

DEGENERATION OF 3-DIMENSIONAL HYPERBOLIC CONE STRUCTURES WITH DECREASING CONE ANGLES

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ABSTRACT. For 3-dimensional hyperbolic cone structures with cone angles θ , local rigidity is known for $0 \leq \theta \leq 2\pi$, but global rigidity is known only for $0 \leq \theta \leq \pi$. The proof of the global rigidity by Kojima is based on the fact that hyperbolic cone structures with cone angles at most π do not degenerate in deformations decreasing cone angles to zero.

In this paper, we give an example of a degeneration of hyperbolic cone structures with decreasing cone angles less than 2π . These cone structures are constructed on a certain alternating link in the thickened torus by gluing four copies of a certain polyhedron. For this construction, we explicitly describe the isometry types on such a hyperbolic polyhedron.

1. INTRODUCTION

A 3-dimensional hyperbolic cone-manifold is a hyperbolic 3-manifold with cone-type singularities. In this paper, we assume that a cone-manifold has finite volume, and cone singularities consist of disjoint closed geodesics.

Let X be a 3-manifold, and let Σ be a link in X . Let $\Sigma_1, \dots, \Sigma_n$ denote the components of Σ . Suppose that $X \setminus \Sigma$ admits an incomplete hyperbolic structure, and the completed metric has the form

$$dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r dz^2$$

in cylindrical coordinates around each component Σ_i for $1 \leq i \leq n$, where r is the distance from the singular locus, z is the distance along the singular locus, and θ is the angle measured modulo $\theta_i > 0$. Then the metric on (X, Σ) is called a *hyperbolic cone structure*. More precisely, an equivalence class of such cone metrics by isometries isotopic to the identity is a cone structure. The angle θ_i is called the *cone angle* at the cone locus Σ_i . If $\theta_i = 2\pi$, the cone locus Σ_i can be regarded as non-singular. Furthermore, a cusp of a hyperbolic 3-manifold can be regarded as a cone locus with cone angle zero. This is justified by the fact that hyperbolic cone structures converge to a cusped hyperbolic structure in the pointed Gromov–Hausdorff topology if the cone angles converge to zero.

From now on, fix a pair (X, Σ) . Let \mathcal{C} denote the set of cone structures on (X, Σ) such that the cone angles are at most 2π . Suppose that there is a finite volume cone structure $g \in \mathcal{C}$. Then any $g \in \mathcal{C}$ has finite volume. The set \mathcal{C}

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admits the pointed Gromov–Hausdorff topology, which is induced by the geometric convergence of metric spaces. The continuous map $\Theta: \mathcal{C} \rightarrow [0, 2\pi]^n$ is defined by $\Theta(g) = (\theta_1, \dots, \theta_n)$, where θ_i is the cone angle at Σ_i in the cone-manifold $(X, \Sigma; g)$. Let g_0 be an element in \mathcal{C} such that $\Theta(g_0) = (0, \dots, 0)$. The cusped hyperbolic structure g_0 is unique for (X, Σ) by the Mostow–Prasad rigidity [11, 12].

Local and global rigidity for hyperbolic cone structures are known as follows.

Theorem 1.1 (The local rigidity by Hodgson and Kerckhoff [6]). *The space \mathcal{C} is Hausdorff, and the map $\Theta: \mathcal{C} \rightarrow [0, 2\pi]^n$ is a local homeomorphism. In other words, the space \mathcal{C} is locally parametrized by the cone angles.*

The local rigidity does not hold in general if cone angles exceed 2π . Izmistiev [7] constructed infinitesimally flexible hyperbolic cone-manifolds with cone angles more than 2π by gluing polyhedra.

Theorem 1.2 (The global rigidity by Kojima [8]). *Let*

$$\mathcal{C}_{[0, \pi]} = \{g \in \mathcal{C} \mid \Theta(g) \in [0, \pi]^n\}.$$

Then the map $\Theta|_{\mathcal{C}_{[0, \pi]}}: \mathcal{C}_{[0, \pi]} \rightarrow [0, \pi]^n$ is injective. In other words, the cone structure is determined by the cone angles if the cone angles do not exceed π .

The global rigidity is not known when cone angles exceed π . Theorem 1.2 follows from the Mostow–Prasad rigidity for g_0 and Theorems 1.1 and 1.3.

Theorem 1.3 (Kojima [8]). *Let $g \in \mathcal{C}$. Suppose that $\Theta(g) = (\theta_1, \dots, \theta_n) \in [0, \pi]^n$. Then there is $A \subset \mathcal{C}$ such that $g \in A$ and $\Theta|_A: A \rightarrow [0, \theta_1] \times \dots \times [0, \theta_n]$ is a homeomorphism. In other words, we can obtain a continuous family of cone structures from g to g_0 by arbitrarily decreasing cone angles.*

Similar results are known for 3-dimensional hyperbolic cone-manifolds with vertices. The local rigidity for cone angles less than 2π was proved by Mazzeo and Montcouquiol [10], and independently Weiss [15]. The global rigidity for cone angles at most π was proved by Weiss [14].

Theorem 1.4 is the main result of this paper. It implies that Theorem 1.3 cannot be generalized for cone angles less than 2π . A *continuous degenerating family* of cone structures on (X, Σ) is a continuous map $\gamma: [0, 1) \rightarrow \mathcal{C}$ such that $\lim_{x \rightarrow 1} \Theta(\gamma(x)) \in [0, 2\pi]^n$ but $\gamma(x)$ does not converge in \mathcal{C} as $x \rightarrow 1$.

