

MAPS ON THE MORSE BOUNDARY

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ABSTRACT. For a proper geodesic metric space, the Morse boundary focuses on the hyperbolic-like directions in the space. The Morse boundary is a quasi-isometry invariant. That is, a quasi-isometry between two spaces induces a homeomorphism on their Morse boundaries. In this paper, we investigate additional structures on the Morse boundary which determine the space up to a quasi-isometry. We prove that a homeomorphism between the Morse boundaries of two proper, cocompact spaces is induced by a quasi-isometry if and only if both the homeomorphism and its inverse are bihölder, or quasi-symmetric, or strongly quasi-conformal.

1. INTRODUCTION

The visual boundaries of hyperbolic metric spaces have been extensively studied. They play a critical role in the study of the geometry, topology and dynamics of hyperbolic spaces. Gromov [Gro87] showed that they are quasi-isometry invariant. That is, a quasi-isometry between two hyperbolic metric spaces induces a homeomorphism between their boundaries. Hence the visual boundary is well-defined for a hyperbolic group. The boundary of a hyperbolic space is metrizable. If we fix visual metrics on the hyperbolic boundaries, the homeomorphism induced by a quasi-isometry between two hyperbolic spaces satisfies a variety of metric properties. It is bihölder, quasi-conformal, quasi-Möbius and power quasi-symmetric. Quasi-Möbius is a condition that bounds the distortion of cross-ratios and quasi-conformal bounds the distortion of metric spheres and annuli. These notions have been studied by Otal, Pansu, Tukia and Väisälä [Ota92, Pan89, TUK80a, TV82, TV84, Väi06]. Quasi-symmetric maps have appeared [TUK80b, Väi81].

A natural question is that to what extent is a hyperbolic space X determined by its boundary ∂X . A result of Paulin [Pau96] and Bourdon [Bou95] answers this question: Let G_1 and G_2 be two word-hyperbolic groups. Suppose $h : \partial G_1 \rightarrow \partial G_2$ is a homeomorphism such that h and h^{-1} are quasi-Möbius or quasi-conformal. Then h extends to a quasi-isometry $f : G_1 \rightarrow G_2$. In our paper, we will use Paulin's idea to approach the problem of extending a map between the Morse boundaries to a quasi-isometry between the spaces. Here the definition of quasi-conformal in the sense of Paulin is different from the one used by Tukia and others.

In the paper [BS00], Bonk and Schramm state a different result using power quasi-symmetric maps on the boundaries. They show that if $f : (\partial X, d_{x_0, \epsilon_X}) \rightarrow (\partial Y, d_{y_0, \epsilon_Y})$ is a power quasi-symmetry on the boundaries of two hyperbolic spaces

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X, Y , then f extends to a quasi-isometry $h : X \rightarrow Y$. Their idea is different from Paulin.

Boundaries can be defined for more general metric spaces. We can define a visual boundary of CAT(0) spaces similarly. But this boundary is not a quasi-isometry invariant. An example of Croke and Kleiner [CK00] shows that there is a group acting geometrically on two CAT(0) spaces with different topological boundaries. In [CS15], Charney and Sultan constructed a new boundary for CAT(0) spaces which is defined by restricting to rays satisfying a contracting property. This boundary is originally called the contracting boundary and it is a quasi-isometry invariant. In CAT(0) spaces, the contracting property is equivalent to the Morse property. Later, Cordes [Cor17] generalized this construction to arbitrary proper geodesic metric spaces using rays satisfying the Morse properties. These boundaries are called Morse boundaries. They are also quasi-isometry invariant. Thus the Morse boundary is well-defined for any finitely generated group. The aim of this boundary is to study hyperbolic behavior in a non-hyperbolic space. Several papers [CD19, CH17, Zal18, Tra18, CCS19, Mur19, CCM19, Liu21, AMST19, PP20, IM20, CRSZ20, Kar21, IMZ21] studied Morse boundaries that many properties and applications of hyperbolic boundaries generalize to Morse boundaries of more general groups.

Denote the Morse boundary of X by ∂_*X . In [CCM19], Charney, Cordes and Murray investigate the question of when a homeomorphism of Morse boundaries is induced by a quasi-isometry of the interior spaces. They show the following result, which is an analogue of Paulin's theorem.

Theorem 1.1 ([CCM19]). *Let $h : \partial_*X \rightarrow \partial_*Y$ be a homeomorphism between the Morse boundaries of two proper, cocompact geodesic metric spaces. Assume that ∂_*X contains at least three points. Then f is induced by a quasi-isometry $h : X \rightarrow Y$ if and only if f and f^{-1} are 2-stable and quasi-Möbius.*

Mousley and Russell also proved Theorem 1.1 in the special case of hierarchically hyperbolic groups [MR19].

The purpose of this paper is to investigate additional structures on Morse boundaries ∂_*X which determine X up to a quasi-isometry. We show Theorem 1.2.

Theorem 1.2 (Main theorem). *Let X and Y be proper, cocompact geodesic metric spaces and assume that ∂_*X contains at least three points. Let $f : \partial_*X \rightarrow \partial_*Y$ be a homeomorphism. Then the following are equivalent.*

- (1) f is induced by a quasi-isometry $h : X \rightarrow Y$.
- (2) f and f^{-1} are bihölder.
- (3) f and f^{-1} are quasi-symmetric.
- (4) f and f^{-1} are strongly quasi-conformal.

Note that in general, the Morse boundary is neither metrizable nor compact, so it seems that there is no obvious way to talk about bihölder maps, quasi-symmetries and quasi-conformal maps on the Morse boundary. In the current paper, we provide an approach to this question, and define these notions on the Morse boundary. The starting step is that, in the paper [CH17], Cordes and Hume show that, for a proper geodesic metric space, the Morse boundary is the direct limit of a collection of Gromov boundaries. We know that Gromov boundaries are metrizable. This provides a possibility of generalizing notions defined in a metric space to the Morse boundary. More precisely, to deal with Morse boundaries, usually we need a basepoint and all the Morse gauges. However, controlling Morse gauges becomes

a critical problem. In our paper, we use a different point of view. Our key idea is that we almost regard the Morse boundary as a collection of Gromov boundaries whose interior spaces are uniformly hyperbolic. In particular, we choose infinitely many basepoints by taking the cocompact action on a fixed basepoint, and choose a sufficiently large Morse gauge N . For each basepoint we can have a Gromov boundary such that the hyperbolicity of the interior space depends only on Morse gauge N . Even though the union of these Gromov boundaries is not the same as the Morse boundary, it contains sufficient information of Morse boundary for us to use.

Combining the Main Theorem 1.2 and the work of Charney, Cordes and Murray [CCM19], we know that these four conditions in the Main Theorem 1.2 are equivalent to the 5th one: f and f^{-1} are 2-stable and quasi-Möbius.

To define these notions, bihölder, quasi-symmetric, strongly quasi-conformal maps, we require a property called basetriangle stable. We will talk about these in section 3. The 2-stable property is equivalent to the basetriangle stable property in our setting by Proposition 3.6. This means we have 2-stable maps for free in the case of bihölder, quasi-symmetric, strongly quasi-conformal homeomorphisms under the assumption of Theorem 1.2.

Also we get Corollary 1.3 in the case of hyperbolic spaces. Please check the definitions in section 3. These notions between the Morse boundaries are different with that in the usual case of metric spaces.

Corollary 1.3. *Let X and Y be proper, cocompact geodesic, hyperbolic spaces. Suppose that ∂X contains at least three points. Let $f : \partial X \rightarrow \partial Y$ be a homeomorphism. Then the following are equivalent.*

- (1) f is induced by a quasi-isometry $h : X \rightarrow Y$.
- (2) f and f^{-1} are bihölder.
- (3) f and f^{-1} are quasi-symmetric.
- (4) f and f^{-1} are strongly quasi-conformal.
- (5) f and f^{-1} are quasi-Möbius and 2-stable.

A paper of Cashen and Mackay [CM19] introduces a metrizable topology on the Morse boundary. It is still interesting to know whether a full analogue of Paulin's theorem holds for this modified Morse boundary. Qing and Rafi [QR22] introduce a new boundary for CAT(0) spaces, called sublinearly Morse boundary, which is strictly larger than the Morse boundary. Qing and Zalloum [QZ19] show that a homeomorphism $f : \partial_K G \rightarrow \partial_K G'$ is induced by a quasi-isometry if and only if f is Morse quasi-Möbius and stable, where G and G' are CAT(0) groups and $\partial_K G$ is the sublinearly Morse boundary of G . It is still interesting to know if there is a way to define quasi-conformal to get an analogue of our main theorem.

The paper is organized as follows. In section 2, we review known properties of the Morse boundary and metrics on the Gromov boundary. We define bihölder maps, quasi-symmetries and strongly quasi-conformal maps on the Morse boundary and prove one direction of the main theorem in section 3. In section 4, under any of these conditions, we extends a map between the boundary to the interior and show that it is a quasi-isometry.

2. PRELIMINARIES

Let X be a metric space. We use $[x, y]$ to represent a geodesic between $x, y \in X$. We say a metric space X is *proper* if any closed ball in X is compact. If A is a subset in X , the r -neighborhood of A in X is denoted by $\mathcal{N}_r(A)$. The *Hausdorff distance* $d_H(A_1, A_2)$ between two subsets A_1 and A_2 is defined by $\inf\{r \mid A_1 \subset \mathcal{N}_r(A_2), A_2 \subset \mathcal{N}_r(A_1)\}$.

Definition 2.1. Let $f : (X, d_X) \rightarrow (Y, d_Y)$ be a map between two metric spaces X and Y , $K \geq 1, C \geq 0$. If for all $x_1, x_2 \in X$,

$$K^{-1}d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq Kd_X(x_1, x_2) + C,$$

then f is called a (K, C) -quasi-isometric embedding. If, in addition, $d_Y(f(x), Y) \leq C$ for all $x \in X$, then f is called a (K, C) -quasi-isometry. If X is an interval of \mathbb{R} , then a (K, C) -quasi-isometric embedding f is called a (K, C) -quasi-geodesic. We use the image of f to describe the quasi-geodesic.

Definition 2.2. Let N be a function from $[1, \infty) \times [0, \infty)$ to $[0, \infty)$. A geodesic γ in X is N -Morse if for any (K, C) -quasi-geodesic α with endpoints on γ , we have $\alpha \subset \mathcal{N}_{N(K, C)}(\gamma)$. The function N is called a Morse gauge.

We know that in a hyperbolic space X , there exists a Morse gauge N such that all geodesics in X are N -Morse. If we consider the Euclidean spaces, there is no infinite Morse geodesic. In more general metric spaces, some infinite geodesics are Morse, others are not. Morse geodesics in a proper geodesic metric space behave similarly to geodesics in a hyperbolic metric space.

2.1. The Morse boundary. Now let us define the Morse boundary. All the details can be found in [Cor17].

Let X be a proper geodesic metric space and x_0 be a basepoint. The *Morse boundary* $\partial_* X$ of X is the set of equivalence classes of all Morse geodesic rays with basepoint x_0 and we say two Morse geodesics rays are *equivalent* if they have finite Hausdorff distance. Fix an N Morse gauge, we give a topology to the set

$$\partial_*^N X_{x_0} = \{[\alpha] \mid \text{there exists an } N\text{-Morse geodesic ray } \beta \in [\alpha] \text{ with basepoint } x_0\}$$

with the compact-open topology.

Consider the set \mathcal{M} , all Morse gauges. We put a partial ordering on \mathcal{M} : $N \leq N'$ if and only if $N(K, C) \leq N'(K, C)$ for all K, C . The Morse boundary is defined to be

$$\partial_* X_{x_0} = \varinjlim_{\mathcal{M}} \partial_*^N X_{x_0}$$

with the induced direct limit topology, i.e., a set V is open in $\partial_* X_{x_0}$ if and only if $V \cap \partial_*^N X_{x_0}$ is open in $\partial_*^N X_{x_0}$ for all N Morse gauges.

Another equivalent way to give the compact-open topology on $\partial_*^N X_{x_0}$ is using a system of neighborhoods. For a proper geodesic space X , the Morse boundary $\partial_* X_{x_0}$ is basepoint independent, we will omit the basepoint from the notation, the Morse boundary is denoted by $\partial_* X$.

Here let us list some properties of Morse triangles and Morse geodesics.

Lemma 2.3 ([Liu21]). *For any N Morse gauge, there exists N' such that any segment of an N -Morse geodesic is N' -Morse.*

Lemma 2.4 says that a geodesic which is close to a Morse geodesic is uniformly Morse. It is an easy exercise we leave to the reader.

Lemma 2.4. *Let $\alpha : [a, b] \rightarrow X$ and β be geodesics in a geodesic metric space X . Suppose that β is N -Morse, and $d(\alpha(a), \beta), d(\alpha(b), \beta) \leq \epsilon$. Then α is N' -Morse where N' depends only on N and ϵ .*

Recall that a geodesic triangle T is called δ -slim if δ -neighborhood of any two sides of T covers the third side.

