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TOTALLY RAMIFIED RATIONAL MAPS

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ABSTRACT. Totally ramified rational maps and regularly ramified rational maps are defined and studied in this paper. We first give a complete classification of regularly ramified rational maps and show that our definition of a regularly ramified rational map is equivalent to a much stronger definition of a map of this kind given by Milnor [*Dynamics in one complex variable*, Princeton University Press, Princeton, NJ, 2006]. Then we show that (1) any totally ramified rational map of degree $d \leq 6$ must be regularly ramified; (2) for any integer $d > 6$, there exists a totally ramified rational map of degree d which is not regularly ramified. Furthermore, we count totally ramified rational maps up to degree 10. Finally, we present explicit formulas for all totally but not regularly ramified rational maps of degree 7 or 8, up to pre- and post-composition by Möbius transformations.

1. INTRODUCTION AND MAIN RESULTS

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree at least 2. A *critical point* of f is a point where f is not injective in any neighborhood of the point. The image of a critical point is called a *critical value* of f. Generally, the trajectories of the critical points under the iteration of f determine the dynamical behaviors of f on $\widehat{\mathbb{C}}$. Sometimes, certain pattern of critical points eliminates certain type of dynamical behavior. For example, it is proved in [9] that if all preimages of every critical value of f are critical points, then there is no Herman ring in the Fatou set of f. Motivated by this result, we investigate in this paper the existence of such rational maps for degree $d \geq 2$. Let us first introduce a name for this type of rational functions.

Definition 1.1 (Totally/regularly ramified maps)**.** Let f be a rational map of degree $d \geq 2$. Then f is said to be *totally ramified* if all the preimages of every critical value of f are critical points. In particular, if for each critical value v, the map f has the same local degree (with value depending on v) at all the preimages of v, then we say that f is *regularly ramified*.

Note that it is equivalent to define f to be regularly ramified by requiring that for every point $q \in \widehat{\mathbb{C}}$, f has the same local degree (index) at the preimages of q. Clearly, the condition "regularly ramified" is stronger than "totally ramified". Regularly ramified rational maps appear in the classical study of Riemann surfaces

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and classification of Kleinian groups (see [4]). Let us first give a brief summary on this type of rational maps.

Let G be a finite group of Möbius transformations acting on $\widehat{\mathbb{C}}$ (a finite Kleinian group). Then the quotient space $\widehat{\mathbb{C}}/G$ is conformally equivalent to $\widehat{\mathbb{C}}$ and hence the projection map $R_G : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}/G$ is a regularly ramified rational map. There are only five types of finite Kleinian groups. More precisely, up to conjugation by a Möbius transformation, a nontrivial finite Kleinian group G is equal to one of the following five groups:

(1) A *cyclic group* G_1 of order ν generated by $z \mapsto e^{\frac{2\pi i}{\nu}} z$, where ν is a positive integer ≥ 2;
(2) A *dihedral group* G_2 generated by the symmetries of a regular polygon of ν sides, where ν is a positive integer ≥ 2;
(3) The *tetrahedral group* G_3 generated by the symmetries of a regular tetrahedron;
(4) The *octahedral group* G_4 generated by the symmetries of a regular octahedron or its dual, a cube;
(5) The *icosahedral group* G_5 generated by the symmetries of a regular icosahedron or its dual, a regular dodecahedron.

The corresponding five types of projection maps are denoted by R_{G_j}, $j = 1, 2, 3, 4, 5$. Then

- R_{G_1} is the power map $z \mapsto z^\nu$; the degree of R_{G_2} is 2ν; the degree of R_{G_3} is 12; the degree of R_{G_4} is 24; the degree of R_{G_5} is 60.
- R_{G_1} has two distinct critical values and two distinct critical points. R_{G_j}, $2 \leq j \leq 5$, has three distinct critical values and more than 3 distinct critical points.
- All R_{G_j}, $j = 1, 2, 3, 4, 5$, are regularly ramified.

In fact, Milnor defines a regularly ramified rational map in [11] as follows.

Definition 1.2 (Milnor)**.** A rational map f is said to be *regularly ramified* if there exists a group G of conformal homeomorphisms (Möbius transformations) of $\widehat{\mathbb{C}}$ such that two points z_1 and z_2 have the same image under f if and only if there is an element g of G with $g(z_1) = z_2$.

