } i \in \{k+2, \ldots, k+n\}.$$

In addition, since $\boldsymbol{x} \in \mathcal{H}_{k,m}$, $\beta x_{k+m} = x_k$, so that $y_{k+m+1} = y_{k+1}$. Thus the claim is true for $n+1$. □

2.2. **The main result about $F_{k,m}$.** Lemma 2.5 and Lemma 2.6 allow us to restrict our study to the dynamics of $F_{k,m}$ on $\mathcal{M}_{k,m}$. With a slight abuse of notations, from now on, we shall denote by $F_{k,m}$ the restriction of $F_{k,m} \colon \mathcal{E} \to \mathcal{E}$ to $\mathcal{M}_{k,m}$. The following result sums up the properties of $F_{k,m} \colon \mathcal{M}_{k,m} \to \mathcal{M}_{k,m}$ which are important for us.

Theorem B'. *Given $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ and let $\boldsymbol{z} \in \mathcal{M}_{k,m} \setminus \{\boldsymbol{0}\}$ be a fixed point of the map $F_{k,m} \colon \mathcal{M}_{k,m} \to \mathcal{M}_{k,m}$. Let k' be the preperiod of the sequence $\boldsymbol{z}$ and m' be its period. Then,*

(1) *the polynomial $P(t) = t^d + z_1 \in \mathbb{C}[t]$ is post-critically finite and we have $\boldsymbol{z} = (P^{\circ j}(0))_{j \geq 1}$; in particular, the critical value z_1 of P is preperiodic of preperiod k' to a cycle of period m' of multiplier λ,*
(2) *there exists a partial order $\preceq$ on $\mathbb{N} \times \mathbb{N}^*$ such that $(k', m') \preceq (k, m)$ if and only if $\mathcal{M}_{k',m'} \subseteq \mathcal{M}_{k,m}$ is a $F_{k,m}$-invariant subspace and the restriction of $F_{k,m}$ to $\mathcal{M}_{k',m'}$ is $F_{k',m'}$,*
(3) *if $k + m - 1 \geq 2$, $F_{k,m} \colon \mathcal{M}_{k,m} \to \mathcal{M}_{k,m}$ and $G_{k,m} \colon \mathbb{C}^{k+m-1} \to \mathbb{C}^{k+m-1}$ are holomorphically conjugate,*
(4) $\operatorname{Spec} D_{\boldsymbol{z}} F_{k',m'} \subset \mathbb{C} \setminus \overline{\mathbb{D}}$,
(5) *If $(k, m) \neq (k', m')$,*

$$\operatorname{Spec} (D_{\boldsymbol{z}} F_{k,m})^* |_{(\mathcal{M}_{k',m'})^0} = \begin{cases} \{0\} & \text{if } k' = 0 \\ \{\mu \mid \mu^m = \lambda^{\frac{m}{m'}}, \mu^{m'} \neq \lambda\} & \text{if } k' \neq 0 \end{cases},$$

where $\mathcal{M}_{k',m'}^0 = \{\omega \in \mathcal{M}_{k,m}^* \mid \omega|_{\mathcal{M}_{k',m'}} \equiv 0\}$ is the annihilator of $\mathcal{M}_{k',m'}$ in $\mathcal{M}_{k,m}$.

In particular, we shall see that Theorem A' is a direct consequence of Theorem B'. The rest of this article is devoted to the proof of Theorem B'. Item 1 is proved in Proposition 4.4. The partial order $\preceq$ will be introduced in Definition 3.2 and item 2 will be proved in Proposition 3.3. Item 3 is proved in Lemma 3.1. Item 4 is due to Koch, and we recall its proof in Proposition 4.9 for the sake of completeness. Our main contribution is the proof of item 5 which will be proved in Proposition 4.12. Finally, we prove Theorem A' by using Theorem B'.

3. Dynamics of $F_{k,m}$

3.1. $F_{k,m}$ is conjugate to $G_{k,m}$. Lemma 3.1 assures that the class of maps $F_{k,m}$ for $(k,m) \in \mathbb{N} \times \mathbb{N}^*$ is a good alternative when one wants to study $G_{k,m}$.

Lemma 3.1. *Let $(k,m) \in \mathbb{N} \times \mathbb{N}^*$ be such that $k+m-1 \geq 2$, the maps $F_{k,m}$ and $G_{k,m}$ are holomorphically conjugate.*

Proof. Recall that when $k=0, m \geq 3$, we have

$$G_{0,m} : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \end{pmatrix} \mapsto \begin{pmatrix} -x_{m-1}^d \\ x_1^d - x_{m-1}^d \\ \vdots \\ x_{m-2}^d - x_{m-1}^d \end{pmatrix},$$

and when $k \geq 1, m \geq 2$, we have

$$G_{k,m} : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+m-1} \end{pmatrix} \mapsto \begin{pmatrix} \left(-\frac{\beta x_{k+m-1} - x_{k-1}}{\beta - 1}\right)^d \\ \left(x_1 - \frac{\beta x_{k+m-1} - x_{k-1}}{\beta - 1}\right)^d \\ \vdots \\ \left(x_{k+m-2} - \frac{\beta x_{k+m-1} - x_{k-1}}{\beta - 1}\right)^d \end{pmatrix}.$$

Let $i_{k,m} \colon \mathcal{M}_{k,m} \to \mathbb{C}^{k+m-1}, i_{k,m}(\boldsymbol{x}) = (x_1, \ldots, x_{k+m-1})$ and set

$$\widetilde{F_{k,m}} = i_{k,m} \circ F_{k,m} \circ i_{k,m}^{-1}.$$

Then we have when $k=0$,

$$\widetilde{F_{0,m}} : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \end{pmatrix} \mapsto \begin{pmatrix} -x_{m-1}^d \\ x_1^d - x_{m-1}^d \\ \vdots \\ x_{m-2}^d - x_{m-1}^d \end{pmatrix},$$

and when $k \geq 1$,

$$\widetilde{F_{k,m}} : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+m-1} \end{pmatrix} \mapsto \begin{pmatrix} -\frac{\beta x_{k+m-1}^d - x_{k-1}^d}{\beta - 1} \\ x_1^d - \frac{\beta x_{k+m-1}^d - x_{k-1}^d}{\beta - 1} \\ \vdots \\ x_{k+m-2}^d - \frac{\beta x_{k+m-1}^d - x_{k-1}^d}{\beta - 1} \end{pmatrix}.$$

It is enough to show that $G_{k,m}$ and $\widetilde{F_{k,m}}$ are conjugate. Indeed, let $\tau \colon \mathbb{C}^{k+m-1} \to \mathbb{C}^{k+m-1}$ be a linear map of the following form

$$\tau \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+m-1} \end{pmatrix} \mapsto \begin{pmatrix} \tau_1 \\ x_1 + \tau_1 \\ \vdots \\ x_{k+m-2} + \tau_1 \end{pmatrix} \text{ with } \tau_1 = \begin{cases} -x_{m-1} \text{ when } k = 0 \\ -\frac{\beta x_{k+m-1} - x_{k-1}}{\beta - 1} \text{ when } k \geq 1 \end{cases}$$

and set

$$\mathfrak{d}(x_1, \ldots, x_{k+m-1}) = (x_1^d, \ldots, x_{k+m-1}^d)$$

then $G_{k,m} = \mathfrak{d} \circ \tau, \widetilde{F_{k,m}} = \tau \circ \mathfrak{d}$. Thus,

$$\tau \circ G_{k,m} = \widetilde{F_{k,m}} \circ \tau.$$

Note that τ is an isomorphism, whence $G_{k,m}$ and $\widetilde{F_{k,m}}$ are conjugate. □

3.2. **Comparing $F_{k,m}$ by a partial order $\preceq$.** Our initial expectation was that for arbitrary pairs (k_1, m_1) and (k_2, m_2) in $\mathbb{N} \times \mathbb{N}^*$, the maps F_{k_1,m_1} and F_{k_2,m_2} would agree on the intersection $\mathcal{M}_{k_1,m_1} \cap \mathcal{M}_{k_2,m_2}$. However this is not true as shown in the following example. Consider the case $d = 2$ and $\beta = -1$, the sequence

$$\boldsymbol{x} := \{2, 0, 0, \ldots, 0, \ldots\}$$

Then $\boldsymbol{x} \in \mathcal{M}_{2,1} \cap \mathcal{M}_{3,1}$ and

$$\mathcal{Q}(\boldsymbol{x}) = \{0, 4, 0, 0, 0, 0, 0, \ldots\}.$$

However,

$$F_{2,1}(\boldsymbol{x}) = \{-2, 2, -2, -2, -2, -2, \ldots\} \text{ and } F_{3,1}(\boldsymbol{x}) = \{0, 4, 0, 0, 0, 0, 0, \ldots\}.$$

We will now see that if some order $(k', m') \preceq (k, m)$ is satisfied, with the fixed d and β, we have $\mathcal{M}_{k',m'} \subseteq \mathcal{M}_{k,m}$ and $F_{k',m'}$ is the restriction of $F_{k,m}$ to $\mathcal{M}_{k',m'}$.