Theorem 1.4. *Let $L = L_1 \sqcup \dots \sqcup L_n \subset T^2 \times I$ be a link in the thickened torus as indicated in Figure 1, where I is an open interval. Then there is a continuous degenerating family of cone structures on $(T^2 \times I, L)$ with decreasing cone angles. In this degeneration, two of the cone loci L intersect transversally. Two simultaneous intersections may occur.*

Links in $T^2 \times I$ have been studied in several situations. One of them concerns the hyperbolic structures on the complements. By projecting a link in $T^2 \times I$ to T^2 , we obtain a diagram of the link in T^2 . The diagram gives the notion of alternating links in $T^2 \times I$. In [1, 4], the hyperbolic structure on the complement of an alternating link in $T^2 \times I$ is constructed by gluing ideal bipyramids given from the diagram. The above L is one of the simplest examples of alternating links, and it was described in detail by Champanerkar, Kofman, and Purcell [3].

We will construct cone structures on $(T^2 \times I, L)$ by polyhedral decomposition, which is a generalization of the above construction for cusped hyperbolic structures.

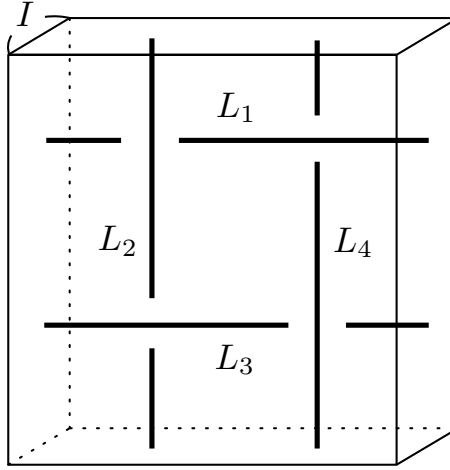


FIGURE 1. A link L in $T^2 \times I$ (A fundamental domain of $T^2 \times I$ is drawn.)

If a hyperbolic cone structure on $(T^2 \times I, L)$ has sufficient symmetry, the hyperbolic cone-manifold can be decomposed into four copies of a certain polyhedron, called a tetragonal trapezohedron (a.k.a. an antibipyramid). A tetragonal trapezohedron is the dual of a square antiprism. Thus we will be reduced to considering isometric types of a hyperbolic tetragonal trapezohedron.

We need to know whether a tetragonal trapezohedron with the assigned dihedral angles can be realized in the hyperbolic space. The most powerful result for our problem is Andreev's theorem [2], which gives the condition by linear inequalities of dihedral angles for a finite volume hyperbolic polyhedron with non-obtuse dihedral angles (see [13] for details of compact cases). For obtuse dihedral angles, however, Diaz [5] gave an example that no linear inequalities of dihedral angles hold. We will obtain another such example. Since there are no general tools for our problem, we need to explicitly describe isometric types of a tetragonal trapezohedron.

2. AN ALTERNATING LINK IN THE THICKENED TORUS

We consider a link $L = L_1 \sqcup \cdots \sqcup L_4 \subset T^2 \times I$ as indicated in Figure 1. Let \mathcal{C} denote the space of cone structures on $(T^2 \times I, L)$ as in Section 1, where the components of $T^2 \times \partial I$ keep to be two cusps. Note that any of the cone angles cannot be equal to 2π . For instance, if the cone angle at L_1 is equal to 2π , the cone loci L_2 and L_4 are parallel to a cusp, which is impossible. The map $\Theta: \mathcal{C} \rightarrow [0, 2\pi]^4$ assigns the cone angles at L_i . If each L_i is a cusp, we obtain $g_0 \in \mathcal{C}$, which is the unique element of \mathcal{C} satisfying that $\Theta(g_0) = (0, \dots, 0)$. Let $\mathcal{C}_0 \subset \mathcal{C}$ denote the component containing g_0 .

We consider symmetry of $(T^2 \times I, L)$. There is an automorphism γ_1 of $(T^2 \times I, L)$ that fixes each point of L_1 and L_3 , and fixes each of L_2 and L_4 as sets, but reverses the orientations of L_2 and L_4 . The fixed point set of γ_1 is two open annuli containing L_1 and L_3 , whose ends are contained in neighborhood of $T^2 \times \partial I$. By changing the roles of (L_1, L_3) and (L_2, L_4) , we obtain an automorphism γ_2 of $(T^2 \times I, L)$. The automorphisms γ_1 and γ_2 are determined up to isotopy. Let Γ denote the group

generated by γ_1 and γ_2 . We choose γ_1 and γ_2 so that the group Γ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

We call a cone structure $g \in \mathcal{C}$ *symmetric* if the Γ -action on $(T^2 \times I, L; g)$ is isotopic to an isometric action. Let \mathcal{C}_{sym} denote the set of symmetric cone structures in \mathcal{C} .

Proposition 2.1. *The component \mathcal{C}_0 is contained in \mathcal{C}_{sym} .*

Proof. The Mostow–Prasad rigidity implies that $g_0 \in \mathcal{C}_{\text{sym}}$. The local rigidity implies that the set \mathcal{C}_{sym} is closed and open subset of \mathcal{C} . Hence \mathcal{C}_{sym} is the union of components of \mathcal{C} , one of which is \mathcal{C}_0 . \square

In fact, the space \mathcal{C}_{sym} is connected. Hence $\mathcal{C}_0 = \mathcal{C}_{\text{sym}}$. This will be shown in Corollary 3.12.

In this paper, we cannot treat non-symmetric cone structures. We do not even know whether there exists a non-symmetric cone structure. It is a really surprising example if it exists. Nonetheless, if the global rigidity for \mathcal{C} holds, then $\mathcal{C}_{\text{sym}} = \mathcal{C}$. This will be shown in Corollary 3.13.

