

MINIMIZING ENTROPY FOR TRANSLATION SURFACES

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ABSTRACT. In this note we consider the entropy by Dankwart [*On the large-scale geometry of flat surfaces*, 2014, PhD thesis. <https://bib.math.uni-bonn.de/downloads/bms/BMS-401.pdf>] of unit area translation surfaces in the $SL(2, \mathbb{R})$ orbits of square tiled surfaces that are the union of squares, where the singularities occur at the vertices and the singularities have a common cone angle. We show that the entropy over such orbits is minimized at those surfaces tiled by equilateral triangles where the singularities occur precisely at the vertices. We also provide a method for approximating the entropy of surfaces in the orbits.

1. INTRODUCTION

We begin by recalling for the purposes of motivation a well known classical result of Katok from 1982 for compact negatively curved surfaces. Let \mathcal{M}_g denote the space of negatively curved C^∞ Riemannian metrics of unit volume on a compact orientable surface of genus $g \geq 2$. The entropy function $h : \mathcal{M}_g \rightarrow \mathbb{R}^+$ can be defined in terms of the growth rate of closed geodesics

$$h(\rho) = \lim_{T \rightarrow +\infty} \frac{1}{T} \log \#\{\gamma : \ell_\rho(\gamma) \leq T\}$$

where $\rho \in \mathcal{M}_g$ and $\ell_\rho(\gamma)$ denotes the length of a closed ρ -geodesic γ . When restricted to metrics of unit volume the entropy is minimized precisely at metrics of constant curvature [10].

In this note we want to formulate a partial analogue of this result for translation surfaces. Informally, a translation surface can be thought of as a closed surface obtained from taking a collection of polygons in the plane and gluing together parallel edges via isometries (see [19] for a good introduction to translation surfaces). A translation surface has a finite number of singularities with cone angles of the form $2\pi(k+1)$ where $k \in \mathbb{N}$. To see what this means, consider the following construction: let $k \in \mathbb{N}$ and take $(k+1)$ copies of the upper half-plane with the usual metric and $(k+1)$ copies of the lower half-plane. Then glue them together along the half infinite rays $[0, \infty)$ and $(-\infty, 0]$ in cyclic order (Figure 1).

There are a few equivalent definitions of translation surfaces that appear in the literature. We will use Definition 1.1 (see [18]).

Definition 1.1. A translation surface is a closed and connected topological surface, X , together with a finite set of points Σ and an atlas of charts to \mathbb{C} on $X \setminus \Sigma$, whose transition maps are translations. Furthermore, we require that for each

Received by the editors November 19, 2021, and, in revised form, March 25, 2022.

2020 *Mathematics Subject Classification*. Primary 37B40, 37C83; Secondary 37D35.

The second author was supported by ERC Grant 833802-Resonances and EPSRC grant EP/T001674/1.

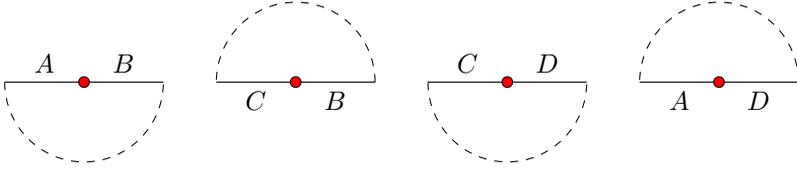


FIGURE 1. Four half-disks glued together cyclically. A singularity/cone-point of angle 4π on a translation surface has a neighborhood isometric to a neighborhood of the the origin in the picture (the red dot).

point $x \in \Sigma$, there exists some $k \in \mathbb{N}$ and a homeomorphism of a neighborhood of x to a neighborhood of the origin in the $2k + 2$ half-plane construction that is an isometry away from x .

Dankwart associated to a translation surface X with a non-empty finite singularity set Σ an analogous notion of entropy [6]. Given $k_1, \dots, k_n \geq 1$ we can denote by $\mathcal{H}^1(k_1, \dots, k_n)$ the space of unit area translation surfaces with n singularities in Σ with cone angles $2\pi(k_1 + 1), \dots, 2\pi(k_n + 1)$. The entropy function $h : \mathcal{H}^1(k_1, \dots, k_n) \rightarrow \mathbb{R}^+$ can be defined in terms of the growth rate of closed geodesics containing a singular point

$$h(X) = \lim_{T \rightarrow +\infty} \frac{1}{T} \log \#\{\gamma : \gamma \cap \Sigma \neq \emptyset, \ell_X(\gamma) \leq T\}$$

where $\ell_X(\gamma)$ denotes the length of a closed geodesic γ on X which includes a singular point from Σ .¹ The entropy is continuous and bounded below, and can become arbitrarily large (when a closed geodesic becomes sufficiently small) even when the total area is normalized (although it will always be finite) [6]. We restrict the type of translation surfaces we will consider as follows:

- (1) Firstly, we fix a unit area square tiled surface X_0 , which is a union of squares where the singularities occur at the vertices and the singularities have a common cone angle; and
- (2) Secondly, we consider the three dimensional orbit $SL(2, \mathbb{R})X_0$ associated to the linear action of the group $SL(2, \mathbb{R})$ [18].

The surfaces described in (1) are known as square tiled surfaces (see [13] for a good introduction to square tiled surfaces).

Note that the area of surfaces in the orbit $SL(2, \mathbb{R})X_0$ coincides with the area of X_0 .