Definition 3.2. Let $\preceq$ be the partial order on $\mathbb{N} \times \mathbb{N}^*$ defined by

$$(k', m') \preceq (k, m) \Leftrightarrow \begin{cases} m' \text{ divides } m, \\ \text{either } k' = k \text{ or } (k' = 0 \text{ and } m' \text{ divides } k). \end{cases}$$

The strict order $\prec$ is defined by

$$(k', m') \prec (k, m) \Leftrightarrow (k', m') \preceq (k, m) \text{ and } (k', m') \neq (k, m).$$

Proposition 3.3. *For two pairs of integers $(k, m), (k', m') \in \mathbb{N} \times \mathbb{N}^*$, we have that*

$$(k', m') \preceq (k, m) \Leftrightarrow \mathcal{M}_{k',m'} \subseteq \mathcal{M}_{k,m}$$

Moreover, if $\mathcal{M}_{k',m'} \subseteq \mathcal{M}_{k,m}$ then $F_{k,m}|_{\mathcal{M}_{k',m'}} = F_{k',m'}$.

Proof. We first prove that

$$(k, m) \preceq (k', m') \Leftrightarrow \mathcal{M}_{k',m'} \subseteq \mathcal{M}_{k,m}.$$

Assume that $(k', m') \preceq (k, m)$. We shall prove that $\mathcal{M}_{k',m'} \subseteq \mathcal{M}_{k,m}$. Assume $\boldsymbol{x} \in \mathcal{M}_{k',m'}$. Then $\boldsymbol{x}$ is preperiodic of period less than k' to a cycle of period dividing m'. Since $k' \leq k$ and $m' \mid m$, we deduce that $\boldsymbol{x} \in \mathcal{P}_{k,m}$. We need to show that $\beta x_{k+m} - x_k = 0$. Indeed,

- if $k' = k$, since $m' \mid m$ and $\boldsymbol{x} \in \mathcal{M}_{k',m'}$, we have $x_{k+m} = x_{k'+m'}$. Thus $\beta x_{k+m} - x_k = \beta x_{k'+m'} - x_{k'} = 0$,
- if $k' = 0$, in that case $m' \mid k$ and $\boldsymbol{x}$ is periodic of period m'. Thus $x_{k+m} = x_k = 0$.

Let us now assume that $\mathcal{M}_{k',m'} \subseteq \mathcal{M}_{k,m}$. We claim that $(k', m') \preceq (k, m)$. Indeed,

- either $k' = 0$; in this case, consider $\boldsymbol{x} \in \mathcal{M}_{0,m'}$ given by $x_i = 0$ if $m' \mid i$ and 1 otherwise. Since $\boldsymbol{x} \in \mathcal{M}_{k,m}$, $\beta x_{k+m} - x_k = 0$ with $\beta \neq 1$. Then necessarily, $x_{k+m} = x_k = 0$ thus m' divides k and m.
- or $k' \geq 1$; in this case, consider $\boldsymbol{x} \in \mathcal{M}_{k',m'}$ given by $x_{k'} = \beta, x_{k'+jm'} = 1$ for $j \geq 1$ and $x_i = 2$ otherwise. If $\beta x_{k+m} - x_k = 0$ then $x_k = \beta$ and $x_{k+m} = 1$. Hence $k = k'$ and $m' \mid m$.

We assume now $\mathcal{M}_{k',m'} \subseteq \mathcal{M}_{k,m}$, or equivalently, $(k',m') \preceq (k,m)$. Let us prove that the restriction of $F_{k,m}$ to $\mathcal{M}_{k',m'}$ is $F_{k',m'}$. Assume $\boldsymbol{x} \in \mathcal{M}_{k',m'}$. Set $\boldsymbol{y} = F_{k',m'}(x)$ and $\boldsymbol{z} = F_{k,m}(\boldsymbol{x})$. According to Lemma 2.3, for all $i \geq 2$, $y_i = x_{i-1}^d + y_1, z_i = x_{i-1}^d + z_1$ where

$$y_1 = \begin{cases} -x_{m'-1}^d & \text{if } k' = 0 \\ -\frac{\beta x_{k'+m'-1}^d - x_{k'-1}^d}{\beta - 1} & \text{if } k' \geq 1 \end{cases} \text{ and } z_1 = \begin{cases} -x_{m-1}^d & \text{if } k = 0 \\ -\frac{\beta x_{k+m-1}^d - x_{k'-1}^d}{\beta - 1} & \text{if } k \geq 1. \end{cases}$$

It is enough to prove that $y_1 = z_1$.

- Case $k' = 0$. In that case, k and m are multiples of m'. If $k = 0$ then

$$y_1 = -x_{m'-1}^d = -x_{m-1}^d = z_1.$$

If $k \neq 0$, $x_{k+m-1} = x_{k-1} = x_{m'-1}$ hence

$$y_1 = -x_{m'-1}^d = -\frac{\beta x_{k+m-1}^d - x_{k-1}^d}{\beta - 1} = z_1.$$

- Case $k' \neq 0$. In that case, $k' = k$ and m is a multiple of m'. Then

$$x_{k+m-1} = \begin{cases} \frac{1}{\beta} x_{k'+m'-1} & \text{if } m' - 1 = 0 \text{ and } m - 1 \geq 1 \\ x_{k'+m'-1} & \text{otherwise.} \end{cases}$$

[1] Then

$$y_1 = -\frac{\beta x_{k'+m'-1}^d - x_{k'-1}^d}{\beta - 1} = -\frac{\beta x_{k+m-1}^d - x_{k-1}^d}{\beta - 1} = z_1. \qquad \square$$

3.3. **The post-critical set of $F_{k,m}$.** In this section, we fix a pair of integers $(k,m) \in \mathbb{N} \times \mathbb{N}^*$. Recall that

$$C(F_{k,m}) := \text{the critical set of } F_{k,m} \colon \mathcal{M}_{k,m} \to \mathcal{M}_{k,m},$$

$$CV(F_{k,m}) := \text{ the critical value set of } F_{k,m} \colon \mathcal{M}_{k,m} \to \mathcal{M}_{k,m}$$

and

$$PC(F_{k,m}) := \text{ the post-critical set of } F_{k,m} \colon \mathcal{M}_{k,m} \to \mathcal{M}_{k,m}.$$

Lemma 3.4. *We have that*

$$C(F_{k,m}) = \{\boldsymbol{x} \in \mathcal{M}_{k,m} \mid x_i = 0 \text{ for some } 1 \leq i \leq k+m-1\}.$$

Proof. Recall that $F_{k,m} = \pi_{k,m} \circ \mathcal{Q}$. Differentiating both sides, we see that for any $\boldsymbol{x} \in \mathcal{M}_{k,m}$ and for any $\boldsymbol{v} \in T_{\boldsymbol{x}}\mathcal{E} = \mathcal{E}$,

$$D_{\boldsymbol{x}} F_{k,m}(\boldsymbol{v}) = \pi_{k,m} \circ D_{\boldsymbol{x}} \mathcal{Q}(\boldsymbol{v}) = \pi_{k,m}(0, dx_1^{d-1} v_1, dx_2^{d-1} v_2, \ldots).$$

On the one hand, assume $\boldsymbol{x} \in C(F_{k,m})$. Then there exists $\boldsymbol{v} \in T_{\boldsymbol{x}}\mathcal{M}_{k,m} \setminus \{\boldsymbol{0}\}$ such that $x_i^{d-1} v_i = 0$ for all $i \geq 1$. Observe that there exists $i \in \{1, \ldots, k+m-1\}$ such that $v_i \neq 0$ whence $x_i = 0$. Indeed otherwise, $v_{k+m} = \frac{1}{\beta} v_k = 0$ and by preperiodicity, $v_i = 0$ for all $i \geq 1$.

