BEST POSSIBILITY OF THE FATOU-SHISHIKURA INEQUALITY FOR TRANSCENDENTAL ENTIRE FUNCTIONS IN THE SPEISER CLASS

MASASHI KISAKA AND HIROTO NABA

Abstract. The Speiser class $S$ is the set of all entire functions with finitely many singular values. Let $S_q \subset S$ be the set of all transcendental entire functions with exactly $q$ distinct singular values. The Fatou-Shishikura inequality for $f \in S_q$ gives an upper bound $q$ of the sum of the numbers of its Cremer cycles and its cycles of immediate attractive basins, parabolic basins, and Siegel disks. In this paper, we show that the inequality for $f \in S_q$ is best possible in the following sense: For any combination of the numbers of these cycles which satisfies the inequality, some $T \in S_q$ realizes it. In our construction, $T$ is a structurally finite transcendental entire function.

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-linear entire function. The Fatou set $F(f) \subset \mathbb{C}$ is the maximal open set where the iterates $f^n$ of $f$ form a normal family. The complement of $F(f)$ is called the Julia set $J(f)$. Both $F(f)$ and $J(f)$ are completely invariant sets in the following sense: $f(F(f)) \subset F(f), f^{-1}(F(f)) \subset F(f), f(J(f)) \subset J(f)$, and $f^{-1}(J(f)) \subset J(f)$.

A point $z \in \mathbb{C}$ is called periodic if there exists the minimum number $p \in \mathbb{N}$ such that $f^p(z) = z$. In particular, $z$ is called fixed if $p = 1$. The set $\{z, f(z), \cdots, f^{p-1}(z)\}$ is called the cycle containing $z$. This $p$ is called the period of $z$ (or the cycle containing $z$). The multiplier of $z$ (or the cycle containing $z$) is $\lambda := (f^p)'(z)$. We say that $z$ (or the cycle containing $z$) is repelling, attracting, rationally indifferent, or irrationally indifferent if $|\lambda| > 1$, $|\lambda| < 1$, $\lambda = e^{2\pi i\theta} (\theta \in \mathbb{Q})$, or $\lambda = e^{2\pi i\theta} (\theta \in \mathbb{R} \setminus \mathbb{Q})$, respectively. We call the last three cases non-repelling. The following facts are well known:

(1) Attracting periodic points are in $F(f)$;
(2) Rationally indifferent periodic points are in $J(f)$;
(3) $J(f)$ is the closure of the set of all repelling periodic points.

(See [Ber] p. 157, p. 160, Theorem 4.) Furthermore, we call an irrationally indifferent periodic point $z$ a Siegel point if $z \in F(f)$, or a Cremer point if $z \in J(f)$. The cycle containing a Siegel point is called a Siegel cycle. Similarly, we define Cremer cycles.

Let $U$ be a connected component of $F(f)$. Then $f^n(U)$ is contained in some component $U_n$ of $F(f)$ for $n = 1, 2, \cdots$. The domain $U$ is called a wandering domain.
Remark 1. Note that every entire function has no Herman rings which rational functions can have. (See [Ber] p. 164.)

Points $c \in \mathbb{C}$ and $f(c)$ are called a critical point and a critical value respectively, if $f'(c) = 0$. A point $\alpha \in \mathbb{C}$ is called an asymptotic value if there exists a continuous curve $\gamma(t)(0 \leq t < 1)$ with $\lim_{t \to 1} \gamma(t) = \infty$ and $\lim_{t \to 1} f(\gamma(t)) = \alpha$. Critical values, asymptotic values, or their accumulation points are called singular values. (Note that singular values of polynomials are critical values.) Let $\text{sing}(f^{-1})$ be the set of all singular values of $f$. Singular values have important relations with periodic components and Cremer cycles (for example, see [BrF] p. 116, Theorem 3.39). Let $a \in \text{sing}(f^{-1})$. We call $a$ a eventually repelling if $f^n(a)$ is a repelling periodic point for some $n \geq 0$.

The Speiser class $S$ is the set of all entire functions with finitely many singular values. Let $S_q \subset S$ be the set of all transcendental entire functions which have exactly $q$ distinct singular values. Let $\text{Pol}_d$ be the set of all polynomials of degree $d \geq 2$. Any $f \in \text{Pol}_d$ has at most $d - 1$ critical values in $\mathbb{C}$.

Here, we define structurally finite transcendental entire functions. Set

$$SF_{k,l} := \left\{ f(z) = \int_0^z (c_k t^k + \cdots + c_0) e^{a_1 t^l + \cdots + a_l t} dt + b \mid b, c_i, a_j \in \mathbb{C} (i = 0, \ldots, k, j = 1, \ldots, l), c_k a_l \neq 0 \right\}$$

for $k \geq 0$ and $l \geq 1$. In addition, put

$$SF := \bigcup_{k \geq 0, l \geq 1} SF_{k,l}.$$

A transcendental entire function $f$ is called structurally finite if $f \in SF$. Every $f \in SF_{k,l}$ has at most $k$ critical values and at most $l$ asymptotic values. (To be more precise, $f$ has exactly $k$ critical points counted with multiplicity and $l$
transcendental singularities. See [T2 and OR p. 347, Theorem 1.1].) Therefore, $SF$ is a subset of the Speiser class $S$. In general, we will restrict ourselves to the functions in $SF_{k,l}$ with exactly $q$ singular values, and hence we will often write $S_q \cap SF_{k,l}$.

Now we introduce the Fatou-Shishikura inequality for $f \in Pol_d$ and that for $f \in S_q$. When $f \in Pol_d \cup S_q$, we define $n_{\text{rat}}(f)$ as the number of attracting cycles of $f$ in $C$. Similarly, we define

$$n_{\text{rat}}(f), n_{\text{SI}}(f), n_{\text{CR}}(f), n_{\text{AB}}(f), n_{\text{PB}}(f), n_{\text{SD}}(f), n_{\text{ER}}(f)$$

as the number of rationally indifferent cycles, Siegel cycles, Cremer cycles, AB-cycles, PB-cycles, SD-cycles, and eventually repelling singular values, respectively. In addition,

$$n_{\text{rat}}(f) = n_{\text{AB}}(f), \quad n_{\text{SI}}(f) = n_{\text{SD}}(f), \quad n_{\text{rat}}(f) \leq n_{\text{PB}}(f)$$

hold among these notations. In fact, $n_{\text{PB}}(f)$ is a multiple of $n_{\text{rat}}(f)$. (The former two equalities are obvious. See [Bea] p. 116, Theorem 6.5.4, p. 122, Theorem 6.5.7 for the last inequality.) The following is the Fatou-Shishikura inequality for $f \in Pol_d$, which is some modification of [Shi p. 5, Corollary 2, p. 6, Theorem 4]):

**Theorem A** ([Shi p. 5, Corollary 2, p. 6, Theorem 4]). Let $f \in Pol_d$. Then

$$n_{\text{AB}}(f) + n_{\text{PB}}(f) + n_{\text{SD}}(f) + n_{\text{CR}}(f) \leq d - 1.$$  