Our main result is Theorem 1.2.

Theorem 1.2. *If X_0 satisfies the hypotheses in (1) then the entropy function*

$$h : SL(2, \mathbb{R})X_0 \rightarrow \mathbb{R}^+$$

is minimized at equilateral translation surfaces, by which we mean translation surfaces tiled by equilateral triangles where the singularities occur precisely at the vertices of the triangles.

¹This avoids the complication of accounting for cylinders of uncountably many parallel geodesics. Alternatively, we could account for these by counting only their free homotopy classes, but then their polynomial growth does not affect the definition of the entropy.

We take the convention that we identify surfaces that are identical up to a rotation (i.e., the action of $SO(2)$).

Theorem 1.2 applies to the following simple example and to the examples listed in §3. Furthermore, it is known that every stratum contains an equilateral translation surface [4].

Example 1.3. Let X_0 be the L -shaped square tiled translation surface made up of three squares (see Figure 2). The surface X_0 has genus 2 and a single singularity of cone angle 6π . The $SL(2, \mathbb{R})$ -orbit, $SL(2, \mathbb{R})X_0$, contains one equilateral translation surface up to isometry. This surface globally minimizes entropy in the orbit space.

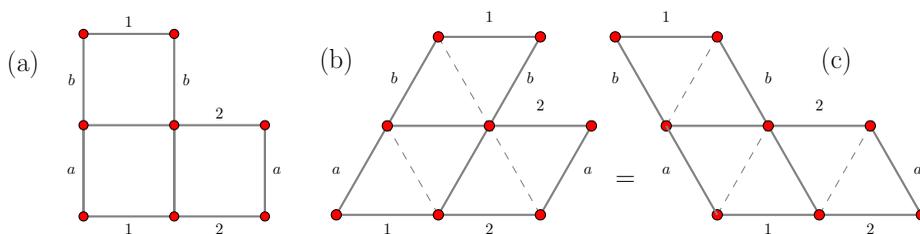


FIGURE 2. (a) The L -shaped translation surface X_0 where the horizontal and vertical sides are identified; (b) and (c) are equivalent surfaces $X_1 = \begin{pmatrix} \sqrt{2/\sqrt{3}} & \frac{1}{\sqrt{2\sqrt{3}}} \\ 0 & \sqrt{\sqrt{3}/2} \end{pmatrix} X_0$ and $X_2 = \begin{pmatrix} \sqrt{2/\sqrt{3}} - \frac{1}{\sqrt{2\sqrt{3}}} \\ 0 & \sqrt{\sqrt{3}/2} \end{pmatrix} X_0$ in the orbit which are triangulated by equilateral triangles.

In Figure 3 we have plotted an approximation² to the entropy of the surfaces.

$$\begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} X_1 \text{ for } -1/2 \leq s \leq 1/2 \text{ and } -\frac{1}{10} \leq u \leq \frac{1}{10}$$

and indicated the point above $X_1 = X_2$. We confirm empirically that the entropy is locally minimized at $X_1 = X_2$. A simple symmetry argument confirms that X_0 is also a critical point.

If we homothetically scale any translation surface by a factor $c > 0$ then the entropy scales by $1/c$, but the area scales by c^2 . Therefore, it is appropriate to consider translation surfaces scaled to have unit area, say. Let $\mathcal{H}^1[k, n]$ denote a stratum of unit area surfaces with n singularities, each with the same cone angle $2\pi(k+1)$, where $n, k \geq 1$. In light of Theorem 1.2 we conjecture³ the following:

Conjecture 1.4. *The entropy function $h : \mathcal{H}^1[k, n] \rightarrow \mathbb{R}^+$ has global minima at equilateral translation surfaces.*

In Sections 2 and 3 we present some preliminary results on entropy. In Section 4 we will present some more examples of surfaces satisfying the hypotheses in (1). In Sections 5 and 6 we introduce the main technical ingredients in the proof:

²This plot was obtained using the method for approximating the entropy of surfaces in the orbit of X_0 derived in Section 8.

³This conjecture may be related to the ‘‘Universal Optimality Conjecture’’ (see Conjecture 9.4 in Section 9 of [5]).

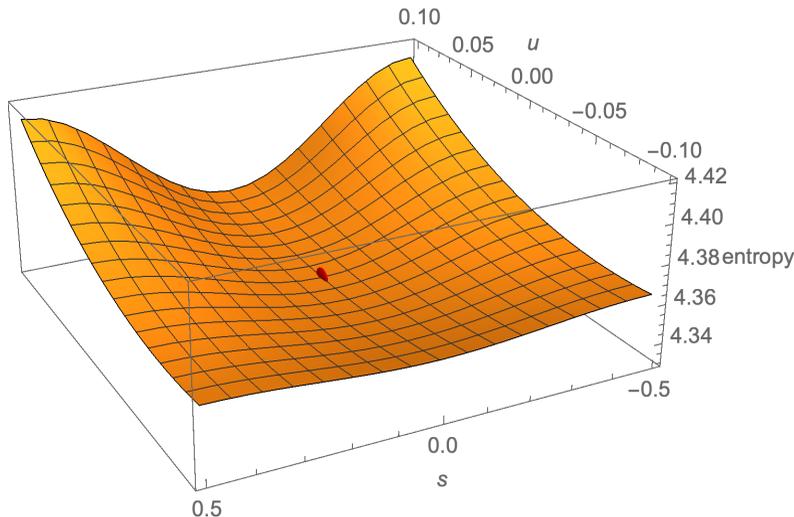


FIGURE 3. A numerical approximation to the plot of the entropy of the surface $\begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} X_1$, where X_1 is represented by the surface in Figure 1(b). The minimum occurring at $s = 0$ and $u = 0$ is illustrated.