On the other hand, given $\boldsymbol{x} \in \mathcal{E}$, define $\boldsymbol{v} \in \mathcal{E}$ by

$$v_j = \begin{cases} \beta & \text{if } x_j = 0 \text{ and } j = k \\ 1 & \text{if } x_j = 0 \text{ and } j \neq k \\ 0 & \text{if } x_j \neq 0 \end{cases}.$$

[1] Note that if $m' = 1$, $\boldsymbol{x} \in \mathcal{M}_{k',m'}$ implies that for all $i \geq 1$, $x_{k'+i} = x_{k'+1}$. In particular, with $k = k'$, $x_{k+m-1} = x_{k'+1} = \frac{1}{\beta} x_{k'}$

Then $x_j^{d-1}v_j = 0$ for all $j \geq 1$ so that $D_{\boldsymbol{x}}F_{k,m}(\boldsymbol{v}) = \mathbf{0}$. Moreover, if $\boldsymbol{x} \in \mathcal{M}_{k,m}$ then $\boldsymbol{v} \in \mathcal{M}_{k,m}$. Finally, if there exists $i \in \{1, \ldots, k+m-1\}$ such that $x_i = 0$ then $\boldsymbol{v} \neq \mathbf{0}$ whence $\boldsymbol{x} \in C(F_{k,m})$. □

Definition 3.5. Denote by

$$\Delta_{k,m} = \{\boldsymbol{x} \in \mathcal{M}_{k,m} \mid \text{ there exists } 1 \leq i < j \leq k+m \text{ such that } x_i = x_j\}$$

The set $\Delta_{k,m}$ consists of $\binom{k+m}{2}$ hyperplanes.

Proposition 3.6. *We have that* $CV(F_{k,m}) \subseteq \Delta_{k,m}$ *and* $F_{k,m}(\Delta_{k,m}) \subseteq \Delta_{k,m}$. *Consequently,* $PC(F_{k,m}) \subseteq \Delta_{k,m}$.

Proof. Let $\boldsymbol{x} \in C(F_{k,m})$ and set $\boldsymbol{y} = F_{k,m}(\boldsymbol{x})$. Then by Lemma 3.4, there exists $i \in \{1, \ldots, k+m-1\}$ such that $x_i = 0$. By Lemma 2.3, we have

$$y_{i+1} = x_i^d + y_1 = y_1$$

Thus $\boldsymbol{y} \in \Delta_{k,m}$, whence $CV(F_{k,m}) \subseteq \Delta_{k,m}$.

Now we prove that $\Delta_{k,m}$ is invariant under $F_{k,m}$. Assume $\boldsymbol{x} \in \Delta_{k,m}$ and set $\boldsymbol{y} = F_{k,m}(\boldsymbol{x})$. Then there exist $1 \leq i < j \leq k+m$ such that $x_i = x_j$. By Lemma 2.3, for every $l \geq 2, y_l = x_{l-1}^d + y_1$. Note that since $\boldsymbol{x} \in \mathcal{M}_{k,m}$, $\beta x_{k+m} - x_k = 0$ with the convention $x_0 := 0$.

- If $j \leq k+m-1$, we have $y_{i+1} = x_i^d + y_1 = x_j^d + y_1 = y_{j+1}$.
- If $j = k+m$, then
  - either $i = k$ so that $x_k = x_i = x_j = x_{k+m}$; since $\beta x_{k+m} - x_k = 0$ and $\beta \neq 1$, $x_k = 0$ whence $\boldsymbol{x} \in C(F_{k,m})$ and $\boldsymbol{y} \in \Delta_{k,m}$;
  - or $i \neq k$ so that $i+1 \neq k+1$; since $x_k = \beta x_{k+m} = \beta x_i$, we have

$$y_{i+1} = x_i^d + y_1 = x_k^d + y_1 = y_{k+1}.$$

Hence, in any case, we have $\boldsymbol{y} \in \Delta_{k,m}$, i.e. $F_{k,m}(\Delta_{k,m}) \subset \Delta_{k,m}$ and the lemma is proved. □

4. Fixed points of Koch maps

In this section, we shall study the eigenvalues of the derivative of $F_{k,m}$ at its fixed points and we will prove Theorem B'. Then, we deduce Theorem A' by using Theorem B'.

4.1. **Relation with post-critically finite polynomials.** There is a close connection between fixed points $F_{k,m}$ and post-critically finite polynomials. More precisely, we will consider monic centered unicritical polynomials of degree $d \geq 2$,

$$P(t) = t^d + c \in \mathbb{C}[t], c \in \mathbb{C}.$$

The critical orbit of such a polynomial is the sequence $\boldsymbol{c}_P \in \mathcal{E}$ defined by

$$\boldsymbol{c}_P = (c_i)_{i \geq 1} \in \mathcal{E} \text{ where } c_i = P^{\circ i}(0).$$

Since the preperiod and the period of a preperiodic sequence will be extensively discussed in this chapter, we introduce the following notions.

Definition 4.1. Given integers $k \geq 0, m \geq 1$, a sequence $\boldsymbol{x} \in \mathcal{E}$ is called *preperiodic of type* (k, m) if for every $i \geq k+1$, $x_{i+m} = x_i$, *preperiodic of **exact** type* (k, m) if, additionally, k and m are the smallest integers satisfying such conditions.

For a sequence of exact type (k,m), the pair (k,m) consists of the preperiod k and the period m. The vector space $\mathcal{P}_{k,m}$ is the space of preperiodic sequences of type (k,m).

Definition 4.2. A *degree d polynomial of (exact) type* (k,m) is a monic centered unicritical polynomial P of degree $d \geq 2$ whose critical orbit $\boldsymbol{c}_P$ is of (exact) type (k,m).

In other words, a polynomial is of type (k,m) if and only if its critical orbit belongs to $\mathcal{P}_{k,m}$. Note that a polynomial of type (k,m) is post-critically finite.

Remark 4.3. Let P be a polynomial of type (k,m) of degree d. If $k=0$, then the critical value c of P is a periodic point of period dividing m, i.e. $P^{\circ m}(c)=c$. In other words, $P^{\circ(m-1)}(c) \in P^{-1}(c)$. However, since P is a unicritical polynomial, $P^{-1}(c)$ consists of exactly one point which is the critical point of P. This means that $P^{\circ(m-1)}(c)$ is in fact the critical point of P. This is the case if and only if the critical point of P is also a periodic point of type $(0,m)$.

Proposition 4.4. *Given $(k,m) \in \mathbb{N}\times\mathbb{N}^*$. Let $\boldsymbol{z} \in \mathcal{M}_{k,m}$ be a fixed point of $F_{k,m}$. Set $P(t)=t^d+z_1$. Then $\boldsymbol{z}=\boldsymbol{c}_P$ and P is of type (k,m). Let (k',m') be the exact type of P. Then $(k',m') \preceq (k,m)$.*

Proof. First, according to Lemma 2.3, for every $i \geq 2$, we have that $z_i = z_{i-1}^d + z_1$ hence

$$z_i = P(z_{i-1}).$$

In other words, $\boldsymbol{z}$ is the sequence of iterates of z_1 under P, or $\boldsymbol{z}=\boldsymbol{c}_P$. The sequence $\boldsymbol{z} \in \mathcal{M}_{k,m} \subset \mathcal{P}_{k,m}$ is of type (k,m). Hence, by definition, the polynomial P is a polynomial of type (k,m).

Second, regarding the exact type (k',m') of P, we prove that $(k',m') \preceq (k,m)$, i.e.

$$\begin{cases} m' \mid m \\ \text{either } k'=k \text{ or } (k'=0 \text{ and } m' \mid k). \end{cases}$$

Since (k',m') is the exact type of the orbit of z_1, $k' \leq k$ and $m' \mid m$. If $k'=k$, we are done. If $k' \neq k$, we need to prove that $k'=0$ and $m' \mid k$. Since (k',m') is the exact type of $\boldsymbol{z}$, we have $\boldsymbol{z} \in \mathcal{P}_{k',m'}$. Thus, $k'+1 \leq k$ and $m' \mid m$; and since $\boldsymbol{z} \in \mathcal{P}_{k',m'}$, this implies that $z_{k+m}=z_k$. Moreover, $\boldsymbol{z} \in \mathcal{M}_{k,m}$ implies that $\beta z_{k+m}-z_k=0$. The last two equalities imply that $z_k=0$. Therefore, $P^{\circ k}(0)=z_k=0$. In other words, 0 is a periodic point of P, i.e. $k'=0$, and the period of 0 is m'. Moreover, $P^{\circ k}(0)=0$ also implies that k is a multiple of m'. Thus, we can conclude that $(k',m') \preceq (k,m)$. □

Remark 4.5. The converse statement of Proposition 4.4 is true under some assumptions on the choice of the root of unity β. More precisely, given a post-critically finite unicritical polynomial $P(t)=t^d+z_1$ of type (k,m), there exists a d-th root of unity $\beta' \neq 1$ such that the critical orbit $\boldsymbol{c}_P$ of P is a fixed point of $F_{k,m}$.

