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COMPLETENESS OF p-WEIL-PETERSSON DISTANCE

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Abstract. Our goal of this paper is to research the completeness of the p-Weil-Petersson distance, which is induced by the p-Weil-Petersson metric on the p-integrable Teichmüller space of hyperbolic Riemann surfaces. As a result, we see that the metric is incomplete for all the hyperbolic Riemann surfaces with Lehner's condition except for the ones that are conformally equivalent to either the unit disk or the punctured unit disk. The proof is based on the one by Wolpert's original paper, which is given in the case of compact Riemann surfaces.

1. Introduction

The Weil-Petersson metric, which is abbreviated to the WP metric in this paper, is originally defined as a Hermitian metric on the Teichmüller space of compact Riemann surfaces. Since the 1960's, many researchers found lots of properties of this metric. In complex analysis, Ahlfors [1, 2] proved that the WP metric is a Kähler metric and has negative holomorphic sectional curvature, negative Ricci curvature and negative scalar curvature. In hyperbolic geometry, Wolpert [12, 13] gave several relations between the WP metric and Fenchel-Nielsen coordinates.

If a Riemann surface is of analytically infinite type, the WP metric generally diverges on its Teichmüller space (cf. [8]). From 2000, some mathematicians started research of the WP metric on a certain subspace of the universal Teichmüller space, which we call the square integrable Teichmüller space. Takhtajan and Teo [11] proved the Kählerity and some kinds of negativity of its WP metric, and Cui [3] showed that the distance induced by this metric is complete. This distance is called the Weil-Petersson distance or the WP distance for short. Recently, Matsuzaki [7] extended the concept to the metric defined on the p-integrable universal Teichmüller space for $p \geq 2$, which we call the p-Weil-Petersson metric or the p-WP metric for short. In addition, the distance induced by the p-WP metric is said to be the p-Weil-Petersson distance, which is abbreviated to the p-WP distance. In fact, this distance is continuous and complete. Our study lies in the stream of this research.

The author [15] introduced the p-integrable Teichmüller space of hyperbolic Riemann surfaces with Lehner's condition and proved that this space has a complex analytic structure modeled on Banach spaces. In [16], the WP metric is defined on the square integrable Teichmüller space and has negative holomorphic sectional curvature and the negative Ricci curvature. Then the recent paper [17] says that

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the p-WP metric is defined on the p-integrable Teichmüller space and that this metric is smooth and strongly pseudoconvex.

This paper deals with the completeness of p-WP distance. The main theorem is the following:

Theorem 1.1 (Wolpert, Cui, Matsuzaki, Y.). *Let R be a hyperbolic Riemann surface with Lehner's condition and $p \geq 2$. If R is conformally equivalent to either the unit disk $\mathbb{D}$ or the punctured unit disk $\mathbb{D}_0 = \mathbb{D} \setminus \{0\}$, then the p-Weil-Petersson distance on the p-integrable Teichmüller space of R is complete; otherwise this distance is incomplete.*

This theorem corresponds to an extension of the theorems given by Wolpert [14], Cui [3] and Matsuzaki [7]. The proof is mainly based on the one given in [14], where the Teichmüller map induced by the Jenkins-Strebel differential on R is essentially used. In our study, we cannot use this map directly because its Beltrami coefficient is not p-integrable with respect to the Poincaré metric on R. In order to overcome this problem, we modify the map such that its Beltrami coefficient has compact support in R. Our contribution of this paper is to verify that the similar argument to the proof in [14] is also valid only by a minor modification.

2. Definition of p-WP distance

In this section, we introduce the p-WP distance on the p-integrable Teichmüller space. First, we refer to the Teichmüller space and the Teichmüller distance. Let R be a hyperbolic Riemann surface. A measurable $(-1,1)$-differential on R is called a *Beltrami differential* if the $\mathcal{L}^\infty$-norm

$$\|\mu\|_\infty = \operatorname*{ess\,sup}_{z\in R} |\mu(z)|$$

is finite. Let $\mathcal{L}^\infty(R)$ be the Banach space of Beltrami differentials on R and $\mathrm{Bel}(R)$ be the open unit ball of $\mathcal{L}^\infty(R)$. Each element in $\mathrm{Bel}(R)$ is called a *Beltrami coefficient* on R. By the measurable Riemann mapping theorem, every $\mu \in \mathcal{L}^\infty(R)$ has a quasiconformal map f on R with Beltrami coefficient μ, that is, the expression

$$\mu(z) = \frac{\bar{\partial} f(z)}{\partial f(z)}$$

holds for a.e. $z \in R$. This map f is uniquely determined up to the post-composition of a conformal map. For $\nu \in \mathcal{L}^\infty(R)$, let g be a quasiconformal map on R with Beltrami coefficient ν. Then μ is *Teichmüller equivalent* to ν if there exists a conformal map $h : f(R) \to g(R)$ such that $h \circ f \circ g^{-1}$ is homotopic to the identity on $g(R)$ and fixes the ideal boundary of $g(R)$. Let $[\mu]$ be the Teichmüller equivalence class represented by μ. *The Teichmüller space of* R is defined as the coset of $\mathcal{L}^\infty(R)$ by the Teichmüller equivalence relation. For $\tau_1, \tau_2 \in T(R)$, *the Teichmüller distance* between τ_1 and τ_2 is defined as

$$d_T(\tau_1, \tau_2) = \frac{1}{2} \inf \log \frac{1 + \|(\mu_1 - \mu_2)/(1 - \mu_1 \bar{\mu_2})\|_\infty}{1 - \|(\mu_1 - \mu_2)/(1 - \mu_1 \bar{\mu_2})\|_\infty},$$

where the infimum is taken over all the representatives $\mu_1 \in \tau_1$, $\mu_2 \in \tau_2$. It is well-known that the metric space $(T(R), d_T)$ is complete.