Montgomery's and Bernstein's Theorems, respectively. In Section 7 we complete the proof of Theorem 1.2. In Section 8 we derive a method for approximating the entropy of the types of surfaces we consider in this note. In the final section we collect together some final comments and questions.

2. TRANSLATION SURFACES AND ENTROPY

Fix a translation surface X with singularity set Σ . A saddle connection s is a straight line on X between two singularities (which does not contain a singularity in its interior).

The entropy can be defined in terms of the growth of saddle connection paths, which are geodesics joining singularities. Let $i(s)$ and $t(s)$ denote the initial and terminal singularity, respectively, of an oriented saddle connection s . Given a translation surface X , let $\underline{s} = s_1 \dots s_n$ denote an oriented saddle connection path of length $\ell(\underline{s})$ where consecutive oriented saddle connections s_i and s_{i+1} form a locally distance minimizing geodesic. In particular, the angle between s_i and s_{i+1} , for $1 \leq i \leq n-1$, should be greater than or equal to π on both sides. We write $\ell(\underline{s}) = \sum_{i=1}^n \ell(s_i)$ where $\ell(s_i)$ denotes the length of s_i . Definition 2.1 is easily seen to be equivalent to the definition from the introduction.

Definition 2.1. The entropy $h(X)$ of a translation surface X is given by the growth rate of saddle connection paths on X

$$h(X) = \lim_{T \rightarrow +\infty} \frac{1}{T} \log \# \{ \underline{s} : \ell(\underline{s}) \leq T \}.$$

Whenever $\Sigma \neq \emptyset$ we have that $h(X) > 0$.

There is a useful alternative formulation which we now present in the next lemma that follows from Definition 2.1 (see [11]).

Lemma 2.2. *We can write*

$$h(X) = \inf \left\{ t > 0: \sum_{\underline{s}} \exp(-t\ell(\underline{s})) < +\infty \right\}$$

where the summation is over all oriented saddle connection paths on X .

Proof. Let h' be the infimal value of t for which $\sum_{\underline{s}} e^{-t\ell(\underline{s})}$ converges. Then for $t > h$,

$$\begin{aligned} \sum_{\underline{s}} e^{-t\ell(\underline{s})} &= \sum_{n=0}^{\infty} \sum_{\underline{s}: n \leq \ell(\underline{s}) \leq n+1} e^{-t\ell(\underline{s})} \\ &\leq \sum_{n=0}^{\infty} \# \{ \underline{s}: \ell(\underline{s}) \leq n+1 \} e^{-tn} \\ &= \sum_{n=0}^{\infty} e^{(h-t+o(1))n}. \end{aligned}$$

It follows that $\sum_{\underline{s}} e^{-t\ell(\underline{s})}$ converges when $t > h$, hence $h \geq h'$.

Suppose that $h > h'$. Then because the set of u for which

$$\sum_{\underline{s}} e^{-u\ell(\underline{s})} < \infty$$

is an interval, we can choose some $t \in (h', h)$ such that

$$\sum_{\underline{s}} e^{-t\ell(\underline{s})} < \infty.$$

Then for $R > 0$,

$$e^{tR} \sum_{\underline{s}} e^{-t\ell(\underline{s})} = \sum_{\underline{s}} e^{t(R-\ell(\underline{s}))} \geq \sum_{\underline{s}: \ell(\underline{s}) \leq R} e^{t(R-\ell(\underline{s}))}.$$

For $\ell(\underline{s}) < R$, we have $e^{t(R-\ell(\underline{s}))} > 1$, hence

$$e^{tR} \sum_{\underline{s}} e^{-t\ell(\underline{s})} \geq \# \{ \underline{s}: \ell(\underline{s}) \leq R \}.$$

Taking the logarithm of both sides and letting R tend to infinity, we obtain $t \geq h$, which gives a contradiction. Hence $h = h'$. \square

We conclude this section by introducing a notational device for certain sequences of saddle connections that will be used in the proof of Lemma 3.2 in the next section.

Definition 2.3. A *singular* connection, e , is a finite sequence of saddle connections, i.e. $e = s_1 \dots s_n$, such that for $1 \leq i < n$, $t(s_i) = i(s_{i+1})$ and the angle formed by starting at s_i and moving clockwise⁴ about $t(s_i)$ to s_{i+1} is equal to π (Figure 4).

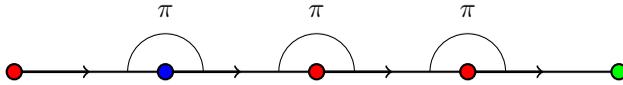


FIGURE 4. Four saddle connections that form a single singular connection.

⁴Translation surfaces have a well defined notion of clockwise orientation at every point.

Let $\underline{s} = s_1 \dots s_n$ be a saddle connection path. Consecutive saddle connections in \underline{s} will join at an angle allowed by the condition that \underline{s} is a geodesic, and so typically it will not be a singular connection. However, exceptionally \underline{s} may have substrings of saddle connections that form singular connections. Assuming that we only consider singular connections e that are maximal with respect to \underline{s} (i.e. there does not exist another substring e' of \underline{s} such that e is a substring of e' and e' is also a singular connection), then \underline{s} has a unique decomposition of the form $\underline{s} = e_1 \dots e_m$, where the e_i are singular connections, and for $1 \leq i < m$, $e_i e_{i+1}$ is not a singular connection (i.e., the clockwise angle between them is greater than π).

