CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 26, Pages 165–181 (October 11, 2022)
https://doi.org/10.1090/ecgd/373

BEST POSSIBILITY OF THE FATOU-SHISHIKURA INEQUALITY FOR TRANSCENDENTAL ENTIRE FUNCTIONS IN THE SPEISER CLASS

MASASHI KISAKA AND HIROTO NABA

Abstract. The Speiser class S is the set of all entire functions with finitely many singular values. Let $S_q \subset S$ be the set of all transcendental entire functions with exactly q distinct singular values. The Fatou-Shishikura inequality for $f \in S_q$ gives an upper bound q of the sum of the numbers of its Cremer cycles and its cycles of immediate attractive basins, parabolic basins, and Siegel disks. In this paper, we show that the inequality for $f \in S_q$ is best possible in the following sense: For any combination of the numbers of these cycles which satisfies the inequality, some $T \in S_q$ realizes it. In our construction, T is a structurally finite transcendental entire function.

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-linear entire function. The *Fatou set* $F(f) \subset \mathbb{C}$ is the maximal open set where the iterates f^n of f form a normal family. The complement of $F(f)$ is called the *Julia set* $J(f)$. Both $F(f)$ and $J(f)$ are completely invariant sets in the following sense: $f(F(f)) \subseteq F(f), f^{-1}(F(f)) \subseteq F(f), f(J(f)) \subseteq J(f)$, and $f^{-1}(J(f)) \subseteq J(f)$.

A point $z \in \mathbb{C}$ is called *periodic* if there exists the minimum number $p \in \mathbb{N}$ such that $f^p(z) = z$. In particular, z is called *fixed* if $p = 1$. The set $\{z, f(z), \cdots, f^{p-1}(z)\}$ is called the *cycle* containing z. This p is called the *period* of z (or the cycle containing z). The *multiplier* of z (or the cycle containing z) is $\lambda := (f^p)'(z)$. We say that z (or the cycle containing z) is *repelling*, *attracting*, *rationally indifferent*, or *irrationally indifferent* if $|\lambda| > 1, |\lambda| < 1$, $\lambda = e^{2\pi i\theta} (\theta \in \mathbb{Q})$, or $\lambda = e^{2\pi i\theta} (\theta \in \mathbb{R} \setminus \mathbb{Q})$, respectively. We call the last three cases *non-repelling*. The following facts are well known:

(1) Attracting periodic points are in $F(f)$;
(2) Rationally indifferent periodic points are in $J(f)$;
(3) $J(f)$ is the closure of the set of all repelling periodic points.

(See [Ber, p. 157, p. 160, Theorem 4].) Furthermore, we call an irrationally indifferent periodic point z a *Siegel point* if $z \in F(f)$, or a *Cremer point* if $z \in J(f)$. The cycle containing a Siegel point is called a *Siegel cycle*. Similarly, we define *Cremer cycles*.

Let U be a connected component of $F(f)$. Then $f^n(U)$ is contained in some component U_n of $F(f)$ for $n = 1, 2, \cdots$. The domain U is called a *wandering domain*

Received by the editors November 17, 2021, and, in revised form, June 4, 2022.
2020 *Mathematics Subject Classification.* Primary 37F10; Secondary 30D05, 37F31.
Key words and phrases. The Fatou-Shishikura inequality, the Speiser class, quasiconformal surgery.

if $U_m \neq U_n$ for any distinct $m, n \in \mathbb{N}$ (for example, see [Ba, p. 567, Example 5.1]). Otherwise, U is called an *eventually periodic component.* The domain U is called a *periodic component* if there exists the minimum number $n \in \mathbb{N}$ such that $U_n = U$. In addition, $\{U, U_1, \cdots, U_{n-1}\}$ is called the *cycle* (*of period* n) containing U. There are the following four possibilities of the periodic component U. (See [BrF, p. 116, Theorem 3.37].)

(AB) There exists an attracting periodic point $z_0 \in U$ of period n and every point $z \in U$ satisfies $f^{nk}(z) \to z_0$ as $k \to \infty$. (We call U an *immediate attractive basin.*)

(PB) There exists a rationally indifferent periodic point $z_0 \in \partial U$ with $f^n(z_0) = z_0$ and $(f^n)'(z_0) = 1$, and every point $z \in U$ satisfies $f^{nk}(z) \to z_0$ as $k \to \infty$. (We call U a *parabolic basin.*)

(SD) There exists a Siegel point $z_0 \in U$ of period n and $f^n|_U$ is analytically conjugate to an irrational rotation $z \mapsto (f^n)'(z_0)z$ of the unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$. (We call U a *Siegel disk.*)

(BD) Every point $z \in U$ satisfies $f^{nk}(z) \to \infty$ as $k \to \infty$. Moreover, ∞ is an essential singularity of f. (We call U a *Baker domain.*)

We call the cycle containing U an *AB-cycle* (resp. a *PB-cycle*, etc.) if U is an immediate attractive basin (resp. a parabolic basin, etc.).

Remark 1. Note that every entire function has no *Herman rings* which rational functions can have. (See [Ber, p. 164].)

Points $c \in \mathbb{C}$ and $f(c)$ are called a *critical point* and a *critical value* respectively, if $f'(c) = 0$. A point $\alpha \in \mathbb{C}$ is called an *asymptotic value* if there exists a continuous curve $\gamma(t)(0 \leq t < 1)$ with $\lim_{t\to 1} \gamma(t) = \infty$ and $\lim_{t \to 1} f(\gamma(t)) = \alpha$. Critical values, asymptotic values, or their accumulation points are called *singular values.* (Note that singular values of polynomials are critical values.) Let $\mathrm{sing}(f^{-1})$ be the set of all singular values of f. Singular values have important relations with periodic components and Cremer cycles (for example, see [BrF, p. 117, Theorem 3.39]). Let $a \in \mathrm{sing}(f^{-1})$. We call *a eventually repelling* if $f^n(a)$ is a repelling periodic point for some $n \geq 0$.

The *Speiser class* S is the set of all entire functions with finitely many singular values. Let $S_q \subset S$ be the set of all transcendental entire functions which have exactly q distinct singular values. Let Pol_d be the set of all polynomials of degree $d \geq 2$. Any $f \in Pol_d$ has at most $d-1$ critical values in $\mathbb{C}$.

Here, we define *structurally finite* transcendental entire functions. Set

$$SF_{k,l} := \Big\{ f(z) = \int_0^z (c_k t^k + \cdots + c_0) e^{a_l t^l + \cdots + a_1 t} dt + b$$

$$\mid b, c_i, a_j \in \mathbb{C}\ (i = 0, \cdots, k, j = 1, \cdots, l), c_k a_l \neq 0 \Big\}$$

for $k \geq 0$ and $l \geq 1$. In addition, put

$$SF := \bigcup_{k \geq 0,\, l \geq 1} SF_{k,l}.$$

A transcendental entire function f is called *structurally finite* if $f \in SF$. Every $f \in SF_{k,l}$ has at most k critical values and at most l asymptotic values. (To be more precise, f has exactly k critical points counted with multiplicity and l

transcendental singularities. See [T2] and [Ok, p. 347, Theorem 1.1].) Therefore, SF is a subset of the Speiser class S. In general, we will restrict ourselves to the functions in $SF_{k,l}$ with exactly q singular values, and hence we will often write $S_q \cap SF_{k,l}$.