The partial order $\preceq$ enables us to study the relative positions of the fixed points of $F_{k,m}$ and $\Delta_{k,m}$.

Lemma 4.6. *Let $\boldsymbol{z}$ be a fixed point of $F_{k,m}$ and let (k',m') be the exact type of $\boldsymbol{z}$. Then, $\boldsymbol{z} \in \Delta_{k,m}$ if and only if $(k',m') \prec (k,m)$.*

Proof. Assume $\boldsymbol{z} \in \Delta_{k,m}$, i.e. there exists $1 \leq i < j \leq k+m$ such that $z_i = z_j$. In particular, $\boldsymbol{z}$ is a preperiodic sequence of preperiod at most $i-1$ and of period dividing $j-i$. Whence, since (k', m') is the exact type of $\boldsymbol{z}$, we have $k' \leq i-1$ and m' divides $j-i$.

- If $i \leq k$ then $k' \leq i-1 < k$.
- If $i \geq k+1$ then $j - i \leq k + m - (k+1) < m$. Since $m' \mid j-i$, we have $m' < m$.

In both cases, we have $(k', m') \neq (k, m)$. Note that, according to Proposition 4.4, $(k', m) \preceq (k, m)$. Hence $(k', m') \prec (k, m)$.

Conversely, assume $(k', m') \prec (k, m)$. In particular, $k' \leq k$, $m' \leq m$ and $(k', m') \neq (k, m)$. Note that $\boldsymbol{z}$ is of exact type (k', m'). Hence,

$$z_{k'+1} = z_{k'+m'+1}.$$

If $k' \neq k$ then $k' < k$. Whence $k'+1$ and $k'+m'+1$ are integers in $\{1, \ldots, k+m\}$. If $k' = k$ then $m' < m$. In this case, $k+1, k+m'+1$ are also in $\{1, \ldots, k+m\}$. Therefore, in both cases, we deduce by that $\boldsymbol{z} \in \Delta_{k,m}$. □

4.2. **Eigenvalues of moduli maps at fixed points.** In order to study the eigenvalues of the derivative of moduli maps at one of its fixed point, we will in fact study its transpose. Note that when $k+m = 1$, $\mathcal{M}_{k,m} = \{\mathbf{0}\}$ and $F_{k,m}$ is trivial. Let us fix $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ such that $k+m \geq 2$. Assume that $\boldsymbol{z} \in \mathcal{M}_{k,m}$ is a fixed point of $F_{k,m}$. We will describe the transpose of the derivative $D_{\boldsymbol{z}} F_{k,m} \colon T_{\boldsymbol{z}}\mathcal{M}_{k,m} \to T_{\boldsymbol{z}}\mathcal{M}_{k,m}$. Since $\mathcal{M}_{k,m}$ is a vector space, there is a canonical identification of $T_{\boldsymbol{z}}\mathcal{M}_{k,m}$ with $\mathcal{M}_{k,m}$, the derivative $D_{\boldsymbol{z}} F_{k,m} \colon T_{\boldsymbol{z}}\mathcal{M}_{k,m} \to T_{\boldsymbol{z}}\mathcal{M}_{k,m}$ identifies with a linear map

$$L : \mathcal{M}_{k,m} \to \mathcal{M}_{k,m},$$

and the transpose identifies with the pull-back map of L

$$L^* \colon \mathcal{M}_{k,m}^* \to \mathcal{M}_{k,m}^*.$$

4.2.1. *The dual space $\mathcal{M}_{k,m}^*$.* For $i \geq 1$, let $\omega_i \in \mathcal{M}_{k,m}^*$ be the linear form defined by for all $\boldsymbol{v} \in \mathcal{M}_{k,m}$,

$$\omega_i(\boldsymbol{v}) := v_i.$$

Lemma 4.7. *The family $\{\omega_i, 1 \leq i \leq k+m-1\}$ is a basis of $\mathcal{M}_{k,m}^*$.*

Proof. Note that $\dim \mathcal{M}_{k,m} = k+m-1$ hence it is enough to prove that

$$\{\omega_1, \ldots, \omega_{k+m-1}\}$$

are linearly independent. Assume that

$$\sum_{1 \leq i \leq k+m-1} \lambda_i \omega_i = 0 \text{ with } \lambda_i \in \mathbb{C}.$$

Let $i \geq 1$. To prove that $\lambda_i = 0$, consider the vector $\boldsymbol{v} \in \mathcal{M}_{k,m}$ defined by

- if $i < k$, $v_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise,} \end{cases}$
- if $i = k$, $v_j = \begin{cases} 1 & \text{if } j = i = k \\ \frac{1}{\beta} & \text{if } j > k \text{ and } j \equiv k \mod m \\ 0 & \text{otherwise,} \end{cases}$

- if $i > k$, $v_j = \begin{cases} 1 & \text{if } j \geq i \text{ and } j \equiv i \mod m \\ 0 & \text{otherwise.} \end{cases}$

In any case, we have

$$0 = \sum_{1 \leq i \leq k+m-1} \lambda_i \omega_i(\boldsymbol{v}) = \lambda_i v_i = \lambda_i. \qquad \square$$

4.2.2. *The transpose of the derivative.* Observe that $L^* \colon \mathcal{M}_{k,m}^* \to \mathcal{M}_{k,m}^*$ is the pull-back of forms, i.e. for all $\omega \in \mathcal{M}_{k,m}^*$,

$$L^* \omega = \omega \circ L.$$

For all $i \geq 1$, set

$$\delta_i = dz_i^{d-1}$$

where $\boldsymbol{z} = (z_1, z_2, \ldots) \in \mathcal{M}_{k,m}$ is the considered fixed point of $F_{k,m}$.

Lemma 4.8. *We have that*

$$L^* \omega_1 = \begin{cases} -\delta_{m-1} \omega_{m-1} & \text{if } k = 0 \\ -\frac{\beta \delta_{k+m-1} \omega_{k+m-1} - \delta_{k-1} \omega_{k-1}}{\beta - 1} & \text{otherwise,} \end{cases}$$

and for all $i \geq 2$,

$$L^* \omega_i = \delta_{i-1} \omega_{i-1} + L^* \omega_1.$$

Proof. Recall that for all $\boldsymbol{v} \in \mathcal{M}_{k,m}$,

$$L(\boldsymbol{v}) \quad = \quad \pi_{k,m} \circ D_{\boldsymbol{z}} \mathcal{Q}(\boldsymbol{v}).$$

Set $\boldsymbol{u} = D_{\boldsymbol{z}} \mathcal{Q}(\boldsymbol{v})$ then

$$u_1 = 0 \quad \text{and for all } i \geq 2, \quad u_i = dz_{i-1}^{d-1} v_{i-1} = \delta_{i-1} v_{i-1}.$$

In addition, if $\boldsymbol{w} := L(\boldsymbol{v}) = \pi_{k,m}(\boldsymbol{u})$ then

$$\text{for all } i \geq 2, \quad w_i = u_i + w_1 \text{ with } w_1 = \begin{cases} -u_m & \text{if } k = 0 \\ -\frac{\beta u_{k+m-1} - u_{k-1}}{\beta - 1} & \text{otherwise,} \end{cases}$$

Combining those formulas, we obtain that $\boldsymbol{w} = L(\boldsymbol{v})$ and for all $i \geq 2$,

$$w_i = \delta_{i-1} v_{i-1} + w_1 \text{ with } w_1 = \begin{cases} -\delta_{m-1} v_{m-1} & \text{if } k = 0 \\ -\frac{\beta \delta_{k+m-1} v_{k+m-1} - \delta_{k-1} v_{k-1}}{\beta - 1} & \text{otherwise,} \end{cases} \tag{4.1}$$

We deduce that for all $\boldsymbol{v} \in \mathcal{M}_{k,m}$,

$$L^* \omega_1(\boldsymbol{v}) = \omega_1 \circ L(\boldsymbol{v}) = w_1 = \begin{cases} -\delta_{m-1} v_{m-1} & \text{if } k = 0 \\ -\frac{\beta \delta_{k+m-1} \omega_{k+m-1} - \delta_{k-1} \omega_{k-1}}{\beta - 1} & \text{otherwise,} \end{cases}$$

hence

$$L^* \omega_1 = \begin{cases} -\delta_{m-1} \omega_{m-1} & \text{if } k = 0 \\ -\frac{\beta \delta_{k+m-1} \omega_{k+m-1} - \delta_{k-1} \omega_{k-1}}{\beta - 1} & \text{otherwise.} \end{cases}$$

In addition, for all $i \geq 2$ and for all $\boldsymbol{v} \in \mathcal{M}_{k,m}$, we have

$$L^* \omega_i(\boldsymbol{v}) = \omega_i \circ L(\boldsymbol{v}) = \delta_{i-1} v_{i-1} + w_1 = \delta_{i-1} v_{i-1} + \omega_1 \circ L(\boldsymbol{v}),$$

hence

$$L^* \omega_i = \delta_{i-1} \omega_{i-1} + L^* \omega_1. \qquad \square$$

4.2.3. *Fixed points outside the post-critical set.* According to Section 3.1, the map $F_{k,m}$ is conjugate to the map $G_{k,m}$ constructed by Koch [8]. By [8, Corollary 7.2], the derivative of $G_{k,m}$ at its fixed points outside the post-critical set has only eigenvalues of modulus strictly greater than 1, whence so does $F_{k,m}$. For the sake of completeness, we give here the proof of this property. For further discussion about the arithmetics of such eigenvalues, we refer to [2]. The main content of this paragraph is the following result.