Next, we introduce the p-integrable Teichmüller space. Most of the contents in this part is also to be found in [17, Section 2]. Let $p \geq 1$ and ρ_R be the hyperbolic metric on R. A measurable $(-1,1)$-differential μ on R is *p-integrable* if the hyperbolic $\mathcal{L}^p$-norm

$$\|\mu\|_p = \left(\int_R |\mu|^p \rho_R^2\right)^{\frac{1}{p}}$$

is finite. Define $\mathcal{L}^p(R)$ as the Banach space of p-integrable $(-1,1)$ differentials on R. Then the set $\mathcal{L}^{p,\infty}(R) = \mathcal{L}^p(R) \cap \mathcal{L}^\infty(R)$ becomes a Banach space with norm $\|\cdot\|_{p,\infty} = \|\cdot\|_p + \|\cdot\|_\infty$. Then *the p-integrable Teichmüller space of* R, which is denoted by $T^p(R)$, is defined as the set of Teichmüller equivalence classes that contain a p-integrable Beltrami coefficient as their representative. If R is of analytically finite type, that is, if R has a finite hyperbolic area, then $T^p(R)$ coincides with the whole space $T(R)$. Therefore our study is essential for Riemann surfaces of analytically infinite type.

We call R satisfies *Lehner's condition* if the infimum of the length of all simple closed geodesics on the hyperbolic surface (R, ρ_R) is positive. If $p \geq 2$ and R satisfies Lehner's condition, then $T^p(R)$ has a complex analytic structure modeled on Banach spaces (cf. [15, 17]).

Let us introduce the p-WP metric on $T^p(R)$. For this purpose, we consider the lift of Beltrami differentials on R to $\mathbb{H}$, which is the universal covering surface of R. Let Γ be the Fuchsian group acting on $\mathbb{H}$ with $R = \mathbb{H}/\Gamma$, and let $\tilde{\nu}$ be a measurable $(-1,1)$-differential on $\mathbb{H}$ for Γ. In other words, $\tilde{\nu}$ satisfies $(\tilde{\nu}\circ\gamma)\overline{\gamma'}/\gamma' = \tilde{\nu}$ for $\gamma \in \Gamma$. Then $\tilde{\nu}$ is said to be a *Beltrami differential on* $\mathbb{H}$ *for* Γ if the $\mathcal{L}^\infty$-norm

$$\|\tilde{\nu}\|_\infty = \operatorname*{ess\,sup}_{z\in\mathbb{H}} |\tilde{\nu}(z)|$$

is finite. In particular, if $\|\tilde{\nu}\|_\infty < 1$, then $\tilde{\nu}$ is called a *Beltrami coefficient on* $\mathbb{H}$ *for* Γ. For $p \geq 1$, $\tilde{\nu}$ is *p-integrable* if the hyperbolic $\mathcal{L}^p$-norm

$$\|\tilde{\nu}\|_p = \left(\int_N |\tilde{\nu}|^p \rho_{\mathbb{H}}^2\right)^{\frac{1}{p}}$$

is finite, where N is a fundamental region of Γ in $\mathbb{H}$ and $\rho_{\mathbb{H}}(z) = 2(\operatorname{Im} z)^{-1}$ is the Poincaré metric on $\mathbb{H}$. For $1 \leq p \leq \infty$, define $\mathcal{L}^p(\mathbb{H},\Gamma)$ as the Banach space of measurable $(-1,1)$-differentials on $\mathbb{H}$ for Γ with finite $\mathcal{L}^p$-norm. Let $\mathrm{Bel}(\mathbb{H},\Gamma)$ be the set of Beltrami coefficients on $\mathbb{H}$ for Γ. Then the set $\mathcal{L}^{p,\infty}(\mathbb{H},\Gamma) = \mathcal{L}^p(\mathbb{H},\Gamma) \cap \mathcal{L}^\infty(\mathbb{H},\Gamma)$ becomes a Banach space with norm $\|\cdot\|_{p,\infty} = \|\cdot\|_p + \|\cdot\|_\infty$. We see clearly that for every $\nu \in \mathcal{L}^p(R)$, its lift $\tilde{\nu}$ belongs to $\mathcal{L}^p(\mathbb{H},\Gamma)$ and satisfies $\|\tilde{\nu}\|_p = \|\nu\|_p$, and vice versa. Moreover, $\mathcal{L}^{p,\infty}(R)$ is isometrically isomorphic to $\mathcal{L}^{p,\infty}(\mathbb{H},\Gamma)$.

An argument similar to the above one also follows for quadratic differentials. A measurable quadratic differential ψ is *bounded* if the hyperbolic sup norm

$$\|\psi\|_\infty = \sup_{z\in R} |\psi(z)|\rho_R(z)^{-2}$$

is finite. For $p \geq 1$, ψ is *p-integrable* if the hyperbolic $\mathcal{L}^p$-norm

$$\|\psi\|_p = \left(\int_R |\psi|^p \rho_R^{2-2p}\right)^{\frac{1}{p}}$$

is finite. For $1 \leq p \leq \infty$, Set $A^p(R)$ as the Banach space of holomorphic quadratic differentials on R with finite $\mathcal{L}^p$-norm. Let $\tilde{\psi}$ be a measurable quadratic differential on $\mathbb{H}$ for Γ, that is, $\tilde{\psi}$ satisfies $(\tilde{\psi}\circ\gamma)(\gamma')^2 = \tilde{\psi}$ for every $\gamma \in \Gamma$. Then $\tilde{\psi}$ is *bounded* if the hyperbolic $\mathcal{L}^\infty$-norm

$$\|\tilde{\psi}\|_\infty = \sup_{z\in\mathbb{H}} |\tilde{\psi}(z)|\rho_{\mathbb{H}}(z)^{-2}$$

is finite. For $p \geq 1$, $\tilde{\psi}$ is *p-integrable* if the hyperbolic $\mathcal{L}^p$-norm

$$\|\tilde{\psi}\|_p = \left(\int_N |\tilde{\psi}|^p \rho_{\mathbb{H}}^{2-2p}\right)^{\frac{1}{p}}$$

is finite. For $1 \leq p \leq \infty$, let $M^p(\mathbb{H},\Gamma)$ be the Banach space of measurable quadratic differentials on $\mathbb{H}$ for Γ with finite hyperbolic $\mathcal{L}^p$-norm, and let $A^p(\mathbb{H},\Gamma)$ be its closed subspace consisting of holomorphic ones. It follows that for $\psi \in A^p(R)$, its lift $\tilde{\psi}$ belongs to $A^p(\mathbb{H},\Gamma)$ and satisfies $\|\tilde{\psi}\|_p = \|\psi\|_p$, and vice versa. In addition, $A^p(R)$ is isometrically isomorphic to $A^p(\mathbb{H},\Gamma)$.