Now we introduce the Fatou-Shishikura inequality for $f \in Pol_d$ and that for $f \in S_q$. When $f \in Pol_d \cup S_q$, we define $n_{\text{att}}(f)$ as the the number of attracting cycles of f in $\mathbb{C}$.[1] Similarly, we define

$$n_{\text{rat}}(f), n_{\text{Si}}(f), n_{\text{Cr}}(f), n_{\text{AB}}(f), n_{\text{PB}}(f), n_{\text{SD}}(f), n_{\text{ER}}(f)$$

as the number of rationally indifferent cycles, Siegel cycles, Cremer cycles, AB-cycles, PB-cycles, SD-cycles, and eventually repelling singular values, respectively. In addition,

$$n_{\text{att}}(f) = n_{\text{AB}}(f), \quad n_{\text{Si}}(f) = n_{\text{SD}}(f), \quad n_{\text{rat}}(f) \leq n_{\text{PB}}(f)$$

hold among these notations. In fact, $n_{\text{PB}}(f)$ is a multiple of $n_{\text{rat}}(f)$. (The former two equalities are obvious. See [Bea, p. 116, Theorem 6.5.4, p. 122, Theorem 6.5.7] for the last inequality.) The following is the Fatou-Shishikura inequality for $f \in Pol_d$, which is some modification of [Shi, p. 5, Corollary 2, p. 6, Theorem 4]):

Theorem A ([Shi, p. 5, Corollary 2, p. 6, Theorem 4]). *Let $f \in Pol_d$. Then*

$$n_{\text{AB}}(f) + n_{\text{PB}}(f) + n_{\text{SD}}(f) + n_{\text{Cr}}(f) \leq d - 1.$$

Moreover, the inequality is best possible in the following sense: If non-negative integers $m_{\text{AB}}, m_{\text{PB}}, m_{\text{SD}}$, and m_{Cr} satisfy

$$m_{\text{AB}} + m_{\text{PB}} + m_{\text{SD}} + m_{\text{Cr}} \leq d - 1,$$

then there exists a polynomial $P \in Pol_d$ with

$$(n_{\text{AB}}(f), n_{\text{PB}}(P), n_{\text{SD}}(P), n_{\text{Cr}}(P)) = (m_{\text{AB}}, m_{\text{PB}}, m_{\text{SD}}, m_{\text{Cr}}).$$

The Fatou-Shishikura inequality for $f \in S_q$ is as follows:

Theorem B ([EL, p. 1005, Theorem 5]). *Let $f \in S_q$. Then*

$$n_{\text{AB}}(f) + n_{\text{PB}}(f) + n_{\text{SD}}(f) + n_{\text{Cr}}(f) \leq q.$$

It is known that every $f \in Pol_d \cup S_q$ has the following important dynamical properties:

(1) It has finitely many singular values;
(2) It has no Herman rings, no Baker domains, and no wandering domains (see [Ber, p. 164], [EL, p. 994, Theorem 1, p. 1004, Theorem 3], and [Su, p. 404, Theorem 1]);
(3) It satisfies the Fatou-Shishikura inequalities (Theorem A and Theorem B).

According to Theorem A, the Fatou-Shishikura inequality for $f \in Pol_d$ is best possible. From these dynamical properties of $f \in S_q$ similar to those of $f \in Pol_d$, we can expect best possibility of the Fatou-Shishikura inequality for $f \in S_q$ analogous to that for $f \in Pol_d$. Our main purpose is to show that this is actually true.

---

[1] When we regard $f \in Pol_d$ as a rational function defined on the Riemann sphere $\widehat{\mathbb{C}}$, ∞ is a super-attracting fixed point, which is a critical point with multiplicity $d-1$. By the Fatou-Shishikura inequality for rational functions of degree d (see [Shi, p. 5, Corollary 2]), the sum of the numbers of AB-cycles, PB-cycles, SD-cycles, and Cremer cycles is less than or equal to the number $2d-2$ of critical points in $\widehat{\mathbb{C}}$ counted with multiplicity. We adopt the definition of $n_{\text{att}}(f)(= n_{\text{AB}}(f))$ so that the right-hand side of the modified inequality in Theorem A becomes the number $d-1$ of critical points in $\mathbb{C}$.

Main Theorem. *The Fatou-Shishikura inequality for* $f \in S_q$ *is best possible in the following sense: If non-negative integers* $m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}$*, and* m_{Cr} *satisfy*

$$m_{\mathrm{AB}} + m_{\mathrm{PB}} + m_{\mathrm{SD}} + m_{\mathrm{Cr}} \leq q,$$

then there exists a $T \in S_q$ *with*

$$(n_{\mathrm{AB}}(T), n_{\mathrm{PB}}(T), n_{\mathrm{SD}}(T), n_{\mathrm{Cr}}(T)) = (m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}}).$$

More precisely, T *satisfies* $n_{\mathrm{PB}}(T) = n_{\mathrm{rat}}(T)$ *and* $T \in SF$*. In addition, every non-repelling periodic point of* T *has the same period relatively prime with* q*.*

The proof of the Main Theorem is based on an analogy of [Shi]. For rational functions, we can tell that irrationally indifferent cycles are Cremer cycles if their multipliers satisfy some condition (see [Cr1]). Shishikura used the result to prove best possibility of the Fatou-Shishikura inequality for rational functions (see [Shi]). On the other hand, we cannot use the result for our transcendental case. Hence the main difference between our proof and [Shi] is the way how to construct T with Cremer cycles (see Remark 2). Moreover, our construction can be also used for the rational case, which leads to a slightly different proof of [Shi, p. 6, Theorem 4].

This paper is organized as follows: We devote Section 2 to preliminaries for proving the Main Theorem. In Section 3, we prove the Main Theorem. Finally, we make some concluding remarks in Section 4.

Remark 2. Thankfully, Walter Bergweiler gave us the information about [Cr2]. According to [Cr2], an irrationally indifferent cycle of a transcendental entire function is a Cremer cycle if its multiplier satisfies some condition. If we use this fact, we can construct Cremer cycles of T by the method similar to that in [Shi]. Actually, we knew it after we finished writing the first version of this paper. Thus our proof does not rely on the result. We will mention the proof of the Main Theorem which uses the result after our original proof.

2. Preliminaries

For entire functions, there are the following criteria which tell whether an irrationally indifferent periodic point is a Siegel point or a Cremer point.

Proposition C ([Si]). *Let* z *be an irrationally indifferent periodic point with multiplier* λ*. If there exist positive constants* M *and* k *such that* λ *satisfies the following condition:*

$$\lceil\text{Siegel}\rfloor \qquad \frac{1}{|\lambda^n - 1|} \leq M n^k \quad (n = 1, 2, \cdots),$$

then z *is a Siegel point. In addition,* ⌈Siegel⌋ *is satisfied by almost every* λ *in the unit circle* $\{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$*.*

Proposition D. *Let* z *be an irrationally indifferent periodic point. If there exist periodic points in any punctured neighborhood of* z*, then* z *is a Cremer point.*

Entire functions f and g are called *topologically equivalent* if there exist homeomorphisms $\varphi, \Psi : \mathbb{C} \to \mathbb{C}$ such that

$$\Psi \circ f = g \circ \varphi.$$

Denote by $M_f \subset S_q$ the set of all entire functions topologically equivalent to $f \in S_q$. We can take M_f as a $(q+2)$-dimensional complex analytic manifold whose topology is locally equivalent to the topology of uniform convergence on compact subsets of $\mathbb{C}$. Also, we can take a local coordinate on any small enough open set $U \subset M_f$

$$\Phi : U \to \mathbb{C}^{q+2}, \qquad \Phi(g) = (\Phi_1(g), \cdots, \Phi_{q+2}(g))$$

such that

$$\{\Phi_1(g), \cdots, \Phi_q(g)\} = \operatorname{sing}(g^{-1});$$

the mapping

$$\Phi(U) \times \mathbb{C} \to \mathbb{C}, \qquad (\Phi(g), z) \mapsto g(z)$$

is analytic. (See [EL, Section 3].) In addition, the following result is known.

Proposition E ([T1, p. 69, Proposition 2]). *Let $f \in SF_{k,l}$. Then every $g \in M_f$ satisfies $g \in SF_{k,l}$.*

We introduce the fundamental lemma for quasiconformal surgery which we use to prove the Main Theorem. The definition of *quasiconformal mappings* is as follows:

Definition. Let D and D' be domains of $\mathbb{C}$. A homeomorphism $\phi : D \to D'$ is a quasiconformal mapping if ϕ satisfies the following conditions:

(1) ϕ is absolutely continuous on almost all lines parallel to real-axis and almost all lines parallel to imaginary-axis;
(2) $|\phi_{\overline{z}}| \leq k|\phi_z|$ holds almost everywhere for some $0 \leq k < 1$.