Proposition 4.9. *Let $(k,m) \in \mathbb{N} \times \mathbb{N}^*$ and $\boldsymbol{z} \notin PC(F_{k,m})$ be a fixed point of the moduli map $F_{k,m}$. Then every eigenvalue of $D_{\boldsymbol{z}} F_{k,m}$ has modulus strictly greater than* 1.

Proof. Since $\mathcal{M}_{k,m}$ has finite dimension, it is suffice to prove that every eigenvalue of the transpose L^* of $D_{\boldsymbol{z}} F_{k,m}$ has modulus strictly bigger than 1.

Recall that, by Lemma 4.7, the family $\{\omega_i \colon \mathcal{M}_{k,m} \to \mathbb{C}\}_{i \in \{1,\dots,k+m-1\}}$ is a basis of $\mathcal{M}_{k,m}^*$. According to Lemma 4.8, setting $\delta_i = dz_i^{d-1}$, we have

$$L^*\omega_1 = \begin{cases} -\delta_{m-1}\omega_{m-1} & \text{if } k = 0 \\ -\dfrac{\beta\delta_{k+m-1}\omega_{k+m-1} - \delta_{k-1}\omega_{k-1}}{\beta - 1} & \text{if } k \geq 1, \end{cases}$$

and for all $i \geq 2$,

$$L^*\omega_i = \delta_{i-1}\omega_{i-1} + L^*\omega_1.$$

According to Lemma 4.6, the point $\boldsymbol{z}$ is a fixed point of $F_{k,m}$ of exact type (k,m). Therefore, $\delta_i \neq 0$ for all $i \in \{1,\dots,k+m-1\}$. Indeed, note that according to Proposition 4.4, the sequence $\boldsymbol{z}$ is the critical orbit of $P(t) = t^d + z_1$. Assume that $dz_i^{d-1} = \delta_i = 0$ for some $i \in \{1,\dots,k+m-1\}$. Then $P^{\circ i}(0) = z_i = 0$. This implies that $k = 0$ and m divides i. However $i \leq k+m-1 = m-1 < m$, hence contradiction.

We may therefore define a linear map $L_* : \mathcal{M}_{k,m}^* \to \mathcal{M}_{k,m}^*$ by

$$\forall i \in \{1,\dots,k+m-1\} \quad L_*(\omega_i) = \frac{\omega_{i+1} - \omega_1}{\delta_i}. \tag{4.2}$$

Lemma 4.10. *The linear map L^* is invertible and its inverse is L_*.*

Proof. Observe that for all $i \in \{1,\dots,k+m-1\}$,

$$\begin{aligned} L^* \circ L_*(\omega_i) = L^*\left(\tfrac{\omega_{i+1}-\omega_1}{\delta_i}\right) &= \tfrac{1}{\delta_i}\left(L^*(\omega_{i+1}) - L^*(\omega_1)\right) \\ &= \tfrac{1}{\delta_i}\big(\delta_i\omega_i + L^*(\omega_1) - L^*(\omega_1)\big) = \omega_i. \end{aligned}$$

By Lemma 4.7, $L^* \circ L_* = \mathrm{Id}_{\mathcal{M}_{k,m}^*}$. Since $\mathcal{M}_{k,m}^*$ is of finite dimension, we deduce that $L_* \circ L^* = L^* \circ L_* = \mathrm{Id}_{\mathcal{M}_{k,m}^*}$.

Thus, the linear map $L_* \colon \mathcal{M}_{k,m}^* \to \mathcal{M}_{k,m}^*$ is indeed the inverse of L^* □

In order to prove Proposition 4.9, it is therefore enough to prove that every eigenvalue of $L_* \colon \mathcal{M}_{k,m}^* \to \mathcal{M}_{k,m}^*$ is contained in the open unit disc $\mathbb{D}$. Inspired by the proof of [8, Corollary 7.2], we will show that L_* is conjugate to a linear transformation on a space of meromorphic quadratic differentials on $\mathbb{C}$, whose eigenvalues are all contained in $\mathbb{D}$.

Consider the quadratic polynomial $P(t) := t^d + z_1$, so that $z_i = P^{\circ i}(0)$ for all $i \geq 1$. Following Milnor [13], denote by $\mathfrak{Q}(\mathbb{C})$ the space of meromorphic quadratic differentials on $\mathbb{C}$ which have at worst simple poles and let us use the notation $Q \in \mathfrak{Q}(\mathbb{C})$ with

$$Q = q(t)\mathrm{d}t^2,$$

and $q(t)$ is a meromorphic function. Let $U \subset \mathbb{C}$ be a sufficiently large disk so that $P^{-1}(U)$ is compactly contained in U and for $Q \in \mathfrak{Q}(\mathbb{C})$, consider the norm

$$\|Q\|_U := \iint_U \left|q(t)\mathrm{d}t^2\right|.$$

The pushforward of Q by P is the quadratic differential $P_*Q \in \mathfrak{Q}(\mathbb{C})$ defined by

$$P_*Q := \sum_{P(u)=t} \frac{q(u)}{\big(P'(u)\big)^2}\mathrm{d}t^2.$$

It follows from the triangle inequality that

$$\|P_*Q\|_U \leq \|Q\|_{P^{-1}(U)} < \|Q\|_U.$$

For $i \geq 1$, let $Q_i \in \mathfrak{Q}(\mathbb{C})$ be the quadratic differential defined by

$$Q_i := \frac{\mathrm{d}t^2}{t - z_i}.$$

Lemma 4.11 generalizes a result due to Milnor, [13, Lemma 1] in the case $d = 2$.

Lemma 4.11. *For all* $i \in \{1, \dots, k+m-1\}$,

$$P_*Q_i = \frac{Q_{i+1} - Q_1}{\delta_i}. \tag{4.3}$$

Proof of Lemma 4.11. Set $\xi := e^{\frac{2\pi i}{d}}$. Observe that for a given $z \in \mathbb{C}$ and for a given $w \in \mathbb{C}$ such that $P(w) = z$, we have $\{P(u) = z\} = \{w, \xi w, \dots, \xi^{d-1}w\}$. Thus, for a given $i \in \{1, \dots, k+m-1\}$,

$$\begin{aligned} P_*Q_i(z) &= \textstyle\sum_{P(u)=z} \frac{1}{u - z_i} \frac{1}{\big(P'(u)\big)^2}\mathrm{d}t^2 \\ &= \textstyle\sum_{j=0}^{d-1} \frac{1}{\xi^j w - z_i} \frac{1}{\left(d\,(\xi^j w)^{d-1}\right)^2}\mathrm{d}t^2 \\ &= \frac{1}{d^2 w^{2d-2}} \left(\textstyle\sum_{j=0}^{d-1} \frac{1}{\xi^{-j}w - \xi^{-2j}z_i}\right)\mathrm{d}t^2. \end{aligned}$$

Note that $\sum_{j=0}^{d-1} \frac{1}{\xi^{-j}w - \xi^{-2j}z_i} = \frac{dz_i w^{d-2}}{w^d - z_i^d}$[2]. Therefore,

$$\begin{aligned} P_*Q_i(z) &= \frac{z_i}{dw^d(w^d - z_i^d)}\mathrm{d}t^2 \\ &= \frac{1}{\delta_i}\frac{z_i^d}{w^d(w^d - z_i^d)}\mathrm{d}t^2 \end{aligned}$$

[2]This equality is equivalent to the equality $\sum_{j=0}^{d-1} \frac{1}{\xi^{-j}\chi - \xi^{-2j}} = \frac{d\chi^{d-2}}{\chi^d - 1}$. The later follows from an elementary computation by comparing the partial fraction decomposition.