For $g \in \mathcal{C}_{\text{sym}}$ and the isometric Γ -action, the quotient space $(T^2 \times I, L; g)/\Gamma$ is isometric to a (tetragonal) trapezohedron in the hyperbolic 3-space as indicated in Figure 2. The edge \hat{L}_i is the image of the cone locus L_i . The two ideal vertices disjoint from \hat{L}_i correspond to the components of $T^2 \times \partial I$. The faces are totally geodesic. If $\Theta(g) = (\theta_1, \dots, \theta_4)$, the dihedral angles are $\pi/2$ except the four angles $\alpha_i = \theta_i/2$ at \hat{L}_i . We remark that \hat{L}_i degenerates to an ideal vertex if $\alpha_i = 0$. We use the term “trapezohedron” also for such a degenerated polyhedron. Thus we decompose a symmetric cone-manifold $(T^2 \times I, L; g)$ into four trapezohedra. The four trapezohedra correspond to the complementary regions of the diagram of L in T^2 in Figure 1.

Conversely, we can obtain a symmetric hyperbolic cone structure in \mathcal{C}_{sym} by gluing four trapezohedra. The way of gluing is as follows. Let T be a hyperbolic trapezohedron with right dihedral angles except at \hat{L}_i . We color the faces of T black and white in a “checkerboard” fashion so that two faces with a common color are adjacent only along \hat{L}_i . Take four copies $T_{00}, T_{01}, T_{10}, T_{11}$ of T . For $j = 0, 1$, glue T_{j0} and T_{j1} along the black faces, and glue T_{0j} and T_{1j} along the white faces. Here corresponding vertices are matched. This construction can be called “double of double”.

The above argument for g_0 gives a decomposition of $T^2 \times I \setminus L$ into four regular ideal octahedra. This decomposition was given in [1, 3], and the “double of double” construction was described in detail in [9]. We remark that $T^2 \times I \setminus L$ is homeomorphic to the complement of the minimally twisted 6-chain link.

We summarize the above argument Proposition 2.2.

Proposition 2.2. *There is a natural one-to-one correspondence between \mathcal{C}_{sym} and the set of hyperbolic trapezohedra with right dihedral angles except at \hat{L}_i .*

3. DIHEDRAL ANGLES OF A TETRAGONAL TRAPEZOHEDRON

We consider hyperbolic trapezohedra (possibly \hat{L}_i degenerates to an ideal vertex) with right dihedral angles except at \hat{L}_i . Let \mathcal{A} denote the image of the map $\frac{1}{2}\Theta: \mathcal{C}_{\text{sym}} \rightarrow [0, \pi)^4$. In other words, $(\alpha_1, \dots, \alpha_4) \in \mathcal{A}$ if and only if there exists

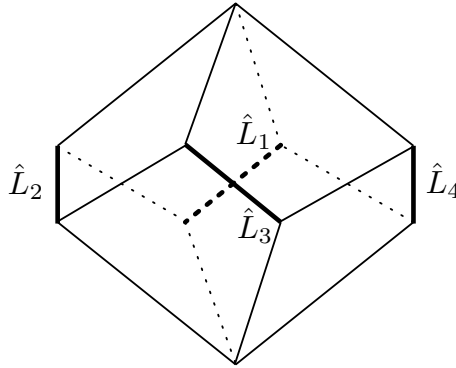


FIGURE 2. A tetragonal trapezohedron

a trapezohedron with dihedral angles α_i at \hat{L}_i and $\pi/2$ at the other edges in the hyperbolic space.

Theorem 3.1. *The isometry class of a hyperbolic trapezohedron is determined by the element of \mathcal{A} . In other words, $\frac{1}{2}\Theta: C_{\text{sym}} \rightarrow \mathcal{A}$ is injective.*

The local rigidity for \mathcal{C} implies that $\mathcal{A} \subset [0, \pi]^4$ is an open subset. Define $\text{cos}: [0, \pi]^4 \rightarrow (-1, 1]^4$ by $\text{cos}(\alpha_1, \dots, \alpha_4) = (\cos \alpha_1, \dots, \cos \alpha_4)$, which is a homeomorphism. We will often write $c_i = \cos \alpha_i$. Let $\mathcal{A}' = \text{cos}(\mathcal{A}) \subset (-1, 1]^4$. We explicitly describe \mathcal{A}' instead of \mathcal{A} . From now on, the indices $i = 1, \dots, 4$ are regarded modulo 4.

Theorem 3.2. *For $1 \leq i \leq 4$, let a function Φ_i be defined by*

$$\begin{aligned} \Phi_i(c_1, \dots, c_4) &= c_i c_{i+1} (c_i c_{i+1} + 1) c_{i+2} c_{i+3} - c_i c_{i+1} (c_i + c_{i+1}) (c_{i+2} + c_{i+3}) \\ &\quad + (c_i + c_{i+1})^2 - c_i c_{i+1} - 1. \end{aligned}$$

Let $\partial\mathcal{A}'$ denote the frontier of \mathcal{A}' in $(-1, 1]^4$. Then

$$\mathcal{A}' = \{(c_1, \dots, c_4) \in (-1, 1]^4 \mid \text{for any } i, \Phi_i(c_1, \dots, c_4) < 0 \text{ or } c_i + c_{i+1} > 0\},$$

and we can write $\partial\mathcal{A}' = \bigcup_i \partial_i \mathcal{A}'$, where

$$\begin{aligned} \partial_i \mathcal{A}' &= \{(c_1, \dots, c_4) \in (-1, 1]^4 \mid \Phi_i(c_1, \dots, c_4) = 0, \quad c_i + c_{i+1} \leq 0, \\ &\quad c_i \leq c_{i+2}, \quad c_{i+1} \leq c_{i+3}\}. \end{aligned}$$

As $(c_1, \dots, c_4) \in \mathcal{A}'$ approaches to $\partial_i \mathcal{A}'$, the edge between \hat{L}_i and \hat{L}_{i+1} degenerates. In particular, $\mathcal{A}' \neq (-1, 1]^4$.