A Fuchsian group Γ satisfies Lehner's condition if the Riemann surface $\mathbb{H}/\Gamma$ satisfies Lehner's condition. This condition induces the inclusion relation between $A^p(\mathbb{H},\Gamma)$:

Proposition 2.1 (Lehner, Rao)**.** *Let $1 \leq p_1 < p_2 \leq \infty$ and Γ be a Fuchsian group acting on $\mathbb{H}$ with Lehner's condition. Then $A^{p_1}(\mathbb{H},\Gamma)$ is contained in $A^{p_2}(\mathbb{H},\Gamma)$. In particular, the inclusion map of $(A^{p_1}(\mathbb{H},\Gamma), \|\cdot\|_{p_1})$ into $(A^{p_2}(\mathbb{H},\Gamma), \|\cdot\|_{p_2})$ is a bounded linear operator.*

For the detail, see [6, 9]. Hereafter, for simplicity of notations, we identify differentials defined on R (ν, ψ etc.) with their lifts to $\mathbb{H}$ ($\tilde{\nu}, \tilde{\psi}$ etc.) unless otherwise noted.

Let us assume that $1 \leq p \leq \infty$. For $\psi \in M^p(\mathbb{H},\Gamma)$, define *the Bergman projection* of ψ as

$$b_2(\psi)(z) = \frac{3}{\pi}\int_{\mathbb{H}} \rho_{\mathbb{H}}(\zeta)^{-2}\psi(\zeta)\overline{K_{\mathbb{H}}(z,\zeta)}\, d\xi d\eta$$

for $z \in \mathbb{H}$, where $K_{\mathbb{H}}(z,\zeta) = (\bar{z}-\zeta)^{-4}$ is the Bergman kernel on $\mathbb{H}$ and $\zeta = \xi + i\eta$. It is known that b_2 is a bounded complex linear operator of $M^p(\mathbb{H},\Gamma)$ onto $A^p(\mathbb{H},\Gamma)$, and $b_2|_{A^p(\mathbb{H},\Gamma)}$ becomes the identity map. In addition, it follows that

$$\|b_2\| \leq 3. \tag{2.1}$$

For the detail, see [5, Chapter 3]. We note that the upper bound 3 is independent of the index p.

For $\nu \in \mathcal{L}^{p,\infty}(\mathbb{H},\Gamma)$, define *the harmonic Beltrami differential* of ν as

$$H[\nu] = \rho_{\mathbb{H}}^{-2}\,\overline{b_2(\rho_{\mathbb{H}}^2\bar{\nu})}.$$

Set $\mathrm{HB}^p(\mathbb{H},\Gamma) = H(\mathcal{L}^{p,\infty}(\mathbb{H},\Gamma))$ and $\mathrm{HB}^p_1(\mathbb{H},\Gamma) = \mathrm{Bel}(\mathbb{H},\Gamma) \cap \mathrm{HB}^p(\mathbb{H},\Gamma)$. By the properties mentioned above, the harmonic Beltrami operator H is a bounded complex linear operator of $\mathcal{L}^{p,\infty}(\mathbb{H},\Gamma)$ onto $\mathrm{HB}^p(\mathbb{H},\Gamma)$, and $H|_{\mathrm{HB}^p(\mathbb{H},\Gamma)}$ becomes the identity map. These results imply the following direct sum decomposition

$$\mathcal{L}^{p,\infty}(\mathbb{H},\Gamma) = \mathrm{HB}^p(\mathbb{H},\Gamma) \oplus \mathrm{Ker}_p\, H, \tag{2.2}$$

where $\mathrm{Ker}_p\, H$ is the kernel of $H : \mathcal{L}^{p,\infty}(\mathbb{H},\Gamma) \to \mathrm{HB}^p(\mathbb{H},\Gamma)$.

For $\mu \in \mathrm{Bel}(\mathbb{H},\Gamma)$, there exists a quasiconfromal self-map on $\mathbb{H}$ with Beltrami coefficient μ. This map is uniquely determined under the condition that its homeomorphic extension to $\overline{\mathbb{H}}$ fixes $0, 1$ and ∞, which is denoted by f^μ. For $\tau \in T(R)$ and $\mu \in \tau$, define $\Gamma^\tau = f^\mu\Gamma(f^\mu)^{-1}$. This definition is well-defined independent of the choice of μ. Let (U_τ, ς_τ) be the chart at $\tau \in T^p(R)$, where U_τ is a certain neighborhood at τ and ς_τ is the local coordinate of U_τ into $\mathrm{HB}_1^p(\mathbb{H}, \Gamma^\tau)$. For $\xi \in U_\tau$ and $\nu \in \mathrm{HB}^p(\mathbb{H}, \Gamma^\tau)$, *the p-Weil-Petersson metric* of the holomorphic tangent vector (ξ, ν) is defined as

$$h^p_{\mathrm{WP}}(\xi, \nu) = \left\| H \circ L^{\varsigma_\tau(\xi)}[\nu] \right\|_p.$$

Here, the map

$$L^\mu[\nu] = \left(\frac{\nu}{1 - |\mu|^2} \frac{\partial f^\mu}{\overline{\partial f^\mu}} \right) \circ (f^\mu)^{-1}$$

is defined for $\mu \in \mathrm{HB}_1^p(\mathbb{H}, \Gamma^\tau)$ and $\nu \in \mathrm{HB}^p(\mathbb{H}, \Gamma^\tau)$. For a C^1-smooth curve $c(t)$: $[a, b] \to T^p(R)$, let $v(t)$ be its velocity vector. Then *the p-Weil-Petersson length* of $c(t)$ is introduced as

$$\ell^p_{\mathrm{WP}}(c) = \int_a^b h^p_{\mathrm{WP}}(c(t), v(t))\, dt.$$

For $\tau_1, \tau_2 \in T^p(R)$, *the p-Weil-Petersson distance* $d^p_{\mathrm{WP}}(\tau_1, \tau_2)$ between τ_1 and τ_2 is defined as the infimum of the p-WP length of all C^1-smooth curves with terminal points τ_1 and τ_2.