Since the degree of f is finite, Milnor's definition implies that the group G in the definition is a finite group (hence a finite Kleinian group). Therefore, up to pre- and post-composition by Möbius transformations, a regularly ramified rational map in the sense of Milnor has to be one of R_{G_j}'s. By now, one may also note that Definition 1.1 for a regularly ramified rational map is possibly weaker than Definition 1.2 since there is no group involved in the first definition. As the first result presented in this paper, we prove that these two definitions are equivalent.

Theorem 1.1. *Let f be a regularly ramified rational map of degree $d \geq 2$ in the sense of Definition* 1.1. *Then up to pre- and post-composition by Möbius transformations, f is equal to R_{G_j} for some $1 \leq j \leq 5$.*

A totally ramified rational map is a more general concept than a regularly ramified rational map and it has more flexibility in the following sense. Let f_1 be one of R_{G_j}'s and f_2 be one of R_{G_j}'s with $2 \leq j \leq 5$, and let M be a Möbius transformation sending the critical values of f_1 to some critical points of f_2. Then $(f_2 \circ M \circ f_1)(z) = f_2(M(f_1(z)))$ is a totally ramified rational map, but it is not necessarily regularly ramified. A natural problem is to classify all totally ramified rational maps, and a specific question is to ask how many distinct critical values a totally ramified rational map can have. By the Riemann-Hurwitz formula, it is proved in [9] that each totally ramified rational map has two or three distinct critical values. Again using the Riemann-Hurwitz formula, one can see that any rational map with two distinct critical values has only two distinct critical points. It follows that any totally ramified rational map with two critical values and of degree $d \geq 2$ must be regularly ramified and it is equal to the power map $z \mapsto z^d$ up to pre- and post-composition by Möbius transformations (see Theorem 3.4). Note that each quadratic rational map has two critical points and two critical values. Thus, every quadratic rational map is regularly ramified. If a cubic rational map is totally ramified, then each critical value has a unique preimage and hence it has two critical values. This means that every totally ramified cubic rational map is regularly ramified. Therefore, it remains to study the existence and classification of totally ramified rational maps with three distinct critical values and of degree $d \geq 4$.

By now, we know that the degree of a regularly ramified rational map with three distinct critical values is even and at least 4. Concerning the degree of the previously mentioned example $f_1 \circ M \circ f_2$, note that if it has three critical values, then its degree is also even. The following problems arise.

(1) For each odd integer $d \geq 4$, is there a totally ramified rational map of degree d with three distinct critical values?
(2) For each integer $d \geq 4$, is there a totally ramified rational map that is not regularly ramified?
(3) For each integer $d \geq 4$, how many totally ramified rational maps of degree d with three distinct critical values are there up to pre- and post-composition by Möbius transformations?

In the paper, we give complete answers to the first two questions by proving the following two theorems. A partial answer to the third question is given in the last theorem.

Theorem 1.2 (Rigidity for degrees ≤ 6)**.** *If f is a totally ramified rational map of degree $d \leq 6$, then f is regularly ramified.*

Theorem 1.3 (Existence of totally but not regularly ramified rational map)**.** *For any integer $d \geq 7$, there exists a totally ramified rational map f of degree d which is not regularly ramified.*

Theorem 1.4. *Up to pre- and post-composition by Möbius transformations, the numbers of totally ramified rational maps up to degree* 10 *are summarized in Table* 1*.*

To prove these theorems, we first establish a one-to-one correspondence between rational maps with three critical values and a type of homogeneous graphs (called the Speiser graphs). Then we show the existence of the corresponding homogeneous graph for each degree $d \geq 3$. This second step occupies the main body of the paper and the classification of such homogeneous graphs becomes more and more complicated as the degree d increases. The proof of Theorem 1.3 is divided into three steps: for each even degree $d \geq 8$, we modify the Speiser graph of a R_{G_2} map to obtain the Speiser graph for a totally but not regularly ramified rational map of

TABLE 1. $T_d(n)$ denotes the collection of the equivalence classes of totally ramified rational maps with n critical values and of degree d under pre- and post-composition by Möbius transformations.

Degree d	2	3	4	5	6	7	8	9	10
$\lvert T_d(2)\rvert$	1	1	1	1	1	1	1	1	1
$\lvert T_d(3)\rvert$	0	0	1	0	1	1	3	3	≥ 7
$\lvert T_d(2) \cup T_d(3)\rvert$	1	1	2	1	2	2	4	4	≥ 8

the same degree; for each odd degree $d \geq 7$, we first manually construct the Speiser graph for a totally ramified rational map of degree 7 and then we use a rule to construct a Speiser graph for degree $d+2$ by modifying the Speiser graph of degree d and adding vertices and edges accordingly; finally, we employ branched covering maps of $\widehat{\mathbb{C}}$ and Stoïlow's Theorem to obtain the realization of the Speiser graph by a totally ramified rational map.