Moreover, the inequality is best possible in the following sense: If non-negative integers $m_{\text{AB}}, m_{\text{PB}}, m_{\text{SD}},$ and $m_{\text{CR}}$ satisfy

$$m_{\text{AB}} + m_{\text{PB}} + m_{\text{SD}} + m_{\text{CR}} \leq d - 1,$$

then there exists a polynomial $P \in Pol_d$ with

$$(n_{\text{AB}}(f), n_{\text{PB}}(P), n_{\text{SD}}(P), n_{\text{CR}}(P)) = (m_{\text{AB}}, m_{\text{PB}}, m_{\text{SD}}, m_{\text{CR}}).$$

The Fatou-Shishikura inequality for $f \in S_q$ is as follows:

**Theorem B** ([EL p. 1005, Theorem 5]). Let $f \in S_q$. Then

$$n_{\text{AB}}(f) + n_{\text{PB}}(f) + n_{\text{SD}}(f) + n_{\text{CR}}(f) \leq q.$$  

It is known that every $f \in Pol_d \cup S_q$ has the following important dynamical properties:

1. It has finitely many singular values;
2. It has no Herman rings, no Baker domains, and no wandering domains (see [Ber] p. 164, [EL] p. 994, Theorem 1, p. 1004, Theorem 3, and [Shi] p. 404, Theorem 1);
3. It satisfies the Fatou-Shishikura inequalities (Theorem A and Theorem B).

According to Theorem A, the Fatou-Shishikura inequality for $f \in Pol_d$ is best possible. From these dynamical properties of $f \in S_q$ similar to those of $f \in Pol_d$, we can expect best possibility of the Fatou-Shishikura inequality for $f \in S_q$ analogous to that for $f \in Pol_d$. Our main purpose is to show that this is actually true.

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1When we regard $f \in Pol_d$ as a rational function defined on the Riemann sphere $\hat{\mathbb{C}}$, $\infty$ is a super-attracting fixed point, which is a critical point with multiplicity $d - 1$. By the Fatou-Shishikura inequality for rational functions of degree $d$ (see [Shi] p. 5, Corollary 2]), the sum of the numbers of AB-cycles, PB-cycles, SD-cycles, and Cremer cycles is less than or equal to the number $2d - 2$ of critical points in $\hat{\mathbb{C}}$ counted with multiplicity. We adopt the definition of $n_{\text{rat}}(f)(= n_{\text{AB}}(f))$ so that the right-hand side of the modified inequality in Theorem A becomes the number $d - 1$ of critical points in $\hat{\mathbb{C}}$. 


Main Theorem. The Fatou-Shishikura inequality for \( f \in S_q \) is best possible in the following sense: If non-negative integers \( m_{AB}, m_{PB}, m_{SD}, \) and \( m_{Cr} \) satisfy
\[
m_{AB} + m_{PB} + m_{SD} + m_{Cr} \leq q,
\]
then there exists a \( T \in S_q \) with
\[
(n_{AB}(T), n_{PB}(T), n_{SD}(T), n_{Cr}(T)) = (m_{AB}, m_{PB}, m_{SD}, m_{Cr}).
\]
More precisely, \( T \) satisfies \( n_{PB}(T) = n_{rat}(T) \) and \( T \in SF \). In addition, every non-repelling periodic point of \( T \) has the same period relatively prime with \( q \).

The proof of the Main Theorem is based on an analogy of \([Shi]\). For rational functions, we can tell that irrationally indifferent cycles are Cremer cycles if their multipliers satisfy some condition (see \([Cr1]\)). Shishikura used the result to prove best possibility of the Fatou-Shishikura inequality for rational functions (see \([Shi]\)). On the other hand, we cannot use the result for our transcendental case. Hence the main difference between our proof and \([Shi]\) is the way how to construct \( T \) with Cremer cycles (see Remark 2). Moreover, our construction can be also used for the rational case, which leads to a slightly different proof of \([Shi\, p.\, 6, \text{Theorem}\, 4]\).

This paper is organized as follows: We devote Section 2 to preliminaries for proving the Main Theorem. In Section 3 we prove the Main Theorem. Finally, we make some concluding remarks in Section 4.

Remark 2. Thankfully, Walter Bergweiler gave us the information about \([Cr2]\). According to \([Cr2]\), an irrationally indifferent cycle of a transcendental entire function is a Cremer cycle if its multiplier satisfies some condition. If we use this fact, we can construct Cremer cycles of \( T \) by the method similar to that in \([Shi]\). Actually, we knew it after we finished writing the first version of this paper. Thus our proof does not rely on the result. We will mention the proof of the Main Theorem which uses the result after our original proof.

2. Preliminaries

For entire functions, there are the following criteria which tell whether an irrationally indifferent periodic point is a Siegel point or a Cremer point.

**Proposition C (\([Si]\)).** Let \( z \) be an irrationally indifferent periodic point with multiplier \( \lambda \). If there exist positive constants \( M \) and \( k \) such that \( \lambda \) satisfies the following condition:
\[
[Siegel] \quad \frac{1}{|\lambda^n - 1|} \leq Mn^k \quad (n = 1, 2, \cdots),
\]
then \( z \) is a Siegel point. In addition, \([Siegel]\) is satisfied by almost every \( \lambda \) in the unit circle \( \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} \).

**Proposition D.** Let \( z \) be an irrationally indifferent periodic point. If there exist periodic points in any punctured neighborhood of \( z \), then \( z \) is a Cremer point.

Entire functions \( f \) and \( g \) are called *topologically equivalent* if there exist homeomorphisms \( \varphi, \Psi : \mathbb{C} \to \mathbb{C} \) such that
\[
\Psi \circ f = g \circ \varphi.
\]
Denote by \( M_f \subset S_q \) the set of all entire functions topologically equivalent to \( f \in S_q \). We can take \( M_f \) as a \((q+2)\)-dimensional complex analytic manifold whose topology
is locally equivalent to the topology of uniform convergence on compact subsets of \( \mathbb{C} \). Also, we can take a local coordinate on any small enough open set \( U \subset M_f \)

\[
\Phi : U \to \mathbb{C}^{q+2}, \quad \Phi(g) = (\Phi_1(g), \cdots, \Phi_{q+2}(g))
\]

such that

\[
\{\Phi_1(g), \cdots, \Phi_q(g)\} = \text{sing}(g^{-1});
\]

the mapping

\[
\Phi(U) \times \mathbb{C} \to \mathbb{C}, \quad (\Phi(g), z) \mapsto g(z)
\]

is analytic. (See [EL, Section 3].) In addition, the following result is known.

**Proposition E** ([T1, p. 69, Proposition 2]). Let \( f \in SF_{k,l} \). Then every \( g \in M_f \) satisfies

\[
g \in SF_{k,l}.
\]

We introduce the fundamental lemma for quasiconformal surgery which we use to prove the Main Theorem. The definition of quasiconformal mappings is as follows:

**Definition.** Let \( D \) and \( D' \) be domains of \( \mathbb{C} \). A homeomorphism \( \phi : D \to D' \) is a quasiconformal mapping if \( \phi \) satisfies the following conditions:

1. \( \phi \) is absolutely continuous on almost all lines parallel to real-axis and almost all lines parallel to imaginary-axis;
2. \(|\phi_\overline{z}| \leq k|\phi_z|\) holds almost everywhere for some \( 0 \leq k < 1 \).