3. Infinitesimal representation of holomorphic tangent spaces

It is known that the holomorphic tangent space $T^{1,0}_\tau T(R)$ of $T(R)$ at $\tau \in T(R)$ is represented by using infinitesimal equivalence relation of Beltrami differentials. This section is devoted to giving the infinitesimal representation of the holomorphic tangent space $T^{1,0}_\tau T^p(R)$ of $T^p(R)$ at $\tau \in T^p(R)$. Then we consider the relation between the Teichmüller distance and p-WP distance.

For $1 \leq p \leq \infty$, let q be the real number with $\frac{1}{p} + \frac{1}{q} = 1$. It follows from Hölder's inequality that we can define the pairing $\int_N \nu\psi$ for $\nu \in \mathcal{L}^{p,\infty}(\mathbb{H}, \Gamma)$ and $\psi \in M^q(\mathbb{H}, \Gamma)$. By the definitions of b_2 and H, we have

$$\int_N \nu b_2(\psi) = \int_N H[\nu]\psi. \tag{3.1}$$

This formula is shown by the same simple computation as [1, p.166].

For $\nu \in \mathcal{L}^{p,\infty}(\mathbb{H}, \Gamma)$, ν is said to be *infinitesimally trivial* if $\int_N \nu\psi = 0$ for every $\psi \in A^q(\mathbb{H}, \Gamma)$. Let $\mathcal{N}_p(\Gamma)$ be the set of infinitesimally trivial Beltrami differentials in $\mathcal{L}^{p,\infty}(\mathbb{H}, \Gamma)$. For $\nu, \lambda \in \mathcal{L}^{p,\infty}(\mathbb{H}, \Gamma)$, ν is *infinitesimally equivalent* to λ if $\nu - \lambda$ belongs to $\mathcal{N}_p(\Gamma)$. In other words, the expression

$$\int_N \nu\psi = \int_N \lambda\psi$$

holds for every $\psi \in A^q(\mathbb{H}, \Gamma)$. This relation becomes an equivalence relation on $\mathcal{L}^{p,\infty}(\mathbb{H}, \Gamma)$. Let $[\nu]^*_p$ be the infinitesimally equivalence class represented by $\nu \in \mathcal{L}^{p,\infty}(\mathbb{H}, \Gamma)$. Then the quotient space $\mathcal{L}^{p,\infty}(\mathbb{H}, \Gamma)/\mathcal{N}_p(\Gamma)$ becomes a Banach space with norm

$$\left\| [\nu]^*_p \right\|^*_p = \sup_{\psi \in A^q(\mathbb{H},\Gamma)\setminus\{0\}} \frac{\left| \int_N \nu\psi \right|}{\|\psi\|_q}.$$

When $p = \infty$, Theorem 3.1 is well-known:

Theorem 3.1 (Teichmüller's lemma)**.** *Let Γ be a Fuchsian group acting on $\mathbb{H}$. Then $\mathcal{N}_\infty(\Gamma)$ coincides with the kernel* $\mathrm{Ker}_\infty H$ *of the harmonic Beltrami operator $H : \mathcal{L}^\infty(\mathbb{H}, \Gamma) \to \mathrm{HB}^\infty(\mathbb{H}, \Gamma)$.*

Thanks to this theorem, Lemma 3.2, which corresponds to the $\mathcal{L}^p$-version of the theorem, immediately holds:

Lemma 3.2. *Let $p \geq 1$ and Γ be a Fuchsian group acting on $\mathbb{H}$ with Lehner's condition. Then $\mathcal{N}_p(\Gamma)$ coincides with the kernel $\mathrm{Ker}_p\, H$ of the harmonic Beltrami operator $H : \mathcal{L}^{p,\infty}(\mathbb{H},\Gamma) \to \mathrm{HB}^p(\mathbb{H},\Gamma)$.*

Proof. Let ν be an arbitrary element in $\mathcal{N}_p(\Gamma)$. Since Γ satisfies Lehner's condition, $A^1(\mathbb{H},\Gamma)$ is contained in $A^q(\mathbb{H},\Gamma)$. Then ν satisfies $\int_N \nu\psi = 0$ for every $\psi \in A^1(\mathbb{H},\Gamma)$. This implies that $\mathcal{N}_p(\Gamma)$ is contained in $\mathcal{N}_\infty(\Gamma)$. By Theorem 3.1, we obtain $\nu \in \mathrm{Ker}_p\, H$, that is, $H[\nu] = 0$. Hence $\mathcal{N}_p(\Gamma)$ is contained in $\mathrm{Ker}_p\, H$. Conversely, take $\nu \in \mathrm{Ker}_p\, H$ arbitrarily. Then it follows from (3.1) that

$$\int_N \nu\psi = \int_N \nu b_2(\psi) = \int_N H[\nu]\psi = 0$$

for every $\psi \in A^q(\mathbb{H},\Gamma)$. Hence $\nu \in \mathcal{N}_p(\Gamma)$. In other words, $\mathcal{N}_p(\Gamma) = \mathrm{Ker}_p\, H$. $\square$

This lemma and (2.2) implies that the map

$$\iota : \mathrm{HB}^p(\mathbb{H},\Gamma) \ni \nu \mapsto \iota(\nu) = [\nu]_p^* \in \mathcal{L}^{p,\infty}(\mathbb{H},\Gamma)/\mathcal{N}_p(\Gamma)$$

is a complex linear isomorphism. Moreover, Proposition 3.3 holds:

Proposition 3.3. *Let $1 \leq p \leq \infty$ and Γ be a Fuchsian group acting on $\mathbb{H}$ with Lehner's condition. Then the map ι is a Banach isomorphism of $(\mathrm{HB}^p(\mathbb{H},\Gamma), \|\cdot\|_p)$ onto $(\mathcal{L}^{p,\infty}(\mathbb{H},\Gamma)/\mathcal{N}_p(\Gamma), \|\cdot\|_p^*)$.*