Example 2.4. For the L -shaped translation surface we can consider the saddle connection path $s_1 s_2 s_3 s_3 s_4$ illustrated in Figure 5. Since the clockwise angle between s_1 and s_2 is π , and the angle between s_3 and itself is π . However, the angle between s_2 and s_3 is $\frac{5}{4}\pi$ and the angle between s_3 and s_4 is $\frac{3}{2}\pi$. Thus we can denote the singular connections $e_1 = s_1 s_2$, $e_2 = s_3 s_3$ and $e_3 = s_4$ and write $s_1 s_2 s_3 s_3 s_4 = e_1 e_2 e_3$.

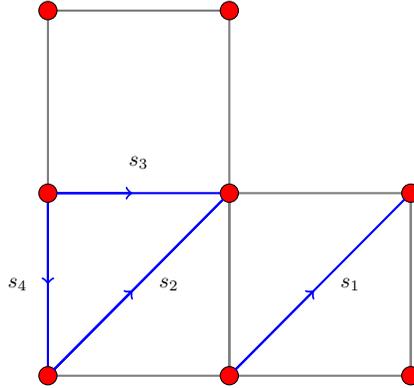


FIGURE 5. The saddle connection path $s_1 s_2 s_3 s_3 s_4 = e_1 e_2 e_3$

3. ENTROPY FORMULA FOR SURFACES IN $\mathcal{H}^1[k, n]$ AND THEIR $SL(2, \mathbb{R})$ -ORBITS

Fix $X_0 \in \mathcal{H}^1[k, n]$, a unit area square tiled surface with n singularities with cone angle $2\pi(k+1)$ satisfying the conditions in (1) in the introduction. Note that X_0 will be tiled by $n(k+1)$ squares.

In order to study surfaces AX_0 , where $A \in SL(2, \mathbb{R})$, we can denote by

$$\Lambda = \left\{ \left(\frac{a}{\sqrt{n(k+1)}}, \frac{b}{\sqrt{n(k+1)}} \right) : (a, b) \in \mathbb{Z}^2 \setminus \{(0, 0)\} \right\}$$

the scaled standard lattice (minus the origin) and $A(\Lambda) = \{(A \begin{pmatrix} p \\ q \end{pmatrix})^T : (p, q) \in \Lambda\}$ its image under the linear action of A . We can then associate functions $f_t : SL(2, \mathbb{R}) \rightarrow$

\mathbb{R}^+ for each $t > 0$ by

$$f_t(A) = \sum_{\underline{v} \in A(\Lambda)} \exp(-t\|\underline{v}\|)$$

where $\|\underline{v}\|$ is the usual Euclidean length of $\underline{v} \in \mathbb{R}^2$.

We will relate the entropy $h(AX_0)$ of AX_0 to $f_t(A)$ by taking advantage of the additional structure of the set of saddle connection paths on AX_0 .

Lemma 3.1. *Let $X_0 \in \mathcal{H}^1[k, n]$ satisfy the hypotheses in (1). Fix $A \in SL(2, \mathbb{R})$. There is a correspondence between the set of oriented saddle connection paths that admit a unique decomposition into m singular connections, and*

$$(x; j_1, (a_1, b_1); j_2, (a_2, b_2); \cdots; j_m, (a_m, b_m)) \\ \in \Sigma \times \underbrace{(\mathbb{Z}/(k+1)\mathbb{Z} \times A(\Lambda)) \times (\mathbb{Z}/k\mathbb{Z} \times A(\Lambda)) \times \cdots \times (\mathbb{Z}/k\mathbb{Z} \times A(\Lambda))}_{m \text{ times}}.$$

Moreover, we can write

$$\ell(\underline{s}) = \sum_{i=1}^m \|(a_i, b_i)\| \text{ where } \|(a_i, b_i)\| = \sqrt{a_i^2 + b_i^2}.$$

Proof. We will proceed by induction on m , the number of singular connections in the saddle connection path.

We begin by looking at the set of saddle connection paths consisting of one singular connection. First note that the square tiled surface AX_0 (which has area 1 and consists of $n(k+1)$ square tiles) covers the torus, $(1/\sqrt{n(k+1)})A(\mathbb{R}^2/\mathbb{Z}^2)$, and around the singularities the covering map can be written in the form $w = z^{k+1}$ using local complex coordinates. Hence, each oriented singular connection based at some $x \in \Sigma$ projects onto an oriented closed geodesic on the aforementioned torus. The set of such oriented closed geodesics (up to homotopy) is in correspondence with $A(\Lambda)$ and their lengths are given by the lengths of the corresponding vectors. Each oriented closed geodesic on the torus lifts to $(k+1)$ oriented singular connections based at x (the angle between any of the lifts based at x will be equal to $2\pi \pmod{1}$). Hence the set of oriented saddle connection paths on X that consist of one singular connection is in correspondence with $\Sigma \times (\mathbb{Z}/(k+1)\mathbb{Z} \times A(\Lambda))$ and the length of the saddle connection path is the length of the corresponding vector in $A(\Lambda)$.