Lemma 2.5 (Slim triangles, [CCM19]). *Let X be a proper geodesic metric space. Let a, b, c be points in $X \cup \partial_* X$. Suppose that two sides of the triangle $T(a, b, c)$ are N -Morse, then the third side is N' -Morse and the triangle is δ_N -slim, where N' and δ_N depend only on N .*

2.2. Visual metrics and topology on hyperbolic boundaries. In this subsection, we will give a quick review about the construction of sequential boundary and visual metrics for a hyperbolic space. All of this can be found in [Ghy90] or [BH13, Chapter III.H].

Let us give the definition of hyperbolic space in the sense of Gromov.

Definition 2.6. Let X be a metric space and let $x, y, z \in X$. The *Gromov product* of x and y with respect to z is defined by

$$(x \cdot y)_z = \frac{1}{2}(d(z, x) + d(z, y) - d(x, y)).$$

Definition 2.7. Let X be a metric space and let $\delta \geq 0$ be a constant. We call X is δ -hyperbolic if for all $w, x, y, z \in X$, we have

$$(x \cdot y)_w \geq \min\{(x \cdot z)_w, (z \cdot y)_w\} - \delta.$$

Let X be a δ -hyperbolic space. Let $p \in X$ be a basepoint and (x_n) be a sequence in X . We say the sequence (x_n) converges at infinity if $(x_i \cdot x_j)_p \rightarrow \infty$ as $i, j \rightarrow \infty$. Two convergent sequences (x_n) , (y_m) are called equivalent if $(x_i \cdot y_j)_p \rightarrow \infty$ as $i, j \rightarrow \infty$. We use $\lim x_n$ to denote the equivalence class of (x_n) .

The sequential boundary of X , $\partial_s X$ is defined to be the set of equivalence classes of convergent sequences. In our paper, we omit the notation s . For a hyperbolic space X , ∂X is denoted as its boundary. We extend the Gromov product to ∂X by:

$$(x \cdot y)_p = \sup \liminf_{n, m \rightarrow \infty} (x_n \cdot y_m)_p,$$

where the supremum is taken over all sequences (x_n) , $(y_m) \in X$ such that $x = \lim x_n$, $y = \lim y_m$.

We have the following properties for the Gromov product.

Lemma 2.8. *Let X be a δ -hyperbolic space and $p \in X$ be a basepoint. Then:*

- (1) $(x \cdot y)_p = \infty$ if and only if $x = y \in \partial X$.
- (2) For all $x, y \in \partial X$ and all sequences (x_n) , $(y_m) \in X$ with $x = \lim x_n$, $y = \lim y_m$, we have

$$(x \cdot y)_p - 2\delta \leq \liminf_{n, m \rightarrow \infty} (x_n \cdot y_m)_p \leq (x \cdot y)_p.$$

- (3) For any $x, y, z \in \partial X \cup X$, we have $(x \cdot y)_p \geq \min\{(x \cdot z)_p, (y \cdot z)_p\} - 2\delta$.

Definition 2.9. For a hyperbolic space X with basepoint p , we say a metric d on ∂X is a *visual metric* with parameter ϵ if there exist constants $k_1, k_2 > 0$ so that

$$k_1 e^{-\epsilon(x \cdot y)_p} \leq d(x, y) \leq k_2 e^{-\epsilon(x \cdot y)_p}$$

for all $x, y \in \partial X$.

Here is the standard construction of the metrics on ∂X . If $x, y \in \partial X, p \in X, \epsilon > 0$, let

$$d_{\partial X, p, \epsilon}(x, y) = d_{p, \epsilon}(x, y) = \inf \left\{ \sum_{i=1}^n e^{-\epsilon(x_{i-1} \cdot x_i)_p} \right\},$$

where the infimum is taken over all finite sequence $x = x_1, x_2, \dots, x_n = y$ in ∂X , no bound on n .

Theorem 2.10 ([Ghy90, Section 7.3]). *Let X be a δ -hyperbolic space. Let ϵ be a positive constant so that $e^{2\delta\epsilon} \leq \sqrt{2}$. Then $(3 - 2e^{2\delta\epsilon})e^{-\epsilon(x \cdot y)_p} \leq d_{p, \epsilon}(x, y) \leq e^{-\epsilon(x \cdot y)_p}$ for any $x, y \in \partial X$*

Remark 2.11. The canonical gauge $\mathcal{G}(X)$ on ∂X is the set of all metrics of the form $d = d_{p, \epsilon}$. Two metrics $(\partial X, d_{p_1, \epsilon_1})$ and $(\partial X, d_{p_2, \epsilon_2})$ are *B-equivalent* if there exists a constant $k > 0$ such that

$$k^{-1} d_{p_2, \epsilon_2}^{\epsilon_1} \leq d_{p_1, \epsilon_1}^{\epsilon_2} \leq k d_{p_2, \epsilon_2}^{\epsilon_1}.$$

Theorem 2.10 says that if $p_1, p_2 \in X$ and $\epsilon_1, \epsilon_2 > 0$ such that $e^{2\delta\epsilon_1}, e^{2\delta\epsilon_2} \leq \sqrt{2}$, then $(\partial X, d_{p_1, \epsilon_1})$ and $(\partial X, d_{p_2, \epsilon_2})$ are B-equivalent. A metric $d_{p, \epsilon}$ in $\mathcal{G}(X)$ induces a topology on ∂X and this topology does not depend on the choice of the metric $d_{p, \epsilon}$ in $\mathcal{G}(X)$.

2.3. Metric Morse boundaries. With the Gromov product in hand, we can define metric Morse boundaries, this is the work of Cordes and Hume in their paper [CH17]. I will give a quick review of their work.

Let (X, d) be a geodesic metric space. For a Morse gauge N , let

$$X_{x_0}^{(N)} = \{x \in X \mid \text{all geodesics between } x \text{ and } x_0 \text{ are } N\text{-Morse}\}.$$

This metric space is not necessarily a geodesic space. However, it is a $32N(3, 0)$ -hyperbolic spaces in the sense of Definition 2.7.

Thus we get a sequential boundary $\partial X_{x_0}^{(N)}$ with a visual metric $d_{x_0, \epsilon_N} \in \mathcal{G}(X_{x_0}^{(N)})$ for some visibility parameter ϵ_N . In our paper, it is convenient to fix a visibility parameter ϵ_N for each N and to work with this visual metric $(\partial X_{x_0}^{(N)}, d_{x_0, \epsilon_N})$.

Definition 2.12. Let $x, y \in \partial X_{x_0}^{(N)}$. The N -Gromov product of x and y is defined by

$$(x \cdot_N y)_{x_0} = \sup \liminf_{n, m \rightarrow \infty} (x_n \cdot y_m)_{x_0},$$

where the supremum is taken over all sequences $(x_n), (y_m)$ in $X_{x_0}^{(N)}$ such that $x = \lim x_n, y = \lim y_m$.

The Gromov products of $x, y \in \partial X_{x_0}^{(N)}$ depend on N . From Lemma 3.11 in [CH17], if $x, y \in \partial X_{x_0}^{(N)}$ and $N \leq N'$, then

$$(x \cdot_N y)_{x_0} \leq (x \cdot_{N'} y)_{x_0} \leq (x \cdot_N y)_{x_0} + 64N'(3, 0).$$

2.4. Relationship between $\partial X_{x_0}^{(N)}$ and $\partial_*^N X_{x_0}$. Cordes and Hume [CH17] gave the definition of the space $X_{x_0}^{(N)}$ and proved many properties. In this subsection, let us review these properties since we will use them in later sections. Also we need to be more careful about the space $X_{x_0}^{(N)}$ since it is not necessarily geodesic.

Now assume that (X, d) is a proper geodesic metric space. We will discuss the relationship between $\partial X_{x_0}^{(N)}$ and $\partial_*^N X_{x_0}$. Note that $\partial X_{x_0}^{(N)}$ is the sequential boundary with respect to the Gromov product and $\partial_*^N X_{x_0}$ is the N -Morse boundary with the compact-open topology.

The Morse gauge of segments of an N -Morse geodesic might be different from N , so the natural map from $\partial_*^N X_{x_0}$ to $\partial X_{x_0}^{(N)}$ does not always exist, see Figure 2.1. This problem is not crucial, we will use $\partial X_{x_0}^{(N')}$ instead of $\partial X_{x_0}^{(N)}$ for some Morse gauge N' . Thus we have Theorem 2.13.

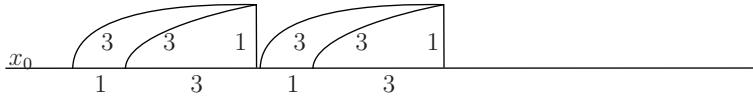


FIGURE 2.1. The horizontal line is $\mathbb{R}_{\geq 0}$. Let $x_0 = 0$. At each triple $(t, t+1, t+4)$ of points, a $(3,3,1)$ -type graph is attached. This horizontal line is a 2-Morse geodesic ray. But $\partial X_{x_0}^{(2)}$ is empty.

Theorem 2.13. *For any N Morse gauge, there exists N' depending only on N such that the following holds.*

Let X be a proper geodesic metric space. Let x_0 be a basepoint. There exist two natural embeddings

$$\begin{aligned} i_N : \partial X_{x_0}^{(N)} &\rightarrow \partial_*^N X_{x_0}, \text{ and} \\ j_{N,N'} : \partial_*^N X_{x_0} &\rightarrow \partial X_{x_0}^{(N')}. \end{aligned}$$

Proof. From Lemma 2.3, any segments of N -Morse geodesics are N' -Morse, where N' depends only on N . We follow the proof of Theorem 3.14 in [CH17]. They constructed two maps i_N and $j_{N,N'}$ and showed that i_N is well-defined. It is not hard to show that these two maps are injections. Following Remark 3.17 in [BH13, III.H.], when X is proper, these injections i_N and $j_{N,N'}$ are embeddings. \square

Remark 2.14. From Theorem 2.13, we conclude that there exists a homeomorphism between $\partial_* X_{x_0}$ and $\varinjlim \partial X_{x_0}^{(N)}$. Lemma 5.4 in [Liu21] tells us that the subspace topology on $\partial_*^N X_{x_0}$ induced by the inclusion map from $\partial_*^N X_{x_0}$ to $\partial_* X_{x_0}$ is the compact-open topology of $\partial_*^N X_{x_0}$. Together with Theorem 2.13, when X is a proper geodesic metric space, the subspace topology of $\partial X_{x_0}^{(N)}$, which is induced by the inclusion map from $\partial X_{x_0}^{(N)}$ to $\partial_* X_{x_0}$, agrees with the topology of $\partial X_{x_0}^{(N)}$ induced by a visual metric $d_{x_0, \epsilon_N} \in \mathcal{G}(X_{x_0}^{(N)})$.

In the proof of Theorem 4.3, we will consider convergence sequences in different topological spaces. We need Lemma 2.15 which follows from Lemma 5.3 in [Liu21] and Theorem 2.13.

Lemma 2.15. *Let x_0 be a basepoint in a proper geodesic metric space X . Let p_n and p be points in $\partial_* X$. Then the following are equivalent:*

- (1) The sequence p_n converges to p in the topology of $\partial_* X_{x_0}$.
- (2) There exists N such that $p_n, p \in \partial_*^N X_{x_0}$ and p_n converges to p in the topology of $\partial_*^N X_{x_0}$.
- (3) There exists N' such that $p_n, p \in \partial X_{x_0}^{(N')}$ and p_n converges to p in the topology of $\partial X_{x_0}^{(N')}$.
- (4) There exists N' such that $p_n, p \in \partial X_{x_0}^{(N')}$ and $d_{x_0, \epsilon_{N'}}(p_n, p) \rightarrow 0$.

With Theorem 2.13, it is not hard to show the following. In the case of a proper geodesic hyperbolic space, we have this [BH13, III.H. 3.18(3)].

Lemma 2.16. *For any Morse gauge N , there exists a constant $C_N > 0$ such that the following holds.*

Let X be a proper geodesic metric space and $x_0 \in X$ be a basepoint. For any $p, q \in \partial X_{x_0}^{(N)}$, let γ be any geodesic between p and q . Then we have $|d_X(p, q) - d_X(x_0, \gamma)| \leq C_N$.

Remark 2.17. In particular, from Lemma 2.16 and Theorem 2.10, it is not hard to have the following observation. Applying the visual metric $(\partial X_{x_0}^{(N)}, d_{x_0, \epsilon_N})$, there exists a positive constant $L_N \geq 1$ depending only on N such that for any $p, q \in \partial X_{x_0}^{(N)}$ and any geodesic γ from p to q ,

$$(2.1) \quad L_N^{-1} e^{-\epsilon_N d_X(x_0, \gamma)} \leq d_{x_0, \epsilon_N}(p, q) \leq L_N e^{-\epsilon_N d_X(x_0, \gamma)}.$$

3. MAPS ON THE MORSE BOUNDARY

For proper metric spaces X, Y , we would like to study bihölder maps, quasi-symmetries and strongly quasi-conformal maps between their Morse boundaries $\partial_* X$ and $\partial_* Y$. Normally, one defines these three types of maps for metric spaces. The fact that Morse boundaries are not metrizable is a problem. But instead, we have a collection of metric Morse boundaries if we choose a basepoint. That is, for any N Morse gauge, basepoint $x_0 \in X$, the boundary $\partial X_{x_0}^{(N)}$ has a visual metric. Here, however, we will consider homeomorphisms between Morse boundaries that, a priori, have nothing to do with the interior points. The following result makes a connection between ideal triangles and interior points. It comes from the work of Charney, Cordes, Murray. Using ideal triangles on the boundary rather than basepoints in the interior is more reasonable in our setting.