Remark 3.3. Clearly $(1, 1, 1, 1) \in \mathcal{A}'$. Since $\Phi_i(1, 1, 1, 1) = 0$, we need the condition $c_i + c_{i+1} > 0$ in the description of \mathcal{A}' .

We consider a hyperbolic trapezohedron T whose dihedral angles are α_i at \hat{L}_i and $\pi/2$ at the other edges. We use the upper half-space model of hyperbolic 3-space. Regard $\partial\mathbb{H}^3 = \mathbb{R}^2 \cup \{\infty\}$. The trapezohedron T has two ideal vertices disjoint from \hat{L}_i . We set them at ∞ and $O = (0, 0)$. We project T to $\mathbb{R}^2 \subset \partial\mathbb{H}^3$ as indicated in Figure 3. The endpoints \tilde{P}_i and \tilde{Q}_i of the edge \hat{L}_i are projected respectively to P_i and Q_i . The images of the faces of T adjacent to O are four quadrilaterals $OQ_{i-1}P_iQ_i$. Their union is a rectangle $P_1P_2P_3P_4$. If $\alpha_i = 0$, then $\hat{L}_i = P_i = Q_i$.

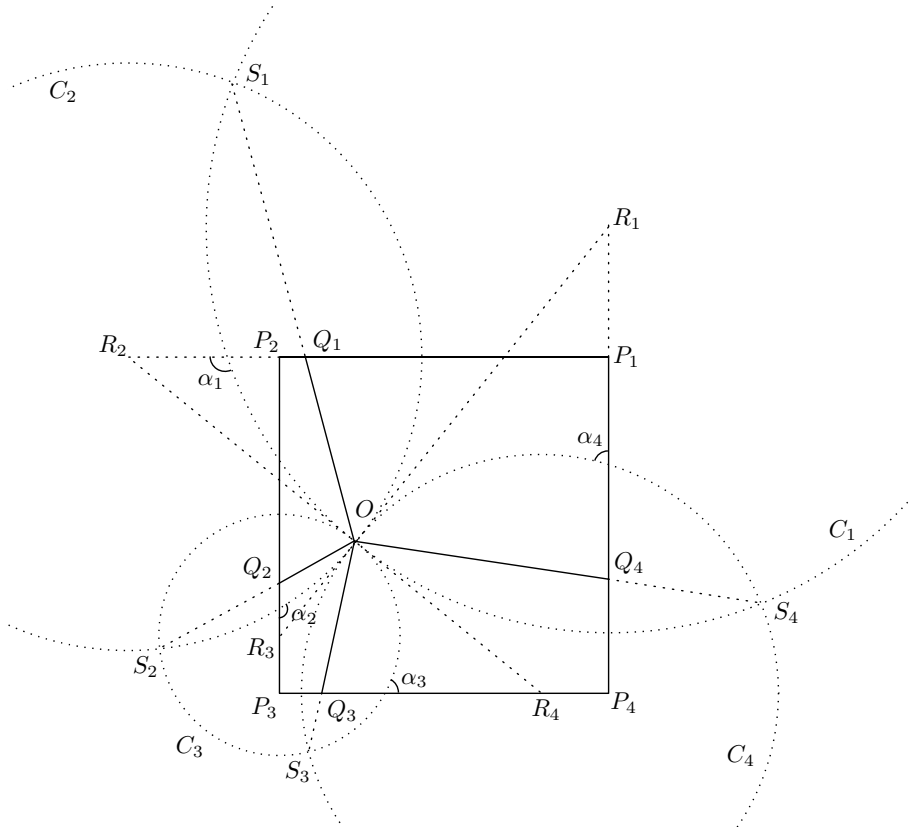


FIGURE 3. Projection of a trapezohedron

Let F_i denote the face of T projected to $OQ_{i-1}P_iQ_i$. We extend F_i to a totally geodesic plane \tilde{C}_i , and let C_i denote the boundary of \tilde{C}_i . By considering the dihedral angles at F_i , we see that the circle C_i is orthogonal to $P_{i-1}P_i$, and the angle between C_i and P_iP_{i+1} is α_i as indicated in Figure 3. Since the dihedral angles at the edges of T around O are $\pi/2$, the circles C_i and C_{i+1} intersect orthogonally. Let R_i denote the center of the circle C_i . Then R_i is contained in the line $P_{i-1}P_i$. The segments OR_i and OR_{i+1} are orthogonal. Let $S_i = C_i \cap C_{i+1} \setminus O$. Then Q_i is the intersection of the segments OS_i and P_iP_{i+1} .

Conversely, take points P_i and R_i in \mathbb{R}^2 such that $P_1P_2P_3P_4$ is a rectangle containing O , and R_i is contained in the line $P_{i-1}P_i$. Let C_i and S_i be as above. Then the condition that the projection of a trapezohedron is obtained is as follows:

- The segments OS_i and P_iP_{i+1} intersect, and
- their intersection Q_i is distinct from P_{i+1} .

The vertices \tilde{P}_i and \tilde{Q}_i are the intersection of the plane \tilde{C}_i and respectively the lines ∞P_i and ∞Q_i . If $Q_i = P_{i+1}$, the edge between \hat{L}_i and \hat{L}_{i+1} degenerates, which corresponds to intersection of L_i and L_{i+1} in $T^2 \times I$.