See [A] for more details about quasiconformal mappings. Let \( U \subset \mathbb{C} \) be a domain of \( \mathbb{C} \). A mapping \( g : U \to \mathbb{C} \) is called quasiregular if \( g \) can be expressed as

\[
g = f \circ \phi,
\]

where \( \phi : U \to \phi(U) \) is a quasiconformal mapping and \( f : \phi(U) \to g(U) \) is a holomorphic function. The following is the fundamental lemma for our quasiconformal surgery, which is some modification of [Shi, p. 7, Lemma 1, p. 9, Lemma 3]):

**Lemma F** ([Shi, p. 7, Lemma 1, p. 9, Lemma 3]). For \( \varepsilon \in \mathbb{C} \) in a neighborhood of 0, set a quasiregular mapping

\[
g_\varepsilon = f \circ \Psi_\varepsilon,
\]

where \( f \) is an entire function and \( \Psi_\varepsilon : \mathbb{C} \to \mathbb{C} \) is a quasiconformal mapping. Suppose that \( g_\varepsilon \) satisfies the following conditions:

1. \( \|(\Psi_\varepsilon)_\overline{z}/(\Psi_\varepsilon)_z\|_{\infty} \to 0(\varepsilon \to 0) \) and \( \Psi_\varepsilon \to \text{Id}_\mathbb{C}(\varepsilon \to 0) \) locally uniformly on \( \mathbb{C} \);
2. There exists an open set \( E_\varepsilon \) such that \( g_\varepsilon(E_\varepsilon) \subset E_\varepsilon \) and \( (g_\varepsilon)_\overline{z} = 0 \) almost everywhere on \( E_\varepsilon \cup (\mathbb{C} \setminus (g_\varepsilon)^{-1}(E_\varepsilon)) \).

Then there exists a quasiconformal mapping \( \varphi_\varepsilon \) with the following properties:

(a) \( \tilde{g}_\varepsilon = \varphi_\varepsilon \circ g_\varepsilon \circ \varphi_\varepsilon^{-1} \) is an entire function;
(b) \( \varphi_\varepsilon \to \text{Id}_\mathbb{C} \) and \( \tilde{g}_\varepsilon \to f \) locally uniformly on \( \mathbb{C} \) as \( \varepsilon \to 0 \);
(c) \( \varphi_\varepsilon \) is conformal on the interior of \( E_\varepsilon \cup (\mathbb{C} \setminus \bigcup_{n=1}^{\infty}(g_\varepsilon)^{-n}(E_\varepsilon)) \).

### 3. Proof of the main theorem

Basically, we follow Shishikura’s method for the rational case in [Shi]. However, there are differences between rational functions and transcendental entire functions. For example, in the rational case, the value at \( \infty \) is defined naturally. On the other hand, in our transcendental case, the value at \( \infty \) cannot be defined naturally, since \( \infty \) is an essential singularity. Thus we have to modify his proof at each step. The
critical difference is in our construction of Cremer cycles. Shishikura constructed Cremer cycles one by one. He realized this by his quasiconformal surgery which converts one Siegel cycle into one Cremer cycle. This is based on the result specific to rational functions by Cremer (see [Cr1]). We cannot use this for our case. Thus we have to make Cremer cycles of $T$ in a different way. We do not construct Cremer cycles one by one because our construction does not guarantee that one Cremer cycle constructed is kept unchanged while we construct another Cremer cycle. Instead, we construct all Cremer cycles of $T$ in the final step.

Here we give the sketch of the proof. If $(m_{AB}, m_{PB}, m_{SD}, m_{Cr}) \neq (0, 0, 0, 0)$, we construct $T$ by the following procedure: First of all, we construct a $T_0 \in S_q \cap SF_{0,q}$ which has $q$ Siegel cycles (Lemma 1). Next, we take $T_0$ as $T$ if $(m_{AB}, m_{PB}, m_{SD}, m_{Cr}) = (0, 0, q, 0)$. Let $(m_{AB}, m_{PB}, m_{SD}, m_{Cr}) \neq (0, 0, q, 0)$. If $m_{Cr} = 0$, we convert $T_0$ into $T$ by making one Siegel cycle repelling, attracting, or rationally indifferent repeatedly. This step by step procedure is done by quasiconformal surgery (Lemma 3) or some argument on analytic sets (Lemma 4). If $m_{Cr} \neq 0$, we convert $T_0$ into a $\tilde{T}$ with

$$(n_{AB}(\tilde{T}), n_{PB}(\tilde{T}), n_{SD}(\tilde{T}), n_{Cr}(\tilde{T})) = (m_{AB}, m_{PB}, m_{SD} + m_{Cr}, 0)$$

in the manner above. Then we convert $\tilde{T}$ into $T$ by making $m_{Cr}$ Siegel cycles of $\tilde{T}$ into $m_{Cr}$ Cremer cycles of $T$ at a time (Lemma 5).

Let $p$ be any positive integer relatively prime with $q$. Put $\lambda = \exp(2\pi i/p)$. Set

$$f_\alpha(z) := (1 + \alpha)\lambda \int_0^z e^{t^p} dt \in SF_{0,q} \quad \text{for } \alpha \in \mathbb{C} \setminus \{-1\}.$$ 

Since $f_\alpha$ has $q$ distinct asymptotic values and no critical values (see [Ne, p. 168, 2.3]), we have $f_\alpha \in S_q$.

Lemma 1. There is an uncountable set $A \subset \mathbb{C} \setminus \{-1\}$ such that $f_\alpha(\alpha \in A)$ has $q$ Siegel cycles of period $p$ whose multipliers satisfy the condition [Siegel] of Proposition 3.

Proof. An easy calculation shows that

$$f_\alpha(z) = (1 + \alpha)\lambda z \left(1 + \frac{z^q}{q + 1} + \frac{z^{2q}}{2(2q + 1)} + \cdots \right).$$

From this and the argument in [Beal, p. 130], we get

$$f_0^p(z) = z\{1 + c_0 z^{pkq} + O(|z|^{(pk+1)q})\} \quad \text{as } z \to 0,$$

where $c_0 \neq 0$, $k \geq 1$. In addition, there are $kq$ PB-cycles of period $p$. By Theorem 1 we have $kq \leq q$. Thus we obtain $k = 1$. It follows that

$$f_\alpha^p(z) = z\{(1 + \alpha)^p + c(\alpha) z^{pq} + O(|z|^{(p+1)q})\} \quad \text{as } \alpha, z \to 0,$$

where $c(\alpha)$ is a holomorphic function of $\alpha$, with $c(0) = c_0$. Set $X = z^q$. Let $f_\alpha^p(z) = zF(X, \alpha)$. Thus we have

$$F(X, \alpha) = (1 + \alpha)^p \left(1 + \frac{c(\alpha)}{(1 + \alpha)^p} X^p + O(|X|^{p+1})\right) \quad \text{as } \alpha, X \to 0.$$ 