Since $w^d = z - z_1, z_i^d = z_{i+1} - z_1$, we have

$$P_*Q_i(z) = \frac{1}{\delta_i}\frac{z_{i+1} - z_1}{(z - z_1)(z - z_{i+1})}\mathrm{d}t^2 = \frac{1}{\delta_i}\left(\frac{1}{z - z_{i+1}} - \frac{1}{z - z_1}\right)\mathrm{d}t^2.$$

Thus, $P_iQ_i = \dfrac{Q_{i+1} - Q_i}{\delta_i}$ □

The quadratic differentials $(Q_i)_{1\le i\le k+m-1}$ span a vector space $\mathcal{Q}_P \subset \mathfrak{Q}(\mathbb{C})$ of dimension $k + m - 1$. According to Equation (4.3), this subspace is invariant by P_*. According to Equations (4.2) and (4.3), the linear map $\iota : \mathcal{Q}_P \to \mathcal{M}_{k,m}$ which sends $Q_i \in \mathfrak{Q}(\mathbb{C})$ to $\omega_i \in \mathcal{M}_{k,m}$ is an isomorphism which conjugates $P_* : \mathcal{Q}_P \to \mathcal{Q}_P$ to $L_* : \mathcal{M}^*_{k,m} \to \mathcal{M}^*_{k,m}$

Since $\|P_*Q\|_U < \|Q\|_U$ for all $Q \in \mathcal{Q}_P$, the spectrum of $P_* : \mathcal{Q}_P \to \mathcal{Q}_P$ is contained in the unit disk. It follows that the spectrum of $L_* : \mathcal{M}^*_{k,m} \to \mathcal{M}^*_{k,m}$ is contained in the unit disk as required. □

4.2.4. *Fixed points inside the post-critical set.* We will now study the derivatives of moduli maps at fixed points which are inside the post-critical set. Let $\boldsymbol{z} \in PC(F_{k,m})$ be a fixed point of $F_{k,m}$ and let (k', m') be the exact type of $\boldsymbol{z}$.

According to Lemma 4.6, $(k', m') \prec (k, m)$ and, by Proposition 3.3, $\mathcal{M}_{k',m'} \subsetneq \mathcal{M}_{k,m}$ is invariant under $F_{k,m}$. Since $\mathcal{M}_{k',m'}$ is invariant under $D_{\boldsymbol{z}}F_{k,m}$, the vector space

$$\mathcal{M}^0_{k',m'} = \{\omega \in \mathcal{M}^*_{k,m} \mid \omega|_{\mathcal{M}_{k',m'}} \equiv 0\},$$

which is called *the annihilator of* $\mathcal{M}_{k',m'}$ in $\mathcal{M}_{k,m}$, is invariant under the transpose L^* of $D_zF_{k,m}$ and we have the following decomposition

$$\operatorname{Spec} L = \operatorname{Spec}(L|_{\mathcal{M}_{k',m'}}) \cup \operatorname{Spec}\left(L^*|_{\mathcal{M}^0_{k',m'}}\right). \tag{4.4}$$

Moreover, according to Proposition 3.3, we have

$$L|_{\mathcal{M}_{k',m'}} = D_{\boldsymbol{z}}F_{k',m'}.$$

Whence, by Proposition 4.9, $L|_{\mathcal{M}_{k',m'}}$ has only eigenvalues of modulus strictly greater than 1. In order to describe $\operatorname{Spec} L$, we need to study $\operatorname{Spec}\left(L^*|_{\mathcal{M}^0_{k',m'}}\right)$. We will prove the following result.

Proposition 4.12. *Let* $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ *and* $\boldsymbol{z} \in PC(F_{k,m})$ *be a fixed point of* $F_{k,m}$ *of exact type* $(k', m') \prec (k, m)$. *Let* λ *be the multiplier of the polynomial* $P(t) = t^d + z_1 \in \mathbb{C}[t]$ *along the cycle of* $P^{\circ k'}(z_1)$. *Then*

$$\operatorname{Spec}\left((D_{\boldsymbol{z}}F_{k,m})^*|_{\mathcal{M}^0_{k',m'}}\right) = \begin{cases}\{0\} & \text{if } k' = 0\\ \{\mu \mid \mu^m = \lambda^{\frac{m}{m'}}, \mu^{m'} \neq \lambda\} & \text{if } k' \neq 0.\end{cases}$$

The rest of this section is devoted to the proof of this proposition. To simplify the notation, we denote by L^* the restriction of $(D_{\boldsymbol{z}}F_{k,m})^*$ to $\mathcal{M}^0_{k',m'}$. The study of the transpose $L^*\colon \mathcal{M}^0_{k',m'} \to \mathcal{M}^0_{k',m'}$ is divided into two cases, $k' = 0$ and $k' \neq 0$, and each case will be treated separately.

Proof of Proposition 4.12 *when* $k' = 0$. Since $(0, m') \preceq (k, m)$, m' divides k and m. It is enough to prove that $L^* : \mathcal{M}^0_{0,m'} \to \mathcal{M}^0_{0,m'}$ is nilpotent. Recall that for $i \ge 1$, the form $\omega_i\colon \mathcal{M}_{k,m} \to \mathbb{C}$ is defined by $\omega_i(\boldsymbol{v}) = v_i$. For $i \ge 1$, set $\alpha_i\colon \mathcal{M}_{k,m} \to \mathbb{C}$

defined by
$$\alpha_i = \omega_i - \omega_{i+m'}.$$
Recall that $\mathcal{M}^0_{0,m'} = \{\omega\colon \mathcal{M}_{k,m} \to \mathbb{C} \text{ such that } \omega|_{\mathcal{M}_{0,m'}} \equiv 0\}$.

Lemma 4.13. *We have* $\mathcal{M}^0_{0,m'} = \operatorname{Span}\{\alpha_i, 1 \leq i \leq k+m\}$.

Proof. By duality, it is equivalent to show that
$$\mathcal{M}_{0,m'} = \bigcap_{1\leq i\leq k+m} \operatorname{Ker}\alpha_i.$$

Assume $\boldsymbol{v} \in \mathcal{M}_{0,m'}$. Then for all $j \geq 1, v_j = v_{j+m'}$. Given $i \in \{1,\ldots,k+m\}$, we have
$$\alpha_i(\boldsymbol{v}) = \omega_i(\boldsymbol{v}) - \omega_{i+m'}(\boldsymbol{v}) = v_i - v_{i+m'} = 0.$$
Hence $\mathcal{M}_{0,m'} \subseteq \bigcap_{1\leq i\leq k+m} \operatorname{Ker}\alpha_i$.

Conversely, assume $\boldsymbol{v} \in \bigcap_{1\leq i\leq k+m} \operatorname{Ker}\alpha_i$, i.e. for all $i \in \{1,\ldots,k+m\}$, $v_i = v_{i+m'}$. In order to prove that $\boldsymbol{v} \in \mathcal{M}_{0,m'}$, we will prove that for all $j \geq k+m+1, v_j = v_{j+m'}$ and that $v_{m'} = 0$. Given $j \geq k+m+1$, there exists an integer $j' \in \{k+1,\ldots,k+m\}$ such that $j \equiv j' \mod m$. Since $\boldsymbol{v} \in \mathcal{M}_{k,m}$, we have $v_j = v_{j'}$ and $v_{j+m'} = v_{j'+m'}$. Moreover, the fact that $\boldsymbol{v} \in \operatorname{Ker}\alpha_{j'}$ implies that $v_{j'} = v_{j'+m'}$. Thus
$$v_j = v_{j'} = v_{j'+m'} = v_{j+m'}.$$
In order to conclude, we need to show that $v_{m'} = 0$. Note that the previous argument shows that $\boldsymbol{v}$ is a periodic sequence of period dividing m'. Since m' divides k and m,
$$v_{m'} = v_k = v_{k+m}.$$
Since $\boldsymbol{v} \in \mathcal{M}_{k,m}$, we have $\beta v_{k+m} - v_k = 0$ with $\beta \neq 1$, whence
$$v_{m'} = v_k = v_{k+m} = 0.$$
□

Lemma 4.14. *We have* $L^*\alpha_1 = 0$ *and for* $i \geq 2$, $L^*\alpha_i = \delta_{i-1}\alpha_{i-1}$.