Proof. It is sufficient to compare $\|\nu\|_p$ and $\|\iota(\nu)\|_p^*$ for $\nu \in \mathrm{HB}^p(\mathbb{H},\Gamma)$. By Hölder's inequality, we have

$$\left|\int_N \nu\psi\right| \leq \|\nu\|_p \|\psi\|_q$$

for every $\psi \in A^q(\mathbb{H},\Gamma)$. This clearly implies $\|\iota(\nu)\|_p^* \leq \|\nu\|_p$. On the other hand, it follows from $b_2(M^q(\mathbb{H},\Gamma)) = A^q(\mathbb{H},\Gamma)$ and (3.1) that

$$\|\nu\|_p = \|H[\nu]\|_p = \sup_{\psi \in M^q(\mathbb{H},\Gamma)\setminus\{0\}} \frac{\left|\int_N H[\nu]\psi\right|}{\|\psi\|_q} \tag{3.2}$$

$$\leq \sup_{\psi \in M^q(\mathbb{H},\Gamma)\setminus\{0\}} \frac{\|b_2(\psi)\|_q}{\|\psi\|_q} \sup_{\psi \in M^q(\mathbb{H},\Gamma)\setminus\{0\}} \frac{\left|\int_N \nu b_2(\psi)\right|}{\|b_2(\psi)\|_q} = \|b_2\| \|\iota(\nu)\|_p^*.$$

$\square$

This proof and (2.1) imply $\frac{1}{3} \leq \|\iota\| \leq 1$. Note that this estimate is independent of p.

Proposition 3.3 implies that $T_0^{1,0}T^p(R)$ is Banach isomorphic to

$$(\mathcal{L}^{p,\infty}(\mathbb{H},\Gamma)/\mathcal{N}_p(\Gamma), \|\cdot\|_p^*).$$

For the proof of main theorem, we represent this result by using differentials on R. For $\nu \in \mathcal{L}^{p,\infty}(R)$, ν is *infinitesimally trivial* if $\int_R \nu\psi = 0$ for every $\psi \in A^q(R)$. Let $\mathcal{N}_p(R)$ be the set of infinitesimally trivial Beltrami differentials in $\mathcal{L}^{p,\infty}(R)$. For $\nu, \lambda \in \mathcal{L}^{p,\infty}(R)$, ν is *infinitesimally equivalent* to λ if $\nu - \lambda$ belongs to $\mathcal{N}_p(R)$. Then the quotient space $\mathcal{L}^{p,\infty}(R)/\mathcal{N}_p(R)$ becomes a Banach space with norm

$$\|[\nu]_p^*\|_p^* = \sup_{\psi \in A^q(R)\setminus\{0\}} \frac{\left|\int_R \nu\psi\right|}{\|\psi\|_q},$$

and $\mathcal{L}^{p,\infty}(R)/\mathcal{N}_p(R)$ is isometrically isomorphic to $\mathcal{L}^{p,\infty}(\mathbb{H},\Gamma)/\mathcal{N}_p(\Gamma)$ with respect to norm $\|\cdot\|_p^*$. By the similar arguments to above, we can clearly see that

$$T_\tau^{1,0}T^p(R) = \mathrm{HB}^p(\mathbb{H},\Gamma^\tau) \simeq \mathcal{L}^{p,\infty}(R_\tau)/\mathcal{N}_p(R_\tau) \tag{3.3}$$

for every $\tau \in T^p(R)$. Here, $R_\tau = \mathbb{H}/\Gamma^\tau$ and the symbol " $\simeq$ " means the Banach isomorphic equivalence relation.

This section will end by stating an important consequence as follows:

Proposition 3.4. *Let $p \geq 2$ and R be a Riemann surface with Lehner's condition. Then the inclusion map of $(T^p(R), d_{\mathrm{WP}}^p)$ into $(T(R), d_T)$ is continuous.*

Proof. In [17, Section 2], $\mathrm{HB}^p(\mathbb{H},\Gamma)$ is isometrically isomorphic to $A^p(\mathbb{H},\Gamma)$. By this result and Proposition 2.1, the inclusion map of $\mathrm{HB}^p(\mathbb{H},\Gamma)$ into $\mathrm{HB}^\infty(\mathbb{H},\Gamma)$ is a bounded linear operator. In [4, Section 2.5], the Teichmüller metric on $T(R)$ is induced by the norm on the Banach space $\mathcal{L}^\infty(\mathbb{H},\Gamma^\tau)/\mathcal{N}_\infty(\Gamma^\tau)$ for each $\tau \in T(R)$. These results and the equivalence relation (3.3) imply that the map

$$\mathrm{HB}^p(\mathbb{H},\Gamma^\tau) \ni \nu \mapsto [\nu]_\infty^* \in \mathcal{L}^\infty(\mathbb{H},\Gamma^\tau)/\mathcal{N}_\infty(\Gamma^\tau)$$

becomes bounded. Since the p-WP metric on $T^p(R)$ is induced by the norm on $\mathrm{HB}^p(\mathbb{H},\Gamma^\tau)$, the proposition holds. $\square$

4. Proof of main theorem

In this section, we will prove Theorem 1.1. Let us first assume that a hyperbolic Riemann surface R is conformally equivalent to $\mathbb{D}$ or $\mathbb{D}_0$. Moreover, we can assume that $R = \mathbb{D}$ or $\mathbb{D}_0$ without loss of generality. In case of $R = \mathbb{D}$, the assertion is true by [7, Theorem 8.3].

We will consider the case of $R = \mathbb{D}_0$. Take an arbitrary quasiconformal map f on $\mathbb{D}_0$. Then we can extend f to a quasiconformal map $\hat{f}$ on $\mathbb{D}$. By the Riemann mapping theorem, there exists a conformal map h of $\hat{f}(\mathbb{D})$ onto $\mathbb{D}$ with $h(\hat{f}(0)) = 0$. By the post-composition of some suitable rotation, we can normalize f such that f is a self-map of $\mathbb{D}_0$ and its homeomorphic extension to $\overline{\mathbb{D}}$ fixes 1. This normalization implies that for every $\tau \in T(\mathbb{D}_0)$, there exists a biholomorphic automorphism ω_τ on $T(\mathbb{D}_0)$ such that $\omega_\tau(\tau) = 0 \in T(\mathbb{D}_0)$, that is, the Teichmüller modular group of $T(\mathbb{D}_0)$ acts transitively. Since ω_τ is a biholomorphic automorphism on $T^p(\mathbb{D}_0)$ for every $\tau \in T^p(\mathbb{D}_0)$, the action of the Teichmüller modular group of $T^p(\mathbb{D}_0)$ is also transitive.