By the construction and Rouché’s theorem, if $\alpha \neq 0$ is small enough, $F(X, \alpha) = 1$ has $p$ different solutions $X = \zeta_1(\alpha), \cdots, \zeta_p(\alpha)$ with $\zeta_j(\alpha) \neq 0$ and $\zeta_j(\alpha) \to 0(\alpha \to 0)$ for $j = 1, \cdots, p$. Therefore, $f_\alpha$ has $q$ cycles $C_1(\alpha), \cdots, C_q(\alpha)$ of period $p$ for small
enough $\alpha \neq 0$. They consist of $pq$ $q$-th roots of $\zeta_j(\alpha)(j = 1, \cdots, p)$. Moreover, it follows that
\[
\frac{\partial F}{\partial X}(\zeta_j(\alpha_0), \alpha_0) \neq 0 \quad (j = 1, \cdots, p)
\]
forsmallenough$\alpha_0 \neq 0$. By the implicit function theorem, $\zeta_j(\alpha)(j = 1, \cdots, p)$ are holomorphic functions of $\alpha$ on some neighborhood of $\alpha_0$. It follows that
\[
\Sigma(\alpha) := \Sigma_{j=1}^p \zeta_j(\alpha)
\]
is a holomorphic function of $\alpha$ on some punctured neighborhood of 0. By the construction, $C_j(\alpha)(j = 1, \cdots, p)$ have the same multiplier $\sigma(\alpha)$. An easy calculation shows that
\[
\sigma(\alpha) = (1 + \alpha)^p e^{\Sigma(\alpha)}.
\]
Thus we have $\sigma(\alpha) \to 1$ as $\alpha \to 0$ and $\sigma(\alpha)$ is holomorphic on some punctured neighborhood of 0. Set $\sigma(0) = 1$. By the Riemann removable singularity theorem, $\sigma(\alpha)$ is holomorphic on some neighborhood $U$ of 0. It follows that $\sigma(U)$ is a neighborhood of 1. Hence there is an uncountable set $A \subset U$ such that $\sigma(\alpha)(\alpha \in A)$ satisfies the condition [Siegel] of Proposition C. Thus $f_\alpha(\alpha \in A)$ has $q$ Siegel cycles of period $p$ with multiplier $\sigma(\alpha)$. \hfill \Box

Lemma 1 shows the existence of $T_0 \in S_q \cap SF_{0,q}$ with $q$ Siegel cycles of period $p$. We convert $T_0$ into $T$ with non-repelling cycles of period $p$. Henceforth, we construct $T$ with $p = 1$ for simplicity. The case $p \geq 2$ is shown exactly in the same way.

We define $C(f)$ for $f \in S_q$ by
\[
C(f) := (n_{AB}(f), n_{PB}(f), n_{SD}(f), n_{Cr}(f), n_{ER}(f)).
\]
The combination $C(f)$ always satisfies
\[
n_{\text{rat}}(f) \leq n_{\text{PB}}(f), \quad n_{AB}(f) + n_{PB}(f) + n_{SD}(f) + n_{Cr}(f) + n_{ER}(f) \leq q.
\]
(See Section 1 for the former. The latter follows from Shishikura’s idea used in the proof of Theorem B. See [Shi] p. 25 for details.) They yield Lemma 2.

**Lemma 2.** Suppose that non-negative integers $n_{AB}, n_{\text{rat}}, n_{SD}, n_{Cr},$ and $n_{ER}$, and $f \in S_q$ satisfy
\[
n_{AB} + n_{\text{rat}} + n_{SD} + n_{Cr} + n_{ER} = q,
n_{AB}(f) \geq n_{AB}, \cdots, n_{ER}(f) \geq n_{ER}.
\]
Then $n_{PB}(f) = n_{\text{rat}}(f)$ and
\[
C(f) = (n_{AB}, n_{\text{rat}}, n_{SD}, n_{Cr}, n_{ER}).
\]

By quasiconformal surgery, we will reduce the number of SD-cycles by one and increase that of AB-cycles (or PB-cycles, eventually repelling singular values) by one. We use Lemma 3 which is applicable to general structurally finite transcendental entire functions:

**Lemma 3.** Let $f \in S_q \cap SF_{k,A}$. We assume the following conditions:

1. Every non-repelling periodic point of $f$ is a fixed point;
2. Every Siegel point of $f$ has the multiplier satisfying the condition [Siegel] of Proposition C.
Then for every neighborhood $\varepsilon_1$ one can construct a polynomial $z$ and has a preimage other than itself;
(d) $g_j \in SF_{k,l},$ $n_{PB}(g_j) = n_{rat}(g_j),$
and
\[
C(g_1) = (n_{AB}(f) + 1, n_{PB}(f), n_{SD}(f) - 1, 0, n_{ER}(f)),
\]
\[
C(g_2) = (n_{AB}(f), n_{PB}(f) + 1, n_{SD}(f) - 1, 0, n_{ER}(f)),
\]
\[
C(g_3) = (n_{AB}(f), n_{PB}(f), n_{SD}(f) - 1, 0, n_{ER}(f) + 1).
\]

Proof. First of all, we show the existence of $g_1$ and $g_2$. There are at least two Siegel points of $f,$ say $z_0$ and $z_1.$ Hence we can assume that $z_0$ is not a Picard exceptional value and has a preimage $z^* \neq z_0.$ By using the Lagrange interpolating polynomial, one can construct a polynomial $P$ such that
\[
P \left( \frac{1}{z_1 - z^*} \right) = 0,
\]
\[
P' \left( \frac{1}{z_1 - z^*} \right) = -(z_1 - z^*)^2;
\]

if $a \neq z_1$ is a non-repelling fixed point or $a = f^n(b),$ where $n \geq 0$ and $b$ is an eventually repelling singular value. Let $\rho$ be an increasing $C^\infty$ function on $[0, \infty)$ satisfying $\rho = 0$ on $[0, 1]$ and $\rho = 1$ on $[2, \infty).$ Let $d$ be the degree of $P.$ We define $H_\varepsilon : \mathbb{C} \rightarrow \mathbb{C}$ for $\varepsilon \in \mathbb{C} \setminus \{0\}$ by
\[
H_\varepsilon(z) := \left\{ \begin{array}{ll}
z + \varepsilon \rho \left( |\varepsilon|^{-1/(3d)} |z - z^*| \right) & \text{if } z \neq z^* \\
z^* & \text{if } z = z^*.
\end{array} \right.
\]