3.1. Triangles and coarse center. Denote by $\partial_* X^{(n, N)}$ the set of n -triples of distinct points (p_1, \dots, p_n) in $\partial_* X$ such that any bi-infinite geodesic from p_i to p_j is N -Morse. For any ideal triangle $T(a, b, c)$, where $(a, b, c) \in \partial_* X^{(3, N)}$, we can define centers of $T(a, b, c)$. The center is not unique, but Lemma 3.1 shows that they are a bounded set.

Lemma 3.1 ([CCM19, Lemma 2.5]). *Let X be a proper geodesic metric space. Let $(a, b, c) \in \partial_* X^{(3, N)}$. Set*

$$E_k(a, b, c) = \{x \in X \mid x \text{ lies within } k \text{ of all three sides of some triangle } T(a, b, c)\}.$$

For any $k \geq \delta_N$, we have the following:

- (1) $E_k(a, b, c)$ is non-empty,
- (2) $E_k(a, b, c)$ has bounded diameter L depending only on N and k .

Set $K(a, b, c) = 1 + \inf\{k \mid E_k(a, b, c) \neq \emptyset\}$. It depends only on the vertices, not on N . From Lemma 3.1 we know that $K(a, b, c)$ is bounded by $1 + \delta_N$.

Any point in $E_{K(a, b, c)}(a, b, c)$ is called a *coarse center* of (a, b, c) . For simplicity, we will suppress the notation $E_{K(a, b, c)}(a, b, c)$ as $E(a, b, c)$.

The following two observations are not hard, but they are useful in the later sections.

Lemma 3.2. *For any Morse gauge N , there exists N' such that the following holds.*

Let X be a proper geodesic metric space. For any $(a, b, c) \in \partial_ X^{(3, N)}$ and coarse center $x_0 \in E(a, b, c)$, we have $a, b, c \in \partial X_{x_0}^{(N')}$.*

Proof. Let x_0 be a coarse center of (a, b, c) . By Lemma 3.1, there exists an N -Morse geodesic γ from a to b and a point $x_1 \in \gamma$ such that $d(x_0, x_1) \leq 1 + \delta_N$. By Lemma 2.3, the geodesic $[x_1, a]$ is N_0 -Morse where N_0 depends only on N . Now Lemma 2.4 implies that any geodesic from x_0 to a is N_1 -Morse for some Morse gauge N_1 depending only on N_0 and $1 + \delta_N$. It means that $a \in \partial_*^{N_1} X_{x_0}$. An analogous argument proves that $b, c \in \partial_*^{N_1} X_{x_0}$. By Theorem 2.13, there exists N' depending only on N_1 such that $a, b, c \in \partial X_{x_0}^{(N')}$. \square

Lemma 3.3 says that if two ideal triangles are sufficiently close to each other, then their coarse center sets are uniformly bounded.

Lemma 3.3. *Let X be a proper geodesic metric space and $x_0 \in X$ be a basepoint. Consider the visual metric on the N -Morse boundary: $(\partial X_{x_0}^{(N)}, d_{x_0, \epsilon_N})$. Choose three distinct points $a, b, c \in \partial X_{x_0}^{(N)}$. There exist constants $\lambda = \lambda(N)$ and $D = D(N)$ such that the following holds.*

Let $a', b', c' \in \partial X_{x_0}^{(N)}$ be three distinct points which satisfy

$$d_{x_0, \epsilon_N}(a, a'), d_{x_0, \epsilon_N}(b, b'), d_{x_0, \epsilon_N}(c, c') \leq \lambda.$$

For any $p \in E(a, b, c), p' \in E(a', b', c')$, we have

$$d_X(p, p') \leq D.$$

Proof. By Lemma 2.5, it is not hard to see that any ideal triangle with vertices in $\partial X_{x_0}^{(N)}$ is an N' -Morse triangle, hence it is δ -slim, where N' and δ depend only on N . Choose a point $p_0 \in E(a, b, c)$. There exists a bi-infinite geodesic α from a to b such that $d_X(p_0, \alpha) \leq 1 + \delta$. That is, we have a point $p_1 \in \alpha$ with $d_X(p_0, p_1) \leq 1 + \delta$. We can choose λ sufficiently small so that the distances from p_0 to any geodesic (a, a') and to any geodesic (b, b') are larger than $\delta + 1$. See Figure 3.1. Since all triangles with vertices (a, b', a') and (a, b, b') are δ -slim, for any geodesic (a, b') and (a', b') , there exist points $p_2 \in (b', a)$ and $p_3 \in (a', b')$ such that

$$d_X(p_1, p_2), d_X(p_2, p_3) \leq \delta.$$

It follows that $d(p_0, p_3) \leq 1 + 3\delta$, where p_3 is a point in a geodesic from a' to b' . A similar argument shows that the point $p_0 \in E_{1+3\delta}(a', b', c')$. From Lemma 3.1, it is easy to find constants L_1, L_2 such that $d_X(p, p_0) \leq L_1$ and $d_X(p_0, p') \leq L_2$ for any $p \in E(a, b, c), p' \in E(a', b', c')$. The constants L_1 and L_2 depend only on $1 + 3\delta$ and N' . Taking $D = L_1 + L_2$, this shows the lemma. \square

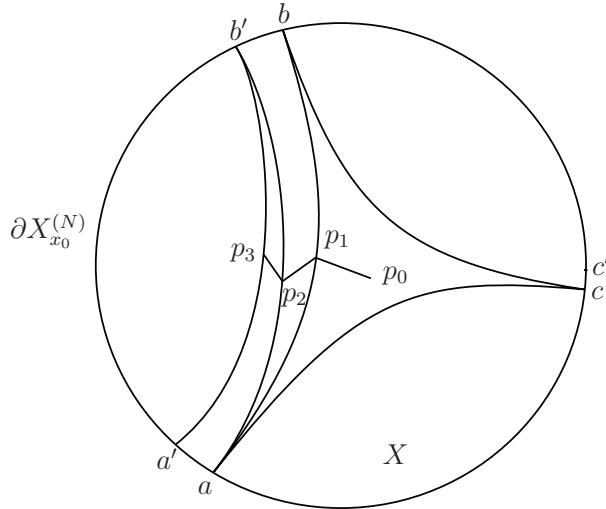


FIGURE 3.1. Point p_0 is a coarse center of the triangle $T(a, b, c)$. Points a' , b' and c' are close to a , b and c , respectively.

3.2. 2-Stable maps and basetriangle stable maps. Let X be a proper geodesic metric space. Let $\partial_* X^{(2,N)} = \{(p, q) \mid p, q \in \partial_* X \text{ and any geodesics between } p \text{ and } q \text{ are } N\text{-Morse}\}$. We say a map $f : \partial_* X \rightarrow \partial_* Y$ is *2-stable* if for any N there exists N' such that $f(\partial_* X^{(2,N)}) \subset \partial_* Y^{(2,N')}$.

From Theorem 3.17 in [CCM19], every quasi-isometry between two proper geodesic metric spaces induces a 2-stable homeomorphism between their Morse boundaries. This homeomorphism does not depend on a choice of basepoint. In our setting, we focus on the maps on the Morse boundary. It does not make sense to use a basepoint in the interior since the map is not defined on the interior. However, using basetriangles on the Morse boundary seems to be reasonable. Indeed, every quasi-isometry induces a *basetriangle stable* map between the Morse boundaries in the following sense. Later we will discuss the relationship between 2-stable maps and basetriangle stable maps.

Definition 3.4 (Basetriangle stable maps). Let X and Y be proper geodesic metric spaces. Let N_0 be a Morse gauge. An embedding $f : \partial_* X \hookrightarrow \partial_* Y$ is N_0 -basetriangle stable if for any N , there exists a Morse gauge $N' = N'(N_0, N)$ such that

- for all 3-triples $(a, b, c) \in \partial_* X^{(3, N_0)}$, and
- for all $x_0 \in E(a, b, c)$, $y_0 \in E(f(a), f(b), f(c))$,

we have

$$f(\partial X_{x_0}^{(N)}) \subset \partial Y_{y_0}^{(N')}.$$

We say that f is *basetriangle stable* if it is N_0 -basetriangle stable for every N_0 .

Remark 3.5. Under the assumptions in Definition 3.4, for simplicity, we usually suppress the notation $f|_{\partial X_{x_0}^{(N)}} : (\partial X_{x_0}^{(N)}, d_{x_0, \epsilon_N}) \hookrightarrow (\partial Y_{y_0}^{(N')}, d_{y_0, \epsilon_{N'}})$ as $f_N : \partial X^{(N)} \hookrightarrow \partial Y^{(N')}$ in Definition 3.11, Definition 3.15 and Definition 3.18. However, for different basetriangles and coarse centers, the embedding f_N is different.

This definition of basetriangle stable map is more complicated than the notion of 2-stable maps. But the motivation comes naturally from the quasi-isometry between two proper geodesic metric spaces. When dealing with quasi-Möbius maps, one focuses on bi-infinite Morse geodesics, so it is convenient to use 2-stable maps. In our setting, we want to study other types of maps, like quasi-symmetric and quasi-conformal maps between the Morse boundaries which are not metrizable in general. Since for any N , the map f_N is between two metric spaces $\partial X_{x_0}^{(N)}$ and $\partial Y_{y_0}^{(N')}$, the basetriangle stable condition provides a way to define these properties for maps on the Morse boundaries. Proposition 3.6 says that basetriangle stable maps are usually weaker than 2-stable maps. However, if the space X has a cocompact group action, they are equivalent. In the following sections, we will switch between these notions.

Proposition 3.6. *Let X and Y be two proper geodesic metric spaces. If $f : \partial_* X \rightarrow \partial_* Y$ is a 2-stable embedding, then f is basetriangle stable. Conversely, if X is cocompact and f is N_0 -basetriangle stable for some N_0 , then f is 2-stable.*

In particular, if $h : X \rightarrow Y$ is a quasi-isometry, then the induced homeomorphism between their Morse boundaries is basetriangle stable.

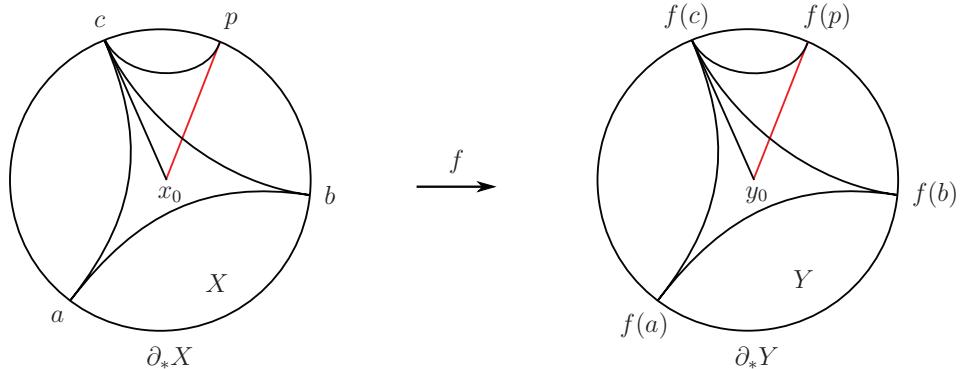


FIGURE 3.2. 2-Stable embedding is basetriangle stable

Proof. First, we assume that $f : \partial_* X \rightarrow \partial_* Y$ is a 2-stable embedding. Given any N_0 , there exists N_1 depending on f and N_0 so that $f(\partial_* X^{(3, N_0)}) \subset \partial_* Y^{(3, N_1)}$. For any N , we are going to find N' such that the following holds. For any $(a, b, c) \in \partial_* X^{(3, N_0)}$, $x_0 \in E(a, b, c)$ and $y_0 \in E(f(a), f(b), f(c))$, we have

$$f(\partial X_{x_0}^{(N)}) \subset \partial Y_{y_0}^{(N')}.$$

By Lemma 3.2, the geodesics

$$[x_0, a] \text{ is } N'_0\text{-Morse and } [y_0, f(a)] \text{ is } N'_1\text{-Morse.}$$

Let $p \in \partial X_{x_0}^{(N)}$. Consider triangles $T(x_0, c, p)$ and $T(y_0, f(c), f(p))$ as in Figure 3.2. The geodesic (c, p) is N_2 -Morse by Lemma 2.5. Since f is 2-stable, the geodesic $(f(c), f(p))$ is N_3 -Morse. Lemma 2.5 implies that the geodesic $[y_0, f(p)]$ is N' -Morse. Hence, $f(p) \in \partial Y_{y_0}^{(N')}$. Note that all Morse gauges N_1, N'_0, N'_1, N_2 and N_3

depend on f , N and N_0 only. So the same holds for N' . Thus f is N_0 -basetriangle stable for every N_0 .