Proof. According to Lemma 4.8, for all $i \geq 2$,
$$L^*\omega_i = \delta_{i-1}\omega_{i-1} + L^*\omega_1.$$
Hence, if $i \geq 2$,
$$L^*\alpha_i = L^*(\omega_i - \omega_{i+m'}) = \delta_{i-1}\omega_{i-1} - \delta_{i+m'-1}\omega_{i+m'-1}.$$
Since $\boldsymbol{z} \in \mathcal{M}_{0,m'}$, we have $\delta_{i-1} = dz^{d-1}_{i-1} = dz^{d-1}_{i+m'-1} = \delta_{i+m'-1}$. Hence
$$L^*\alpha_{i-1} = \delta_{i-1}(\omega_{i-1} - \omega_{i+m'-1}) = \delta_{i-1}\alpha_{i-1}.$$
If $i = 1$, since $m' \geq 1$, we have $1 + m' \geq 2$ so that
$$L^*\omega_{1+m'} = \delta_{m'}\omega_{m'} + L^*\omega_1.$$
Hence
$$L^*\alpha_1 = L^*(\omega_1 - \omega_{1+m'}) = -\delta_{m'}\omega_{m'}.$$
Since $\boldsymbol{z} \in \mathcal{M}_{0,m'}$, we have $z_{m'} = 0$. Therefore, $\delta_{m'} = dz^{d-1}_{m'} = 0$ and $L^*\alpha_1 = 0$. □

It follows from Lemma 4.13 and Lemma 4.14 that $L^*\colon \mathcal{M}_{0,m'} \to \mathcal{M}_{0,m'}$ is nilpotent. □

Proof of Proposition 4.12 *when* $k' \neq 0$. In this case, since $(k', m') \prec (k, m)$, we have

$$k' = k \text{ and } m = pm' \text{ with } p \geq 2.$$

Let λ be the multiplier of $P(t) = t^d + z_1$ at $P^{\circ k}(0)$. Note that, according to Proposition 4.4, $\boldsymbol{z}$ is the critical orbit of P. Since $\boldsymbol{z}$ is preperiodic of preperiod $k > 0$, the critical point 0 of P is preperiodic, i.e. $\lambda \neq 0$. We will show that

$$\operatorname{Spec}(L^* : \mathcal{M}^0_{k,m'} \to \mathcal{M}^0_{k,m'}) = \{\mu \mid \mu^m = \lambda^p, \mu^{m'} \neq \lambda\}.$$

Given $j \in \mathbb{Z}/m\mathbb{Z}$, denote by $\underline{j}$ the representative of j in $\{k+1, \dots, k+m\}$, define a linear form $\beta_j \colon \mathcal{M}_{k,m} \to \mathbb{C}$ by

$$\beta_j := \omega_{\underline{j}} - \omega_{\underline{j+m'}}.$$

Note that for all $j \in \mathbb{Z}/m\mathbb{Z}$, $\beta_j \colon \mathcal{M}_{k,m} \to \mathbb{C}$ is non-trivial. Indeed, for a given $j \in \mathbb{Z}/m\mathbb{Z}$, define $\boldsymbol{u} \in \mathcal{M}_{k,m}$ by

$$u_i = \begin{cases} 1 & \text{if } i \geq k+1 \text{ and } i \equiv \underline{j} \mod m \\ \frac{1}{\beta} & \text{if } i = k \text{ and } \underline{j} = k+m \\ 0 & \text{otherwise.} \end{cases}$$

Since $m' < m$, $\beta_j(\boldsymbol{u}) = u_{\underline{j}} = 1 \neq 0$.

We will show that these forms span $\mathcal{M}^0_{k,m'} \subsetneq \mathcal{M}_{k,m}$ and use them to study the linear map $L^* : \mathcal{M}^0_{k,m'} \to \mathcal{M}^0_{k,m'}$. The properties we need are provided by Lemmas 4.15–4.17.

Lemma 4.15. *We have* $\mathcal{M}^0_{k,m'} = \operatorname{Span}\{\beta_j, j \in \mathbb{Z}/m\mathbb{Z}\}$.

Proof. By duality, it is equivalent to show that

$$\mathcal{M}_{k,m'} = \bigcap_{j \in \mathbb{Z}/m\mathbb{Z}} \operatorname{Ker} \beta_j.$$

Assume $\boldsymbol{v} \in \mathcal{M}_{k,m'}$. Then for all $i \geq k+1, v_i = v_{i+m'}$. Since $\underline{j} \in j$ and m' divides m, we have $\underline{j} \equiv \underline{j+m'} \mod m'$. Moreover, $\underline{j} \geq k+1$ and $\underline{j+m'} \geq k+1$, whence

$$\beta_j(\boldsymbol{v}) = v_{\underline{j}} - v_{\underline{j+m'}} = 0.$$

This shows that $\mathcal{M}_{k,m'} \subseteq \bigcap_{j \in \mathbb{Z}/m\mathbb{Z}} \operatorname{Ker} \beta_j$.

Conversely, assume $\boldsymbol{v} \in \bigcap_{j \in \mathbb{Z}/m\mathbb{Z}} \operatorname{Ker} \beta_j$. We want to prove that for all integer $i \geq k+1, v_i = v_{i+m'}$ and that $\beta v_{k+m'} - v_k = 0$. First, assume $i \geq k+1$ and let $j \in \mathbb{Z}/m\mathbb{Z}$ be the congruence class of i. Since $\underline{j} \in j$, we have $i \equiv \underline{j} \mod m$. Moreover $\boldsymbol{v} \in \operatorname{Ker} \beta_j$, and so

$$v_{\underline{j}} - v_{\underline{j+m'}} = \beta_j(\boldsymbol{v}) = 0.$$

From the fact that $\boldsymbol{v} \in \mathcal{M}_{k,m}$, we therefore deduce that $v_i = v_{\underline{j}}$ and $v_{\underline{j+m'}} = v_{i+m'}$. Thus,

$$v_i = v_{i+m'}.$$

Second, let us show $\beta v_{k+m'} - v_k = 0$. The previous argument shows that $\boldsymbol{v}$ is preperiodic of preperiod at most k and of period dividing m'. Since m' divides m, we deduce that $v_{k+m'} = v_{k+m}$. Since $\boldsymbol{v} \in \mathcal{M}_{k,m}$, we have $\beta v_{k+m} - v_k = 0$. Thus $\beta v_{k+m'} - v_k = 0$. □

Therefore, it is important to understand how L^* acts on $\{\beta_j, j \in \mathbb{Z}/m\mathbb{Z}\}$. Given $j \in \mathbb{Z}/m\mathbb{Z}$, set

$$\sigma_j = \delta_{\underline{j}} = dz_{\underline{j}}^{d-1}.$$

Lemma 4.16. *For $j \in \mathbb{Z}/m\mathbb{Z}$, we have $L^*\beta_j = \sigma_{j-1}\beta_{j-1}$.*

Proof. For $j \in \mathbb{Z}/m\mathbb{Z}$, recall that $\underline{j}$ is the representative of j in $[\![k+1, k+m]\!]$. Let us first prove that for all $j \in \mathbb{Z}/m\mathbb{Z}$,

$$L^*\omega_{\underline{j}} = \sigma_{j-1}\omega_{\underline{j-1}} + L^*\omega_1. \tag{4.5}$$

Indeed, if $\underline{j} = k+1$ then $\underline{j-1} = k+m$, whence $\sigma_{j-1}\omega_{\underline{j-1}} = \delta_{k+m}\omega_{k+m}$. According to Lemma 4.8, we have $L^*\omega_{k+1} = \delta_k\omega_k + L^*\omega_1$. Since $\boldsymbol{z} \in \mathcal{M}_{k,m}$, we have

$$\delta_k = dz_k^{d-1} = d\beta^{d-1}z_{k+m}^{d-1} = \beta^{d-1}\delta_{k+m}.$$

Moreover, $\omega_k = \beta\omega_{k+m}$, whence $\delta_k\omega_k = \beta^d\delta_{k+m}\omega_{k+m}$. Since $\beta^d = 1$,

$$L^*\omega_{k+1} = \delta_k\omega_k + L^*\omega_1 = \delta_{k+m}\omega_{k+m} + L^*\omega_1.$$

If $\underline{j} \neq k+1$ then $\underline{j-1} = \underline{j} - 1$. According to Lemma 4.8, we have

$$L^*\omega_{\underline{j}} = \delta_{\underline{j}-1}\omega_{\underline{j}-1} + L^*\omega_1 = \delta_{\underline{j-1}}\omega_{\underline{j-1}} + L^*\omega_1 = \sigma_{j-1}\omega_{\underline{j-1}} + L^*\omega_1.$$

In any case, we have the equality (4.5). Hence

$$L^*\beta_j = L^*(\omega_{\underline{j}} - \omega_{\underline{j+m'}}) = \sigma_{j-1}\omega_{\underline{j-1}} - \sigma_{j+m'-1}\omega_{\underline{j+m'-1}}.$$

Note that $\underline{j-1}$ and $\underline{j+m'-1}$, which are congruence modulo m', are two integers at least $k+1$. Since $\boldsymbol{z} \in \mathcal{M}_{k,m'}$, we have $\sigma_{j-1} = \delta_{\underline{j-1}} = \delta_{\underline{j+m'-1}} = \sigma_{j+m'-1}$. Thus

$$L^*\beta_j = \sigma_{j-1}(\omega_{\underline{j-1}} - \omega_{\underline{j+m'-1}}) = \sigma_{j-1}\beta_{j-1}. \qquad \square$$

Recall that λ is the multiplier of $P(t) = t^d + z_1$ at the periodic point z_{k+1} of period $m' = \frac{m}{p}$.