Let $H_0 : \mathbb{C} \rightarrow \mathbb{C}$ be the identity. An easy calculation shows that $H_\varepsilon : \mathbb{C} \rightarrow \mathbb{C}$ is a quasiconformal mapping for $\varepsilon$ small enough. Set a quasiregular mapping $F_\varepsilon := f \circ H_\varepsilon$
for $\varepsilon$ small enough. The mappings $H_\varepsilon$ and $F_\varepsilon$ have the following properties:
(i) $\| (H_\varepsilon)z/(H_\varepsilon)_z \|_\infty \rightarrow 0 (\varepsilon \rightarrow 0)$ and $H_\varepsilon \rightarrow H_0 (\varepsilon \rightarrow 0)$ locally uniformly on $\mathbb{C};$
(ii) $H_\varepsilon$ is conformal on $V_\varepsilon := \{z \in \mathbb{C} \mid |z - z^*| > 2|\varepsilon|^{1/(3d)} \},$
and hence $F_\varepsilon$ is holomorphic there and we can define the multipliers of periodic points of $F_\varepsilon$ there as in the case of entire functions;
(iii) There is a neighborhood $U_\varepsilon$ of $z_0$ such that $F_\varepsilon(U_\varepsilon) = U_\varepsilon$ and $\mathbb{C} \setminus V_\varepsilon \subset F_{\varepsilon}^{-1}(U_\varepsilon);$ 
(iv) $z_1$ is a fixed point of $F_\varepsilon$ with multiplier $(1 + \varepsilon)f'(z_1);$
(v) If \( z \neq z_1 \) is a non-repelling fixed point of \( f \), then \( z \) is a fixed point of \( F_\varepsilon \) with multiplier \( f'(z) \);
(vi) If \( z \) is an eventually repelling singular value of \( f \), then
\[
F_\varepsilon^n(z) = f^n(z) \quad (n = 1, 2, \ldots)
\]
and
\[
F'_\varepsilon(\tilde{z}) = f'(\tilde{z}) \quad \text{for any } \tilde{z} \in \{f^n(z) \mid n = 0, 1, \ldots\}.
\]
By the construction, \( F'_\varepsilon(z_0) = f'(z_0) \) satisfies the condition [Siegel] of Proposition C. This yields the property (iii) (see [Si] and [Shi, p. 26, STEP 2]). The other properties follow directly from the construction. From (i), (ii), and (iii), we can apply Lemma F to \( \varepsilon \) that some \( F \) with multiplier \( f'(z) \);
(v) If \( z \) is an eventually repelling singular value of \( f \), then \( \phi_\varepsilon(z) \) is an eventually repelling singular value of \( G_\varepsilon \).

From (i)', we have \( G_\varepsilon = \phi_\varepsilon \circ f \circ \phi_\varepsilon^{-1} \), where \( \phi_\varepsilon \) and \( H_\varepsilon \circ \phi_\varepsilon^{-1} \) are quasiconformal mappings. Therefore, \( G_\varepsilon \) and \( f \) are topologically equivalent. By Proposition E, we have \( G_\varepsilon \in SF_{k,i} \). In addition, we obtain \( G_\varepsilon \in N \) from (ii)'. By the construction, \( G_\varepsilon \) has a Siegel point \( \phi_\varepsilon(z_0) \) with a preimage \( \phi_\varepsilon(z^*) \neq \phi_\varepsilon(z_0) \). It follows from (iii)' that some \( G_{\varepsilon_1} \) (resp. \( G_{\varepsilon_2} \)) is a Siegel point of \( G_\varepsilon \) (resp. a rationally indifferent fixed point \( \phi_{\varepsilon_2}(z_1) \)) whose multiplier is not 1). From (iv)' and (v)', we see that
\[
\begin{align*}
n_{SD}(G_\varepsilon) & \geq n_{SD}(f) - 1, \\
n_{AB}(G_\varepsilon) & \geq n_{AB}(f), \\
n_{rat}(G_\varepsilon) & \geq n_{rat}(f), \\
n_{ER}(G_\varepsilon) & \geq n_{ER}(f).
\end{align*}
\]
It follows from the construction and Lemma 2 that \( G_{\varepsilon_1} \) (resp. \( G_{\varepsilon_2} \)) satisfies the properties (a)~(d) of \( g_1 \) (resp. \( g_2 \)).

Finally, we show the existence of \( g_3 \). Suppose that \( n_{ER}(G_\varepsilon) = n_{ER}(f) \) for any \( \varepsilon \) small enough. Let \( \tilde{J}(F_\varepsilon) \subset \overline{\mathbb{C}} \) be the closure of the set of all repelling periodic points of \( F_\varepsilon \). (Note that all periodic points of \( F_\varepsilon \) are in \( \mathbb{V}_\varepsilon \) for \( \varepsilon \) small enough.) By following Shishikura's idea in [Shi, p. 27, STEP 3], one can show that there exist a neighborhood \( N_0 \) of 0 and a continuous mapping
\[
\Pi : N_0 \times (J(f) \cup \{\infty\}) \to \tilde{J}(F_\varepsilon),
\]
where \( \varepsilon \in N_0 \). Hence we have \( z_1 \notin \tilde{J}(F_\varepsilon) \) for \( \varepsilon \in \mathbb{C} \) small enough. (Recall that \( z_1 \) is a Siegel point of \( f \).) However, from (iv), we can vary the multiplier of \( z_1 \) and make \( z_1 \) into a repelling fixed point of \( F_\varepsilon \). This is a contradiction. Therefore, we have \( n_{ER}(G_{\varepsilon_3}) \geq n_{ER}(F) + 1 \) for some \( \varepsilon_3 \). It follows from the construction and Lemma 2 that \( G_{\varepsilon_3} \) satisfies the properties of \( g_3 \). \( \square \)
Remark 3. In the proof of Lemma 3 we constructed an attracting fixed point \( \phi_{\varepsilon_1}(z_1) \) of \( G_{\varepsilon_1} \) near \( z_1 \) with multiplier \( (1 + \varepsilon_1)f'(z_1) \) \( (\varepsilon_1 \in \mathbb{C}) \). On the other hand, there exists a similar way to make attracting cycles. More precisely, suppose that \( f \) has an irrationally (or a rationally) indifferent fixed point \( z_1' \) with a preimage other than itself. As in [Shi, Section 4], some modification of our surgery enables us to perturb \( f \) so that the fixed point near \( z_1' \) has multiplier \( (1 - \varepsilon)f'(z_1') \) \( (\varepsilon > 0) \). (If \( z_1' \) is rationally indifferent, there is no problem without the condition that \( z_1' \) has a preimage other than itself.)

The assumption of Lemma 3 requires \( n_{SD}(f) \geq 2 \). On the other hand, when \( n_{SD}(f) \geq 1 \), we can reduce the number of SD-cycles by one and increase that of AB-cycles (or PB-cycles) by one as follows:

**Lemma 4.** Let \( f \in S_q \cap SF_{k,t} \). We assume the following conditions:

1. Every non-repelling periodic point of \( f \) is a fixed point whose multiplier is not 1;
2. Every Siegel point of \( f \) has the multiplier satisfying the condition [Siegel] of Proposition C;
3. \( f \) has a Siegel point with a preimage other than itself;
4. \( f \) satisfies \( n_{PB}(f) = n_{rat}(f), n_{C_1}(f) = 0 \), and
   \[
   n_{AB}(f) + n_{PB}(f) + n_{SD}(f) + n_{ER}(f) = q.
   \]

Then for every neighborhood \( N \subset M_f \) of \( f \), there exist \( g_j \in N(j = 1, 2) \) with the following properties:

1. Every non-repelling periodic point of \( g_j \) is a fixed point whose multiplier is not 1;
2. Every Siegel point of \( g_j \) has the multiplier satisfying the condition [Siegel] of Proposition C;
3. \( g_j \in SF_{k,t}, n_{PB}(g_j) = n_{rat}(g_j) \),
4. and
   \[
   C(g_1) = (n_{AB}(f) + 1, n_{PB}(f), n_{SD}(f) - 1, 0, n_{ER}(f)),
   C(g_2) = (n_{AB}(f), n_{PB}(f) + 1, n_{SD}(f) - 1, 0, n_{ER}(f)).
   \]