Now assume that X is cocompact, and f is N_0 -basetriangle stable. Fix a base-triangle $(a, b, c) \in \partial_* X^{(3, N_0)}$, $x_0 \in E(a, b, c)$. Let N be a Morse gauge and for any $(p, q) \in \partial_* X^{(2, N)}$. It is enough to show that the geodesic between $f(p)$ and $f(q)$ is N' -Morse where N' depends only on N and f . As in Figure 3.3, choose a point o in the geodesic (p, q) . Lemma 2.3 tells us that $[o, p]$ and $[o, q]$ are N' -Morse where N' depends on N . Since X is cocompact, there exists a group G acting on X isometrically and there exist $R > 0$ and $g \in G$ such that $d_X(g(x_0), o) \leq R$.

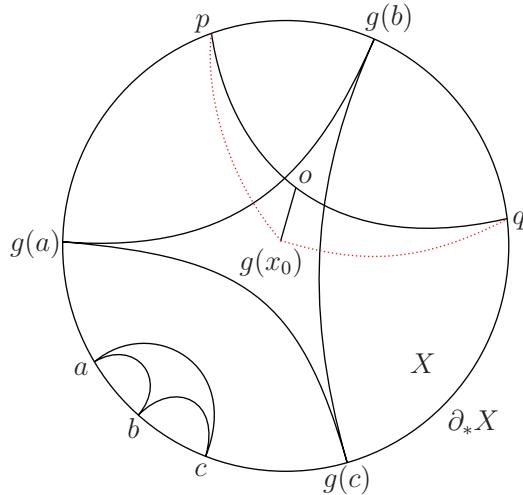


FIGURE 3.3. Geodesic (p, q) is N -Morse. The distance $d(o, g(x_0)) \leq R$

By Lemma 2.4 we have the geodesics $[g(x_0), p]$ and $[g(x_0), q]$ are N_1 -Morse, where N_1 depends on N' and R . That is,

$$p, q \in \partial X_{g(x_0)}^{N_1} \text{ and } g(x_0) \in E(g(a), g(b), g(c)).$$

By Definition 3.4, there exists Morse gauge N'_1 depending on f and N_1 such that for any point $y_0 \in E(f(g(a)), f(g(b)), f(g(c)))$, we have

$$f(p), f(q) \in \partial Y_{y_0}^{N'_1}.$$

Now consider the triangle $T(y_0, f(p), f(q))$. By Lemma 2.5 again, there exists N' depending on N'_1 so that $(f(p), f(q)) \in \partial_* Y^{(2, N')}$. The Morse gauge N' depends only on N , R and f . Thus f is 2-stable. \square

The condition cocompactness is important in the converse part. An N_0 -basetriangle stable homeomorphism f does not guarantee that f is 2-stable. See Examples 3.7 and 3.8.

Example 3.7. Let X_0 be a Euclidean plane \mathbb{R}^2 . Choose two points $(\epsilon, 0)$ and $(-\epsilon, 0)$ for some constant ϵ . At points $(m, 0) \in \mathbb{Z}^2 \subset \mathbb{R}^2$, a vertical ray r_m is attached. At the point $(\epsilon, 0)$ (resp. $(-\epsilon, 0)$), a vertical ray r' (resp. r'') is attached.

Let X be the space X_0 with these rays attached. The Morse boundary of X is the discrete set of the vertical rays.

The contracting property is easier to use than the Morse property in this case since X is a $CAT(0)$ space. We can choose ϵ sufficiently small so that there are exactly three 2ϵ -contracting bi-infinite geodesics in X . They are (r_0, r') , (r_0, r'') and (r', r'') . See Figure 3.4. The constant ϵ is $\frac{1}{4}$.

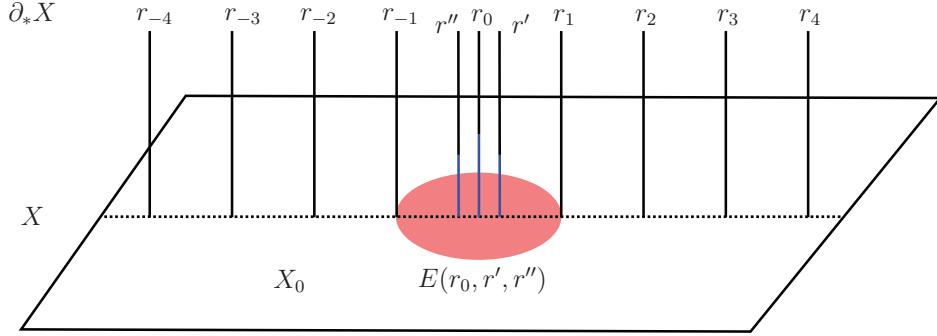


FIGURE 3.4. f is $\frac{1}{2}$ -basetriangle stable but is not 2-stable

Construct a homeomorphism $f : \partial_*X \rightarrow \partial_*X$ as follows. $f(r_{2n}) = r_{-2n}$ for any $n \in \mathbb{Z}$ and f fixes all other points in ∂_*X . For any n , consider the geodesic (r_{2n}, r_{2n+1}) which is 1-contracting, but the geodesic $(f(r_{2n}), f(r_{2n+1})) = (r_{-2n}, r_{2n+1})$ is $4n + 1$ -contracting. This implies f is not a 2-stable map.

But we will see that f is a $\frac{1}{2}$ -basetriangle stable map. Note that the coarse center set $E(r_0, r', r'')$ is a closed unit ball centered at point $(0, 0)$ in X . More precisely, it is the union of three blue segments in the figure and the closed red unit ball in X_0 centered at $(0, 0)$. The map f fixes the triangle $T(r_0, r', r'')$. For any non-negative constant C and $x_0, y_0 \in E(r_0, r', r'')$, we have $f(\partial X_{x_0}^{(C)}) \subset \partial X_{y_0}^{(C+1)}$. This means that f is a $\frac{1}{2}$ -basetriangle stable map. But for any constant $C \geq 1$, this map f is not C -basetriangle stable.

If we choose the points in the real line carefully, we have Example 3.8. A basetriangle stable homeomorphism f does not guarantee that f is 2-stable.

Example 3.8. We can change the points in Example 3.7. Let X_0 be a Euclidean plane \mathbb{R}^2 . For any $m \in \mathbb{Z}_{\geq 0}$, at every point $(\frac{1}{2}m(m+1), 0) \in \mathbb{Z}^2 \subset \mathbb{R}^2$, a vertical ray r_m is attached. For any $m \in \mathbb{Z}_{\leq 0}$, at points $(\frac{1}{2}m(1-m), 0) \in \mathbb{Z}^2 \subset \mathbb{R}^2$, a vertical ray r_m is attached. The space X is X_0 together with these rays. As in Example 3.7, the Morse boundary is the discrete set of the vertical rays.

Construct a homeomorphism $f : \partial_*X \rightarrow \partial_*X$ as follows. $f(r_{2n}) = r_{-2n}$ for any $n \in \mathbb{Z}$ and f fixes all other points in ∂_*X . By a similar argument, the map f is not a 2-stable map. But it is basetriangle-stable. The reason is the following.

Given any positive constant C , there are only finitely many C -contracting bi-infinite geodesics in X . It follows that the union of the coarse center sets of all $(a, b, c) \in \partial_*X^{(3,C)}$ has bounded diameter. An analogous argument shows that the map f is C -basetriangle stable. Thus, it is basetriangle stable and not 2-stable.

3.3. Bihölder maps, quasi-symmetries and strongly quasi-conformal maps. Now we are ready to define bihölder maps, quasi-symmetries and strongly quasi-conformal maps on the Morse boundaries. The motivations for these definitions come naturally from the fact that every quasi-isometry between two proper geodesic metric spaces induces a bihölder map, a quasi-symmetry and a strongly quasi-conformal map on their Morse boundaries. This is shown at the end of this section.

Definition 3.9 (Bihölder maps with respect to basepoints and metrics). Let X, Y be two Gromov hyperbolic spaces. Let $x_0 \in X, y_0 \in Y$ be basepoints. Let d_{x_0, ϵ_X} and d_{y_0, ϵ_Y} be two metrics on the boundaries. A map $f : (\partial_* X_{x_0}, d_{x_0, \epsilon_X}) \rightarrow (\partial_* Y_{y_0}, d_{y_0, \epsilon_Y})$ is *bihölder* with respect to these metrics and basepoints if there exist positive constants $C \geq 1, \alpha_1$, and α_2 such that for all $x_1, x_2 \in \partial_M X_{x_0}^{(N)}$,

$$\frac{1}{C} d_{x_0, \epsilon_X}(x_1, x_2)^{\frac{1}{\alpha_1}} \leq d_{y_0, \epsilon_Y}(f(x_1), f(x_2))^{\alpha_2} \leq C d_{x_0, \epsilon_X}(x_1, x_2)^{\alpha_1}.$$

We call C, α_1 and α_2 *bihölder constants* of f .

Remark 3.10. Bihölder maps can be considered between any two metric spaces. The constant α_2 is usually taken to be 1. In our setting, the metric $d_{x_0, \epsilon}$ on the Gromov boundaries has a parameter ϵ . From Remark 2.11, for different ϵ and ϵ' , the metrics $d_{x_0, \epsilon}$ and $d_{x_0, \epsilon'}$ are B-equivalent. For this reason, we need a positive power α_2 in Definition 3.9.

Now let us give the definition of bihölder maps between Morse boundaries. We use notations as in Definition 3.4 and Remark 3.5.

Definition 3.11 (Bihölder maps between Morse boundaries). Let X and Y be two proper geodesic metric spaces. An N_0 -basetriangle stable map $f : \partial_* X \rightarrow \partial_* Y$ is (N_0, N) -*bihölder* if

- $f_N : \partial_* X^{(N)} \hookrightarrow \partial_* Y^{(N')}$ is bihölder for all choices of base data, and
- the bihölder constants depend only on f, N and N_0 .

We say that f is *bihölder* if it is (N_0, N) -böhölder for every N and every N_0 .

Remark 3.12. From Definition 3.4 and Remark 3.5, the map $f_N : \partial X^{(N)} \hookrightarrow \partial Y^{(N')}$ depends on the base data including the basetriangle and the center of basetriangle. Once we choose some basetriangle and its center, the map f_N is bihölder in the sense of Definition 3.9. This remark also applies to Definition 3.15 and Definition 3.18,

Remark 3.13. In the case of hyperbolic spaces, a bihölder map f between Gromov boundaries in Definition 3.9 is different from a bihölder map in Definition 3.11. The former one is defined in terms of the fixed basepoints and metrics. But the latter one is considering all basetriangles and all centers. The relevant constants do not depend on the choices of basetriangle and center.

Now let us give the definition of a quasi-symmetric map.

Definition 3.14. Let $(X_1, d_1), (X_2, d_2)$ be two metric spaces. A homeomorphism $f : X_1 \rightarrow X_2$ is said to be *quasi-symmetric* if there exists an increasing homeomorphism $\psi_f : (0, \infty) \rightarrow (0, \infty)$ such that for any three distinct points $a, b, c \in X_1$ we have

$$\frac{d_2(f(a), f(b))}{d_2(f(a), f(c))} \leq \psi_f \left(\frac{d_1(a, b)}{d_1(a, c)} \right).$$

Let $\alpha > 0, \lambda \geq 1$. The quasi-symmetry f is called a *power quasi-symmetry* if the homeomorphism ψ is given by the following

$$\psi(t) = \psi_{\alpha, \lambda}(t) = \begin{cases} \lambda t^{\frac{1}{\alpha}} & \text{if } 0 < t < 1, \\ \lambda t^\alpha & \text{if } t \geq 1. \end{cases}$$

Definition 3.15 (Quasi-symmetries between Morse boundaries). Let X and Y be proper geodesic metric spaces. An N_0 -basetriangle stable homeomorphism $f : \partial_* X \rightarrow \partial_* Y$ is (N_0, N) -quasi-symmetric if

- $f_N : \partial_* X^{(N)} \hookrightarrow \partial_* Y^{(N')}$ is quasi-symmetric onto its image for all choices of base data, and
- the homeomorphism ψ_{f_N} depends only on f , N and N_0 .

We say that f is *quasi-symmetric* if it is (N_0, N) -quasi-symmetric for every N and N_0 .

Recall that in a metric space (X, d) , an r -annulus A is defined by

$$A = A(x_0, a, ar) = \{x \in X \mid a \leq d(x, x_0) \leq ar\},$$

where $r \geq 1$, $a > 0$ and x_0 is the center of the annulus. Sometimes we use $A(x_0, r)$ to emphasize the center and the ratio of radii of two concentric spheres.