Suppose we have an oriented saddle connection path $\underline{s} = e_1 \dots e_n$ consisting of n singular connections, e_i . Let e be an oriented singular connection. Then $\underline{s}e$ is a saddle connection path if, $t(\underline{s}) = i(e)$, the anticlockwise angle between e_n and e is greater than or equal to π (see the beginning of Section 2), and the clockwise angle between them is greater than π (see the end of Section 2). We have seen that the set of oriented singular connections that begin at a given singularity corresponds to $(\mathbb{Z}/(k+1)\mathbb{Z} \times A(\Lambda))$, where the oriented singular connections can be grouped in $(k+1)$ -tuples (corresponding to vectors in $A(\Lambda)$) such that each pair in the tuple forms an angle of $2\pi \pmod{1}$ about the singularity. The two angle conditions together eliminate exactly one singular connection from every $(k+1)$ -tuple and so the set of oriented singular connections that form a saddle connection path $\underline{s}e$ with \underline{s} is in correspondence with $(\mathbb{Z}/k\mathbb{Z} \times A(\Lambda))$.

By using the induction hypothesis on $m = n$ and the above reasoning, it follows that the correspondence in the statement of the lemma and expression for $\ell(\underline{s})$ hold for $m = n + 1$ and so we are done by induction. \square

The functions $f_t(\cdot)$ have the following useful properties.

Lemma 3.2. *Let $X_0 \in \mathcal{H}^1[k, n]$ satisfy the hypotheses in (1). Fix $A \in SL(2, \mathbb{R})$.*

- (1) *The function $f_t(A)$ is well defined and $\frac{\partial f_t(A)}{\partial t} < 0$ for all $A \in SL(2, \mathbb{R})$ and $t > 0$.*
- (2) *The entropy $h = h(AX_0) > 0$ is the unique solution to $f_h(A) = \frac{1}{k}$*

Proof. Part 1 follows from the definition of $f_t(A)$.

For part 2, we can use Lemma 2.2 and Lemma 3.1 to write

$$\begin{aligned} \sum_{\underline{s}} \exp(-t\ell(\underline{s})) &= n(k+1) \sum_{\underline{v} \in A(\Lambda)} \exp(-t\ell(\underline{v})) \sum_{m=0}^{\infty} \left(k \sum_{\underline{v} \in A(\Lambda)} \exp(-t\ell(\underline{v})) \right)^m \\ &= n(k+1)f_t(A) \sum_{m=0}^{\infty} (kf_t(A))^m \\ &= n \frac{(k+1)f_t(A)}{1 - kf_t(A)} \end{aligned}$$

provided $f_t(A) < \frac{1}{k}$. Moreover, since $f_t(A)$ is strictly decreasing in t , by part 1 of Lemma 3.2, then Lemma 2.2 and the above identity, the proof is completed. \square

Lemma 3.2 gives a particularly useful characterization of the entropy. As a first application we have the following.

Corollary 3.3. *Let $X_0 \in \mathcal{H}^1[k, n]$ satisfy the hypotheses in (1). Then the entropy function $h : SL(2, \mathbb{R})X_0 \rightarrow \mathbb{R}^+$ is real analytic when restricted to the orbit $SL(2, \mathbb{R})X_0$.*

Proof. In the definition of $f_t(A)$ the dependence of the saddle connections on A is real analytic. The function also has an analytic dependence on t . By Lemma 3.2 we see that $h(AX_0)$ satisfies $f_{h(AX_0)}(AX_0) = \frac{1}{k}$ and then applying the Implicit Function Theorem gives the result. \square

4. EXAMPLES

We are interested in square tiled surfaces where the vertex of each square is a singularity with a common cone angle. We note that any square tiled surface consisting of N square tiles can be represented by a pair of permutations $(h, v) \in \text{Sym}(N) \times \text{Sym}(N)$, where h, v represent the gluings of the horizontal and vertical edges of the squares, respectively (see [16]). We recall that for translation surfaces $X \in \mathcal{H}^1(k_1, \dots, k_n)$ the genus g satisfies $2g - 2 = \sum_{i=1}^n k_i$ [18]. We can consider a few simple examples.

Example 4.1 (O_k , $k \geq 2$, $h = (1, 2, \dots, 2k)$, $v = (1, 2)(3, 4) \dots (2k-1, 2k)$, see Figure 6). This is a translation surface of genus k with two singularities each with cone angle $2\pi k$ (see [16], Definition 5.3, p.53).

We can next consider two different types of stair examples.

Example 4.2 ($St_k = E_{2k-1}$, $k \geq 2$, $h = (1, 2) \dots (2k-3, 2k-2)$, $v = (2, 3) \dots (2k-2, 2k-1)$). This is a translation surface of genus k with one singularity with cone angle $2\pi(2k-1)$ (see [16], Definition 5.10, p.61).

The special case $k = 2$ corresponds to Example 1.3.

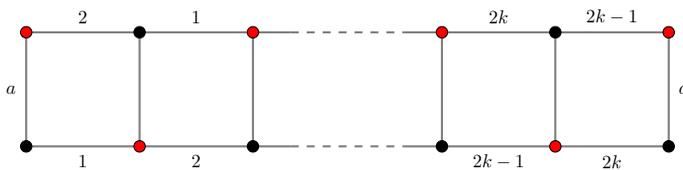


FIGURE 6. A translation surface of genus k and two singularities

Example 4.3 ($G_k = E_{2k}$, $k \geq 2$, $h = (1, 2) \dots (2k - 1, 2k)$, $v = (2, 3) \dots (2k - 2, 2k - 1)$). This is a translation surface of genus k with two singularities each with cone angle $2\pi k$ (see [16], Definition 5.8, p.59).