Remark 1.1. The question on the existence of a totally ramified rational map of degree $d \geq 4$ with three distinct critical values was first raised in [9]. Explicit formulas of five types of regularly ramified rational maps are constructed in [7] and dynamics of some one-parameter families of those maps are explored in [7] and studied in [8].

Remark 1.2. Speiser graphs and their dual graphs and Stoïlow's theorem are used in the study of the existence of entire functions with given tree structures in [2], which motivates our work for this paper. As well-known examples, the sine function $\sin(z)$ is a totally (also regularly) ramified transcendental entire function and the Weierstrass elliptic function is a totally (also regularly) ramified transcendental meromorphic function. From the theory of Nevanlinna, any transcendental meromorphic function has at most four totally ramified values (page 284, [12]).

The paper is organized as follows. In Section 2, we provide background on Speiser graphs and Stoïlow's theorem; in Section 3, we prove that up to pre- and post-composition by Möbius transformations, each totally ramified rational map with three distinct critical values is uniquely determined by a Speiser graph; in Sections 4, 5 and 6, we show Theorems 1.1, 1.2 and 1.3 respectively; in Section 7, we prove Theorem 1.4; finally in Section 8, we work out the formulas for totally but not regularly ramified rational maps of degree 7 or 8, up to pre- and post-compositions by Möbius transformations.

2. Preliminaries

We provide some background in this section.

2.1. **Branched coverings.** Let $\mathcal{S}^2$ be a topological sphere and $f : \mathcal{S}^2 \to \mathcal{S}^2$ be an orientation preserving branched covering map. Then for any point $z \in \mathcal{S}^2$, there is a neighborhood U of z such that for any Jordan curve α surrounding z and oriented in the counterclockwise direction, the winding number of $f(\alpha)$ around $f(z)$ is a constant positive integer, which is denoted by $k(z)$ and called the *index or local degree* of f at z. If $k(z) \geq 2$, then z is called a *branch point* of f, $f(z)$ is called a *branched value* of f, and $k(z) - 1$ is called the *multiplicity* of the branch point z.

Since $\mathcal{S}^2$ is compact, f has finitely many branch points. Denote by $\mathcal{C}(f)$ the set of the branched values of f. Then $f : \mathcal{S}^2 \setminus f^{-1}(\mathcal{C}(f)) \to \mathcal{S}^2 \setminus \mathcal{C}(f)$ is a covering map.

Clearly, rational maps are branched coverings from $\mathcal{S}^2$ to itself.

Definition 2.1 (Totally/regularly ramified branched covering maps)**.** Let $f : \mathcal{S}^2 \to \mathcal{S}^2$ be a branched covering. Then f is said to be *totally ramified* if all preimages of each branched value are branch points. Furthermore, f is said to be *regularly ramified* if for every point $q \in \mathcal{S}^2$, f has the same index at all preimages of q, where the value of the index depends on q.

Clearly, totally (resp. regularly) ramified rational maps are totally (resp. regularly) ramified branched covering maps from $\mathcal{S}^2$ to itself.

Definition 2.2 (Equivalence)**.** Let $f_1,\, f_2 : \mathcal{S}^2 \to \mathcal{S}^2$ be two branched covering maps. We say that f_1 is *topologically equivalent* to f_2 if there exist two orientation preserving homeomorphisms $\varphi,\, \psi : \mathcal{S}^2 \to \mathcal{S}^2$ such that

$$\varphi \circ f_1 = f_2 \circ \psi. \tag{2.1}$$

Similarly, we say that two rational maps are *topologically equivalent* if there are two homeomorphisms $\varphi,\, \psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that the equation (2.1) holds.

2.2. **Speiser graph.** A graph Γ is a triple $(V,\, E,\, I)$, where V is the set of vertices, E is the set of edges, and I is the incidence relation between elements of V and elements of E such that any edge $e \in E$ is incident either to two different vertices $v_1,\, v_2 \in V$ or "twice" to the same vertex $v \in V$ (the latter case is called a loop) (see [3]). A graph Γ is said to be *finite* if the cardinality of E or V is finite. Unless otherwise stated, all graphs considered are connected and finite.