P. Pansu gave the following notion of a quasi-conformal map which says that a quasi-conformal map does not distort annuli too much.

Definition 3.16. A map f between two metric spaces (X_1, d_1) and (X_2, d_2) is said to be quasi-conformal if there exists a function $\phi_f : [1, \infty) \rightarrow [1, \infty)$ such that f maps every r -annulus of X_1 into some $\phi_f(r)$ -annulus of X_2 .

In our setting, we will use a slightly different notion which we call *strongly quasi-conformal* maps, which take account of the center of the annulus.

Definition 3.17. A map f between two metric spaces (X_1, d_1) and (X_2, d_2) is said to be *strongly quasi-conformal* if there exists a function $\phi_f : [1, \infty) \rightarrow [1, \infty)$ such that f maps every r -annulus $A(x, r)$ of X_1 into some $\phi_f(r)$ -annulus $A(f(x), \phi_f(r))$ of X_2 .

Definition 3.18 (Strongly quasi-conformal maps between Morse boundaries). Let X and Y be two proper geodesic metric spaces. An N_0 -basetriangle stable map $f : \partial_* X \rightarrow \partial_* Y$ is (N_0, N) -strongly quasi-conformal if

- $f_N : \partial_* X^{(N)} \rightarrow \partial_* Y^{(N')}$ is strongly quasi-conformal for all base data, and
- the function ϕ_{f_N} depends only on f , N and N_0 .

We say that f is *strongly quasi-conformal* if it is (N_0, N) -strongly quasi-conformal for every N and every N_0 .

It is not hard to check the following. Let X, Y, Z be proper geodesic metric spaces. If $f : \partial_* X \rightarrow \partial_* Y$ and $g : \partial_* Y \rightarrow \partial_* Z$ are bihölder (resp. quasi-symmetric, strongly quasi-conformal), then $g \circ f : \partial_* X \rightarrow \partial_* Z$ is bihölder (resp. quasi-symmetric, strongly quasi-conformal).

It is helpful to know what happens to N -Morse Gromov products under quasi-isometries. In the case of hyperbolic spaces, we have the following result which will be generalized to our case.

Proposition 3.19. *Given $K \geq 1, C, \delta \geq 0$, there is a constant $A = A(K, C, \delta)$ such that the following holds.*

Let X, Y be two geodesic δ -hyperbolic metric spaces and let $h : X \rightarrow Y$ be a (K, C) -quasi-isometry. If $x_0, x_1, x_2 \in X$, then

$$K^{-1}(x_1 \cdot x_2)_{x_0} - A \leq (h(x_1) \cdot h(x_2))_{h(x_0)} \leq K(x_1 \cdot x_2)_{x_0} + A.$$

Proposition 3.20 is a special case of Proposition 3.18 in [CH17].

Proposition 3.20. *Given a Morse gauge N and constants $K \geq 1, C \geq 0$, there exist a constant $A = A(N, K, C)$ and a Morse gauge N' such that the following holds.*

Let X, Y be two proper geodesic metric spaces and let $h : X \rightarrow Y$ be a (K, C) -quasi-isometry, then h induces an embedding $\partial_ h : \partial X_{x_0}^{(N)} \rightarrow \partial Y_{h(x_0)}^{(N')}$. For any $x_1, x_2 \in X_{x_0}^{(N)}$, we have*

$$K^{-1}(x_1 \cdot x_2)_{x_0} - A \leq (h(x_1) \cdot h(x_2))_{h(x_0)} \leq K(x_1 \cdot x_2)_{x_0} + A.$$

In particular, there exists a constant $A' = A'(N, K, C)$ such that for any $x_1, x_2 \in \partial X_{x_0}^{(N)}$, we have

$$K^{-1}(x_1 \cdot_N x_2)_{x_0} - A' \leq (\partial_* h(x_1) \cdot_{N'} \partial_* h(x_2))_{h(x_0)} \leq K(x_1 \cdot_N x_2)_{x_0} + A'.$$

Note that Proposition 3.20 used the Gromov product with respect to the basepoints x_0 and $h(x_0)$. But we would like to use basetriangles in the Morse boundary instead of basepoints in the interior. The following basic proposition describes a connection between the basepoint and the coarse center of a basetriangle. Fix a basetriangle T on the Morse boundary $\partial_* X$, and choose a coarse center of T as a basepoint in X . Proposition 3.21 tells us that, under a quasi-isometry h , the image of this basepoint is not too far away from coarse centers of the image of this basetriangle T .

Proposition 3.21. *Let N_0 be a Morse gauge. For any constants $K \geq 1, C \geq 0$, there exists a constant $A = A(N_0, K, C)$ such that the following holds.*

Let X, Y be two proper geodesic metric spaces and let $h : X \rightarrow Y$ be a (K, C) -quasi-isometry. Then for any

$$(a, b, c) \in \partial_* X^{(3, N_0)}, x_0 \in E(a, b, c) \text{ and } y_0 \in E(\partial_* h(a), \partial_* h(b), \partial_* h(c)),$$

we have $d_Y(h(x_0), y_0) \leq A$.

Proof. By Proposition 3.6, the quasi-isometry h induces a basetriangle stable and 2-stable homeomorphism $\partial_* h$ between their Morse boundaries. For any $(a, b, c) \in \partial_* X^{(3, N_0)}$, there exists N_1 such that $\partial_* h(\partial_* X^{(3, N_0)}) \subset \partial_* Y^{(3, N_1)}$. Since x_0 is a coarse center of (a, b, c) , then there exists a geodesic α joining a and b , such that $d_X(x_0, \alpha) \leq 1 + \delta_{N_0}$. Since h is a quasi-isometry, then $d_Y(h(x_0), h(\alpha)) \leq L_1$, where L_1 depends only on K, C and N_0 . Note that $h(\alpha)$ is a (K, C) -quasi-geodesic between $\partial_* h(a)$ and $\partial_* h(b)$. Let α' be a geodesic between $\partial_* h(a)$ and $\partial_* h(b)$ in Y . By the Morse property of α' , the Hausdorff distance between $h(\alpha)$ and α' is bounded by L_2 , where L_2 depends only on K, C and N_1 . Thus $d_Y(h(x_0), \alpha') \leq L_1 + L_2$. It follows that $h(x_0) \in E_{L_1+L_2}(\partial_* h(a), \partial_* h(b), \partial_* h(c))$.

Setting $L_0 = \max\{L_1 + L_2, 1 + \delta_{N_1}\}$, we have $y_0, h(x_0) \in E_{L_0}(\partial_* h(a), \partial_* h(b), \partial_* h(c))$. From Lemma 3.1, the set $E_{L_0}(\partial_* h(a), \partial_* h(b), \partial_* h(c))$ has bounded diameter A , where A depends only on L_0 and N_1 . Thus $d_Y(h(x_0), y_0) \leq A$ and $A = A(K, C, N_0)$. \square

We will now discuss the properties of the Gromov products with respect to coarse centers of basetriangles under quasi-isometries.

Proposition 3.22. *Let N_0, N be Morse gauges. Let $K \geq 1, C \geq 0$. There exist a constant A and a Morse gauge N' depending only on N_0, N, K, C such that the following holds.*

Let X, Y be two proper geodesic metric spaces and let $h : X \rightarrow Y$ be a (K, C) -quasi-isometry. For any basetriangle and coarse centers

$$(a, b, c) \in \partial_* X^{(3, N_0)}, x_0 \in E(a, b, c), y_0 \in E(\partial_* h(a), \partial_* h(b), \partial_* h(c)),$$

we have

$$\partial_* h(\partial X_{x_0}^{(N)}) \subset \partial Y_{y_0}^{(N')}.$$

And for any $x_1, x_2 \in \partial X_{x_0}^{(N)}$, we have

$$K^{-1}(x_1 \cdot_N x_2)_{x_0} - A \leq (\partial_* h(x_1) \cdot_{N'} \partial_* h(x_2))_{y_0} \leq K(x_1 \cdot_N x_2)_{x_0} + A.$$

Proof. The first part comes from the fact that h induces a basetriangle stable homeomorphism between their Morse boundaries. That is, for any N , there exists $N' = N'(K, C, N)$ such that $\partial_* h(\partial X_{x_0}^{(N)}) \subset \partial Y_{y_0}^{(N')}$. We can choose N' sufficiently large, such that

$$\partial_* h(\partial X_{x_0}^{(N)}) \subset \partial Y_{h(x_0)}^{(N')}.$$

This is possible since $d(y_0, h(x_0)) \leq A_0 = A_0(K, C, N_0)$ by Proposition 3.21. With Lemma 2.16, it is not hard to show that

$$|(\partial_* h(x_1) \cdot_{N'} \partial_* h(x_2))_{y_0} - (\partial_* h(x_1) \cdot_{N'} \partial_* h(x_2))_{h(x_0)}| \leq 2C'_N + A_0,$$

for all $x_1, x_2 \in \partial X_{x_0}^{(N)}$, where C'_N is a constant depending only on N' . The rest of the proof follows from Proposition 3.20. \square

Now we are ready to show Theorem 3.23.

Theorem 3.23. *Let $h : X \rightarrow Y$ be a (K, C) -quasi isometry between two proper geodesic metric spaces. Assume that $\partial_* X$ contains at least three points. Then the induced map $\partial_* h : \partial_* X \rightarrow \partial_* Y$ is a biholder, quasi-symmetric, strongly quasi-conformal homeomorphism.*

Proof. The quasi-isometry $h : X \rightarrow Y$ induces a homeomorphism $f := \partial_* h$ between their Morse boundaries. Let N_0 and N be Morse gauges. Then from Proposition 3.22, there exist A and N' such that the following holds.

For any

$$(a, b, c) \in \partial_* X^{(3, N_0)}, x_0 \in E(a, b, c), y_0 \in E(f(a), f(b), f(c)),$$

there exists a map

$$f|_{\partial X_{x_0}^{(N)}} : (\partial X_{x_0}^{(N)}, d_{x_0, \epsilon_N}) \rightarrow (\partial Y_{y_0}^{(N')}, d_{y_0, \epsilon_{N'}})$$

between two metric spaces. And for any $x_1, x_2 \in \partial X_{x_0}^{(N)}$, we have

$$(3.1) \quad K^{-1}(x_1 \cdot_N x_2)_{x_0} - A \leq (f(x_1) \cdot_{N'} f(x_2))_{y_0} \leq K(x_1 \cdot_N x_2)_{x_0} + A.$$

We will show that the map $f|_{\partial X_{x_0}^{(N)}}$ is bihölder, quasi-symmetric and strongly quasi-conformal. Equation (3.1) implies that

$$\begin{aligned} e^{-K(x_1 \cdot_N x_2)_{x_0} - A} &\leq e^{-(f(x_1) \cdot_{N'} f(x_2))_{y_0}} \leq e^{-K^{-1}(x_1 \cdot_N x_2)_{x_0} + A} \iff \\ (e^{-\epsilon_N(x_1 \cdot_N x_2)_{x_0}})^{K \frac{\epsilon_{N'}}{\epsilon_N}} \cdot e^{-A\epsilon_{N'}} &\leq e^{-\epsilon_{N'}(f(x_1) \cdot_{N'} f(x_2))_{y_0}} \leq (e^{-\epsilon_N(x_1 \cdot_N x_2)_{x_0}})^{K^{-1} \frac{\epsilon_{N'}}{\epsilon_N}} \cdot e^{A\epsilon_{N'}}. \end{aligned}$$

From Theorem 2.10, we have a constant $c = 3 - 2\sqrt{2}$ such that

$$\begin{aligned} ce^{-\epsilon_N(x_1 \cdot_N x_2)_{x_0}} &\leq d_{x_0, \epsilon_N}(x_1, x_2) \leq e^{-\epsilon_N(x_1 \cdot_N x_2)_{x_0}}, \\ ce^{-\epsilon_{N'}(f(x_1) \cdot_{N'} f(x_2))_{y_0}} &\leq d_{y_0, \epsilon_{N'}}(f(x_1), f(x_2)) \leq e^{-\epsilon_{N'}(f(x_1) \cdot_{N'} f(x_2))_{y_0}}. \end{aligned}$$

It is not hard to see that there exist positive constants $C_0 \geq 1, \alpha_1, \alpha_2$ such that the following holds.

$$\frac{1}{C_0} d_{x_0, \epsilon_N}(x_1, x_2)^{\frac{1}{\alpha_1}} \leq d_{f(x_0), \epsilon_{N'}}(f(x_1), f(x_2))^{\alpha_2} \leq C_0 d_{x_0, \epsilon_N}(x_1, x_2)^{\alpha_1},$$

for all $x_1, x_2 \in \partial_* X_{x_0}^{(N)}$. These constants C_0, α_1, α_2 depend only on K, C, N, N_0 . So the map $f|_{\partial X_{x_0}^{(N)}}$ is bihölder. This shows that $f : \partial_* X \rightarrow \partial_* Y$ is bihölder.