Finally, we can consider a well known example of Forni [7] and Herrlich-Schmithüsen [8].

Example 4.4 (Eierlegende Wollmilchsau⁵, $h = (1, 2, 3, 4)(5, 6, 7, 8)$, $v = (1, 6)(2, 5)(3, 8)(4, 7)$, see Figure 7). This is a translation surface of genus 3 with four singularities each with cone angle 4π .

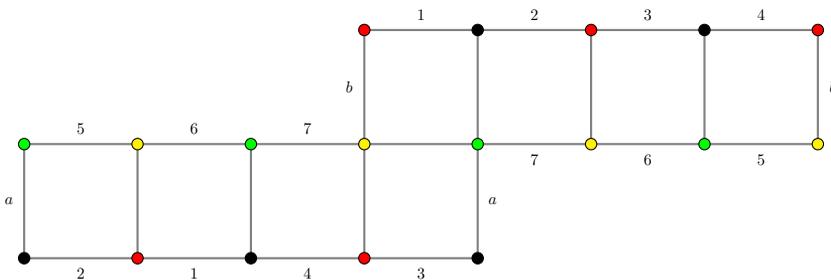


FIGURE 7. A translation surface of genus 3 and 4 singularities

5. MONTGOMERY'S THEOREM

In order to analyze $f_t(\cdot)$, and thus use Lemma 3.2 to study the entropy, it is convenient to first study a related function $F_t(L)$, where L is a unimodular lattice, i.e., of the form $A(\Lambda)$ for some $A \in SL(2, \mathbb{R})$. Throughout we consider the lattices up to rotation. In particular, this will allow us to use a result of Montgomery.

Definition 5.1. We can associate to a unimodular lattice L and $t > 0$ the function

$$F_t(L) = \sum_{p \in L} \exp(-t\|p\|^2)$$

where $\|p\|$ denotes the Euclidean norm.

We see that $F_t(L)$ is finite provided $t > 0$. Moreover, on this domain the function $F_t(L)$ has a smooth dependence on t and L . The next result describes lattices which minimize $F_t(L)$ [15] (see also [1], Appendix A). Let L_Δ denote the equilateral triangular lattice.

⁵This literally translates as “egg-laying wool-milk-pig” and is a reference to the many different useful properties this example has.

Proposition 5.2 (Montgomery's Theorem). *For each $t > 0$ and all (unimodular) lattices L we have that $F_t(L) \geq F_t(L_\Delta)$, with equality iff $L = L_\Delta$.*

Remark 5.3 (Comment on the proof of Proposition 5.2). There is a standard correspondence between unimodular lattices L in \mathbb{R}^2 and the standard Modular domain, i.e., $z = x + iy$ with $-\frac{1}{2} \leq x < \frac{1}{2}$ and $|z| \geq 1$, with suitable identifications on the boundary. Let us denote $L = L_z$. Let $L_\Delta = \mathbb{Z} + z\mathbb{Z}$ be the equilateral triangular lattice with $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. The work of Montgomery established, in particular, the following properties:

- (1) If $0 < x < \frac{1}{2}$ and $y \geq \frac{1}{2}$ then $\frac{\partial F_t(L_z)}{\partial x} < 0$ for all $t > 0$; and
- (2) If $0 \leq x \leq \frac{1}{2}$ and $x^2 + y^2 \geq 1$ then $\frac{\partial F_t(L_z)}{\partial y} \geq 0$ for all $t > 0$ with equality iff $(x, y) = (0, 1)$ or $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, i.e., the ramification points on the modular surface.

By the definitions we have $F_t(L_{x+iy}) = F_t(L_{-x+iy})$ and so we can assume without loss of generality that $0 \leq x \leq \frac{1}{2}$. Thus given a lattice L_z we consider a path consisting of a straight line path from $z = x + iy$ to $\frac{1}{2} + iy$ and then a straight line path from $\frac{1}{2} + iy$ to $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ along which $F_t(L_{x+iy})$ decreases for any $t > 0$ (by (1) and (2), respectively).

6. BERNSTEIN'S THEOREM

To proceed we need to relate $F_t(\cdot)$ and $f_t(\cdot)$. This requires a result of Bernstein on completely monotone functions.

Definition 6.1. We call a smooth function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ *completely monotone* if for all $x > 0$:

$$(-1)^n \frac{d^n \psi}{dx^n}(x) < 0.$$

We will be particularly interested in the following example.

Example 6.2. The function $\psi(x) = \exp(-\sqrt{x})$ is completely monotone (see the corollary on p. 391 of [14]). More generally, given $\psi_1(x)$ and $\psi_2(x)$ with ψ_1 and ψ_2' completely monotone one has that the composition $\psi_1 \circ \psi_2$ is completely monotone (see Theorem 1 in [14]). We can apply this result with $\psi_1(x) = \exp(-x)$ and $\psi_2(x) = x^{\frac{1}{2}}$.

The interest in completely monotone functions is that they are the Laplace transform of positive functions, as is shown in the following classical theorem [3].