**Proof.** Let \( z_0 \) be a Siegel point of \( f \) with a preimage other than itself. Let \( \{\zeta_1, \cdots, \zeta_n\} \) be the set of all non-repelling fixed points of \( f \) other than \( z_0 \), if any. By the assumption, we have \( f'((\zeta_j) \neq 1 \) for \( j = 1, \cdots, n \). By the implicit function theorem, there exist a neighborhood \( W \subset N \) of \( f \) and neighborhoods \( U_{\zeta_j} \) of \( \zeta_j(j = 1, \cdots, n) \) such that every \( g \in W \) has a unique fixed point \( \alpha_j(g) \) in \( U_{\zeta_j} \) and \( \alpha_j(g) \) is a holomorphic function on \( W \). Thus

\[
A_{\zeta_j} := \{ g \in W \mid g'((\alpha_j(g)) = f'((\zeta_j)) \}
\]

is an analytic set in \( W \) when it is expressed by a local coordinate on \( W \). Let \( \{\eta_1, \cdots, \eta_{n_{ER}(f)}\} \) be the set of all eventually repelling singular values, when \( n_{ER}(f) \geq 1 \). Then there exist some integers \( n_t \geq 0 \) and \( m_t \geq 1 \) such that \( f^{n_t}(\eta_t) \) is a repelling periodic point of \( f \) with period \( m_t \) for \( t = 1, \cdots, n_{ER}(f) \). If we take small enough \( W \), there exist neighborhoods \( U_{\eta_t} \) of \( \eta_t(t = 1, \cdots, n_{ER}(f)) \) such that every \( g \in W \) has a unique singular value \( \beta_t(g) \) in \( U_{\eta_t} \). Moreover, there exists some \( 1 \leq t' \leq q \) such that \( \beta_t(g) = \Phi_{t'}(g), \) where \( \Phi(g) = (\Phi_1(g), \cdots, \Phi_{q+2}(g)) \) is a local
coordinate on $W$. (See Section 2 for local coordinates on $M_f$.) Hence $\beta_t(g)$ is holomorphic on $W$. Thus

$$A_{n_t} := \{ g \in W \mid g^{m_t+n_t}(\beta_t(g)) = g^n(\beta_t(g)) \}$$

is an analytic set in $W$. If we take small enough $W$, every $g \in A_{n_t}$ has an eventually repelling singular value $\beta_t(g)$. We define $Z$ by

$$Z := \left\{ \left( \bigcap_{i=1}^{n_t} A_{\zeta_i} \right) \cap \left( \bigcap_{i=1}^{n_{\text{ER}}(f)} A_{n_t} \right) \cup \bigcup_{i=1}^{n_{\text{ER}}(f)} A_{n_t} \cup W \right\}$$

By definition, $Z$ is an analytic set in $W$ and every $g \in Z$ satisfies

$$n_{\text{SD}}(g) \geq n_{\text{SD}}(f) - 1, \quad n_{\text{AB}}(g) \geq n_{\text{AB}}(f), \quad n_{\text{rat}}(g) \geq n_{\text{rat}}(f), \quad n_{\text{ER}}(g) \geq n_{\text{ER}}(f).$$

In addition, by Proposition 5 every $g \in Z$ satisfies $g \in SF_{k,l}$.

If we take small enough $W$, the implicit function theorem shows that there exists a holomorphic function $x(g)$ on $W$ such that

$$g(x(g)) = x(g), \quad x(f) = z_0.$$

(Recall that $z_0$ is a Siegel point of $f$ with a preimage other than itself.) Consider a holomorphic function

$$\lambda(g) := g'(x(g))$$

on $W$. As in Remark 3 some modification of the proof of Lemma 3 enables us to convert $f$ into some $g_0 \in Z$ with $|\lambda(g_0)| < 1$. Thus $\lambda$ is not constant on $Z$. This shows that $\lambda(Z)$ is a neighborhood of $f'(z_0)$ (see [Na, p. 54, Proposition 10 (Maximum Principle)]). Hence $x(\tilde{g}_1)$ and $x(\tilde{g}_2)$ are an attracting fixed point of $\tilde{g}_1$ and a rationally indifferent fixed point of $\tilde{g}_2$ whose multiplier is not 1 respectively, for some $\tilde{g}_1, \tilde{g}_2 \in Z$. It follows from the construction and Lemma 2 that $\tilde{g}_1$ and $\tilde{g}_2$ satisfy the properties of $g_1$ and those of $g_2$ respectively.

**Remark 4.** From the proof of Lemma 4 we can convert $f$ into some $g \in Z$ so that the multiplier of the fixed point $x(g)$ near $z_0$ becomes any value in some open set containing $f'(z_0)$. Also, if $f$ satisfies the assumption of Lemma 4 other than (3) and has one rationally indifferent fixed point $z_0$, similar argument goes well. More precisely, we can perturb $f$ so that the multiplier of the fixed point near $z_0$ becomes any value in some open set containing $f'(z_0)$.

Lemma 4 does not require $n_{\text{SD}}(f) \geq 2$. Therefore, one may think that Lemma 3 is not needed. However, by Lemma 3 we can convert a Siegel cycle without preimages other than itself or increase the number of eventually repelling singular values. This is an advantage of Lemma 3.

When $n_{\text{SD}}(f) \geq 1$, we can convert some Siegel cycles into Cremer cycles at a time as follows:

**Lemma 5.** Suppose that $f \in S_q \cap SF_{k,l}$ satisfies the assumption of Lemma 4. Then for every neighborhood $N \subset M_f$ of $f$ and every $m$ with $1 \leq m \leq n_{\text{SD}}(f)$, there exists a $g_* \in N$ with the following properties:

(a) Every non-repelling periodic point of $g_*$ is a fixed point;
than itself. By the implicit function theorem, there exist a neighborhood of \( f \)
where 0 can convert and holomorphic functions \( x_j(g)(j = 1, \ldots, m) \) on \( W' \) satisfying
\[
g(x_j(g)) = x_j(g), \quad x_j(f) = z_j.
\]

As in the proof of Lemma 4 we can construct an analytic set \( Z' \) in \( W' \) such that every \( g \in Z' \) satisfies \( g \in SF_{k,l} \) and
\[
n_{SD}(g) \geq n_{SD}(f) - m, \quad n_{AB}(g) \geq n_{AB}(f),
\]
\[
n_{rat}(g) \geq n_{rat}(f), \quad n_{ER}(g) \geq n_{ER}(f).
\]

Now we define \( R \subset Z' \) by
\[
R := \{ g \in Z' \mid x_j(g)(j = 1, \ldots, m) \text{ are rationally indifferent fixed points of } g \}.
\]

We construct \( g_* \) as a limit of some sequence of functions in \( R \).