We may assume that

$$P_1 = (p_1, p_2), P_2 = (-p_3, p_2), P_3 = (-p_3, -p_4), P_4 = (p_1, -p_4),$$

where $p_i > 0$. Let t denote the slope of the line OR_1 . Then

$$R_1 = (p_1, tp_1), R_2 = (-tp_2, p_2), R_3 = (-p_3, -tp_3), R_4 = (tp_4, -p_4).$$

Let $q_i = \frac{p_{i+1}}{p_i}$. Since a positive constant multiple on \mathbb{R}^2 extends an isometry of \mathbb{H}^3 , the isometry type of T is determined by q_i and t .

Lemma 3.4.

$$\cos \alpha_i = \frac{q_i - t}{\sqrt{1 + t^2}}.$$

Proof. The radius of the circle C_i is equal to $p_i\sqrt{1+t^2}$. The signed length P_iR_i is equal to $p_{i+1} - tp_i$ (it is positive if R_i is contained inside of the segment $P_{i-1}P_i$). Then $\cos \alpha_i$ is given by their ratio. \square

Lemma 3.5. *The condition that the segments OS_i and P_iP_{i+1} intersect and $Q_i \neq P_{i+1}$ is equivalent to the following inequalities:*

$$t \geq \frac{1}{2}(q_i - q_i^{-1}), \quad (1 - q_iq_{i+1})t < q_i + q_{i+1}.$$

Proof. We prove it for $i = 1$. By calculating the coordinates of S_1 from the ones of O, R_1 , and R_2 , we have

$$S_1 = \left(\frac{2p_1p_2(p_2 - tp_1)}{p_1^2 + p_2^2}, \frac{2p_1p_2(tp_2 + p_1)}{p_1^2 + p_2^2} \right).$$

The slope of OS_1 is equal to $\frac{tp_2 + p_1}{p_2 - tp_1}$. Since $Q_i = OS_1 \cap P_1P_2$, we have

$$Q_1 = \left(\frac{p_2 - tp_1}{tp_2 + p_1} p_2, p_2 \right).$$

Therefore the condition holds if and only if

$$\frac{2p_1p_2(tp_2 + p_1)}{p_1^2 + p_2^2} \geq p_2, \quad -p_3 < \frac{p_2 - tp_1}{tp_2 + p_1} p_2 \leq p_1.$$

The first inequality is equivalent to

$$t \geq \frac{1}{2}(q_1 - q_1^{-1}).$$

The first inequality implies that $tp_2 + p_1 > 0$. Under this condition, the right of second inequality is also equivalent to

$$t \geq \frac{1}{2}(q_1 - q_1^{-1}),$$

and the left of second inequality is equivalent to

$$(1 - q_1q_2)t < q_1 + q_2.$$

\square

Remark 3.6. Suppose that the above condition holds for all i . Since $\prod_{i=1}^4 q_i = 1$, there is i such that $q_i \geq 1$. Hence $t \geq 0$. Thus the second inequality is vacuous if $p_i \leq p_{i+2}$.

Lemma 3.7. *Let*

$$\mathcal{B} = \{(q_1, \dots, q_4, t) \in \mathbb{R}_{>0}^4 \times \mathbb{R}_{\geq 0} \mid \prod_{i=1}^4 q_i = 1, t \geq \frac{1}{2}(q_i - q_i^{-1})\}.$$

Define $f: \mathcal{B} \rightarrow \mathbb{R}^4$ by

$$f(q_1, \dots, q_4, t) = \left(\frac{q_1 - t}{\sqrt{1 + t^2}}, \dots, \frac{q_4 - t}{\sqrt{1 + t^2}} \right).$$

Then f is injective, and the image of f is $(-1, 1]^4$.

Proof. For fixed q_i , the function $f_i(t) = \frac{q_i - t}{\sqrt{1 + t^2}}$ is monotonically decreasing. Since $f_i(\frac{1}{2}(q_i - q_i^{-1})) = 1$ and $\lim_{t \rightarrow \infty} f_i(t) = -1$, the image of f is contained in $(-1, 1]^4$. Take any $(c_1, \dots, c_4) \in (-1, 1]^4$. It is sufficient to show that there is a unique element $(q_1, \dots, q_4, t) \in \mathcal{B}$ such that $f(q_1, \dots, q_4, t) = (c_1, \dots, c_4)$.

Let $g_i(t) = t + c_i\sqrt{1 + t^2}$. Since $-1 < c_i \leq 1$, the function $g_i(t)$ is monotonically increasing. Note that $\lim_{t \rightarrow \infty} g_i(t) = \infty$. Let $g(t) = \prod_{i=1}^4 g_i(t)$. If some c_i is negative, then we take $t' > 0$ to be the maximum of t satisfying $g_i(t) = 0$ for some i . Then $g(t') = 0$ and $g(t)$ is monotonically increasing for $t \geq t'$. If no c_i is negative, then $g(0) = \prod_{i=1}^4 c_i \leq 1$ and $g(t)$ is monotonically increasing for $t \geq 0$. In both cases, there is a unique $t_0 \geq 0$ such that $g(t_0) = 1$ and $g_i(t_0) > 0$. By setting $q_i = g_i(t_0)$, we have $\prod_{i=1}^4 q_i = 1$ and $f(q_1, \dots, q_4, t_0) = (c_1, \dots, c_4)$. Since $f_i(t_0) = c_i \leq 1 = f_i(\frac{1}{2}(q_i - q_i^{-1}))$ and f_i is monotonically decreasing, we have $t_0 \geq \frac{1}{2}(q_i - q_i^{-1})$. Thus we have a unique solution. \square

Let $\mathcal{B}_0 = \{(q_1, \dots, q_4, t) \in \mathcal{B} \mid (1 - q_i q_{i+1})t < q_i + q_{i+1}\}$. Lemma 3.5 implies that an element of \mathcal{B}_0 corresponds to a hyperbolic trapezohedron with right dihedral angles except at \hat{L}_i . Therefore $\mathcal{A}' = f(\mathcal{B}_0)$ by Lemma 3.4. Now $f: \mathcal{B} \rightarrow (-1, 1]^4$ is a homeomorphism. Since $\mathcal{B}_0 \neq \mathcal{B}$, we have $\mathcal{A}' \neq (-1, 1]^4$.