The local valence of Γ at a vertex v is the number of vertices incident to v; that is the number of edges attached at v. A graph Γ is said to be *homogeneous* if the local valences of Γ at all vertices are the same. We say a graph Γ is *bipartite* if the set V of the vertices can be divided into two disjoint sets in such a way that each edge connects two vertices from different sets. For convenience, if Γ is bipartite, the vertices from one set are denoted by $\circ$ and the ones from the other set are denoted by $\times$.

When a graph Γ is embedded on $\mathcal{S}^2$, the components of the complementary of the embedded graph are called the *faces* of Γ, and the embedded edges enclosing a face are the *sides* of the face. In this case, the *dual graph* Γ^* of a graph Γ is defined by identifying each face of Γ to vertices and adding an edge between two vertices as long as the two faces, represented by the two vertices, share a side.

Given a homogeneous and bipartite graph of local valence q embedded on $\mathcal{S}^2$, one can label the faces by $1, 2, \cdots, q$ in a way such that around each $\circ$-vertex $\circ$, the q faces with a vertex at $\circ$ are labeled by $1, 2, \cdots, q$ in the counterclockwise order, and for each $\times$-vertex $\times$, the q faces with a vertex at $\times$ are labeled by $1, 2, \cdots, q$ in the clockwise order. This can be achieved by labelling the faces around a vertex inductively as follows:

(1) One chooses a $\circ$-vertex $\circ$ and label the faces around $\circ$ by $1, 2, \cdots, q$ in the counterclockwise order.
(2) For each $\times$-vertex $\times$ connected to $\circ$ by an edge e, the two faces sharing the edge e are already labeled by two adjacent integers (modulus q) among $1, 2, \cdots, q$ in the counterclockwise order. So we can label the other faces

around $\times$ by the other $q-2$ integers such that all q faces are labeled by $1, 2, \cdots, q$ in the clockwise order.

(3) For each $\circ$-vertex $\circ$ connected to a $\times$-vertex in (2) except the one chosen in (1), we repeat the process of (2) so that the faces of $\circ$ are labeled by $1, 2, \cdots, q$ in the counterclockwise order.

(4) Inductively, we repeat Steps (2) and (3) until the faces around every vertex are labeled.

Now we introduce the name for a homogeneous and bipartite graph with such a labeling on faces.

Definition 2.3 (Speiser graphs)**.** Let Γ be a connected and finite graph embedded on the sphere $\mathcal{S}^2$ with a face marking. We say that Γ is a *Speiser graph* if it satisfies the following conditions:

(1) *Homogeneity*: Γ is a homogeneous graph of valence q.
(2) *Biparity*: every vertex is either a $\circ$-vertex or a $\times$-vertex and vertices connected by an edge have different parities.
(3) *Orientation*: The face labels around $\circ$-vertices are in the counterclockwise order and the ones around $\times$-vertices are in the clockwise order.

Because of homogeneity and biparity, every face of a Speiser graph has an even number of edges on its boundary.

Definition 2.4. Two Speiser graphs Γ_1 and Γ_2 are equivalent if there is a homeomorphism $\varphi : \mathcal{S}^2 \to \mathcal{S}^2$ such that $\varphi(\Gamma_1) = \Gamma_2$ with vertices and edges being mapped to vertices and edges, respectively. In this sense, we also say that Γ_1 is *ambiently homeomorphic* to Γ_2.

Remark 2.1. In the literature, a connected infinite graph embedded on the complex plane and satisfying the first two conditions (Homogeneity and Biparity) in Definition 2.3 is called a Speiser graph (for example, see [1]), and is used to study Nevanlinna's class of surfaces spread over the sphere (see [12]). Also in the literature, a connected finite graph embedded on a Riemann surface and satisfying the Biparity condition is called a dessin d'enfant (for example, see [5] or [10]), which is used to verify Belyi's criterion for a compact Riemann surface to be defined over $\overline{Q}$. Following Definition 2.3 and the definition of a dessin d'enfant given in [5] or [10], one can see the difference and connection between these two types of graphs. The Speiser graphs in Definition 2.3 can be constructed for a branched covering map from a Riemann surface to a Riemann sphere with any finite number of branched values. Even in the case when there are three or two branched values, the resulting Speiser graph of a branched covering map is different from the dessin d'enfant resulting from the same covering map constructed in [5] (resp. [10]), and the latter is the due graph of the Speiser graph (resp. a subgraph of the due graph). This difference is the reason for us to adopt the name of Speiser graph for a connect finite graph satisfying the three conditions in Definition 2.3.