Lemma 4.17. *For all $j \in \mathbb{Z}/m\mathbb{Z}$,*

$$(L^*)^{m'}(\beta_j) = \lambda\beta_{j-m'} \quad \textit{and} \quad (L^*)^m(\beta_j) = \lambda^p\beta_j.$$

Proof. The second equality is the straightforward consequence of the first one. Hence, it is enough to prove the first equality. According to Proposition 4.4, $\boldsymbol{z}$ is the critical orbit of the polynomial $P(t) = t^d + z_1$, i.e. $z_i = P^i(0)$. In particular, for any $j \in \mathbb{Z}/m\mathbb{Z}$, we have $P(z_{\underline{j-1}}) = z_{\underline{j}}$ and $P'(z_{\underline{j-1}}) = dz_{\underline{j-1}}^{d-1} = \delta_{\underline{j-1}}$. Since $\boldsymbol{z}$ is of type (k, m'), the multiplier λ of the cycle of $z_{\underline{j}}$ is

$$\lambda = \prod_{i\in[\![1,m']\!]} P'(z_{\underline{j-i}}) = \prod_{i\in[\![1,m']\!]} \delta_{\underline{j-i}} = \prod_{i\in[\![1,m']\!]} \sigma_{j-i}.$$

Hence, by Lemma 4.16, we have

$$(L^*)^{m'}(\beta_j) = \left(\prod_{i\in[\![1,m']\!]} \sigma_{j-i}\right)\beta_{j-m'} = \lambda\beta_{j-m'},$$

and

$$(L^*)^m(\beta_j) = (L^*)^{m'p}(\beta_j) = \lambda^p\beta_j. \qquad \square$$

Let ν be a m'-th root of λ. Set

$$T = \frac{L^*}{\nu} : \mathcal{M}^0_{k,m'} \to \mathcal{M}^0_{k,m'}.$$

We shall prove that T is diagonalizable with simple eigenvalues and the eigenvalues of T are m-th roots of unity except 1. According to Lemma 4.17, for all $j \in \mathbb{Z}/m\mathbb{Z}$, $T^{m'}(\beta_j) = \beta_{j-m'}$. In addition,

$$\begin{array}{rcl} \sum_{n\in m'\mathbb{Z}/m\mathbb{Z}} \beta_n & = & \sum_{n\in m'\mathbb{Z}/m\mathbb{Z}} (\omega_{\underline{n}} - \omega_{\underline{n+m'}}) = 0 \\ & = & \sum_{n\in m'\mathbb{Z}/m\mathbb{Z}} \omega_{\underline{n}} - \sum_{n\in m'\mathbb{Z}/m\mathbb{Z}} \omega_{\underline{n+m'}} = 0. \end{array}$$

Recall that $p = \frac{m}{m'}$. Hence

$$\beta_0 + T^{m'}(\beta_0) + \ldots + T^{m'(p-1)}(\beta_0) = 0.$$

Applying $m' - 1$ times T and adding the results, we deduce that

$$\beta_0 + T(\beta_0) + T^2(\beta_0) + \ldots + T^{m'p-1}(\beta_0) = 0.$$

According to Lemma 4.15 and Lemma 4.16, the set $\{\beta_0, L^*(\beta_0), (L^*)^2(\beta_0), \ldots\}$ generates $\mathcal{M}^0_{k,m'}$. Hence $\{\beta_0, T(\beta_0), T^2(\beta_0), \ldots\}$ also generates $\mathcal{M}^0_{k,m'}$. Therefore,

$$\operatorname{Id} + T + T^2 + \ldots + T^{m-1} = 0. \tag{4.6}$$

This means that the minimal polynomial of T divides the polynomial $1 + X + X^2 + \ldots + X^{m-1}$. Consequently, T is diagonalizable and the eigenvalues of T are roots of unity which are not 1. We now show that T has only simple eigenvalues. Assume $\zeta \in \operatorname{Spec} T$. Let $v \in \mathcal{M}_{k,m'}$ be an eigenvector associated to ζ. Set

$$H_\zeta = \frac{1}{m}\left(\operatorname{Id} + \frac{T}{\zeta} + \ldots + \frac{T^{m-1}}{\zeta^{m-1}}\right).$$

The equality (4.6) implies that $T^m = \operatorname{Id}$. Additionally, $\zeta^m = 1$. Hence

$$H_\zeta \circ \frac{T}{\zeta} = H_\zeta \quad \text{so that} \quad H_\zeta \circ T^j = \zeta^j H_\zeta \quad \forall j \geq 1.$$

In addition, $\{\beta_0, T(\beta_0), \ldots\}$ generates $\mathcal{M}^0_{k,m'}$, hence

$$\operatorname{Im} H_\zeta \subseteq \operatorname{Span} H_\zeta(\beta_0).$$

Note that $H_\zeta(v) = v$. Hence

$$v \subseteq \operatorname{Im} H_\zeta \subseteq \operatorname{Span} H_\zeta(\beta_0).$$

Thus the eigenspace associated to the eigenvalue ζ of T has dimension 1, i.e. T has only simple eigenvalues.

Since $T = \frac{L^*}{\nu}$, L^* is diagonalizable with simple eigenvalues which are m-th roots of $\nu^m = \lambda^p$. In addition, 1 is not an eigenvalue of T hence ν is not an eigenvalue of L^*. Since ν is an arbitrary m'-th root of λ, we deduce that

$$\operatorname{Spec} L^* \subseteq \{\mu^m = \lambda^p, \mu^{m'} \neq \lambda\}.$$

Since L^* has only simple eigenvalues, $\# \operatorname{Spec} L^* = \dim \mathcal{M}^0_{k,m'} = m - m'$. Hence

$$\operatorname{Spec}(L^* : \mathcal{M}^0_{k,m'} \to \mathcal{M}^0_{k,m'}) = \{\mu^m = \lambda^p, \mu^{m'} \neq \lambda\} \qquad \square$$

Proof of Theorem A'. According to Lemma 3.1, $F_{k,m}$ and $G_{k,m}$ are conjugate. Hence it is enough to consider an eigenvalue μ of $F_{k,m}$ at a fixed point $\boldsymbol{z} \in \mathcal{M}_{k,m}$. Denote by (k', m') the exact type of $\boldsymbol{z}$. Then by Proposition 4.4, we have $(k', m') \preceq (k, m)$, or by Definition 3.2, we have

$$m' \mid m, (\text{either } k' = 0 \text{ or } k' = k, m' \mid k).$$

Regarding μ as an eigenvalue of $F_{k,m}$ at $\boldsymbol{z}$. We recall argument at the beginning of 4.2.4, according to Lemma 4.6, $(k', m') \preceq (k, m)$ and, by Proposition 3.3, $\mathcal{M}_{k',m'} \subseteq \mathcal{M}_{k,m}$ is invariant under $F_{k,m}$. Since $\mathcal{M}_{k',m'}$ is invariant under $D_{\boldsymbol{z}}F_{k,m}$, the annihilator $\mathcal{M}^0_{k',m'}$ of $\mathcal{M}_{k',m'}$ in $\mathcal{M}_{k,m}$ is invariant under the transpose L^* of $D_z F_{k,m}$ and we have the following decomposition

$$\operatorname{Spec} L = \operatorname{Spec}(L|_{\mathcal{M}_{k',m'}}) \cup \operatorname{Spec}\left(L^*|_{\mathcal{M}^0_{k',m'}}\right). \tag{4.7}$$

Moreover, according to Proposition 3.3, we have

$$L|_{\mathcal{M}_{k',m'}} = D_{\boldsymbol{z}}F_{k',m'}.$$

If $\mu \in \operatorname{Spec}(L|_{\mathcal{M}_{k',m'}})$, note that $L|_{\mathcal{M}_{k',m'}} = D_{\boldsymbol{z}}F_{k',m'}$ and $\boldsymbol{z} \notin PC_{k',m'}$, thus μ is an eigenvalue of $G_{k',m'}$ at a fixed point outside its post-critical set.

If $\mu \in \operatorname{Spec}\left(L^*|_{\mathcal{M}^0_{k',m'}}\right)$, then we are done by Proposition 4.12. □

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