See [A] for more details about quasiconformal mappings. Let $U \subset \mathbb{C}$ be a domain of $\mathbb{C}$. A mapping $g : U \to \mathbb{C}$ is called *quasiregular* if g can be expressed as

$$g = f \circ \phi,$$

where $\phi : U \to \phi(U)$ is a quasiconformal mapping and $f : \phi(U) \to g(U)$ is a holomorphic function. The following is the fundamental lemma for our quasiconformal surgery, which is some modification of [Shi, p. 7, Lemma 1, p. 9, Lemma 3]):

Lemma F ([Shi, p. 7, Lemma 1, p. 9, Lemma 3]). *For $\varepsilon \in \mathbb{C}$ in a neighborhood of 0, set a quasiregular mapping*

$$g_\varepsilon = f \circ \Psi_\varepsilon,$$

where f is an entire function and $\Psi_\varepsilon : \mathbb{C} \to \mathbb{C}$ is a quasiconformal mapping. Suppose that g_ε satisfies the following conditions:

(1) $\|(\Psi_\varepsilon)_{\overline{z}}/(\Psi_\varepsilon)_z\|_\infty \to 0(\varepsilon \to 0)$ *and* $\Psi_\varepsilon \to \mathrm{Id}_{\mathbb{C}}(\varepsilon \to 0)$ *locally uniformly on* $\mathbb{C}$*;*
(2) *There exists an open set E_ε such that $g_\varepsilon(E_\varepsilon) \subset E_\varepsilon$ and $(g_\varepsilon)_{\overline{z}} = 0$ almost everywhere on $E_\varepsilon \cup (\mathbb{C} \setminus (g_\varepsilon)^{-1}(E_\varepsilon))$.*

Then there exists a quasiconformal mapping φ_ε with the following properties:

(a) $\tilde{g_\varepsilon} = \varphi_\varepsilon \circ g_\varepsilon \circ \varphi_\varepsilon^{-1}$ *is an entire function;*
(b) $\varphi_\varepsilon \to \mathrm{Id}_{\mathbb{C}}$ *and* $\tilde{g_\varepsilon} \to f$ *locally uniformly on* $\mathbb{C}$ *as* $\varepsilon \to 0$*;*
(c) φ_ε *is conformal on the interior of* $E_\varepsilon \cup (\mathbb{C} \setminus \bigcup_{n=1}^{\infty} (g_\varepsilon)^{-n}(E_\varepsilon))$.

3. Proof of the main theorem

Basically, we follow Shishikura's method for the rational case in [Shi]. However, there are differences between rational functions and transcendental entire functions. For example, in the rational case, the value at ∞ is defined naturally. On the other hand, in our transcendental case, the value at ∞ cannot be defined naturally, since ∞ is an essential singularity. Thus we have to modify his proof at each step. The critical difference is in our construction of Cremer cycles. Shishikura constructed Cremer cycles one by one. He realized this by his quasiconformal surgery which converts one Siegel cycle into one Cremer cycle. This is based on the result specific to rational functions by Cremer (see [Cr1]). We cannot use this for our case. Thus we have to make Cremer cycles of T in a different way. We do not construct Cremer cycles one by one because our construction does not guarantee that one Cremer cycle constructed is kept unchanged while we construct another Cremer cycle. Instead, we construct all Cremer cycles of T in the final step.

Here we give the sketch of the proof. If $(m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}}) \neq (0,0,0,0)$, we construct T by the following procedure: First of all, we construct a $T_0 \in S_q \cap SF_{0,q}$ which has q Siegel cycles (Lemma 1). Next, we take T_0 as T if $(m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}}) = (0,0,q,0)$. Let $(m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}}) \neq (0,0,q,0)$. If $m_{\mathrm{Cr}} = 0$, we convert T_0 into T by making one Siegel cycle repelling, attracting, or rationally indifferent repeatedly. This step by step procedure is done by quasiconformal surgery (Lemma 3) or some argument on analytic sets (Lemma 4). If $m_{\mathrm{Cr}} \neq 0$, we convert T_0 into a $\tilde{T}$ with

$$(n_{\mathrm{AB}}(\tilde{T}), n_{\mathrm{PB}}(\tilde{T}), n_{\mathrm{SD}}(\tilde{T}), n_{\mathrm{Cr}}(\tilde{T})) = (m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}} + m_{\mathrm{Cr}}, 0)$$

in the manner above. Then we convert $\tilde{T}$ into T by making m_{Cr} Siegel cycles of $\tilde{T}$ into m_{Cr} Cremer cycles of T at a time (Lemma 5).

Let p be any positive integer relatively prime with q. Put $\lambda = \exp(2\pi i/p)$. Set

$$f_\alpha(z) := (1+\alpha)\lambda \int_0^z e^{t^q}\, dt \in SF_{0,q} \quad \text{for } \alpha \in \mathbb{C} \setminus \{-1\}.$$

Since f_α has q distinct asymptotic values and no critical values (see [Ne, p. 168, 2.3]), we have $f_\alpha \in S_q$.

Lemma 1. *There is an uncountable set $A \subset \mathbb{C} \setminus \{-1\}$ such that $f_\alpha (\alpha \in A)$ has q Siegel cycles of period p whose multipliers satisfy the condition* ⌈Siegel⌋ *of Proposition* C.

Proof. An easy calculation shows that

$$f_\alpha(z) = (1+\alpha)\lambda z \left(1 + \frac{z^q}{q+1} + \frac{z^{2q}}{2(2q+1)} + \cdots\right).$$

From this and the argument in [Bea, p. 130], we get

$$f_0^p(z) = z\{1 + c_0 z^{pkq} + O(|z|^{(pk+1)q})\} \quad \text{as } z \to 0,$$

where $c_0 \neq 0$, $k \geq 1$. In addition, there are kq PB-cycles of period p. By Theorem B, we have $kq \leq q$. Thus we obtain $k = 1$. It follows that

$$f_\alpha^p(z) = z\{(1+\alpha)^p + c(\alpha) z^{pq} + O(|z|^{(p+1)q})\} \quad \text{as } \alpha, z \to 0,$$

where $c(\alpha)$ is a holomorphic function of α, with $c(0) = c_0$. Set $X = z^q$. Let $f_\alpha^p(z) = zF(X,\alpha)$. Thus we have

$$F(X,\alpha) = (1+\alpha)^p \left(1 + \frac{c(\alpha)}{(1+\alpha)^p} X^p + O(|X|^{p+1})\right) \quad \text{as } \alpha, X \to 0.$$

By the construction and Rouché's theorem, if $\alpha \neq 0$ is small enough, $F(X,\alpha) = 1$ has p different solutions $X = \zeta_1(\alpha), \cdots, \zeta_p(\alpha)$ with $\zeta_j(\alpha) \neq 0$ and $\zeta_j(\alpha) \to 0 (\alpha \to 0)$ for $j = 1, \cdots, p$. Therefore, f_α has q cycles $C_1(\alpha), \cdots, C_q(\alpha)$ of period p for small enough $\alpha \neq 0$. They consist of pq q-th roots of $\zeta_j(\alpha)(j = 1, \cdots, p)$. Moreover, it follows that

$$\frac{\partial F}{\partial X}|_{(\zeta_j(\alpha_0),\alpha_0)} \neq 0 \quad (j = 1, \cdots, p)$$

for every small enough $\alpha_0 \neq 0$. By the implicit function theorem, $\zeta_j(\alpha)(j = 1, \cdots, p)$ are holomorphic functions of α on some neighborhood of α_0. It follows that

$$\Sigma(\alpha) := \Sigma_{j=1}^{p} \zeta_j(\alpha)$$

is a holomorphic function of α on some punctured neighborhood of 0. By the construction, $C_j(\alpha)(j = 1, \cdots, q)$ have the same multiplier $\sigma(\alpha)$. An easy calculation shows that

$$\sigma(\alpha) = (1 + \alpha)^p e^{\Sigma(\alpha)}.$$

Thus we have $\sigma(\alpha) \to 1$ as $\alpha \to 0$ and $\sigma(\alpha)$ is holomorphic on some punctured neighborhood of 0. Set $\sigma(0) = 1$. By the Riemann removable singularity theorem, $\sigma(\alpha)$ is holomorphic on some neighborhood U of 0. It follows that $\sigma(U)$ is a neighborhood of 1. Hence there is an uncountable set $A \subset U$ such that $\sigma(\alpha)(\alpha \in A)$ satisfies the condition ⌈Siegel⌋ of Proposition C. Thus $f_\alpha(\alpha \in A)$ has q Siegel cycles of period p with multiplier $\sigma(\alpha)$. □

Lemma 1 shows the existence of $T_0 \in S_q \cap SF_{0,q}$ with q Siegel cycles of period p. We convert T_0 into T with non-repelling cycles of period p. Henceforth, we construct T with $p = 1$ for simplicity. The case $p \geq 2$ is shown exactly in the same way.