2.3. **Stoïlow's Theorem.** Let S_1 and S_2 be two oriented topological surfaces. We say that a branched covering map $f : S_1 \to S_2$ is a *topological holomorphic map* if it is open, continuous and discrete, where by being *discrete* we mean that $f^{-1}(w)$ is a discrete set for any $w \in S_2$. The classical Stoïlow theorem says that a topological holomorphic map $f : S_1 \to S_2$ defined in this way is locally modelled by power maps. More precisely, for each $z \in S_1$, there exist an integer $k(z) \in \mathbb{N}$, a neighborhood U of z and two homeomorphisms $\phi : U \to \mathbb{D}$ and $\psi : f(U) \to \mathbb{D}$ such that $\psi(f(z)) = \phi(z)^{k(z)}$. This result implies that by equipping S_1 and S_2 with appropriate conformal structures, a topological holomorphic map $f : S_1 \to S_2$ becomes a holomorphic map between two Riemann surfaces in a traditional sense ([13]). Using Stoïlow's theorem, we reduce the construction of a rational map to a branched covering of the sphere. In the next section, we show a branched covering of the sphere is determined by a Speiser graph.

3. One-to-one Correspondence between branched covering maps and Speiser graphs

In this section, we develop a one-to-one correspondence between the branched coverings of the sphere and the Speiser graphs.

3.1. **Speiser graphs of branched coverings and applications.** Let f be a branched covering of the sphere with degree $d \geq 2$. Denote the branched values of f by $a_1, a_2, \ldots, a_q$. We fix an oriented Jordan curve L on $\mathcal{S}^2$ passing through $a_1, a_2, \ldots, a_q$ consecutively, which is called a *base curve*. The curve L divides $\mathcal{S}^2$ into two components A and B with A at the left-hand side of the positive orientation of L. Then we choose a point $\circ \in A$ and a point $\times \in B$ and connect them by q Jordan arcs γ_j $(j = 1, \ldots, q)$ such that $\gamma_j \cap \gamma_k = \emptyset$ whenever $j \neq k$ and γ_j intersects L only once with the intersecting point in the segment (a_j, a_{j+1}), where $a_{q+1} = a_1$. Now we define

$$\Gamma_f := f^{-1}\left(\bigcup_{j=1}^{q} \gamma_j\right).$$

Then Γ_f is a Speiser graph. Since the two base points $\circ$ and $\times$ are not branched values, f has d local inverse maps well defined on a neighborhood of $\circ$ and d local inverse maps on a neighborhood of $\times$. Each local inverse is extended to the component of $\mathcal{S}^2 \setminus L$ containing $\circ$ or $\times$. It follows that Γ_f has $2d$ vertices and the local valences of Γ_f at all vertices are equal to q. Thus, Γ_f is a homogeneous graph. Furthermore, since each Jordan arc γ_j contains no branched value, each preimage of γ_j is a Jordan arc connecting a preimage of $\circ$ to a preimage of $\times$; that is, it connects a $\circ$-vertex to a $\times$-vertex. Therefore, Γ_f is a bipartite graph. Finally, it is clear that the faces of Γ_f around each $\circ$-vertex can be labeled by $1, 2, \cdots, q$ in the counterclockwise order and the faces around each $\times$-vertex are labeled by $1, 2, \cdots, q$ in the clockwise order.

Before giving some examples of Γ_f, we first introduce two immediate applications of the graph Γ_f. The first one shows a well known corollary of the Riemann-Hurwitz formula (see [11, Theorem 7.2]).

Theorem 3.1. *If the degree of a branched covering map $f : \mathcal{S}^2 \to \mathcal{S}^2$ is d, then the number of branch points of f is equal to $2d - 2$, counted by multiplicities.*

Proof. Let Γ_f be the Speiser graph constructed from f in the previous paragraph. Clearly, the number of the vertices $\#(V) = 2d$. Assume that each branched value v_j has i_j distinct preimages and the local degrees at these preimages are denoted by $m_1^j, m_2^j, \cdots, m_{i_j}^j$ where $j = 1, 2, \cdots q$. Then the number $\#(F)$ of the faces of Γ_f is equal to $\sum_{j=1}^{q} i_j$, the faces containing the preimages of v_j are enclosed by