Next we show that $f|_{\partial X_{x_0}^{(N)}}$ is bihölder, which implies that $f|_{\partial X_{x_0}^{(N)}}$ is strongly quasi-conformal. By an easy argument, there exist a function $\phi : [1, \infty) \rightarrow [1, \infty)$ and positive constants $\alpha = \alpha(K, C, N, N_0) > 0$ and $\beta = \beta(K, C, N, N_0) \leq 1$ such that the following holds:

For all $p_0 \in \partial X_{x_0}^{(N)}$, $a > 0$ and $r \geq 1$, we have

$$f|_{\partial X_{x_0}^{(N)}}(A(p_0, a, ra)) \subset A(f(p_0), \beta a^\alpha, \phi(r)\beta a^\alpha),$$

where the r -annulus $A(p_0, a, ar) = \{p \in \partial X_{x_0}^{(N)} \mid a \leq d_{x_0, \epsilon_N}(p, p_0) \leq ar\}$ and the $f(r)$ -annulus $A(f(p_0), \beta a^\alpha, \phi(r)\beta a^\alpha) = \{q \in \partial Y_{y_0}^{(N')} \mid \beta a^\alpha \leq d_{y_0, \epsilon_{N'}}(q, f(p_0)) \leq \phi(r)\beta a^\alpha\}$. This shows that $f : \partial_* X \rightarrow \partial_* Y$ is strongly quasi-conformal.

Cordes and Hume [CH17, Theorem 3.16] show that there is some $N_1 = N_1(K, C, N)$ such that $f : (\partial X_{x_0}^{(N)}, d_{x_0, \epsilon_N}) \rightarrow (\partial Y_{h(x_0)}^{(N_1)}, d_{h(x_0), \epsilon_{N_1}})$ is a quasi-symmetry onto its image. We will change the basepoint $h(x_0)$ to y_0 . By Proposition 3.21, $d(h(x_0), y_0) \leq A(K, C, N_0)$. By Lemma 2.4, there exists $N' = N'(N_1, A)$ such that changing the basepoint induces a natural map $i_{h(x_0), y_0} : \partial Y_{h(x_0)}^{(N_1)} \rightarrow \partial Y_{y_0}^{(N')}$. It is a quasi-symmetric map. Thus

$$f|_{\partial X_{x_0}^{(N)}} : (\partial X_{x_0}^{(N)}, d_{x_0, \epsilon_N}) \rightarrow (\partial Y_{y_0}^{(N')}, d_{y_0, \epsilon_{N'}})$$

is a quasi-symmetry. This shows that $f : \partial_* X \rightarrow \partial_* Y$ is bihölder. \square

Remark 3.24. Cordes and Hume actually show more: the map $f : (\partial X_{x_0}^{(N)}, d_{x_0, \epsilon_N}) \rightarrow (\partial Y_{y_0}^{(N')}, d_{y_0, \epsilon_{N'}})$ is a power quasi-symmetry onto its image and the power quasi-symmetry constants depend only on N, N', K, C .

4. FROM BOUNDARIES TO THE INTERIOR

In this section we will prove the reverse implication in Theorem 1.2 . That is, if $f : \partial_* X \rightarrow \partial_* Y$ is a homeomorphism such that f and f^{-1} are bihölder or strongly quasi-conformal or quasi-symmetric, then f is induced by a quasi-isometry between X and Y . We assume that $\partial_* X$ contains at least three points and that both X

and Y are cocompact. From Definition 3.11, Definition 3.15 and Definition 3.18, if a homeomorphism between Morse boundaries is biholder or quasi-symmetric or strongly quasi-conformal, it must be basetriangle stable. By Proposition 3.6, we know that f and f^{-1} are 2-stable maps in our setting.

4.1. From the Morse boundary to the interior. Now given a homeomorphism f between ∂_*X and ∂_*Y , we would like to extend f to a map Φ_f between the interiors.

Let $\Theta_3(\partial_*X)$ be the set of distinct triples in ∂_*X . For any three distinct points $a, b, c \in \partial_*X$, we define a map:

$$\pi_X : \Theta_3(\partial_*X) \rightarrow X,$$

$$\pi_X((a, b, c)) = p,$$

where p is a choice of coarse center in $E(a, b, c)$. We say $\pi_X((a, b, c))$ is the *projection* of (a, b, c) .

Now for any N , let us define a map:

$$\pi_{X,N} : \partial_*X^{(3,N)} \rightarrow X,$$

$$\pi_{X,N}((a, b, c)) = \pi_X((a, b, c)),$$

where $(a, b, c) \in \partial_*X^{(3,N)}$. That is, $\pi_{X,N} = \pi_X|_{\partial_*X^{(3,N)}}$. Sometimes we want to emphasize the Morse gauge, we will use $\pi_{X,N}$ instead of π_X .

Fix a Morse gauge N_0 such that $\partial_*X^{(3,N_0)} \neq \emptyset$. Choose an ideal triangle $(a, b, c) \in \partial_*X^{(3,N_0)}$. Let $x_0 = \pi_X((a, b, c)) \in E(a, b, c)$. By hypothesis X is cocompact. That is, there exists a group G acting cocompactly by isometries on X . Isometries of X preserve the Morse gauges of bi-infinite geodesics. Hence, for any $g \in G$, we have that $g(x_0) \in E(ga, gb, gc)$. By Lemma 3.1, the distance

$$d_X(g(x_0), \pi_{X,N_0}(ga, gb, gc)) \leq L,$$

where L depends on N_0 . Since X is cocompact, there exists $R_0 > 0$ such that for any $x \in X$ there exists $g \in G$ satisfying

$$d_X(g(x_0), x) \leq R_0.$$

Let $R = R_0 + L$, we have that

$$\pi_{X,N_0}^{-1}(B(x, R)) \neq \emptyset$$

for every $x \in X$.

We know that f is 2-stable, which implies $f(\partial_*X^{(3,N_0)}) \subset \partial_*Y^{(3,N_1)}$ for some N_1 . Let

$$\pi_Y : \Theta_3(\partial_*Y) \rightarrow Y \text{ and } \pi_{Y,N_1} : \partial_*Y^{(3,N_1)} \rightarrow Y$$

be the analogous map for Y , where $\pi_{Y,N_1} = \pi_Y|_{\partial_*Y^{(3,N_1)}}$. Now we define

$$\Phi_f : X \rightarrow Y$$

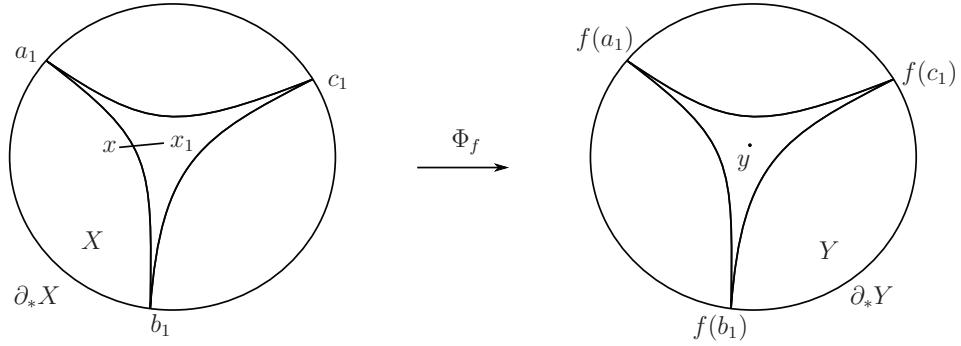
by $\Phi_f(x) = y$, where y is a point in the set

$$\pi_{Y,N_1}(f(\pi_{X,N_0}^{-1}(B(x, R)))) \subset Y.$$

This map Φ_f is called an *extension* of f from ∂_*X to X . The definition of Φ_f depends on choices of N_0, N_1, R, π_X and π_Y .

From the above construction, it is easy to see that

$$\Phi_f(x) = y$$

FIGURE 4.1. Φ_f is an extension of f . $\Phi_f(x) = y$

implies that there exists $(a_1, b_1, c_1) \in \partial_*X^{(3, N_0)}$ such that

$$d_X(\pi_X((a_1, b_1, c_1)), x) \leq R \text{ and } y = \pi_Y(f(a_1), f(b_1), f(c_1)).$$

See Figure 4.1, where $x_1 = \pi_X((a_1, b_1, c_1))$.

We postpone the proof of Proposition 4.1 till the end of this section. It is quite useful for controlling the function Φ_f .

Proposition 4.1. *Let X, Y be two proper geodesic metric spaces. Suppose that X is cocompact and ∂_*X contains at least three points. Let $f : \partial_*X \rightarrow \partial_*Y$ be a homeomorphism which is quasi-symmetric or bihölder or strongly quasi-conformal. Then for any Morse gauge N , there exists a function $\eta_N : (0, \infty) \rightarrow (0, \infty)$ such that for all $(a, b, c), (a', b', c')$ in $\partial_*X^{(3, N)}$,*

$$\begin{aligned} d_X(\pi_X(a, b, c), \pi_X(a', b', c')) &\leq \theta \\ \Rightarrow d_Y(\pi_Y(f(a), f(b), f(c)), \pi_Y(f(a'), f(b'), f(c'))) &\leq \eta_N(\theta). \end{aligned}$$

Now suppose we have Proposition 4.1.

Proposition 4.2. *Fix N_0, N_1 and R as above. Then for any $v \geq 0$, there exists D such that for any $x_1, x_2 \in X$,*

$$d_X(x_1, x_2) \leq v \Rightarrow d_Y(\Phi_f(x_1), \Phi_f(x_2)) \leq D.$$

Proof. For $i = 1, 2$, choose $(a_i, b_i, c_i) \in \pi_{X, N_0}^{-1}(B(x_i, R))$ such that

$$\pi_Y(f(a_i), f(b_i), f(c_i)) = \Phi_f(x_i).$$

Since $d_X(x_1, x_2) \leq v$ and $d_X(x_i, \pi_X(a_i, b_i, c_i)) \leq R$, we have

$$d_X(\pi_X(a_1, b_1, c_1), \pi_X(a_2, b_2, c_2)) \leq v + 2R.$$

By Proposition 4.1, we choose $D = \eta_{N_0}(v + 2R)$ and then

$$d_Y(\pi_Y(f(a_1), f(b_1), f(c_1)), \pi_Y(f(a_2), f(b_2), f(c_2))) \leq D.$$

This proves the proposition. \square

4.2. Main theorem. Now we are ready to prove our main theorem. It suffices to show Theorem 4.3.

Theorem 4.3. *Let X and Y be proper, cocompact geodesic metric spaces and assume that ∂_*X contains at least three points. Let $f : \partial_*X \rightarrow \partial_*Y$ be a homeomorphism. Suppose that f and f^{-1} arebihölder or quasi-symmetric or strongly quasi-conformal. Then there exists a quasi-isometry $h : X \rightarrow Y$ with $\partial_*h = f$.*

Proof. We assume that the Morse boundary of X contains at least three points, similarly for Y . We can choose a Morse gauge N_0 so that

$$\partial_*X^{(3,N_0)} \neq \emptyset \text{ and } \partial_*Y^{(3,N_0)} \neq \emptyset.$$

Note that by Proposition 3.6, homeomorphisms f and f^{-1} are 2-stable. There exists N_1 such that

$$f(\partial_*X^{(2,N_0)}) \subset \partial_*Y^{(2,N_1)} \text{ and } f^{-1}(\partial_*Y^{(2,N_0)}) \subset \partial_*X^{(2,N_1)}.$$

By the construction above, there exists constant $R > 0$ so that

$$\pi_{X,N_0}^{-1}(B(x,R)) \neq \emptyset \text{ and } \pi_{Y,N_0}^{-1}(B(y,R)) \neq \emptyset$$

for all $x \in X$ and $y \in Y$.

Recall that Φ_f is an extension of f from ∂_*X to X and $\Phi_{f^{-1}}$ is an extension of f^{-1} from ∂_*Y to Y . Now let us prove that Φ_f is a quasi-isometry. It is enough to show the following.

- There exist constants A, B such that for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, we have
- $$d_Y(\Phi_f(x_1), \Phi_f(x_2)) \leq Ad_X(x_1, x_2) + B \text{ and } d_X(\Phi_{f^{-1}}(y_1), \Phi_{f^{-1}}(y_2)) \leq Ad_Y(y_1, y_2) + B.$$
- Φ_f and $\Phi_{f^{-1}}$ are quasi-inverses.