We define $C(f)$ for $f \in S_q$ by

$$C(f) := (n_{\mathrm{AB}}(f), n_{\mathrm{PB}}(f), n_{\mathrm{SD}}(f), n_{\mathrm{Cr}}(f), n_{\mathrm{ER}}(f)).$$

The combination $C(f)$ always satisfies

$$n_{\mathrm{rat}}(f) \leq n_{\mathrm{PB}}(f), \quad n_{\mathrm{AB}}(f) + n_{\mathrm{PB}}(f) + n_{\mathrm{SD}}(f) + n_{\mathrm{Cr}}(f) + n_{\mathrm{ER}}(f) \leq q.$$

(See Section 1 for the former. The latter follows from Shishikura's idea used in the proof of Theorem B. See [Shi, p. 25] for details.) They yield Lemma 2:

Lemma 2. *Suppose that non-negative integers* $n_{\mathrm{AB}}, n_{\mathrm{rat}}, n_{\mathrm{SD}}, n_{\mathrm{Cr}}$, *and* n_{ER}, *and* $f \in S_q$ *satisfy*

$$n_{\mathrm{AB}} + n_{\mathrm{rat}} + n_{\mathrm{SD}} + n_{\mathrm{Cr}} + n_{\mathrm{ER}} = q,$$
$$n_{\mathrm{AB}}(f) \geq n_{\mathrm{AB}}, \cdots, n_{\mathrm{ER}}(f) \geq n_{\mathrm{ER}}.$$

Then $n_{\mathrm{PB}}(f) = n_{\mathrm{rat}}(f)$ *and*

$$C(f) = (n_{\mathrm{AB}}, n_{\mathrm{rat}}, n_{\mathrm{SD}}, n_{\mathrm{Cr}}, n_{\mathrm{ER}}).$$

By quasiconformal surgery, we will reduce the number of SD-cycles by one and increase that of AB-cycles (or PB-cycles, eventually repelling singular values) by one. We use Lemma 3 which is applicable to general structurally finite transcendental entire functions:

Lemma 3. *Let* $f \in S_q \cap SF_{k,l}$. *We assume the following conditions:*

1. *Every non-repelling periodic point of* f *is a fixed point;*
2. *Every Siegel point of* f *has the multiplier satisfying the condition* ⌈Siegel⌋ *of Proposition* C*;*

(3) *f satisfies $n_{\mathrm{SD}}(f) \geq 2, n_{\mathrm{PB}}(f) = n_{\mathrm{rat}}(f)$, $n_{\mathrm{Cr}}(f) = 0$, and*

$$n_{\mathrm{AB}}(f) + n_{\mathrm{PB}}(f) + n_{\mathrm{SD}}(f) + n_{\mathrm{ER}}(f) = q.$$

Then for every neighborhood $N \subset M_f$ of f, there exist $g_j \in N (j = 1, 2, 3)$ with the following properties:

(a) *Every non-repelling periodic point of g_j is a fixed point whose multiplier is not 1;*

(b) *Every Siegel point of g_j has the multiplier satisfying the condition* ⌈Siegel⌋ *of Proposition* C*;*

(c) *g_j has a Siegel point with a preimage other than itself;*

(d)
$$g_j \in SF_{k,l}, \quad n_{\mathrm{PB}}(g_j) = n_{\mathrm{rat}}(g_j),$$
and
$$\begin{aligned} C(g_1) &= (n_{\mathrm{AB}}(f) + 1, n_{\mathrm{PB}}(f), n_{\mathrm{SD}}(f) - 1, 0, n_{\mathrm{ER}}(f)), \\ C(g_2) &= (n_{\mathrm{AB}}(f), n_{\mathrm{PB}}(f) + 1, n_{\mathrm{SD}}(f) - 1, 0, n_{\mathrm{ER}}(f)), \\ C(g_3) &= (n_{\mathrm{AB}}(f), n_{\mathrm{PB}}(f), n_{\mathrm{SD}}(f) - 1, 0, n_{\mathrm{ER}}(f) + 1). \end{aligned}$$

Proof. First of all, we show the existence of g_1 and g_2. There are at least two Siegel points of f, say z_0 and z_1. Hence we can assume that z_0 is not a Picard exceptional value and has a preimage $z^* \neq z_0$. By using the Lagrange interpolating polynomial, one can construct a polynomial P such that

$$P\left(\frac{1}{z_1 - z^*}\right) = 0, \qquad P'\left(\frac{1}{z_1 - z^*}\right) = -(z_1 - z^*)^2;$$
$$P\left(\frac{1}{a - z^*}\right) = 0, \qquad P'\left(\frac{1}{a - z^*}\right) = 0$$

if $a \neq z_1$ is a non-repelling fixed point or $a = f^n(b)$, where $n \geq 0$ and b is an eventually repelling singular value. Let ρ be an increasing C^∞ function on $[0, \infty)$ satisfying $\rho = 0$ on $[0, 1]$ and $\rho = 1$ on $[2, \infty)$. Let d be the degree of P. We define $H_\varepsilon : \mathbb{C} \to \mathbb{C}$ for $\varepsilon \in \mathbb{C} \setminus \{0\}$ by

$$H_\varepsilon(z) := \begin{cases} z + \varepsilon\rho\left(|\varepsilon|^{-1/(3d)}|z - z^*|\right) P\left(1/(z - z^*)\right) & (z \neq z^*) \\ z^* & (z = z^*). \end{cases}$$

Let $H_0 : \mathbb{C} \to \mathbb{C}$ be the identity. An easy calculation shows that $H_\varepsilon : \mathbb{C} \to \mathbb{C}$ is a quasiconformal mapping for ε small enough. Set a quasiregular mapping

$$F_\varepsilon := f \circ H_\varepsilon$$

for ε small enough. The mappings H_ε and F_ε have the following properties:

(i) $\|(H_\varepsilon)_{\overline{z}}/(H_\varepsilon)_z\|_\infty \to 0 (\varepsilon \to 0)$ and $H_\varepsilon \to H_0 (\varepsilon \to 0)$ locally uniformly on $\mathbb{C}$;

(ii) H_ε is conformal on
$$V_\varepsilon := \{z \in \mathbb{C} \mid |z - z^*| > 2|\varepsilon|^{1/(3d)}\},$$
and hence F_ε is holomorphic there and we can define the multipliers of periodic points of F_ε there as in the case of entire functions;

(iii) There is a neighborhood U_ε of z_0 such that $F_\varepsilon(U_\varepsilon) = U_\varepsilon$ and $\mathbb{C} \setminus V_\varepsilon \subset F_\varepsilon^{-1}(U_\varepsilon)$;

(iv) z_1 is a fixed point of F_ε with multiplier $(1 + \varepsilon)f'(z_1)$;

(v) If $z \neq z_1$ is a non-repelling fixed point of f, then z is a fixed point of F_ε with multiplier $f'(z)$;

(vi) If z is an eventually repelling singular value of f, then

$$F_\varepsilon^n(z) = f^n(z) \quad (n = 1, 2, \cdots)$$

and

$$F_\varepsilon'(\tilde{z}) = f'(\tilde{z}) \quad \text{for any } \tilde{z} \in \{f^n(z) \mid n = 0, 1, \cdots\}.$$

By the construction, $F_\varepsilon'(z_0) = f'(z_0)$ satisfies the condition $\lceil$Siegel$\rfloor$ of Proposition C. This yields the property (iii) (see [Si] and [Shi, p. 26, STEP 2]). The other properties follow directly from the construction. From (i), (ii), and (iii), we can apply Lemma F to $g_\varepsilon = F_\varepsilon$ and $E_\varepsilon = U_\varepsilon$ for ε small enough. It follows from this, (iv), (v), and (vi) that there exists a quasiconformal mapping $\phi_\varepsilon : \mathbb{C} \to \mathbb{C}$ with the following properties:

(i)′

$$G_\varepsilon := \phi_\varepsilon \circ F_\varepsilon \circ \phi_\varepsilon^{-1}$$

is an entire function;

(ii)′ $\phi_\varepsilon \to \mathrm{Id}_{\mathbb{C}}$ and $G_\varepsilon \to f$ locally uniformly on $\mathbb{C}$ as $\varepsilon \to 0$;

(iii)′ $\phi_\varepsilon(z_1)$ is a fixed point of G_ε with multiplier $(1+\varepsilon)f'(z_1)$;

(iv)′ If $z \neq z_1$ is a non-repelling fixed point of f, then $\phi_\varepsilon(z)$ is a fixed point of G_ε with multiplier $f'(z)$;

(v)′ If z is an eventually repelling singular value of f, then $\phi_\varepsilon(z)$ is an eventually repelling singular value of G_ε.