$2m_1^j, 2m_2^j, \cdots, 2m_{i_j}^j$ edges respectively. So the number $\#(E)$ of the edges of Γ_f is equal to

$$\frac{1}{2}\sum_{j=1}^{q}\sum_{k=1}^{i_j} 2m_k^j = \sum_{j=1}^{q}\sum_{k=1}^{i_j} m_k^j.$$

By applying the Euler characteristic formula to Γ_f, we obtain

$$2d + \sum_{j=1}^{q} i_j - \sum_{j=1}^{q}\sum_{k=1}^{i_j} m_k^j = 2.$$

Thus,

$$\sum_{j=1}^{q}\sum_{k=1}^{i_j}(m_k^j - 1) = 2d - 2. \tag{3.1}$$

This means that the number of the branch points of f, counted by multiplicities, is equal to $2d-2$. □

Theorem 3.2. *If a branched covering map $f : \mathcal{S}^2 \to \mathcal{S}^2$ is totally ramified, then the number of the branched values of f is 2 or 3.*

Proof. Let us use the same notation introduced in the proof of the previous theorem. The proof is as the same as the one given in [9] to show the same conclusion for a totally ramified rational map, which goes as follows. We rewrite (3.1) as

$$\sum_{j=1}^{q}(d - i_j) = 2d - 2.$$

Then

$$q = 2 + \frac{1}{d}\sum_{j=1}^{q} i_j - \frac{2}{d} < 2 + \frac{1}{d}\sum_{j=1}^{q} i_j.$$

If f is totally ramified, then for each j, $i_j \le \frac{d}{2}$ since all $m_1^j, m_2^j, \cdots, m_{i_j}^j$ are bigger than or equal to 2. Thus, $q < 2 + \frac{q}{2}$. Hence, $q < 4$. Since the degree of a ramified branched map f is at least 2, it follows that it has at least two branched values; otherwise, the number of the branch points is $\le d-1 < 2d-2$. Therefore, q is equal to 2 or 3. □

The Speiser graphs of the maps $f(z) = z^3$ and $f(z) = z^2 + \dfrac{1}{z^2}$ are given in Figure 1. In Figure 2, we can see that there are 2-gons in the Speiser graphs of the Chebyshev polynomial of degree 5 and the map $f(z) = \frac{(z-1)^2(z+2)}{3z-2}$.

Theorem 3.3 is obvious.

Theorem 3.3. *A branched covering map $f : \mathcal{S}^2 \to \mathcal{S}^2$ is totally ramified if and only if the Speiser graph Γ_f contains no 2-gon face.*

The next theorem can be proved by using Theorem 3.1. Here we give a proof by using the Speiser graph again.

Theorem 3.4 (Bicriticality)**.** *If a branched covering map $f : \mathcal{S}^2 \to \mathcal{S}^2$ has exactly two branched values, then f has exactly two branch points and hence f is regularly ramified.*

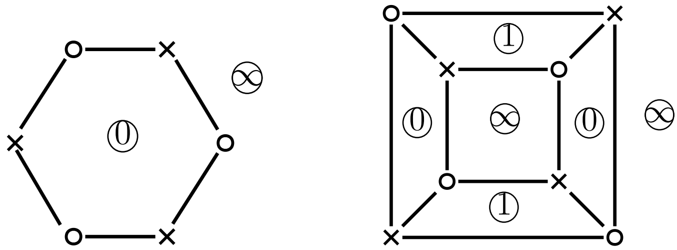

FIGURE 1. Speiser graphs of $f(z) = z^3$ (left) and $f(z) = z^2 + \dfrac{1}{z^2}$ (right)

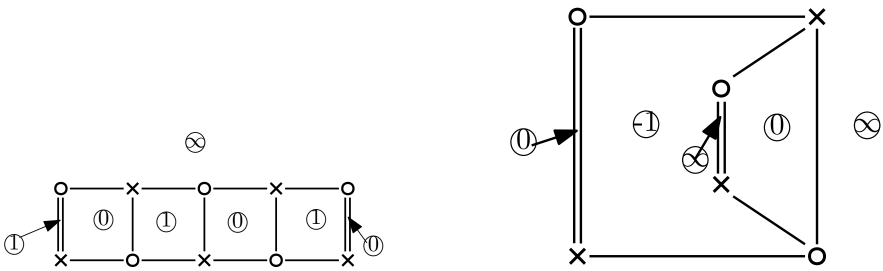

FIGURE 2. Speiser graphs of the Chebyshev polynomial of degree 5 (left) and the rational map $f(z) = \frac{(z-1)^2(z+2)}{3z-2}$ (right). In the drawing, each pair of double lines represents a 2-gon with the face between the lines labeled by $\circ$ or $\times$.