Because of this result, the completeness follows by using the similar argument of the proof of [7, Theorem 8.3]. In fact, as in [7, Corollary 8.2], there exists a constant $C \geq 1$ such that for every τ_1, τ_2 in a neighborhood U_0 at the base point $0 \in T^p(\mathbb{D}_0)$, we have

$$\frac{1}{C}\|\mu_1 - \mu_2\|_p \leq d_{\mathrm{WP}}^p(\tau_1,\tau_2) \leq C\|\mu_1 - \mu_2\|_p, \tag{4.1}$$

where $\mu_1 = \varsigma_0(\tau_1)$ and $\mu_2 = \varsigma_0(\tau_2)$. An essentially different point in the proof of this inequality from the one of Cororally 8.2 is the domain of integration in the hyperbolic $\mathcal{L}^p$-norm $\|\cdot\|_p$, which is changed from the unit disk $\mathbb{D}$ to a fundamental region N of the Fuchsian group Γ with $\mathbb{D}_0 = \mathbb{H}/\Gamma$ (see [7, Theorem 8.1]).

Given an arbitrary Cauchy sequence $\{\tau_n\}$ in the metric space $(T^p(\mathbb{D}_0), d_{\mathrm{WP}}^p)$, take a number N such that for every $n \geq N$, τ_n lies in an open neighborhood V at τ_N whose closure is contained in U_{τ_N}. By the definition of the complex analytic structure on $T^p(\mathbb{D}_0)$, ω_{τ_N} maps U_{τ_N} onto U_0. This implies that the closure of $\omega_{\tau_N}(V)$ is contained in U_0. Set $\mu_n = \varsigma_0(\omega_{\tau_N}(\tau_n))$. Since ω_{τ_N} is an isometric automorphism on the metric space $(T^p(\mathbb{D}_0), d^p_{\mathrm{WP}})$, it follows from (4.1) that

$$\|\mu_k - \mu_\ell\|_p \leq C d^p_{\mathrm{WP}}(\omega_{\tau_N}(\tau_k), \omega_{\tau_N}(\tau_\ell)) = C d^p_{\mathrm{WP}}(\tau_k, \tau_\ell)$$

for $k, \ell \geq N$. Hence, the sequence $\{\mu_n\}$ becomes a Cauchy sequence in $\mathrm{HB}^p(\mathbb{H}, \Gamma)$. By the completeness of $\mathrm{HB}^p(\mathbb{H}, \Gamma)$, μ_n converges to an element μ in the closure of $\varsigma_0(\omega_{\tau_N}(V))$, which is contained in $\varsigma_0(U_0)$. Hence, μ is a Beltrami coefficient and $[\mu]$ belongs to $T^p(\mathbb{D}_0)$. If we set $\tau = \omega_{\tau_N}^{-1}([\mu])$, then it follows from (4.1) that

$$d^p_{\mathrm{WP}}(\tau_n, \tau) = d^p_{\mathrm{WP}}(\omega_{\tau_N}(\tau_n), [\mu]) \leq C\|\mu_n - \mu\|_p$$

for $n \geq N$. Since $\|\mu_n - \mu\|_p \to 0$ as $n \to \infty$, $\{\tau_n\}$ is a convergent sequence in d^p_{WP}, that is, $(T^p(\mathbb{D}_0), d^p_{\mathrm{WP}})$ is complete.

Next, assume that R is a hyperbolic Riemann surface with Lehner's condition that is conformally equivalent to neither $\mathbb{D}$ nor $\mathbb{D}_0$. The key point is to compose a C^1-smooth curve $c(t)$ on $T^p(R)$ for $t \geq 1$ which diverges in the Teichmüller distance but converges in the p-WP one. Under the assumption that such a curve exists, we first give the proof of Theorem 1.1. Then we will prove the existence of such a curve.

Since c has an infinite length in d_T, there exists an increasing infinite sequence $1 = t_1 < t_2 < \cdots < t_n < \cdots$ such that

$$d_T(c(t_n), c(t_{n+1})) = 1 \tag{4.2}$$

for every $n \in \mathbb{N}$. Set $\tau_n = c(t_n)$. On the other hand, it follows from the assumption and the definition of the p-WP length ℓ^p_{WP} that

$$\sum_{n=1}^{\infty} d^p_{\mathrm{WP}}(\tau_n, \tau_{n+1}) \leq \ell^p_{\mathrm{WP}}(c) < \infty.$$

Then we have $d^p_{\mathrm{WP}}(\tau_n, \tau_{n+1}) \to 0$ as $n \to \infty$, that is, $\{\tau_n\}$ becomes a Cauchy sequence in d^p_{WP}.

Suppose that $(T^p(R), d^p_{\mathrm{WP}})$ is complete. Then $\{\tau_n\}$ converges to a point $\tau \in T^p(R)$ in d^p_{WP}. It follows from Proposition 3.4 that this sequence also converges to τ in d_T. Moreover, (4.2) implies that

$$1 = d_T(\tau_n, \tau_{n+1}) \leq d_T(\tau_n, \tau) + d_T(\tau, \tau_{n+1})$$

for every n. On the other hand, we have $d_T(\tau_n, \tau) + d_T(\tau, \tau_{n+1}) \to 0$ as $n \to \infty$. Then this inequality does not hold for sufficiently large n. Hence $(T^p(R), d^p_{\mathrm{WP}})$ is incomplete by contradiction.

The rest of the proof is to verify the existence of a smooth curve $c(t)$ with such conditions that we assumed above. The reader may refer to Figure 1 for the composition of $c(t)$ depending on the situation.