Proof of Theorem 3.1. The map $f: \mathcal{B}_0 \rightarrow \mathcal{A}'$ is injective by Lemma 3.7. Hence the isometry class of a trapezohedron T is determined by the dihedral angles. \square

Proof of Theorem 3.2. Let $\partial\mathcal{B}_0$ be the frontier of \mathcal{B}_0 in \mathcal{B} . Then $f(\partial\mathcal{B}_0) = \partial\mathcal{A}'$, and $f(1, \dots, 1, 0) = (1, \dots, 1)$. Let us describe $\partial\mathcal{A}'$. Define

$$\partial_i\mathcal{A}' = \{(c_1, \dots, c_4) \in \partial\mathcal{A}' \mid c_i \leq c_{i+2}, c_{i+1} \leq c_{i+3}\}.$$

Then $\partial\mathcal{A}' = \bigcup \partial_i\mathcal{A}'$.

We consider $\partial_1\mathcal{A}'$. Recall that $c_i = \cos \alpha_i = \frac{q_i - t}{\sqrt{1 + t^2}}$. Since $c_1 \leq c_3$ and $c_2 \leq c_4$, we have $q_1 \leq q_3$ and $q_2 \leq q_4$. Since $\prod_{i=1}^4 q_i = 1$, we have $q_1 q_2 \leq 1 \leq q_3 q_4$. If $q_4 q_1 < 1$, then $\frac{q_1 + q_2}{1 - q_1 q_2} \leq \frac{q_4 + q_1}{1 - q_4 q_1}$. If $q_2 q_3 < 1$, then $\frac{q_1 + q_2}{1 - q_1 q_2} \leq \frac{q_2 + q_3}{1 - q_2 q_3}$. Hence it is sufficient to consider only the condition that $t < \frac{q_1 + q_2}{1 - q_1 q_2}$. Substitute $q_i = t + c_i\sqrt{1 + t^2}$ in $(1 - q_1 q_2)t < q_1 + q_2$. Then it is equivalent to

$$(c_1 c_2 + 1)t > -(c_1 + c_2)\sqrt{1 + t^2}.$$

If $c_1 + c_2 > 0$, the inequality holds trivially. Suppose that $c_1 + c_2 \leq 0$. Note that $c_1, c_2 \neq 1$. Then (by taking squares of both sides) it is equivalent to

$$t > \frac{-(c_1 + c_2)}{\sqrt{(1 - c_1^2)(1 - c_2^2)}},$$

which is also equivalent to

$$\sqrt{1 + t^2} > \frac{c_1 c_2 + 1}{\sqrt{(1 - c_1^2)(1 - c_2^2)}}.$$

Then they are also equivalent to each of

$$\begin{aligned} q_1 q_2 &= t \left((c_1 c_2 + 1)t + (c_1 + c_2)\sqrt{1 + t^2} \right) + c_1 c_2 > c_1 c_2, \\ q_3 q_4 &= t^2 + (c_3 + c_4)t\sqrt{1 + t^2} + c_3 c_4(1 + t^2) \\ &> \frac{(c_1 + c_2)^2 - (c_1 + c_2)(c_1 c_2 + 1)(c_3 + c_4) + (c_1 c_2 + 1)^2 c_3 c_4}{(1 - c_1^2)(1 - c_2^2)}. \end{aligned}$$

Since $\prod_{i=1}^4 q_i = 1$, we have

$$c_1 c_2 \left((c_1 + c_2)^2 - (c_1 + c_2)(c_1 c_2 + 1)(c_3 + c_4) + (c_1 c_2 + 1)^2 c_3 c_4 \right) < (1 - c_1^2)(1 - c_2^2).$$

Since $c_1 c_2 (c_1 + c_2)^2 - (1 - c_1^2)(1 - c_2^2) = (c_1 c_2 + 1) \left((c_1 + c_2)^2 - c_1 c_2 - 1 \right)$ and $c_1 c_2 + 1 > 0$, we have

$$\begin{aligned} \Phi_1(c_1, \dots, c_4) &= c_1 c_2 (c_1 c_2 + 1) c_3 c_4 - c_1 c_2 (c_1 + c_2)(c_3 + c_4) \\ &\quad + (c_1 + c_2)^2 - c_1 c_2 - 1 \\ &< 0. \end{aligned}$$

After all, $t < \frac{q_1 + q_2}{1 - q_1 q_2}$ is equivalent to $c_1 + c_2 > 0$ or $\Phi_1(c_1, \dots, c_4) < 0$. Under the condition that $c_1 \leq c_3$ and $c_2 \leq c_4$, the frontier of \mathcal{A}' is given by $\Phi_1 = 0$ and $c_1 + c_2 \leq 0$. In this case $Q_i = P_{i+1}$, which means that the edge between \hat{L}_i and \hat{L}_{i+1} degenerates. Therefore \mathcal{A}' and $\partial_i \mathcal{A}'$ are described as in the assertion. \square

Let us see the shape of \mathcal{A} more explicitly.