Proof. We consider the Speiser graph Γ_f of f. Since f has two branched values, the local valence of Γ_f at every vertex is 2. Take one branch point c of f, which is contained in a face of the graph with an even number of edges on the boundary. Using the connectivity of Γ_f and the local valence 2 at each vertex, we conclude that the boundary of this face is actually the entire Speiser graph Γ_f. Thus, f has exactly two branch points which are contained in the two faces of Γ_f. It follows that the local degrees of f at the two branch points are equal to the degree of f and hence f is regularly ramified. □

For a convenience to state the next result, we introduce Definition 3.1.

Definition 3.1 (Totally ramified value)**.** Let $f : \mathcal{S}^2 \to \mathcal{S}^2$ be a branched covering map. A branched value v of f is said to be totally ramified, if every preimage of v is a branch point. If, in addition, the local degrees of f at the preimages of v are equal, then we say that v is regularly ramified.

Theorem 3.5. *Let $f : \mathcal{S}^2 \to \mathcal{S}^2$ be a totally ramified covering map. If a branched value v of f has exactly two preimages, then v is regularly ramified and hence the degree of f is even.*

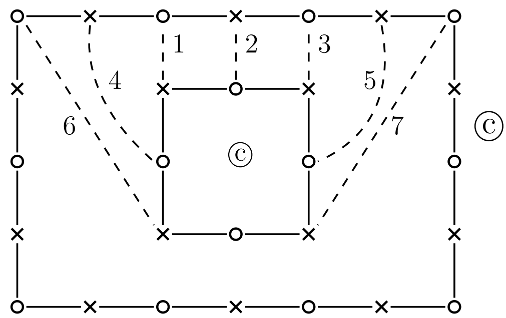

FIGURE 3. An illustration for the idea used in the proof of Theorem 3.5

Proof. Because of Theorems 3.2 and 3.4, it remains to prove the result when f has three branched values. Denote by d the degree of f and by z_1 and z_2 the two preimages of v. Assume that the local degrees of f at z_i are d_1 and d_2 respectively, where we may assume that $d_1 \geq d_2$. Then on the Speiser graph Γ_f of f, there are two faces F_1 and F_2 of the same label bounded by $2d_1$ edges and $2d_2$ edges respectively. In Figure 3, F_1 and F_2 are labeled by the symbol with c in a circle. By the orientation condition satisfied by a Speiser graph, F_1 and F_2 cannot share any edge or any vertex. We claim that except the edges forming the boundaries of F_1 and F_2, Γ_f has no other edges connecting two vertices on the boundary of F_1 or F_2. Otherwise, without loss of generality, we assume that e is such an edge connecting two vertices a and b (with different labels) on the boundary of F_1. Now denote by P the polygon bounded by the edge e and the edges on the boundary of F_1 between a and b. Suppose that P contains more that two vertices. Because the local valence of Γ_f is 3, Γ_f has another edge e' in the interior of P connecting two vertices a' and b' (with different labels) on the boundary of F_1. Now we replace the polygon P by a smaller polygon P' bounded by e' and the edges on the boundary of F_1 between a' and b'. This replacement can be carried on as soon as P' has more than two vertices. So we end up with a polygon face P' with only two vertices, that is, a 2-gon. This is a contradiction to Theorem 3.3. Thus, each vertex on the boundary of F_1 is connected to a vertex on the boundary of F_2 and vice versa, which imply that $d_1 = d_2$. Since $d = d_1 + d_2$, the degree of f is even. □

3.2. **From Speiser graphs to branched coverings.** As we have discussed in the previous subsection, the Speiser graph of a branched covering map contains information on the branching pattern. In this subsection, we show how to recover a branched covering map from a given Speiser graph.

Let Γ be a Speiser graph with valence q. Using the orientation condition of a Speiser graph, we know the faces around each $\circ$-vertex are labeled by q markings $a_1, \ldots a_q$ in the counterclockwise order and the faces around each $\times$-vertex are labeled by the same set of markings in the clockwise order.