Since R is conformally equivalent to neither $\mathbb{D}$ nor $\mathbb{D}_0$, we can take a simple closed curve γ on R which is homotopically non-trivial and cannot be shrinked homotopically to any puncture of R. It follows from [10, Theorem 4.1] that there exists a Jenkins-Strebel differential φ as the solution of the height problem for the simple curve system (γ) and height $b = 1$. The characteristic ring domain R_0 for γ is conformal equivalent to the annulus $A = \{1 < |z| < e^\alpha\}$ in the z-plane, where $\alpha = 2\pi M(R_0)$ and $M(R_0)$ is the modulus of R_0. Hereafter, we identify R_0 with A. In addition, we can say that $\varphi = (\alpha z)^{-2}dz^2$ on A. Given $t \geq 1$, we will compose

$$
\begin{tikzcd}
 & S \arrow[r, "\tilde{f}^t"'] \arrow[d, "\sigma"] & S_t \arrow[d, "\sigma_t"] & \\
R \arrow[r, "\text{cutting}"'] & A \arrow[r, "f^t"'] & A_t \arrow[r, "\text{gluing}"'] & R_t
\end{tikzcd}
$$

FIGURE 1

a quasiconformal map f^t on R which stretches a certain relatively compact ring domain with the same homotopy type as γ. Let $S = \{\zeta = \xi + i\eta \,|\, 0 < \xi < \alpha\}$ be a vertical strip in the ζ-plane. Then the map $z = \sigma(\zeta) = e^{\zeta}$ is a covering map of S onto A. Set $\alpha_1 = \frac{1}{4}\alpha$, $\alpha_2 = \frac{3}{4}\alpha$ and $S' = \{\alpha_1 < \xi < \alpha_2\}$. It clearly follows that S' is a vertical strip contained in S. Define the map $\tilde{f}^t$ as

$$
\tilde{f}^t(\zeta) = \begin{cases} \zeta & (0 < \xi < \alpha_1) \\ t\xi + i\eta - \dfrac{1}{4}(t-1)\alpha & (\alpha_1 \leq \xi \leq \alpha_2) \\ \zeta + \dfrac{1}{2}(t-1)\alpha & (\alpha_2 < \xi < \alpha) \end{cases}
$$

for $\zeta = \xi + i\eta \in S$. Then we see that $\tilde{f}^t$ is a homeomorphism of S onto its image $S_t = \{\zeta_t = \xi_t + i\eta_t \,|\, 0 < \xi_t < \frac{t+1}{2}\alpha\}$ in the ζ_t-plane. Let $\alpha_2(t) = \frac{2t+1}{4}\alpha$, $\alpha(t) = \frac{t+1}{2}\alpha$ and $S'_t = \{\alpha_1 < \xi_t < \alpha_2(t)\}$. Then we have $S'_t = \tilde{f}^t(S')$. In fact, $\tilde{f}^t$ stretches S' with respect to the ξ-axis, and the width of S'_t is t times as long as that of S_t. Let $\tilde{\mu}_t$ be the Beltrami coefficient of $\tilde{f}^t$. By a simple computation, we have

$$
\tilde{\mu}_t(\zeta) = \begin{cases} \dfrac{t-1}{t+1} & (\zeta \in S') \\ 0 & (\zeta \in S \setminus \overline{S'}), \end{cases}
$$

where $\overline{S'}$ is the closure of S'. Since $t \geq 1$, it follows that $\|\tilde{\mu}_t\|_\infty < 1$ and that $\tilde{f}^t$ is a quasiconformal map of S onto S_t.

The map $z_t = \sigma_t(\zeta_t) = e^{\zeta_t}$ is a covering map of S_t onto the annulus $A_t = \{1 < |z_t| < e^{\alpha(t)}\}$ in the z_t-plane. Then there exists a quasiconformal map f^t of A onto A_t satisfying $\sigma_t \circ \tilde{f}^t = f^t \circ \sigma$. Set the annulus $A' = \{e^{\alpha_1} < |z_t| < e^{\alpha_2}\}$, which corresponds to the projection of S' by σ. In particular, the Beltrami coefficient μ_t of f^t is represented as

$$
\mu_t(z) = \begin{cases} \dfrac{t-1}{t+1}\dfrac{z}{\bar{z}} & (z \in A') \\ 0 & (z \in A \setminus \overline{A'}). \end{cases}
$$

Take an arbitrary connected component δ in the boundary of R_0. Then δ is a critical trajectory of φ, which means that δ is a maximal horizontal arc with respect to coordinates induced by φ and that all endpoints of δ are zeros of φ (see [10, p. 25]). We define δ^+ and δ^- as the two arcs on ∂A which correspond to the two slits obtained by cutting R along δ. Assume that δ^+ lies in the inner boundary component of ∂A and δ^- lies in the outer one. Since f^t can be extended to a homeomorphism of $\overline{A}$ onto $\overline{A_t}$, the positional relation of all the cut lines on ∂A are preserved. Hence we can obtain a new Riemann surface R_t by glueing $f^t(\delta^+)$ and $f^t(\delta^-)$ for each critical trajectory δ. We identify A_t with the ring domain of R_t that is conformally equivalent to A_t. Since f^t is conformal on $A \setminus \overline{A'}$, it is extended to a quasiconformal map of R onto R_t, which corresponds to the desired map.

Set $c(t) = [\mu_t]$. Since μ_t is p-integrable, $c(t)$ is a curve in $T^p(R)$. Let us compute the tangent vector $v(t)$ of $c(t)$. We have to notice the complex analytic structure of $T^p(R)$. Fix $t \geq 1$ and take a real number s sufficiently close to t. By simple computation, the Beltrami coefficient $\mu_{s,t}$ of $f^s \circ (f^t)^{-1}$ is represented as

$$\mu_{s,t}(z_t) = \begin{cases} \dfrac{s-t}{s+t}\dfrac{z_t}{\bar{z}_t} & (z_t \in A'_t) \\ 0 & (z_t \in A_t \setminus \overline{A'_t}), \end{cases}$$

where $A'_t = \{e^{\alpha_1} < |z_t| < e^{\alpha_2(t)}\}$. Then we obtain

$$\nu(t) := \frac{d}{ds}\big(\mu_{s,t}(z_t)\big)\bigg|_{s=0} = \begin{cases} \dfrac{1}{2t}\dfrac{z_t}{\bar{z}_t} & (z_t \in A'_t) \\ 0 & (z_t \in A_t \setminus \overline{A'_t}) \end{cases}$$

and the representation $v(t) = [\nu(t)]^*_p$. The definition of $\nu(t)$ implies that $c(t)$ is a C^1-smooth curve in $T^p(R)$.