From (i)′, we have $G_\varepsilon = \phi_\varepsilon \circ f \circ (H_\varepsilon \circ \phi_\varepsilon^{-1})$, where ϕ_ε and $H_\varepsilon \circ \phi_\varepsilon^{-1}$ are quasiconformal mappings. Therefore, G_ε and f are topologically equivalent. By Proposition E, we have $G_\varepsilon \in SF_{k,l}$. In addition, we obtain $G_\varepsilon \in N$ from (ii)′. By the construction, G_ε has a Siegel point $\phi_\varepsilon(z_0)$ with a preimage $\phi_\varepsilon(z^*) \neq \phi_\varepsilon(z_0)$. It follows from (iii)′ that some G_{ε_1}(resp. G_{ε_2}) $\in N$ has an attracting fixed point $\phi_{\varepsilon_1}(z_1)$ (resp. a rationally indifferent fixed point $\phi_{\varepsilon_2}(z_1)$ whose multiplier is not 1). From (iv)′ and (v)′, we see that

$$n_{\mathrm{SD}}(G_\varepsilon) \geq n_{\mathrm{SD}}(f) - 1, \qquad n_{\mathrm{AB}}(G_\varepsilon) \geq n_{\mathrm{AB}}(f),$$
$$n_{\mathrm{rat}}(G_\varepsilon) \geq n_{\mathrm{rat}}(f), \qquad n_{\mathrm{ER}}(G_\varepsilon) \geq n_{\mathrm{ER}}(f).$$

It follows from the construction and Lemma 2 that G_{ε_1}(resp. G_{ε_2}) satisfies the properties (a)∼(d) of g_1 (resp. g_2).

Finally, we show the existence of g_3. Suppose that $n_{\mathrm{ER}}(G_\varepsilon) = n_{\mathrm{ER}}(f)$ for any ε small enough. Let $\tilde{J}(F_\varepsilon) \subset \widehat{\mathbb{C}}$ be the closure of the set of all repelling periodic points of F_ε. (Note that all periodic points of F_ε are in V_ε for ε small enough.) By following Shishikura's idea in [Shi, p. 27, STEP 3], one can show that there exist a neighborhood N_0 of 0 and a continuous mapping

$$\Pi : N_0 \times (J(f) \cup \{\infty\}) \to \tilde{J}(F_\varepsilon),$$

where $\varepsilon \in N_0$. Hence we have $z_1 \notin \tilde{J}(F_\varepsilon)$ for $\varepsilon \in \mathbb{C}$ small enough. (Recall that z_1 is a Siegel point of f.) However, from (iv), we can vary the multiplier of z_1 and make z_1 into a repelling fixed point of F_ε. This is a contradiction. Therefore, we have $n_{\mathrm{ER}}(G_{\varepsilon_3}) \geq n_{\mathrm{ER}}(f) + 1$ for some ε_3. It follows from the construction and Lemma 2 that G_{ε_3} satisfies the properties of g_3. □

Remark 3. In the proof of Lemma 3, we constructed an attracting fixed point $\phi_{\varepsilon_1}(z_1)$ of G_{ε_1} near z_1 with multiplier $(1+\varepsilon_1)f'(z_1)$ $(\varepsilon_1 \in \mathbb{C})$. On the other hand, there exists a similar way to make attracting cycles. More precisely, suppose that f has an irrationally (or a rationally) indifferent fixed point z_1' with a preimage other than itself. As in [Shi, Section 4], some modification of our surgery enables us to perturb f so that the fixed point near z_1' has multiplier $(1-\varepsilon)f'(z_1')$ $(\varepsilon > 0)$. (If z_1' is rationally indifferent, there is no problem without the condition that z_1' has a preimage other than itself.)

The assumption of Lemma 3 requires $n_{\mathrm{SD}}(f) \geq 2$. On the other hand, when $n_{\mathrm{SD}}(f) \geq 1$, we can reduce the number of SD-cycles by one and increase that of AB-cycles (or PB-cycles) by one as follows:

Lemma 4. *Let $f \in S_q \cap SF_{k,l}$. We assume the following conditions:*

1. *Every non-repelling periodic point of f is a fixed point whose multiplier is not* 1*;*
2. *Every Siegel point of f has the multiplier satisfying the condition* ⌈Siegel⌋ *of Proposition* C*;*
3. *f has a Siegel point with a preimage other than itself;*
4. *f satisfies $n_{\mathrm{PB}}(f) = n_{\mathrm{rat}}(f), n_{\mathrm{Cr}}(f) = 0$, and*

$$n_{\mathrm{AB}}(f) + n_{\mathrm{PB}}(f) + n_{\mathrm{SD}}(f) + n_{\mathrm{ER}}(f) = q.$$

Then for every neighborhood $N \subset M_f$ of f, there exist $g_j \in N (j = 1, 2)$ with the following properties:

- (a) *Every non-repelling periodic point of g_j is a fixed point whose multiplier is not* 1*;*
- (b) *Every Siegel point of g_j has the multiplier satisfying the condition* ⌈Siegel⌋ *of Proposition* C*;*
- (c)

$$g_j \in SF_{k,l}, \quad n_{\mathrm{PB}}(g_j) = n_{\mathrm{rat}}(g_j),$$

and

$$C(g_1) = (n_{\mathrm{AB}}(f)+1, n_{\mathrm{PB}}(f), n_{\mathrm{SD}}(f)-1, 0, n_{\mathrm{ER}}(f)),$$
$$C(g_2) = (n_{\mathrm{AB}}(f), n_{\mathrm{PB}}(f)+1, n_{\mathrm{SD}}(f)-1, 0, n_{\mathrm{ER}}(f)).$$

Proof. Let z_0 be a Siegel point of f with a preimage other than itself. Let $\{\zeta_1, \cdots, \zeta_n\}$ be the set of all non-repelling fixed points of f other than z_0, if any. By the assumption, we have $f'(\zeta_j) \neq 1$ for $j = 1, \cdots, n$. By the implicit function theorem, there exist a neighborhood $W \subset N$ of f and neighborhoods U_{ζ_j} of $\zeta_j (j = 1, \cdots, n)$ such that every $g \in W$ has a unique fixed point $\alpha_j(g)$ in U_{ζ_j} and $\alpha_j(g)$ is a holomorphic function on W. Thus

$$A_{\zeta_j} := \{g \in W \mid g'(\alpha_j(g)) = f'(\zeta_j)\}$$

is an analytic set in W when it is expressed by a local coordinate on W. Let $\{\eta_1, \cdots, \eta_{n_{\mathrm{ER}}(f)}\}$ be the set of all eventually repelling singular values, when $n_{\mathrm{ER}}(f) \geq 1$. Then there exist some integers $n_t \geq 0$ and $m_t \geq 1$ such that $f^{n_t}(\eta_t)$ is a repelling periodic point of f with period m_t for $t = 1, \cdots, n_{\mathrm{ER}}(f)$. If we take small enough W, there exist neighborhoods U_{η_t} of $\eta_t (t = 1, \cdots, n_{\mathrm{ER}}(f))$ such that every $g \in W$ has a unique singular value $\beta_t(g)$ in U_{η_t}. Moreover, there exists some $1 \leq t' \leq q$ such that $\beta_t(g) = \Phi_{t'}(g)$, where $\Phi(g) = (\Phi_1(g), \cdots, \Phi_{q+2}(g))$ is a local coordinate on W. (See Section 2 for local coordinates on M_f.) Hence $\beta_t(g)$ is holomorphic on W. Thus