Corollary 3.8. *For any $\alpha, \beta \in [0, \pi)$, it holds that $(\alpha, \beta, \alpha, \beta) \in \mathcal{A}$. Consequently, $\partial_i \mathcal{A}' \cap \partial_{i+2} \mathcal{A}' = \emptyset$.*

Proof. If $(c_1, \dots, c_4) \in \partial_i \mathcal{A}' \cap \partial_{i+2} \mathcal{A}'$, then $c_1 = c_3$ and $c_2 = c_4$. If $-1 < c_1, c_2 < 1$, then $\Phi_1(c_1, c_2, c_1, c_2) = -(1 - c_1 c_2)(1 - c_1^2)(1 - c_2^2) < 0$. By the same argument for all Φ_i 's, we have $(c_1, c_2, c_1, c_2) \in \mathcal{A}'$ for any $c_1, c_2 \in (-1, 1]$. \square

Corollary 3.9. *It holds that $\partial_i \mathcal{A}' \cap \partial_{i+1} \mathcal{A}' \neq \emptyset$. In other words, there is a degeneration in \mathcal{C} in which two intersections of cone loci occur.*

Proof. Since $\Phi_1(c, c, c, 1) = \Phi_2(c, c, c, 1) = (c - 1)^2(c + 1)(c^2 - c - 1)$, we have

$$\left(\frac{1 - \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}, 1 \right) \in \partial_1 \mathcal{A}' \cap \partial_2 \mathcal{A}'.$$

\square

Theorem 3.10. *It holds that $[0, \arccos(1 - \sqrt{2})]^4 \subset \mathcal{A}$. Furthermore, $(\arccos(1 - \sqrt{2}), \arccos(1 - \sqrt{2}), 0, 0) \notin \mathcal{A}$. Hence the value $\arccos(1 - \sqrt{2})$ is best possible.*

Proof. Since $\Phi_1(c, c, 1, 1) = (c-1)^2(c^2 - 2c - 1)$, we have $(1 - \sqrt{2}, 1 - \sqrt{2}, 1, 1) \in \partial_1 \mathcal{A}'$. Hence $(\arccos(1 - \sqrt{2}), \arccos(1 - \sqrt{2}), 0, 0) \notin \mathcal{A}$.

Without loss of generality, it is sufficient to show that $\Phi_1(c_1, \dots, c_4) < 0$ or $c_1 + c_2 > 0$ if $1 - \sqrt{2} < c_1 \leq c_3 \leq 1$ and $1 - \sqrt{2} < c_2 \leq c_4 \leq 1$. Suppose that $1 - \sqrt{2} < c_1 \leq c_3 \leq 1, 1 - \sqrt{2} < c_2 \leq c_4 \leq 1$, and $c_1 + c_2 \leq 0$. Note that $c_1, c_2 \neq 1$. If $c_1 = 0$, then $\Phi_1(c_1, \dots, c_4) = c_2^2 - 1 < 0$. The same argument holds if $c_2 = 0$. Hence we may assume that $c_1 c_2 \neq 0$.

For fixed c_1 and c_2 , the equation

$$\begin{aligned} \Phi_1(c_1, \dots, c_4) &= c_1 c_2 (c_1 c_2 + 1) \left(c_3 - \frac{c_1 + c_2}{c_1 c_2 + 1} \right) \left(c_4 - \frac{c_1 + c_2}{c_1 c_2 + 1} \right) \\ &\quad - \frac{(1 - c_1^2)(1 - c_2^2)}{c_1 c_2 + 1} \\ &= 0 \end{aligned}$$

gives a hyperbola H in the (c_3, c_4) -plane. The asymptotic lines of H are given by $c_3 = \frac{c_1 + c_2}{c_1 c_2 + 1}$ and $c_4 = \frac{c_1 + c_2}{c_1 c_2 + 1}$. Since $-\frac{(1 - c_1^2)(1 - c_2^2)}{c_1 c_2 + 1} < 0$, the inequality $\Phi_1(c_1, \dots, c_4) < 0$ gives the complementary region of H containing the two asymptotic lines.

Suppose that $c_1 c_2 < 0$. Then the hyperbola H is contained in the upper left and lower right complementary regions of the two asymptotic lines. Now $\Phi_1(c_1, c_2, -1, 1) = \Phi_1(c_1, c_2, 1, -1) = -(1 - c_1^2)(1 - c_2^2) < 0$. Therefore $\Phi_1(c_1, \dots, c_4) < 0$ for any $-1 \leq c_3, c_4 \leq 1$.

Suppose that $c_1 c_2 > 0$. Then the hyperbola H is contained in the upper right and lower left complementary regions of the two asymptotic lines. Let us consider $\Phi_1(c_1, c_2, 1, 1) = (c_1 c_2 - c_1 - c_2)^2 - 1$. Since $1 - \sqrt{2} < c_1, c_2 < 0$, we have $1 < (1 - c_1)(1 - c_2) < 2$. Hence $0 < c_1 c_2 - c_1 - c_2 < 1$. Thus $\Phi_1(c_1, c_2, 1, 1) < 0$. Furthermore, $\Phi_1(c_1, c_2, c_1, c_2) < 0$ by Corollary 3.8. Therefore $\Phi_1(c_1, \dots, c_4) < 0$ for any $c_1 \leq c_3 \leq 1$ and $c_2 \leq c_4 \leq 1$. \square

Remark 3.11. Andreev's theorem immediately implies that $[0, \pi/2]^4 \subset \mathcal{A}$. For deformation in \mathcal{C} , cone loci with cone angles less than π do not intersect unless the volumes converge to zero or a 2-dimensional Euclidean sub-cone-manifold appears, as shown by Kojima [8].