Let $x_1, x_2 \in X$, choose integer n such that $n \leq d_X(x_1, x_2) < n + 1$. Choose a sequence of $n + 2$ points $x_1 = p_0, p_1, \dots, p_{n+1} = x_2$ on the geodesic $[x_1, x_2]$ such that

$$d_X(p_i, p_{i+1}) \leq 1,$$

where $0 \leq i \leq n$. By Proposition 4.2, there exists $D = \eta_{N_0}(1 + 2R)$ such that for all $0 \leq i \leq n$,

$$d_Y(\Phi_f(p_i), \Phi_f(p_{i+1})) \leq D.$$

It follows that

$$d_Y(\Phi_f(x_1), \Phi_f(x_2)) \leq D(n + 1) \leq D(d_X(x_1, x_2) + 1).$$

A similar argument for $\Phi_{f^{-1}}$ gives us that for all $y_1, y_2 \in Y$, we have

$$d_X(\Phi_{f^{-1}}(y_1), \Phi_{f^{-1}}(y_2)) \leq D(d_Y(y_1, y_2) + 1).$$

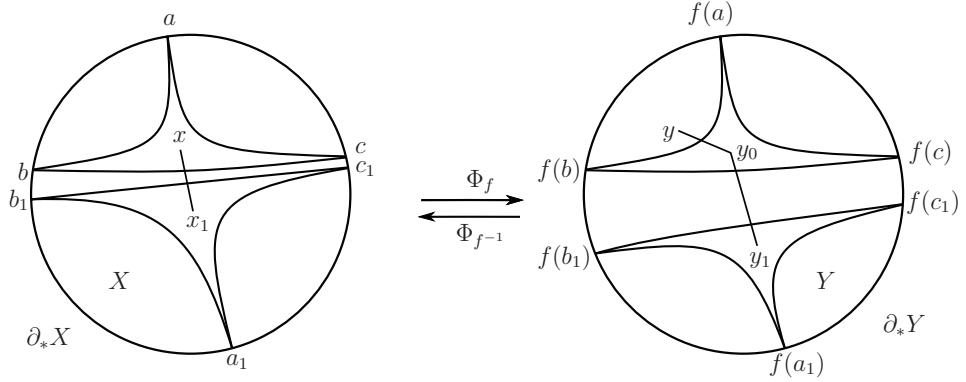
Thus the first bullet point holds.

Next we show that Φ_f and $\Phi_{f^{-1}}$ are quasi-inverses. Let $y \in Y$. Set $x = \Phi_{f^{-1}}(y)$ and $y_1 = \Phi_f(x)$. Choose $(a_1, b_1, c_1) \in \pi_{X,N_0}^{-1}(B(x, R))$ and $(f(a), f(b), f(c)) \in \pi_{Y,N_0}^{-1}(B(y, R))$ such that

$$\pi_{Y,N_1}(f(a_1), f(b_1), f(c_1)) = y_1 \text{ and } \pi_{X,N_1}(a, b, c) = x.$$

Denote $y_0 = \pi_Y(f(a), f(b), f(c))$ and $x_1 = \pi_X(a_1, b_1, c_1)$. See Figure 4.2. Note that

$$d_Y(y, y_0) \leq R \text{ and } d_X(x, x_1) \leq R.$$

FIGURE 4.2. $d_Y(y, \Phi_f(\Phi_{f^{-1}}(y)))$ is bounded uniformly

Let $N = \max\{N_0, N_1\}$. By Proposition 4.1, the distance

$$d_Y(y_0, y_1) \leq \eta_N(R).$$

Then we have

$$d_Y(y, y_1) \leq d_Y(y, y_0) + d_Y(y_0, y_1) \leq R + \eta_N(R).$$

Hence, for any $y \in Y$, we have

$$d_Y(y, \Phi_f(\Phi_{f^{-1}}(y))) \leq R + \eta_N(R).$$

Using the same argument we have that $d_X(x, \Phi_{f^{-1}}(\Phi_f(x)))$ is bounded by $R + \eta_N(R)$ for all $x \in X$. Thus Φ_f is a quasi-isometry.

Next we will prove that the quasi-isometry Φ_f induces f on the Morse boundary. Choose a basepoint x_0 in X . Let p be a point in $\partial_* X$. Let γ be an N -Morse geodesic from x_0 to p such that $\gamma(0) = x_0$. Let $x_n = \gamma(n)$. Setting $y_n = \Phi_f(x_n)$. Choose y_0 as a basepoint in Y . By the construction of Φ_f , for each n , there exists $(a_n, b_n, c_n) \in \pi_{X, N_0}^{-1}(B(x_n, R))$ such that $\pi_Y(f(a_n), f(b_n), f(c_n)) = y_n$. It is not hard to see that $x_n \in E_{\delta_{N_0}+1+R}(a_n, b_n, c_n)$.

Using the slimness property of Morse triangles and Lemma 2.4, we can find a Morse gauge N' depending only on R, N_0 and N such that all the points x_n , and all geodesics between a_n, b_n and c_n are subsets of $X_{x_0}^{(N')}$, and the points $a_n, b_n, c_n \in \partial X_{x_0}^{(N')} \subset \partial_*^{N'} X_{x_0}$.

From the compactness of $\partial_*^{N'} X_{x_0}$ and completeness of $\partial X_{x_0}^{(N')}$, passing to a subsequence, all these three sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ converge to points p_1, p_2 and p_3 in $\partial X_{x_0}^{(N')}$, respectively.

Claim. Two of p_1, p_2, p_3 will be the point p .

Proof of Claim. Firstly suppose that these three points are distinct, we have the coarse center set $E(p_1, p_2, p_3)$. When n is sufficiently large, the sequences of points $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sufficiently close to points p_1, p_2, p_3 respectively in the visual metric $(\partial X_{x_0}^{(N')}, d_{x_0, \epsilon_{N'}})$. From Lemma 3.3, we can see that the point x_n lies within a uniformly bounded distance from $E(p_1, p_2, p_3)$ when n is sufficiently large. But the distance $d_X(x_n, x_0)$ goes to infinity. We get a contradiction. Without loss of generality, we may assume that $p_1 = p_2$.

Since $x_n \in E_{\delta_{N_0}+1+R}(a_n, b_n, c_n)$, there exists a geodesic α_n between a_n and b_n for every n and a point $x'_n \in \alpha_n \subset X_{x_0}^{(N')}$ such that

$$d_X(x_n, x'_n) \leq \delta_{N_0} + 1 + R.$$

Denote by α'_n the segment of the geodesic α_n from x'_n to a_n . We have

$$d_X(x_0, \alpha_n) \leq d_X(x_0, \alpha'_n).$$

Note that $(a_n \cdot_{N'} b_n)_{x_0} \rightarrow \infty$ as $n \rightarrow \infty$. From Lemma 2.16, $d_X(x_0, \alpha_n)$ tends to infinity as $n \rightarrow \infty$. This implies that

$$d_X(x_0, \alpha'_n) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

By Lemma 2.16 again, we have

$$(a_n \cdot_{N'} x'_n)_{x_0} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Applying Lemma 2.8 twice, we get

$$(p_1 \cdot_{N'} x_n)_{x_0} \geq \min\{(p_1 \cdot_{N'} a_n)_{x_0}, (a_n \cdot_{N'} x'_n)_{x_0}, (x'_n \cdot_{N'} x_n)_{x_0}\} - C_{N'},$$

where $C_{N'}$ is a constant depending only on N' . It is easy to see that $(x'_n \cdot_{N'} x_n)_{x_0}$ and $(p_1 \cdot_{N'} a_n)_{x_0}$ tend to infinity as $n \rightarrow \infty$. This implies that $p_1 = \lim x_n$. Note that $p = \lim x_n$, so $p = p_1$. This proves the claim. \square

Thus by passing to a subsequence, two sequences $\{a_n\}$ and $\{b_n\}$ converge to the point p in the topology of $\partial X_{x_0}^{(N')}$. It follows they also converge to the point p in the topology of $\partial_* X_{x_0}$.

Now consider two sequences $\{f(a_n)\}$ and $\{f(b_n)\}$, they converge to $f(p)$ in the topology of $\partial_* Y_{y_0}$ since f is a homeomorphism between $\partial_* X_{x_0}$ and $\partial_* Y_{y_0}$. From Lemma 2.15, we can find a Morse gauge N_2 such that all points $f(a_n), f(b_n), f(p) \in \partial Y_{y_0}^{(N_2)}$ for every n , and these two sequences $\{f(a_n)\}$ and $\{f(b_n)\}$ converge to $f(p)$ in the topology of $\partial Y_{y_0}^{(N_2)}$. Also for N_2 sufficiently large, an analogue of the argument used to prove the Claim shows that $f(p) = \lim y_n = \lim \Phi_f(x_n)$ in the topology of $\partial Y_{y_0}^{(N_2)}$. From Theorem 2.13, it is not hard to get that the Hausdorff distance between $\Phi_f(\gamma)$ and β is finite, where β is some geodesic from y_0 to $f(p)$. Thus we conclude that $\partial_* \Phi_f(p) = f(p)$ for any $p \in \partial_* X$. This means that the quasi-isometry $\Phi_f : X \rightarrow Y$ induces the homeomorphism $f : \partial_* X \rightarrow \partial_* Y$. \square

4.3. Proof of Proposition 4.1.

Proof. Let N be a Morse gauge and $(a_1, b_1, c_1), (a_2, b_2, c_2) \in \partial_* X^{(3, N)}$. As in Figure 4.3, set $x_i = \pi_X((a_i, b_i, c_i))$ and $y_i = \pi_Y((f(a_i), f(b_i), f(c_i)))$, where $i = 1, 2$.

If $d_X(x_1, x_2) \leq \theta$, there exists $N_1 = N_1(N, \theta)$ such that for any $i, j \in \{1, 2\}$

$$a_i, b_i, c_i \in \partial X_{x_j}^{(N_1)}.$$

Since f is basetriangle stable, there exists $N_2 = N_2(N, f)$ so that for any $j = 1, 2$, we have

$$f : \partial X_{x_j}^{(N_1)} \rightarrow \partial Y_{y_j}^{(N_2)}.$$

There are four metric spaces:

$$(\partial X_{x_j}^{(N_1)}, d_{x_j, \epsilon_{N_1}}), (\partial Y_{y_j}^{(N_2)}, d_{y_j, \epsilon_{N_2}}), \text{ where } j = 1, 2.$$

In the following argument, we will use d_{x_j} and d_{y_j} to simplify notations $d_{x_j, \epsilon_{N_1}}$ and $d_{y_j, \epsilon_{N_2}}$, where $j \in \{1, 2\}$.

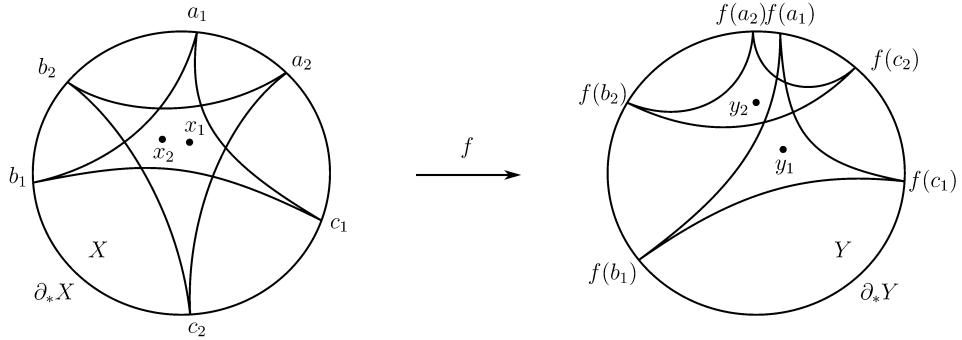


FIGURE 4.3. The homeomorphism f between the Morse boundaries $\partial_* X$ and $\partial_* Y$ induces a map $f : \partial X_{x_j}^{(N_1)} \rightarrow \partial Y_{y_j}^{(N_2)}$, where $j = 1, 2$

Since x_1, x_2 are coarse centers and distance between x_1 and x_2 is bounded by θ , we have that both points x_1 and x_2 lie uniformly bounded distance $\theta + 1 + \delta_N$ from some geodesics (a_i, b_i) , (a_i, c_i) and (b_i, c_i) where $i = 1, 2$. Hence, by Equation (2.1), there exists a positive constant $u = u(\epsilon_{N_1}, \delta_N, \theta)$ such that for any $i, j \in \{1, 2\}$,

$$(4.1) \quad d_{x_j}(a_i, b_i), d_{x_j}(a_i, c_i), d_{x_j}(b_i, c_i) \geq u.$$

For coarse center y_i , it lies uniformly bounded distance $1 + \delta_N$ from some geodesics $(f(a_i), f(b_i))$, $(f(a_i), f(c_i))$ and $(f(b_i), f(c_i))$ where $i = 1, 2$. Again by Equation (2.1), there exists positive constant $u' = u'(\epsilon_{N_2}, \delta_{N_2})$ such that for $j = 1, 2$, we have

$$(4.2) \quad d_{y_j}(f(a_j), f(b_j)), d_{y_j}(f(a_j), f(c_j)), d_{y_j}(f(b_j), f(c_j)) \geq u'.$$

We would like to show a similar equation by switching the basepoints under certain conditions which is the following claim.

Claim. If f is a biholder or quasi-symmetric or strongly quasi-conformal homeomorphism, there exists a lower bound u_0 of

$$d_{y_1}(f(a_2), f(b_2)), d_{y_1}(f(a_2), f(c_2)), d_{y_1}(f(b_2), f(c_2)),$$

where u_0 is a positive constant and depends only on N, f and θ .