$$A_{\eta_t} := \{g \in W \mid g^{m_t+n_t}(\beta_t(g)) = g^{n_t}(\beta_t(g))\}$$

is an analytic set in W. If we take small enough W, every $g \in A_{\eta_t}$ has an eventually repelling singular value $\beta_t(g)$. We define Z by

$$Z := \begin{cases} (\bigcap_{j=1}^{n} A_{\zeta_j}) \cap (\bigcap_{t=1}^{n_{\mathrm{ER}}(f)} A_{\eta_t}) & (n \geq 1, n_{\mathrm{ER}}(f) \geq 1) \\ \bigcap_{j=1}^{n} A_{\zeta_j} & (n \geq 1, n_{\mathrm{ER}}(f) = 0) \\ \bigcap_{t=1}^{n_{\mathrm{ER}}(f)} A_{\eta_t} & (n = 0, n_{\mathrm{ER}}(f) \geq 1) \\ W & (n = 0, n_{\mathrm{ER}}(f) = 0). \end{cases}$$

By definition, Z is an analytic set in W and every $g \in Z$ satisfies

$$n_{\mathrm{SD}}(g) \geq n_{\mathrm{SD}}(f) - 1, \qquad n_{\mathrm{AB}}(g) \geq n_{\mathrm{AB}}(f),$$
$$n_{\mathrm{rat}}(g) \geq n_{\mathrm{rat}}(f), \qquad n_{\mathrm{ER}}(g) \geq n_{\mathrm{ER}}(f).$$

In addition, by Proposition E, every $g \in Z$ satisfies $g \in SF_{k,l}$.

If we take small enough W, the implicit function theorem shows that there exists a holomorphic function $x(g)$ on W such that

$$g(x(g)) = x(g), \qquad x(f) = z_0.$$

(Recall that z_0 is a Siegel point of f with a preimage other than itself.) Consider a holomorphic function

$$\lambda(g) := g'(x(g))$$

on W. As in Remark 3, some modification of the proof of Lemma 3 enables us to convert f into some $g_0 \in Z$ with $|\lambda(g_0)| < 1$. Thus λ is not constant on Z. This shows that $\lambda(Z)$ is a neighborhood of $f'(z_0)$ (see [Na, p. 54, Proposition 10 (Maximum Principle)]). Hence $x(\tilde{g}_1)$ and $x(\tilde{g}_2)$ are an attracting fixed point of $\tilde{g}_1$ and a rationally indifferent fixed point of $\tilde{g}_2$ whose multiplier is not 1 respectively, for some $\tilde{g}_1, \tilde{g}_2 \in Z$. It follows from the construction and Lemma 2 that $\tilde{g}_1$ and $\tilde{g}_2$ satisfy the properties of g_1 and those of g_2 respectively. ☐

Remark 4. From the proof of Lemma 4, we can convert f into some $g \in Z$ so that the multiplier of the fixed point $x(g)$ near z_0 becomes any value in some open set containing $f'(z_0)$. Also, if f satisfies the assumption of Lemma 4 other than (3) and has one rationally indifferent fixed point z_0', similar argument goes well. More precisely, we can perturb f so that the multiplier of the fixed point near z_0' becomes any value in some open set containing $f'(z_0')$.

Lemma 4 does not require $n_{\mathrm{SD}}(f) \geq 2$. Therefore, one may think that Lemma 3 is not needed. However, by Lemma 3, we can convert a Siegel cycle without preimages other than itself or increase the number of eventually repelling singular values. This is an advantage of Lemma 3.

When $n_{\mathrm{SD}}(f) \geq 1$, we can convert some Siegel cycles into Cremer cycles at a time as follows:

Lemma 5. *Suppose that $f \in S_q \cap SF_{k,l}$ satisfies the assumption of Lemma 4. Then for every neighborhood $N \subset M_f$ of f and every m with $1 \leq m \leq n_{\mathrm{SD}}(f)$, there exists a $g_* \in N$ with the following properties:*

(a) *Every non-repelling periodic point of g_* is a fixed point;*

(b)

$$g_* \in SF_{k,l}, \qquad n_{\mathrm{PB}}(g_*) = n_{\mathrm{rat}}(g_*),$$

and

$$C(g_*) = (n_{\mathrm{AB}}(f), n_{\mathrm{PB}}(f), n_{\mathrm{SD}}(f) - m, m, n_{\mathrm{ER}}(f)).$$

Proof. Let $z_1, \cdots, z_m$ be m Siegel points containing a point with a preimage other than itself. By the implicit function theorem, there exist a neighborhood $W' \subset M_f$ of f and holomorphic functions $x_j(g)(j = 1, \cdots, m)$ on W' satisfying

$$g(x_j(g)) = x_j(g), \qquad x_j(f) = z_j.$$

As in the proof of Lemma 4, we can construct an analytic set Z' in W' such that every $g \in Z'$ satisfies $g \in SF_{k,l}$ and

$$n_{\mathrm{SD}}(g) \geq n_{\mathrm{SD}}(f) - m, \qquad n_{\mathrm{AB}}(g) \geq n_{\mathrm{AB}}(f),$$
$$n_{\mathrm{rat}}(g) \geq n_{\mathrm{rat}}(f), \qquad n_{\mathrm{ER}}(g) \geq n_{\mathrm{ER}}(f).$$

Now we define $R \subset Z'$ by

$$R := \{g \in Z' \mid x_j(g)(j = 1, \cdots, m) \text{ are rationally indifferent fixed points of } g\}.$$

We construct g_* as a limit of some sequence of functions in R.

First of all, we convert f into some $g_1 \in R$ by applying Lemma 3 or Lemma 4 repeatedly. Set

$$A_{a,b}(x) := \{z \in \mathbb{C} \mid a < |z - x| < b\},$$

where $0 < a < b$ and $x \in \mathbb{C}$. In addition, we define $0 \leq \theta_j(g) < 1$ for $g \in W'$ and every $j (1 \leq j \leq m)$ by

$$g'(x_j(g)) = |g'(x_j(g))| e^{2\pi i \theta_j(g)}.$$

Recall that rationally indifferent periodic points are in the Julia set, which is the closure of the set of all repelling periodic points. Hence g_1 has periodic points in any punctured neighborhood of each of $x_j(g_1)(j = 1, \cdots, m)$. Thus there exist $r_1 > 0$ and $0 < r_2 < r_1/2$ such that g_1 has some p_j-periodic point in each of annuli $A_{r_2,r_1}(x_j(g_1))(j = 1, \cdots, m)$. By applying Rouché's theorem to $g_1^{p_j}(z) - z$ and $g^{p_j}(z) - g_1^{p_j}(z)$, there exists a closed neighborhood $U_1 \subset W'$ of g_1 such that every $g \in U_1$ has some (p_j-)periodic point in each of annuli $A_{r_2,r_1}(x_j(g))(j = 1, \cdots, m)$. In addition, if we take U_1 small enough, it follows from the continuity of $\theta_j(g)$ at g_1 that every $g \in U_1$ satisfies

$$|\theta_j(g_1) - \theta_j(g)| = \left| \frac{p_{1,j}}{q_{1,j}} - \theta_j(g) \right| < \frac{1}{2(q_{1,j})^2} \qquad (j = 1, \cdots, m),$$

where $p_{1,j}/q_{1,j} = \theta_j(g_1)$ and $p_{1,j}, q_{1,j} \in \mathbb{N}$ are mutually prime. From Remark 4, we can convert g_1 into some $g \in Z'$ so that the multiplier of each of $x_j(g)(j = 1, \cdots, m)$ becomes any value in some neighborhood of $g_1'(x_j(g_1))$. Thus we can get some $g_2 \in (U_1 \setminus \{g_1\}) \cap R$ such that

$$g_1'(x_j(g_1)) \neq g_2'(x_j(g_2)) \qquad (j = 1, \cdots, m).$$

From the construction similar to that of U_1, there exist a closed neighborhood $U_2 \subset U_1$ of g_2 and $0 < r_3 < r_2/2$ such that:

(1) Every $g \in U_2$ has some periodic point in each of annuli $A_{r_3,r_2}(x_j(g))$ $(j = 1, \cdots, m)$;

(2) Every $g \in U_2$ satisfies

$$|\theta_j(g_2) - \theta_j(g)| = \left|\frac{p_{2,j}}{q_{2,j}} - \theta_j(g)\right| < \frac{1}{2(q_{2,j})^2} \qquad (j = 1, \cdots, m),$$

where $p_{2,j}/q_{2,j} = \theta_j(g_2)$ and $p_{2,j}, q_{2,j} \in \mathbb{N}$ are mutually prime.