Proposition 6.3 (Bernstein's Theorem). *If ψ is completely monotone, then there exists a finite positive Borel measure μ on $[0, +\infty)$ such that*

$$\psi(r) = \int_0^\infty \exp(-ru) d\mu(u).$$

An account appears, for example, in the book of Widder (see Chapter IV, §12 [17]).

7. PROOF OF THEOREM 1.2

We want to use Proposition 6.3 to convert Proposition 5.2 for $F_t(\cdot)$ into the corresponding result for $f_t(\cdot)$, in Proposition 7.1.

Proposition 7.1 (Bétermin). *For each $t > 0$ and all lattices L we have that $f_t(L) \geq f_t(L_\Delta)$ with equality iff $L = L_\Delta$.*

For completeness we recall the elegant short proof of Bétermin.

Proof. First one uses Proposition 6.3 to write

$$\exp(-t\|p\|) = \int_0^\infty \exp(-ut\|p\|^2) \rho(u) du$$

for $t > 0$ and $p \in L$. Summing gives

$$f_t(L) - f_t(L_\Delta) = \int_0^\infty (F_{ut}(L) - F_{ut}(L_\Delta)) \rho(u) du \geq 0$$

with $F_{ut}(L) - F_{ut}(L_\Delta) \geq 0$ and with equality iff $L = L_\Delta$, using Proposition 5.2. \square

We can now complete the proof of Theorem 1.2.

Let $A \in SL(2, \mathbb{R})$ be chosen so that AX_0 has a triangulation by equilateral triangles and let $B \in SL(2, \mathbb{R})$ be an element in the group which does *not* correspond to triangulation by equilateral triangles.

We can use the functions $f_t(A)$ and $f_t(B)$ to compare the entropies $h(A)$ and $h(B)$ of AX_0 and BX_0 , respectively. By Proposition 7.1 we know that $f_t(A) < f_t(B)$ for all $t > 0$. By part 2 of Lemma 3.2 the entropy $h(A)$ for the surface AX_0 is the unique value such that $f_{h(A)}(A) = \frac{1}{k}$. However, by part 1 of Lemma 3.2 the function $t \mapsto f_t(B)$ is monotone decreasing so the solution $f_{h(B)}(B) = \frac{1}{k}$ implies that $h(B) > h(A)$.

8. APPROXIMATING $h(X_0)$

Let X_0 denote a square tiled surface in $\mathcal{H}^1[k, n]$. In this section we present a method for finding arbitrarily good approximations to $h(A)$, for a given $A \in SL(2, \mathbb{R})$, and calculate the error terms of these approximations. We will then use this method to approximate $h(\Delta)$, the entropy of the equilateral surface in Example 1.3.

Let $N \in \mathbb{N}$ and define $\mathbb{Z}_N = \{n \in \mathbb{Z} : |n| \leq N\}$. We define the finite square lattice

$$\Lambda_N = \left\{ \left(\frac{a}{\sqrt{n(k+1)}}, \frac{b}{\sqrt{n(k+1)}} \right) : (a, b) \in \mathbb{Z}_N^2 \setminus \{(0, 0)\} \right\}.$$

Fix $t > 0$. We can then define an N^{th} approximation to $f_t(A)$ by considering a truncation of the infinite series in the definition of $f_t(A)$:

$$f_t^{(N)}(A) = \sum_{\underline{v} \in A(\Lambda_N)} \exp(-t\ell(\underline{v})).$$

Note that the first few derivatives of $f_t^{(N)}(A)$ also give approximations to the respective derivatives of $f_t(A)$. Finally, we note that the region

$$P(A) = \left\{ A(x, y) : \frac{1}{2\sqrt{n(k+1)}} \leq x, y \leq \frac{1}{2\sqrt{n(k+1)}} \right\}$$

has area $\text{Area}(P(A)) = \frac{1}{n(k+1)}$ and its translates by $A \left(\frac{1}{\sqrt{n(k+1)}} \mathbb{Z}^2 \right)$ tile the plane \mathbb{R}^2 .

The following simple lemma allows us to bound the error in the approximations.

Lemma 8.1. *Fix $A \in SL(2, \mathbb{R})$. We define*

$$d(A) := \inf_{\underline{x} \in \mathbb{R}^2: \|\underline{x}\|_2=1} \|A(\underline{x})\|_2 \text{ and } D(A) := \text{diam}(P(A)).$$

Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the function $g(R) = \exp(-tR)$. Then the following inequalities hold:

(1)

$$\sum_{\underline{v} \in A(\Lambda)} g(\ell(\underline{v})) \leq 2\pi n(k+1) \exp(tD(A)) \int_0^\infty Rg(R)dR; \text{ and}$$

(2) *For $N \in \mathbb{N}$,*

$$\sum_{\underline{v} \in A(\Lambda)} g(\ell(\underline{v})) - \sum_{\underline{v} \in A(\Lambda_N)} g(\ell(\underline{v})) \leq 2\pi n(k+1) \exp(tD(A)) \int_{d(A)N/\sqrt{n(k+1)}}^\infty Rg(R)dR.$$

Proof. The proof follows easily by bounding each term

$$g(\ell(\underline{v})) \leq \frac{\exp(tD(A))}{\text{Area}(P(A))} \int_{\underline{v}+P(A)} g(\sqrt{x^2+y^2})dx dy.$$

For part (1) we integrate over \mathbb{R}^2 and for part (2) over $\left\{ (x, y) : x^2 + y^2 \geq \frac{d(A)^2 N^2}{n(k+1)} \right\}$, in both cases using polar coordinates. □

We will now use the $f_t^N(A)$ and Lemma 8.1 to approximate $h(\Delta)$, where $h(\Delta)$ denotes the entropy of the equilateral surface in Example 1.3 (the unique equilateral surface in $\mathcal{H}^1[2, 1]$).