First of all, we convert \( f \) into some \( g_1 \in R \) by applying Lemma 3 or Lemma 4 repeatedly. Set
\[
A_{a,b}(x) := \{ z \in \mathbb{C} \mid a < |z - x| < b \},
\]
where \( 0 < a < b \) and \( x \in \mathbb{C} \). In addition, we define \( 0 \leq \theta_j(g) < 1 \) for \( g \in W' \) and every \( j(1 \leq j \leq m) \) by
\[
g'(x_j(g)) = |g'(x_j(g))|e^{2\pi i \theta_j(g)}.
\]
Recall that rationally indifferent periodic points are in the Julia set, which is the closure of the set of all repelling periodic points. Hence \( g_1 \) has periodic points in any punctured neighborhood of each of \( x_j(g)(j = 1, \ldots, m) \). Thus there exist \( r_1 > 0 \) and \( 0 < r_2 < r_1/2 \) such that \( g_1 \) has some \( p_j \)-periodic point in each of annuli \( A_{r_2,r_1}(x_j(g_1)) \). By applying Rouché’s theorem to \( g_1^{p_j}(z) - z \) and \( g^{p_j}(z) - g_1^{p_j}(z) \), there exists a closed neighborhood \( U_1 \subset W' \) of \( g_1 \) such that every \( g \in U_1 \) has some \( (p_j^-) \)-periodic point in each of annuli \( A_{r_2,r_1}(x_j(g)) \). In addition, if we take \( U_1 \) small enough, it follows from the continuity of \( \theta_j(g) \) at \( g_1 \) that every \( g \in U_1 \) satisfies
\[
|\theta_j(g_1) - \theta_j(g)| = \left| \frac{p_{1,j}}{q_{1,j}} - \theta_j(g) \right| < \frac{1}{2(q_{1,j})^2} \quad (j = 1, \ldots, m),
\]
where \( p_{1,j}/q_{1,j} = \theta_j(g_1) \) and \( p_{1,j}, q_{1,j} \in \mathbb{N} \) are mutually prime. From Remark 4 we can convert \( g_1 \) into some \( g \in Z' \) so that the multiplier of each of \( x_j(g)(j = 1, \ldots, m) \) becomes any value in some neighborhood of \( g_1'(x_j(g_1)) \). Thus we can get some \( g_2 \in (U_1 \setminus \{ g_1 \}) \cap R \) such that
\[
g_1'(x_j(g_1)) \neq g_2'(x_j(g_2)) \quad (j = 1, \ldots, m).
\]
From the construction similar to that of \( U_1 \), there exist a closed neighborhood \( U_2 \subset U_1 \) of \( g_2 \) and \( 0 < r_3 < r_2/2 \) such that:

1. Every \( g \in U_2 \) has some periodic point in each of annuli \( A_{r_3,r_2}(x_j(g)) \) \( (j = 1, \ldots, m) \);
(2) Every $g \in U_2$ satisfies

$$|\theta_j(g_2) - \theta_j(g)| = \left| \frac{p_{2,j}}{q_{2,j}} - \theta_j(g) \right| < \frac{1}{2(q_{2,j})^2} \quad (j = 1, \ldots, m),$$

where $p_{2,j}/q_{2,j} = \theta_j(g_2)$ and $p_{2,j}, q_{2,j} \in \mathbb{N}$ are mutually prime.

By repeating this procedure, we get functions $g_n \in R$, closed neighborhoods $U_n \subset W'$ of $g_n$, and $r_n > 0$, for $n = 1, 2, \ldots$, such that:

(i) $g_n(x_j(g_n)) \neq g_n(x_j(g_n))$ if $n_1 \neq n_2$;

(ii) $U_n \supset U_{n+1}$;

(iii) $r_{n+1} < r_n/2^n$;

(iv) Every $g \in U_n$ has some periodic point in each of annuli $A_{r_{n+1}, r_n}(x_j(g))$

$(j = 1, \ldots, m)$;

(v) Every $g \in U_n$ satisfies

$$|\theta_j(g_n) - \theta_j(g)| = \left| \frac{p_{n,j}}{q_{n,j}} - \theta_j(g) \right| < \frac{1}{2(q_{n,j})^2} \quad (j = 1, \ldots, m),$$

where $p_{n,j}/q_{n,j} = \theta_j(g_n)$ and $p_{n,j}, q_{n,j} \in \mathbb{N}$ are mutually prime.

Set

$$K := \{ g \in \mathbb{Z}' \mid |g'(x_j(g))| = 1(j = 1, \ldots, m) \}.$$ 

From (ii) and a standard argument, we get some $g_\infty \in (\bigcap_{n=1}^{\infty} U_n) \cap K$. It follows from this, (iii), and (iv) that $g_\infty$ has some periodic points in any punctured neighborhood of each of $x_j(g_\infty)(j = 1, \ldots, m)$. Next, we show that $x_j(g_\infty)(j = 1, \ldots, m)$ are irrationally indifferent fixed points of $g_\infty$. It follows from $g_\infty \in (\bigcap_{n=1}^{\infty} U_n) \cap K$ and (v) that

$$|\theta_j(g_n) - \theta_j(g_\infty)| = \left| \frac{p_{n,j}}{q_{n,j}} - \theta_j(g_\infty) \right| < \frac{1}{2(q_{n,j})^2}$$

for every $n \geq 1$ and every $j (1 \leq j \leq m)$. Then an easy calculation shows that rational numbers $\theta_j(g_n)(n = 1, 2, \ldots)$ are best approximations (of the second kind) of $\theta_j(g_\infty)$ in the sense of Khinchin (see [K Section 6] and [Sho p. 130]). In addition, it follows from (i) that $\theta_j(g_n)(n = 1, 2, \ldots)$ are different from each other. Thus $\theta_j(g_\infty)$ is an irrational number, since any rational number has at most a finite number of such approximations (see [K] and [O] for basic facts of continued fractions). Therefore, $x_j(g_\infty)(j = 1, \ldots, m)$ are irrationally indifferent fixed points. By Proposition 1, there are $m$ Cremer fixed points. It follows from the construction and Lemma 2 that $g_\infty$ satisfies the properties of $g_\ast$. \hfill \Box

Remark 5. Let $f_\alpha$ be as in Lemma 1. As the referee pointed out, any function $f \in M_{f_\alpha}$ with a Siegel fixed point always satisfies the assumption (3) of Lemma 4 and Lemma 5 that the point has a preimage other than itself. (Hence in Lemma 3 if $f \in M_{f_\alpha}$, then the property (c) of $g_j$ is obvious.) This is due to the following reason: The function $f_\alpha$ does not have any exceptional point with only one preimage. Indeed, if $f_\alpha$ has such a point $b$, then it needs to have the form $(z - \beta)e^{h(z)} + b$, where $\beta$ is the preimage and $h(z)$ is an entire function. Since $f_\alpha$ is of finite order, $h(z)$ must be a polynomial. This is a contradiction. In addition, any function $f \in M_{f_\alpha}$ also satisfies the property, since covering properties are preserved in $M_{f_\alpha}$.

Here, we are ready to prove the Main Theorem.
Proof of the Main Theorem. By Lemma 11 there exists a $T_0 := f_0 \in S_q \cap SF_{0,q}$ with $q$ Siegel points of period 1. Hence we have already shown the Main Theorem when $(m_{AB},m_{PB},m_{SD},m_{Cr}) = (0,0,q,0)$. Thus we show the Main Theorem when $(m_{AB},m_{PB},m_{SD},m_{Cr}) \neq (0,0,q,0)$. We construct $T$ whose non-repelling cycles have the same period 1.