Corollary 3.12. *The space \mathcal{A} is connected. Consequently, it holds that $\mathcal{C}_0 = \mathcal{C}_{\text{sym}}$, and $\frac{1}{2}\Theta: \mathcal{C}_0 \rightarrow \mathcal{A}$ is a homeomorphism.*

Proof. We show that \mathcal{A}' is path-connected. Take $(c_1, \dots, c_4) \in \mathcal{A}'$. Without loss of generality, we may assume that $c_1 \leq c_3$ and $c_2 \leq c_4$. As in the proof of Theorem 3.10, c_1 and c_2 are regarded to be fixed. Consider the slice

$$\mathcal{A}'_1(c_1, c_2) = \{(x, y) \in [c_1, 1] \times [c_2, 1] \mid (c_1, c_2, x, y) \in \mathcal{A}'\}.$$

If $c_1 c_2 \leq 0$, then $\mathcal{A}'_1(c_1, c_2) = [c_1, 1] \times [c_2, 1]$. If $c_1 c_2 > 0$, then

$$\mathcal{A}'_1(c_1, c_2) = [c_1, 1] \times [c_2, 1] \cup \{(x, y) \in \mathbb{R}^2 \mid \Phi_1(c_1, c_2, x, y) < 0\}.$$

In both cases, there is a path joining (c_1, \dots, c_4) and (c_1, c_2, c_1, c_2) by the proof of Theorem 3.10. Moreover, there is a path joining (c_1, c_2, c_1, c_2) and $(1, \dots, 1)$ by Corollary 3.8. Thus we obtain a path joining (c_1, \dots, c_4) and $(1, \dots, 1)$. \square

Corollary 3.13. *The global rigidity for \mathcal{C} holds if and only if $\mathcal{C}_0 = \mathcal{C}$.*

Proof. If $\mathcal{C}_0 = \mathcal{C}_{\text{sym}} = \mathcal{C}$, the global rigidity for \mathcal{C} holds by Theorem 3.1.

If $\mathcal{C}_0 = \mathcal{C}_{\text{sym}} \neq \mathcal{C}$, there is a non-symmetric cone structure $g \in \mathcal{C} \setminus \mathcal{C}_{\text{sym}}$. Then the Γ -action on $(T^2 \times I, L)$ gives distinct cone structures $g, g' \in \mathcal{C}$ such that $\Theta(g) = \Theta(g')$. Therefore the global rigidity for \mathcal{C} fails. \square

Finally, we prove the main theorem.

Proof of Theorem 1.4. There is $(\alpha_1, \dots, \alpha_4) \in [0, \pi)^4$ which does not belong to \mathcal{A} . Take $\max_i \{\alpha_i\} < \alpha < \pi$. Corollary 3.8 implies that $(\alpha, \alpha, \alpha, \alpha) \in \mathcal{A}$. While we decrease the cone angles from (α, \dots, α) to $(\alpha_1, \dots, \alpha_4)$, the trapezohedron degenerates. This corresponds to a degeneration in \mathcal{C} with one or two intersections of L_i .

More explicitly, we construct two paths in \mathcal{A}' whose terminals correspond to degenerations. Firstly, let $\mathbf{c}_1(x) = (1 - \sqrt{2}, 1 - \sqrt{2}, 1, x) \in (-1, 1]^4$ for $0 \leq x \leq 1$. Then

$$\begin{aligned}\Phi_1(\mathbf{c}_1(x)) &= \Phi_2(\mathbf{c}_1(x)) = 2(1 - \sqrt{2})^2(x - 1), \\ \Phi_3(\mathbf{c}_1(x)) &= 2x(x + 1), \\ \Phi_4(\mathbf{c}_1(x)) &= -2(1 - \sqrt{2})(\sqrt{2}x + 1)(x - 1).\end{aligned}$$

For $0 \leq x < 1$, we have $\Phi_1(\mathbf{c}_1(x)) < 0$, $\Phi_2(\mathbf{c}_1(x)) < 0$, $\Phi_4(\mathbf{c}_1(x)) < 0$, and $1 + x > 0$. Hence $\mathbf{c}_1(x) \in \mathcal{A}'$ for $0 \leq x < 1$ by Theorem 3.2. Moreover, $\mathbf{c}_1(1) \in \partial_1 \mathcal{A}'$, and it does not belong to the other $\partial_i \mathcal{A}'$. In the corresponding deformation of cone structures, the cone angle at L_4 decreases. The degeneration is due to an intersection of the cone loci L_1 and L_2 .

Secondly, let $\mathbf{c}_2(x) = \left(\frac{1 - \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}, x\right) \in (-1, 1]^4$ for $0 \leq x \leq 1$.

Then

$$\begin{aligned}\Phi_1(\mathbf{c}_2(x)) &= \Phi_2(\mathbf{c}_2(x)) = \left(\frac{1 - \sqrt{5}}{2}\right)^4 (x - 1), \\ \Phi_3(\mathbf{c}_2(x)) &= \Phi_4(\mathbf{c}_2(x)) = \left(\frac{1 - \sqrt{5}}{2}\right)^2 \left(x^2 - x - \frac{1 + \sqrt{5}}{2}\right).\end{aligned}$$

They are negative for $0 \leq x < 1$. Hence $\mathbf{c}_2(x) \in \mathcal{A}'$ for $0 \leq x < 1$ by Theorem 3.2. Moreover, $\mathbf{c}_2(1) \in \partial_1 \mathcal{A}' \cap \partial_2 \mathcal{A}'$ as shown in the proof of Corollary 3.9. In the corresponding deformation of cone structures, the cone angle at L_4 decreases. The degeneration is due to two intersections of cone loci: one of them is of L_1 and L_2 , and the other is of L_2 and L_3 . \square

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