It follows from Lemma 3.2 that $h(\Delta)$ is the unique $t > 0$ such that $f_t(\Delta) = 1/2$.

By applying inequality (2) from Lemma 8.1 to $f_t(\Delta) - f_t^{(N)}(\Delta)$, we obtain an upper bound for $f_t(\Delta) - f_t^{(N)}(\Delta)$ which we denote by $E_t^{(N)}(\Delta)$. Next observe the following inequalities:

$$f_t^{(N)}(\Delta) \leq f_t(\Delta) \leq f_t^{(N)}(\Delta) + E_t^{(N)}(\Delta),$$

where each of the terms are decreasing in t .

Let $h_L^{(N)}(\Delta)$ denote the unique $t > 0$ such that $f_t^{(N)}(\Delta) = 1/2$ and let $h_U^{(N)}(\Delta)$ denote the unique $t > 0$ such that $f_t^{(N)}(\Delta) + E_t^{(N)}(\Delta) = 1/2$.

It follows from the previous inequality that for all $N \in \mathbb{N}$, $h_L^{(N)}(\Delta) \leq h(\Delta) \leq h_U^{(N)}(\Delta)$. Because $f_t^N(\Delta)$ converges to $f_t(\Delta)$ and $E_t^{(N)}(\Delta)$ converges to 0 as $N \rightarrow \infty$, we obtain arbitrarily close bounds to $h(\Delta)$ by taking N sufficiently large.

We will first calculate $E_t^{(N)}(\Delta)$ using Lemma 8.1 and then compute the bounds for N sufficiently large. Note that $d(\Delta)$ is the smallest singular values of Δ , i.e. the square roots of the smallest eigenvalue of $\Delta^* \Delta$, where Δ^* denotes the adjoint of Δ . By a standard calculation, one can show that $d(\Delta) = 0.759836\dots$. In the present setting the diameter estimate can be taken to be $D(\Delta) = 1.07457\dots$. Then we can

apply Lemma 8.1 to get

$$\begin{aligned} E_t^{(N)}(\Delta) &= 2\pi \exp(tD(\Delta)) \int_{d(\Delta)N/\sqrt{n(k+1)}}^{\infty} R \exp(-tR) dR \\ &= 2\pi \frac{\exp(tD(\Delta))}{t^2} \exp\left(-\frac{td(\Delta)N}{\sqrt{n(k+1)}}\right) \left(\frac{td(\Delta)N}{\sqrt{n(k+1)}} + 1\right). \end{aligned}$$

Using Mathematica's NSolve with working precision equal to 30, we solve $f_t^{(N)}(\Delta) = 1/2$ for t , with $N = 100$ to obtain

$$h_L(\Delta) = 4.34934504614150290303138902137\dots$$

Again, using NSolve, we numerically solve $f_t^{(N)}(\Delta) + E_t^{(N)}(\Delta) = 1/2$ for t , using the expression for $E_t^{(N)}(\Delta)$ with $N = 100$ to also get

$$h_U(\Delta) = 4.34934504614150290303138902137\dots$$

Hence we see that $h(\Delta) = 4.34934504614150290303138902137\dots$ (up to 29 decimal places).

Remark 8.2. We can use the same method as in the proof of Lemma 8.1 to deduce other properties of f_t . For instance, by setting $N = 100$ and approximating the partial derivatives of f_t we can show that the Hessian of $f_t(A)$ at the equilateral surface with $t = 4.349\dots$ is non-degenerate by showing that the determinant of the Hessian is approximately equal to $0.0825337\dots > 0$.

9. FINAL COMMENTS AND QUESTIONS

- (1) We can also consider the entropy function on general strata, without the additional restriction that the singularities share the same cone angle. In this broader context it is not clear what the correct candidates for the global minima are, let alone how to prove they minimize the entropy. We suspect that they won't be equilateral surfaces by analogy with the minimization problem for finite metric graphs where metrics that minimize entropy have edge lengths proportional to the logarithm of the product of the valencies of the edge's vertices (see [12]).
- (2) It is natural to ask if the entropy function is smooth, by the analogy with Riemannian manifolds with negative sectional curvature [9].

We are grateful to the referee for suggesting the following questions.

- (3) Let X_0 be a square-tiled surface satisfying the hypotheses in (1). Is the number of equilateral surfaces in $SL(2, \mathbb{R})X_0$ related to the Veech group of X_0 in $SL(2, \mathbb{Z})$? The Veech groups for our examples in §3 are computed in [16]. This question could potentially be studied by looking at the $SL(2, \mathbb{Z})$ -orbit of X_0 using SageMath (see <http://www.sagemath.org>).
- (4) Since the entropy on the $SL(2, \mathbb{R})$ -orbit does not depend on rotating the square tiled surface, it descends to a function on \mathbb{H}/Γ , where Γ is the Veech group of the surface. A natural question is how this function behaves on this quotient, for example, how does it behave in the cusp?

ACKNOWLEDGMENTS

The authors would like to thank L. Bétermin for suggesting the connection with his work [2] and Samuel Lelièvre for helpful comments. We are also grateful to the referee for their helpful comments and suggestions.

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