By repeating this procedure, we get functions $g_n \in R$, closed neighborhoods $U_n \subset W'$ of g_n, and $r_n > 0$, for $n = 1, 2, \cdots$, such that:

(i) $g'_{n_1}(x_j(g_{n_1})) \neq g'_{n_2}(x_j(g_{n_2}))(j = 1, \cdots, m)$ if $n_1 \neq n_2$;
(ii) $U_n \supset U_{n+1}$;
(iii) $r_{n+1} < r_1/2^n$;
(iv) Every $g \in U_n$ has some periodic point in each of annuli $A_{r_{n+1}, r_n}(x_j(g))$ $(j = 1, \cdots, m)$;
(v) Every $g \in U_n$ satisfies

$$|\theta_j(g_n) - \theta_j(g)| = \left|\frac{p_{n,j}}{q_{n,j}} - \theta_j(g)\right| < \frac{1}{2(q_{n,j})^2} \qquad (j = 1, \cdots, m),$$

where $p_{n,j}/q_{n,j} = \theta_j(g_n)$ and $p_{n,j}, q_{n,j} \in \mathbb{N}$ are mutually prime.

Set

$$K := \{g \in Z' \mid |g'(x_j(g))| = 1 (j = 1, \cdots, m)\}.$$

From (ii) and a standard argument, we get some $g_\infty \in (\bigcap_{n=1}^{\infty} U_n) \cap K$. It follows from this, (iii), and (iv) that g_∞ has some periodic points in any punctured neighborhood of each of $x_j(g_\infty)(j = 1, \cdots, m)$. Next, we show that $x_j(g_\infty)(j = 1, \cdots, m)$ are irrationally indifferent fixed points of g_∞. It follows from $g_\infty \in (\bigcap_{n=1}^{\infty} U_n) \cap K$ and (v) that

$$|\theta_j(g_n) - \theta_j(g_\infty)| = \left|\frac{p_{n,j}}{q_{n,j}} - \theta_j(g_\infty)\right| < \frac{1}{2(q_{n,j})^2}$$

for every $n \geq 1$ and every $j (1 \leq j \leq m)$. Then an easy calculation shows that rational numbers $\theta_j(g_n)(n = 1, 2, \cdots)$ are best approximations (of the second kind) of $\theta_j(g_\infty)$ in the sense of Khinchin (see [K, Section 6] and [Sho, p. 130]). In addition, it follows from (i) that $\theta_j(g_n)(n = 1, 2, \cdots)$ are different from each other. Thus $\theta_j(g_\infty)$ is an irrational number, since any rational number has at most a finite number of such approximations (see [K] and [Ol] for basic facts of continued fractions). Therefore, $x_j(g_\infty)(j = 1, \cdots, m)$ are irrationally indifferent fixed points. By Proposition D, they are m Cremer fixed points. It follows from the construction and Lemma 2 that g_∞ satisfies the properties of g_*. ☐

Remark 5. Let f_α be as in Lemma 1. As the referee pointed out, any function $f \in M_{f_\alpha}$ with a Siegel fixed point always satisfies the assumption (3) of Lemma 4 and Lemma 5 that the point has a preimage other than itself. (Hence in Lemma 3, if $f \in M_{f_\alpha}$, then the property (c) of g_j is obvious.) This is due to the following reason: The function f_α does not have any exceptional point with only one preimage. Indeed, if f_α has such a point b, then it needs to have the form $(z - \beta)e^{h(z)} + b$, where β is the preimage and $h(z)$ is an entire function. Since f_α is of finite order, $h(z)$ must be a polynomial. This is a contradiction. In addition, any function $f \in M_{f_\alpha}$ also satisfies the property, since covering properties are preserved in M_{f_α}.

Here, we are ready to prove the Main Theorem.

Proof of the Main Theorem. By Lemma 1, there exists a $T_0 := f_\alpha \in S_q \cap SF_{0,q}$ with q Siegel points of period 1. Hence we have already shown the Main Theorem when $(m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}}) = (0, 0, q, 0)$. Thus we show the Main Theorem when $(m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}}) \neq (0, 0, q, 0)$. We construct T whose non-repelling cycles have the same period 1.

First of all, suppose that $q = 1$ and $(m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}}) \neq (0, 0, 0, 0)$. From Remark 5, $T_0 = f_\alpha$ satisfies the assumptions of Lemma 4 and Lemma 5. We get T by applying Lemma 4 or Lemma 5 to T_0. More precisely, we convert one Siegel cycle of T_0 into one attracting (or rationally indifferent, Cremer) cycle of T.

Next, suppose that $q \geq 2$ and $(m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}}) \neq (0, 0, 0, 0)$.

(i) When $m_{\mathrm{Cr}} = 0$, we convert T_0 into T by decreasing the number of Siegel cycles and increasing that of attracting cycles (or rationally indifferent cycles, eventually repelling singular values). To be more precise, we apply Lemma 3 repeatedly until we get a function with only one SD-cycle. Since Lemma 3 ensures that the function satisfies the assumption of Lemma 4, we can apply Lemma 4 to it for the last step.

(ii) When $m_{\mathrm{Cr}} \neq 0$, we construct T by the following steps:

Step 1. As in the case (i), we construct a $\tilde{T} \in SF_{0,q}$ with $n_{\mathrm{PB}}(\tilde{T}) = n_{\mathrm{rat}}(\tilde{T})$ and

$$C(\tilde{T}) = (m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}} + m_{\mathrm{Cr}}, 0, q - \Sigma),$$

where $\Sigma = m_{\mathrm{AB}} + m_{\mathrm{PB}} + m_{\mathrm{SD}} + m_{\mathrm{Cr}}$. Note that $\tilde{T}$ has a Siegel point with a preimage other than itself.

Step 2. By the construction, $\tilde{T}$ satisfies the assumption of Lemma 5. We get T by applying Lemma 5 to $\tilde{T}$ so that m_{Cr} Siegel cycles of $\tilde{T}$ become m_{Cr} Cremer cycles of T.

Finally, suppose that $(m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}}) = (0, 0, 0, 0)$. If $q \geq 2$, set

$$h_\varepsilon(z) := \varepsilon z e^{z^{q-1}} \in S_q \cap SF_{q-1,q-1} \qquad \text{for } \varepsilon \in \mathbb{C} \setminus \{0\}.$$

An easy calculation shows that h_ε has an asymptotic value 0 and $q-1$ critical values $z_j(\varepsilon) (j = 1, \cdots, q-1)$ expressed as

$$z_j(\varepsilon) = \varepsilon \sqrt[q-1]{\frac{1}{(q-1)e}}\, e^{i\theta_j},$$

where

$$\theta_j = \frac{(2j-1)\pi}{q-1}.$$

Consider the equation on ε

$$h_\varepsilon(z_1(\varepsilon)) = z_1(\varepsilon).$$

This yields

$$F(\varepsilon) := h_\varepsilon(z_1(\varepsilon))/z_1(\varepsilon) = \varepsilon e^{z_1(\varepsilon)^{q-1}} = 1.$$