We will prove the claim later. Now with this claim, we are ready to find an upper bound for $d_Y(y_1, y_2)$. Since

$$d_{y_1}(f(a_2), f(b_2)), d_{y_1}(f(a_2), f(c_2)), d_{y_1}(f(b_2), f(c_2)) \geq u_0 > 0,$$

there exists a constant $\theta' = \theta'(u_0, N_2, \epsilon_{N_2})$ by Equation (2.1), so that

$$y_1 \in E_{\theta'}((f(a_2), f(b_2), f(c_2))).$$

Let $\theta_1 = \max\{\theta', \delta_{N_2}\}$. We know that

$$y_1, y_2 \in E_{\theta_1}((f(a_2), f(b_2), f(c_2))).$$

Lemma 3.1 says that the set $E_{\theta_1}((f(a_2), f(b_2), f(c_2)))$ has bounded diameter $L = L(\theta_1, N_2)$. Finally, setting $\eta_N(\theta) = L$, we get a constant, which depends only on N, f and θ , such that

$$d_Y(y_1, y_2) \leq \eta_N(\theta).$$

Now it remains to prove the claim.

Proof of Claim. There are three cases.

Case 1 (When f is a bihölder homeomorphism). Suppose that f is a bihölder homeomorphism between $\partial_* X$ and $\partial_* Y$. From the discussion above, we have a bihölder map

$$f : (\partial X_{x_1}^{(N_1)}, d_{x_1}) \rightarrow (\partial Y_{y_1}^{(N_2)}, d_{y_1})$$

and there exist positive constants $C \geq 1$, α_1 , and α_2 such that

$$\frac{1}{C} d_{x_1}(a, b)^{\frac{1}{\alpha_1}} \leq d_{y_1}(f(a), f(b))^{\alpha_2},$$

for any $a, b \in \partial X_{x_1}^{(N_1)}$. Note that C, α_1 and α_2 depend only on f, N_1 and N_2 . From Equation (4.1), we have

$$d_{y_1}(f(a_2), f(b_2)), d_{y_1}(f(a_2), f(c_2)), d_{y_1}(f(b_2), f(c_2)) \geq u_0,$$

where $u_0 = (\frac{1}{C} u^{\frac{1}{\alpha_1}})^{\frac{1}{\alpha_2}}$ is a positive constant depending only on f, N, θ .

Case 2 (When f is a quasi-symmetric homeomorphism). Now let f be a quasi-symmetric homeomorphism between $\partial_* X$ and $\partial_* Y$.

We have a quasi-symmetric homeomorphism

$$f : (\partial X_{x_1}^{(N_1)}, d_{x_1}) \rightarrow (\partial Y_{y_1}^{(N_2)}, d_{y_1})$$

and an increasing homeomorphism $\psi : (0, \infty) \rightarrow (0, \infty)$ such that for any three distinct points $a, b, c \in X_{x_1}^{(N_1)}$ we have

$$\frac{d_{y_1}(f(a), f(b))}{d_{y_1}(f(a), f(c))} \leq \psi \left(\frac{d_{x_1}(a, b)}{d_{x_1}(a, c)} \right).$$

The map ψ depends only on f, N_1 and N_2 .

There are three cases.

- If the set $\{a_1, b_1, c_1\} = \{a_2, b_2, c_2\}$, then the claim follows from Equation (4.2).
- If the set $\{a_1, b_1, c_1\} \cap \{a_2, b_2, c_2\}$ contains two elements. Without loss of generality, we may assume that $a_1 = a_2, b_1 = b_2, c_1 \neq c_2$.

Note that a_2, c_2, b_1 are three distinct points. From Equation (4.1), we have

$$\frac{d_{x_1}(c_2, b_1)}{d_{x_1}(c_2, a_2)} \leq \frac{1}{u}.$$

Thus

$$\frac{d_{y_1}(f(c_2), f(b_1))}{d_{y_1}(f(c_2), f(a_2))} \leq \psi \left(\frac{d_{x_1}(c_2, b_1)}{d_{x_1}(c_2, a_2)} \right) \leq \psi \left(\frac{1}{u} \right).$$

A similar argument for a_2, c_2, c_1 gives

$$\frac{d_{y_1}(f(c_2), f(c_1))}{d_{y_1}(f(c_2), f(a_2))} \leq \psi \left(\frac{d_{x_1}(c_2, c_1)}{d_{x_1}(c_2, a_2)} \right) \leq \psi \left(\frac{1}{u} \right).$$

It follows that

$$2d_{y_1}(f(c_2), f(a_2)) \geq \frac{1}{\psi(\frac{1}{u})} (d_{y_1}(f(c_2), f(b_1)) + d_{y_1}(f(c_2), f(c_1))).$$

With the triangle inequality and Equation (4.2), we have

$$d_{y_1}(f(c_2), f(b_1)) + d_{y_1}(f(c_2), f(c_1)) \geq d_{y_1}(f(b_1), f(c_1)) \geq u'.$$

Finally we deduce that

$$(4.3) \quad d_{y_1}(f(c_2), f(a_2)) \geq \frac{u'}{2\psi(\frac{1}{u})}.$$

Now consider b_2, c_2, a_1 and b_2, c_2, c_1 , applying a similar proof, we get

$$d_{y_1}(f(c_2), f(b_2)) \geq \frac{u'}{2\psi(\frac{1}{u})}.$$

Since $a_1 = a_2, b_1 = b_2$ and from Equation (4.2) we get

$$d_{y_1}(f(a_2), f(b_2)) = d_{y_1}(f(a_1), f(b_1)) \geq u'.$$

Let $u_0 = \min\{u', \frac{u'}{2\psi(\frac{1}{u})}\}$. We have thus shown that

$$d_{y_1}(f(a_2), f(b_2)), d_{y_1}(f(a_2), f(c_2)), d_{y_1}(f(b_2), f(c_2)) \geq u_0.$$

- If the set $\{a_1, b_1, c_1\} \cap \{a_2, b_2, c_2\}$ contains at most one element.

By an analogous argument used to prove Equation (4.3), we can show that

$$d_{y_1}(f(a_2), f(b_2)), d_{y_1}(f(a_2), f(c_2)), d_{y_1}(f(b_2), f(c_2)) \geq \frac{u'}{2\psi(\frac{1}{u})}.$$

Thus we conclude that

$$d_{y_1}(f(a_2), f(b_2)), d_{y_1}(f(a_2), f(c_2)), d_{y_1}(f(b_2), f(c_2)) \geq u_0,$$

where $u_0 = \min\{u', \frac{u'}{2\psi(\frac{1}{u})}\}$ is a positive constant depending only on N, f and θ .

Case 3 (When f is a strongly quasi-conformal homeomorphism). Assume that f is a strongly quasi-conformal homeomorphism between ∂_*X and ∂_*Y .

That is, there exist a strongly quasi-conformal map

$$f : (\partial X_{x_1}^{(N_1)}, d_{x_1}) \rightarrow (\partial Y_{y_1}^{(N_2)}, d_{y_1})$$

and a function $\phi : [1, \infty) \rightarrow [1, \infty)$ such that f maps every r -annulus $A(x, r)$ of $\partial X_{x_1}^{(N_1)}$ into some $\phi(r)$ -annulus $A(f(x), \phi(r))$ of $\partial Y_{y_1}^{(N_2)}$. The map ϕ depends only on f, N_1 and N_2 . Recall that in a metric space (X, d) , an r -annulus A is defined by

$$A(x_0, r) = A(x_0, a, ar) = \{x \in X \mid a \leq d(x, x_0) \leq ar\}.$$

Now we will show that

$$d_{y_1}(f(a_2), f(c_2)), d_{y_1}(f(b_2), f(c_2)), d_{y_1}(f(a_2), f(b_2)) \geq u_0$$

for some positive constant u_0 .

We regard the point c_2 as the center. Let u be the constant in Equation (4.1). Consider the $\frac{3}{u}$ -annulus

$$A(c_2, \frac{3}{u}) = A(c_2, \frac{u}{3}, 1) \subset \partial X_{x_1}^{(N_1)}.$$

There exists some $\phi(\frac{3}{u})$ -annulus $A(f(c_2), v, \phi(\frac{3}{u})v) \subset \partial Y_{y_1}^{(N_2)}$ such that

$$f(A(c_2, \frac{u}{3}, 1)) \subset A(f(c_2), v, \phi(\frac{3}{u})v)$$

for some v .

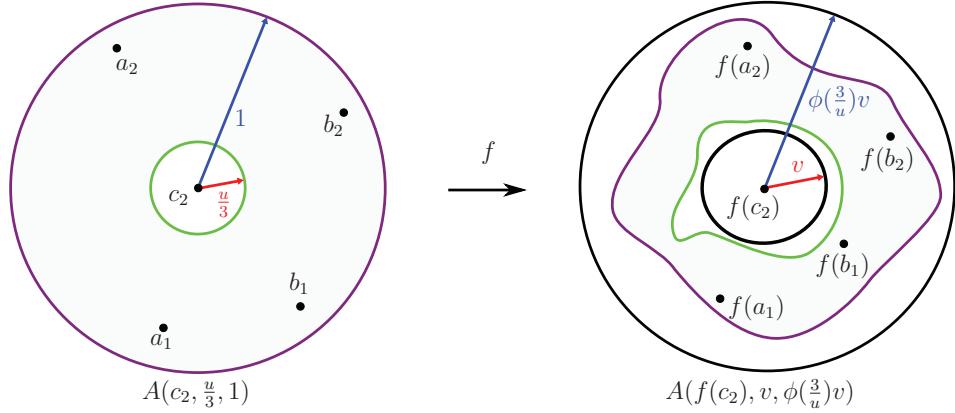


FIGURE 4.4. Point c_2 is the center of a $\frac{3}{u}$ -annulus and $f(A(c_2, \frac{u}{3}, 1)) \subset A(f(c_2), v, \phi(\frac{3}{u})v)$

Note that $a_2, b_2 \in A(c_2, \frac{u}{3}, 1)$. It follows that

$$f(a_2), f(b_2) \in A(f(c_2), v, \phi(\frac{3}{u})v).$$

Hence

$$d_{y_1}(f(a_2), f(c_2)), d_{y_1}(f(b_2), f(c_2)) \geq v.$$

It suffice to find a lower bound for v .

From Equation (4.1), there is at most one point of $\{a_1, b_1, c_1\}$ in the ball $B(c_2, \frac{u}{3})$. See Figure 4.4. Note that the diameter of $\partial_* X_{x_1}^{(N_1)}$ is bounded by 1. Without loss of generality, let us say $a_1, b_1 \in A(c_2, \frac{u}{3}, 1)$. Since $f(A(c_2, \frac{u}{3}, 1)) \subset A(f(c_2), v, \phi(\frac{3}{u})v)$, we have

$$d_{y_1}(f(a_1), f(c_2)), d_{y_1}(f(b_1), f(c_2)) \leq \phi(\frac{3}{u})v.$$

By the triangle inequality, we get

$$d_{y_1}(f(a_1), f(b_1)) \leq d_{y_1}(f(a_1), f(c_2)) + d_{y_1}(f(b_1), f(c_2)) \leq 2\phi(\frac{3}{u})r.$$

From Equation (4.2), it is easy to see that

$$u' \leq 2\phi(\frac{3}{u})v,$$

that is

$$v \geq \frac{u'}{2\phi(\frac{3}{u})}.$$

Let $u_0 = \frac{u'}{2\phi(\frac{3}{u})}$, we have shown that

$$d_{y_1}(f(a_2), f(c_2)), d_{y_1}(f(b_2), f(c_2)) \geq u_0.$$

If we regard the point a_2 as the center and use the similar argument as above, we will get that

$$d_{y_1}(f(a_2), f(b_2)) \geq u_0.$$

This proves the claim. □

□

From the proofs in section 4, we also get Theorem 4.4.

Theorem 4.4. *Let X and Y be proper, cocompact geodesic metric spaces. Suppose that ∂_*X contains at least three points. Let $f : \partial_*X \rightarrow \partial_*Y$ be a homeomorphism. There exist Morse gauges N_0 and N such that the following are equivalent.*

- (1) f is induced by a quasi-isometry $h : X \rightarrow Y$.
- (2) f and f^{-1} are (N_0, N) -bihölder.
- (3) f and f^{-1} are (N_0, N) -quasi-symmetric.
- (4) f and f^{-1} are (N_0, N) -strongly quasi-conformal.

As noted in Remark 3.13, our definitions of bihölder map, quasi-symmetric map, strongly quasi-conformal map are different from the usual definitions. Thus, it is interesting to have Corollary 4.5 in the case of hyperbolic spaces.

Corollary 4.5. *Let X and Y be proper, cocompact geodesic, hyperbolic spaces. Suppose that ∂X contains at least three points. Let $f : \partial X \rightarrow \partial Y$ be a homeomorphism. Then the following are equivalent.*

- (1) f is induced by a quasi-isometry $h : X \rightarrow Y$.
- (2) f and f^{-1} are bihölder.
- (3) f and f^{-1} are quasi-symmetric.
- (4) f and f^{-1} are strongly quasi-conformal.

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