Now we will show that the p-WP length of c is finite. Let q be the index with $\frac{1}{p} + \frac{1}{q} = 1$. By (3.2) and (3.3), we have

$$h^p_{\mathrm{WP}}\big(c(t), v(t)\big) \leq 3\|v(t)\|^*_p = 3 \sup_{\psi \in A^q(R_t)\setminus\{0\}} \frac{\left|\int_{R_t} \nu(t)\psi\right|}{\|\psi\|_q}. \tag{4.3}$$

Let $\tilde{\nu}(t)$ be a lift of $\nu(t)$ for the covering map σ_t, and let

$$\chi_t(\zeta_t) = \begin{cases} 1 & (\zeta_t \in S'_t) \\ 0 & (\zeta_t \in S_t \setminus S'_t) \end{cases}$$

be the characteristic function of S'_t. Then we have $\tilde{\nu}(t) = \frac{1}{2t}\chi_t$. For $\psi \in A^q(R_t)$, let $\tilde{\psi}$ be a lift of ψ for σ_t. The set $N_t = \{\zeta_t \in S_t \mid |\eta_t| < \pi\}$ is a fundamental region for the covering map σ_t. Then we have

$$\int_{R_t} \nu(t)\psi = \int_{N_t} \tilde{\nu}(t)\tilde{\psi} = \frac{1}{2t}\int_{N'_t} \tilde{\psi}, \tag{4.4}$$

where $N'_t = S'_t \cap N_t$. Since A_t is contained in R_t, we have $\rho_{A_t} \geq \rho_{R_t}$ on A_t. Hence, it follows that

$$\|\psi\|^q_q \geq \int_{A_t} |\psi|^q \rho^{2-2q}_{A_t} = \int_{N_t} |\tilde{\psi}|^q \rho^{2-2q}_{S_t} \geq \int_{N'_t} |\tilde{\psi}|^q \rho^{2-2q}_{S_t}, \tag{4.5}$$

where

$$\rho_{S_t}(\zeta_t) = \frac{\pi}{2\alpha(t)} \operatorname{cosec}\left(\frac{\pi}{\alpha(t)}\xi_t\right)$$

is the Poincaré metric on S_t. Then we see that ρ_{S_t} is a function depending only on $\xi_t = \operatorname{Re}\zeta_t$. Let us rewrite $\rho_{S_t}(\zeta_t) = \rho_{S_t}(\xi_t)$. Hence we have

$$\int_{N'_t} \tilde{\psi}\rho_{S_t}^{2-2q} = \frac{2}{\alpha t}\left(\int_{\alpha_1}^{\alpha_2(t)} \rho_{S_t}(\xi_t)^{2-2q}\, d\xi_t\right)\left(\int_{N'_t} \tilde{\psi}\right).$$

It follows from Hölder's inequality and (4.5) that

$$\begin{aligned}\left|\int_{N'_t} \tilde{\psi}\rho_{S_t}^{2-2q}\right| &\leq \left(\int_{N'_t} \rho_{S_t}^{2-2q}\right)^{\frac{1}{p}}\left(\int_{N'_t} |\tilde{\psi}|^q\rho_{S_t}^{2-2q}\right)^{\frac{1}{q}} \\ &\leq (2\pi)^{\frac{1}{p}}\left(\int_{\alpha_1}^{\alpha_2(t)} \rho_{S_t}(\xi_t)^{2-2q}\, d\xi_t\right)^{\frac{1}{p}} \|\psi\|_q.\end{aligned}$$

A simple computation implies that

$$\begin{aligned}\int_{\alpha_1}^{\alpha_2(t)} \rho_{S_t}(\xi_t)^{2-2q}\, d\xi_t &= \left(\frac{\pi}{2\alpha(t)}\right)^{2-2q}\int_{\alpha_1}^{\alpha_2(t)} \sin^{2q-2}\left(\frac{\pi}{\alpha(t)}\xi_t\right) d\xi_t \\ &= \left(\frac{t+1}{\pi}\alpha\right)^{2q-1}\int_{k(t)}^{\frac{\pi}{2}} \sin^{2q-2} x\, dx.\end{aligned} \tag{4.6}$$

where $k(t) = \frac{\pi}{2(t+1)}$. Since $k(t)$ is monotonically decreasing for $t \geq 1$ and $\sin x \geq 0$ on the closed interval $[k(t), \frac{\pi}{2}]$, we have

$$\int_{k(t)}^{\frac{\pi}{2}} \sin^{2q-2} x\, dx \geq \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^{2q-2} x\, dx.$$

Let us denote the last integral by I_q. Then it follows from (4.6) that

$$\int_{\alpha_1}^{\alpha_2(t)} \rho_{S_t}(\xi_t)^{2-2q}\, d\xi_t \geq \left(\frac{t+1}{\pi}\alpha\right)^{2q-1} I_q. \tag{4.7}$$

By connecting from (4.4) to (4.7), we obtain

$$\begin{aligned}\frac{\left|\int_{R_t} \nu(t)\psi\right|}{\|\psi\|_q} &\leq \frac{1}{2t}\left|\int_{N'_t} \tilde{\psi}\right|\left(\int_{N'_t} |\tilde{\psi}|^q\rho_{S_t}^{2-2q}\right)^{-\frac{1}{q}} \\ &\leq \frac{\alpha}{4}\cdot(2\pi)^{\frac{1}{p}}\left(\int_{\alpha_1}^{\alpha_2(t)} \rho_{S_t}(\xi_t)^{2-2q}\, d\xi_t\right)^{-\frac{1}{q}} \leq \frac{J_p}{(t+1)^{1+\frac{1}{p}}},\end{aligned} \tag{4.8}$$

where

$$J_p = \left(\frac{\pi^{p+2}}{2^{2p-1}I_q^{p-1}\alpha}\right)^{\frac{1}{p}}$$

is the constant independent of t. By (4.3) and (4.8), we have

$$\ell_{\mathrm{WP}}^p(c) = \int_1^\infty h_{\mathrm{WP}}^p\big(c(t), v(t)\big)\, dt \leq 3\int_1^\infty \frac{J_p}{(1+t)^{1+\frac{1}{p}}}\, dt = 3\cdot 2^{-\frac{1}{p}} pJ_p < \infty.$$

Therefore, we obtain the desired result and complete the proof.

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