Obviously, $F(\varepsilon)$ is a holomorphic function of ε. It is easy to see that some ε_0 satisfies $F(\varepsilon_0) = 1$. Thus $z_1(\varepsilon_0)$ is a fixed point of h_{ε_0}. In addition, the other critical values $z_j(\varepsilon_0) (j = 2, \cdots, q-1)$ are fixed points because

$$h_{\varepsilon_0}(z_j(\varepsilon_0)) = e^{i(\theta_j - \theta_1)} h_{\varepsilon_0}(z_1(\varepsilon_0)) = e^{i(\theta_j - \theta_1)} z_1(\varepsilon_0) = z_j(\varepsilon_0).$$

Thus all singular values of h_{ε_0} are fixed. If a transcendental entire function f has a non-repelling cycle, f has a singular value a such that $\{f^n(a)\}_{n\in\mathbb{N}}$ is an infinite set (see [BrF, p. 117, Theorem 3.39]). It follows from this fact that we can take h_{ε_0} as T. Also, if $q = 1$,

$$w(z) := 2\pi i e^z \in S_1 \cap SF_{0,1}$$

can be taken as T. Indeed, w has an asymptotic value 0 with

$$w(0) = 2\pi i, \quad w(2\pi i) = 2\pi i. \qquad \square$$

Remark 6. Here we note the proof of the Main Theorem by constructing Cremer cycles one by one as in [Shi]. We use Proposition G:

Proposition G ([Cr2, p. 299]). *Let g be a non-linear entire function such that:*

(1) *The origin is an irrationally indifferent fixed point with multiplier λ;*

(2) *It satisfies*

$$\max_{|z|\le r} |g(z)| \le F(r)$$

for all large enough $r > 0$ and a positive function F defined for all positive real numbers.

If λ satisfies the following condition for every large enough $r > 0$:

$$\lceil\mathrm{Cremer}(F)\rfloor \qquad \liminf_{n\to\infty} \sqrt[\log F^n(r)]{\lambda^n - 1} = 0,$$

then the origin is a Cremer fixed point. Moreover, the set $\Lambda(F)$ of all λ satisfying $\lceil\mathrm{Cremer}(F)\rfloor$ is uncountable and dense in the unit circle $\{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

By definition, $\Lambda(F)$ depends only on F. For entire functions of finite order, we may take $E(r) := e^{e^r}$ as $F(r)$. Thus for such functions (containing structurally finite transcendental entire functions), Proposition G implies that irrationally indifferent fixed points with multipliers in $\Lambda(E)$ are Cremer fixed points. It follows from this and the proofs of Lemma 3 and Lemma 4 that we can convert one Siegel cycle into one Cremer cycle with multiplier in $\Lambda(E)$. Moreover, our construction can keep the multiplier unchanged. Hence we can also construct Cremer cycles of T one by one as in [Shi]. Even when $p \ge 2$, we can also construct Cremer cycles of T with period p by a similar argument. In this case, we can construct Cremer cycles with multipliers in $\Lambda(E^p)$.

4. Concluding remarks

The set $S \setminus SF$ is not empty. For example, $f(z) = \sin z \in S \setminus SF$. We constructed a $T \in S_q \cap SF$ to prove best possibility of the Fatou-Shishikura inequality for $f \in S_q$. Thus there is the following question:

Question. Suppose that non-negative integers $m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}$, and m_{Cr}, and a positive integer q satisfy

$$m_{\mathrm{AB}} + m_{\mathrm{PB}} + m_{\mathrm{SD}} + m_{\mathrm{Cr}} \le q.$$

Is there a $T \in S_q \setminus SF$ such that

$$(n_{\mathrm{AB}}(T), n_{\mathrm{PB}}(T), n_{\mathrm{SD}}(T), n_{\mathrm{Cr}}(T)) = (m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}})?$$

Acknowledgments

The authors would like to thank Walter Bergweiler for giving the information about [Cr2], providing the paper, and his valuable discussions. The authors would also like to thank Mitsuhiro Shishikura for his valuable discussions. The authors would like to thank the referee for his or her helpful comments in improving this paper.

References

[A] Lars V. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand Mathematical Studies, No. 10, D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966. Manuscript prepared with the assistance of Clifford J. Earle, Jr. MR0200442

[Ba] I. N. Baker, *Wandering domains in the iteration of entire functions*, Proc. London Math. Soc. (3) **49** (1984), no. 3, 563–576, DOI 10.1112/plms/s3-49.3.563. MR759304

[Bea] Alan F. Beardon, *Iteration of rational functions*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag, New York, 1991. Complex analytic dynamical systems, DOI 10.1007/978-1-4612-4422-6. MR1128089

[Ber] Walter Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), no. 2, 151–188, DOI 10.1090/S0273-0979-1993-00432-4. MR1216719

[BrF] Bodil Branner and Núria Fagella, *Quasiconformal surgery in holomorphic dynamics*, Cambridge Studies in Advanced Mathematics, vol. 141, Cambridge University Press, Cambridge, 2014. With contributions by Xavier Buff, Shaun Bullett, Adam L. Epstein, Peter Haïssinsky, Christian Henriksen, Carsten L. Petersen, Kevin M. Pilgrim, Tan Lei and Michael Yampolsky. MR3445628

[Cr1] Hubert Cremer, *Zum Zentrumproblem* (German), Math. Ann. **98** (1928), no. 1, 151–163, DOI 10.1007/BF01451586. MR1512397

[Cr2] Hubert Cremer, *Über die Schrödersche Funktionalgleichung und das Schwarzsche Eckenabbildungsproblem* (German), Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **84** (1932), 291–324.

[EL] A. È. Erëmenko and M. Yu. Lyubich, *Dynamical properties of some classes of entire functions* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 989–1020. MR1196102

[K] A. Ya. Khinchin, *Continued fractions*, Translated from the third (1961) Russian edition, Dover Publications, Inc., Mineola, NY, 1997. With a preface by B. V. Gnedenko; Reprint of the 1964 translation. MR1451873

[Na] Raghavan Narasimhan, *Introduction to the theory of analytic spaces*, Lecture Notes in Mathematics, No. 25, Springer-Verlag, Berlin-New York, 1966. MR0217337

[Ne] Rolf Nevanlinna, *Analytic functions*, Die Grundlehren der mathematischen Wissenschaften, Band 162, Springer-Verlag, New York-Berlin, 1970. Translated from the second German edition by Phillip Emig. MR0279280

[Ok] Yûsuke Okuyama, *Linearization problem on structurally finite entire functions*, Kodai Math. J. **28** (2005), no. 2, 347–358, DOI 10.2996/kmj/1123767015. MR2153922

[Ol] C. D. Olds, *Continued fractions*, Random House, New York, 1963. MR0146146

[Shi] Mitsuhiro Shishikura, *On the quasiconformal surgery of rational functions*, Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 1, 1–29. MR892140

[Sho] Ian Short, *Ford circles, continued fractions, and rational approximation*, Amer. Math. Monthly **118** (2011), no. 2, 130–135, DOI 10.4169/amer.math.monthly.118.02.130. MR2795580

[Si] Carl Ludwig Siegel, *Iteration of analytic functions*, Ann. of Math. (2) **43** (1942), 607–612, DOI 10.2307/1968952. MR7044

[Su] Dennis Sullivan, *Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains*, Ann. of Math. (2) **122** (1985), no. 3, 401–418, DOI 10.2307/1971308. MR819553

[T1] Masahiko Taniguchi, *Explicit representation of structurally finite entire functions*, Proc. Japan Acad. Ser. A Math. Sci. **77** (2001), no. 4, 68–70. MR1829377

[T2] Masahiko Taniguchi, *Synthetic deformation space of an entire function*, Value distribution theory and complex dynamics (Hong Kong, 2000), Contemp. Math., vol. 303, Amer. Math. Soc., Providence, RI, 2002, pp. 107–136, DOI 10.1090/conm/303/05238. MR1943529

Department of Mathematical Sciences, Graduate School of Human and Environmental Studies, Kyoto University, Kyoto 606-8501, Japan
Email address: `kisaka@math.h.kyoto-u.ac.jp`

Department of Mathematical Sciences, Graduate School of Human and Environmental Studies, Kyoto University, Kyoto 606-8501, Japan
Email address: `naba.hiroto.78r@st.kyoto-u.ac.jp`