First of all, suppose that $q = 1$ and $(m_{AB},m_{PB},m_{SD},m_{Cr}) \neq (0,0,0,0)$. From Remark 5, $T_0 = f_0$ satisfies the assumptions of Lemma 4 and Lemma 5. We get $T$ by applying Lemma 4 or Lemma 5 to $T_0$. More precisely, we convert one Siegel cycle of $T_0$ into one attracting (or rationally indifferent, Cremer) cycle of $T$.

Next, suppose that $q \geq 2$ and $(m_{AB},m_{PB},m_{SD},m_{Cr}) \neq (0,0,0,0)$.

(i) When $m_{Cr} = 0$, we convert $T_0$ into $T$ by decreasing the number of Siegel cycles and increasing that of attracting cycles (or rationally indifferent cycles, eventually repelling singular values). To be more precise, we apply Lemma 3 repeatedly until we get a function with only one SD-cycle. Since Lemma 3 ensures that the function satisfies the assumption of Lemma 4, we can apply Lemma 4 to it for the last step.

(ii) When $m_{Cr} \neq 0$, we construct $T$ by the following steps:

Step 1. As in the case (i), we construct a $\tilde{T} \in SF_{0,q}$ with $n_{PB}(\tilde{T}) = n_{rat}(\tilde{T})$ and

$$C(\tilde{T}) = (m_{AB},m_{PB},m_{SD} + m_{Cr},0,q - \Sigma),$$

where $\Sigma = m_{AB} + m_{PB} + m_{SD} + m_{Cr}$. Note that $\tilde{T}$ has a Siegel point with a preimage other than itself.

Step 2. By the construction, $\tilde{T}$ satisfies the assumption of Lemma 5. We get $T$ by applying Lemma 5 to $\tilde{T}$ so that $m_{Cr}$ Siegel cycles of $\tilde{T}$ become $m_{Cr}$ Cremer cycles of $T$.

Finally, suppose that $(m_{AB},m_{PB},m_{SD},m_{Cr}) = (0,0,0,0)$. If $q \geq 2$, set

$$h_\varepsilon(z) := \varepsilon z e^{z^{q-1}} \in S_q \cap SF_{q-1,q-1} \quad \text{for } \varepsilon \in \mathbb{C} \setminus \{0\}.$$

An easy calculation shows that $h_\varepsilon$ has an asymptotic value 0 and $q - 1$ critical values $z_j(\varepsilon)(j = 1, \cdots, q - 1)$ expressed as

$$z_j(\varepsilon) = \varepsilon^{q^{-1}} \sqrt{\frac{1}{(q-1)e^{i\theta_j}}},$$

where

$$\theta_j = \frac{(2j - 1)\pi}{q - 1}.$$

Consider the equation on $\varepsilon$

$$h_\varepsilon(z_1(\varepsilon)) = z_1(\varepsilon).$$

This yields

$$F(\varepsilon) := h_\varepsilon(z_1(\varepsilon))/z_1(\varepsilon) = \varepsilon e^{z_1(\varepsilon)^{q-1}} = 1.$$ 

Obviously, $F(\varepsilon)$ is a holomorphic function of $\varepsilon$. It is easy to see that some $\varepsilon_0$ satisfies $F(\varepsilon_0) = 1$. Thus $z_1(\varepsilon_0)$ is a fixed point of $h_{\varepsilon_0}$. In addition, the other critical values $z_j(\varepsilon_0)(j = 2, \cdots, q - 1)$ are fixed points because

$$h_{\varepsilon_0}(z_j(\varepsilon_0)) = e^{i(\theta_j - \theta_1)} h_{\varepsilon_0}(z_1(\varepsilon_0)) = e^{i(\theta_j - \theta_1)} z_1(\varepsilon_0) = z_j(\varepsilon_0).$$
Thus all singular values of $h_{\varepsilon_0}$ are fixed. If a transcendental entire function $f$ has a non-repelling cycle, $f$ has a singular value $a$ such that $\{f^n(a)\}_{n \in \mathbb{N}}$ is an infinite set (see [BrF] p. 117, Theorem 3.39). It follows from this fact that we can take $h_{\varepsilon_0}$ as $T$. Also, if $q = 1$,

$$w(z) := 2\pi i e^z \in S_1 \cap SF_{0,1}$$

can be taken as $T$. Indeed, $w$ has an asymptotic value 0 with

$$w(0) = 2\pi i, \quad w(2\pi i) = 2\pi i.$$

\[ \square \]

**Remark 6.** Here we note the proof of the Main Theorem by constructing Cremer cycles one by one as in [Shi]. We use Proposition G.

**Proposition G** ([Cr2, p. 299]). Let $g$ be a non-linear entire function such that:

1. The origin is an irrationally indifferent fixed point with multiplier $\lambda$;
2. It satisfies

$$\max_{|z| \leq r} |g(z)| \leq F(r)$$

for all large enough $r > 0$ and a positive function $F$ defined for all positive real numbers.

If $\lambda$ satisfies the following condition for every large enough $r > 0$:

$$\lfloor \text{Cremer}(F) \rfloor \liminf_{n \to \infty} \frac{\log F^n(\sqrt[\lambda^n]{-1})}{\lambda^n - 1} = 0,$$

then the origin is a Cremer fixed point. Moreover, the set $\Lambda(F)$ of all $\lambda$ satisfying $\lfloor \text{Cremer}(F) \rfloor$ is uncountable and dense in the unit circle $\{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

By definition, $\Lambda(F)$ depends only on $F$. For entire functions of finite order, we may take $E(r) := e^{e^r}$ as $F(r)$. Thus for such functions (containing structurally finite transcendental entire functions), Proposition G implies that irrationally indifferent fixed points with multipliers in $\Lambda(E)$ are Cremer fixed points. It follows from this and the proofs of Lemma 3 and Lemma 4 that we can convert one Siegel cycle into one Cremer cycle with multiplier in $\Lambda(E)$. Moreover, our construction can keep the multiplier unchanged. Hence we can also construct Cremer cycles of $T$ one by one as in [Shi]. Even when $p \geq 2$, we can also construct Cremer cycles of $T$ with period $p$ by a similar argument. In this case, we can construct Cremer cycles with multipliers in $\Lambda(E^p)$.

4. Concluding remarks

The set $S \backslash SF$ is not empty. For example, $f(z) = \sin z \in S \backslash SF$. We constructed a $T \in S_{q} \cap SF$ to prove best possibility of the Fatou-Shishikura inequality for $f \in S_{q}$. Thus there is the following question:

**Question.** Suppose that non-negative integers $m_{AB}, m_{PB}, m_{SD},$ and $m_{Cr}$, and a positive integer $q$ satisfy

$$m_{AB} + m_{PB} + m_{SD} + m_{Cr} \leq q.$$

Is there a $T \in S_{q} \backslash SF$ such that

$$(n_{AB}(T), n_{PB}(T), n_{SD}(T), n_{Cr}(T)) = (m_{AB}, m_{PB}, m_{SD}, m_{Cr})?$$
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Department of Mathematical Sciences, Graduate School of Human and Environmental Studies, Kyoto University, Kyoto 606-8501, Japan

*Email address:* kisaka@math.h.kyoto-u.ac.jp

Department of Mathematical Sciences, Graduate School of Human and Environmental Studies, Kyoto University, Kyoto 606-8501, Japan

*Email address:* naba.hiroto.78r@st.kyoto-u.ac.jp