Let us consider the dual graph Γ^* of Γ constructed as follows. We choose one point on each face of Γ and denote it by the marking of the face. All of these points are the vertices of Γ^*. There is an edge between any two vertices of Γ^* as long as the two faces F_1 and F_2 of Γ containing them share an edge, which is represented by an arc connecting the two vertices of Γ^* and intersecting once the edge of Γ shared by F_1 and F_2. Furthermore, the arcs representing the edges of Γ^* can be chosen to have pairwise disjoint interiors. Using the conditions satisfied by the Speiser graph Γ, we can see that the dual graph Γ^* has the property that if two faces F_1^* and F_2^* of Γ^* share an edge e^*, then the labels of the vertices of one face are in the counterclockwise order and the labels of the vertices of the other face are in the clockwise order.

Now we choose an oriented Jordan curve L on the sphere $\mathcal{S}^2$ marked by q points $b_1, \dots b_q$ in the counterclockwise order by using the right-hand rule with the thumb pointing outside of the sphere. Then $\mathcal{S}^2 \setminus L$ has two components which are denoted by A and B with A on the left-hand side of L when one talks on through L in the counterclockwise order.

By now, we define a map π mapping γ onto L such that it maps each edge between the two vertices a_j and a_{j+1} homeomorphically onto the subarc on L between b_j and b_{j+1}, where $a_{q+1} = a_1$ and $b_{q+1} = b_1$. Then we extend π to a map from $\mathcal{S}^2$ onto $\mathcal{S}^2$ satisfying: if a face of Γ^* contains a $\circ$-vertex of Γ, then π maps this face homeomorphically onto A; otherwise π maps the face homeomorphically onto B. The resulting extension map $\pi : \mathcal{S}^2 \to \mathcal{S}^2$ is a branched covering map. If Γ is a 2-gon, then $\pi : \mathcal{S}^2 \to \mathcal{S}^2$ is a homeomorphism of $\mathcal{S}^2$; otherwise, Γ^* has faces more than 2 and the set of the branch points of π is equal to $\{b_1, \dots b_q\}$.

The branched covering map $\pi : \mathcal{S}^2 \to \mathcal{S}^2$ constructed from a given Speiser Γ is unique in the sense of Definition 2.2. Therefore, we have established the following result.

Theorem 3.6 (Correspondence theorem)**.** *There is a one-to-one correspondence between the equivalence classes of the branched covering maps $f : \mathcal{S}^2 \to \mathcal{S}^2$ and the equivalence classes of the Speiser graphs.*

In particular, we are more concerned with the following one-to-one correspondence.

Theorem 3.7. *There is a one-to-one correspondence between the equivalence classes of the totally ramified maps $f : \mathcal{S}^2 \to \mathcal{S}^2$ with three branched values and the equivalence classes of the Speiser graphs with valence $q = 3$ and having no 2-gon face.*

3.3. **Rational maps and Speiser graphs.** Clearly, each rational map is a branched covering map from $\mathcal{S}^2$ to $\mathcal{S}^2$. Vice versa, if $f : \mathcal{S}^2 \to \mathcal{S}^2$ is a branched covering map, then f is open, continuous and discrete. As a consequence of Stoïlow's theorem, we obtain the following statement.

Theorem 3.8 (Consequence of Stoïlow's theorem, [13])**.** *Each branched covering map $f : \mathcal{S}^2 \to \mathcal{S}^2$ of degree d is topologically equivalent to a rational map R in the sense that there exist a rational map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree d and two homeomorphisms ϕ and ψ from $\mathcal{S}^2$ to $\hat{\mathbb{C}}$ such that $\psi \circ f = R \circ \phi$.*

We finish this section by emphasizing that all the results proved in the first two subsections of this section hold for rational maps now.

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Let G_j, $j = 1, 2, 3, 4, 5$, be one of the five types of finite Kleinian groups and R_{G_j} be the projection map from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}/G_j$. Let us first have a good sense of the Speiser graph of R_{G_j}.

Clearly, if the order of G_1 is ν, then the Speiser graph of R_{G_1} is a convex polygon of 2ν sides on $\widehat{\mathbb{C}}$ with 0 in the interior and ∞ in the exterior. The left one in Figure 1 is the Speiser graph for R_{G_1} when $\nu = 3$.

If the order of G_2 is 2ν, then up to pre- and post-composition by Möbius transformations,

$$R_{G_2}(z) = \frac{1}{2}\left(z^\nu + \frac{1}{z^\nu}\right).$$

It follows that 0, 1 and ∞ are the three distinct critical values of R_{G_2}. There are two preimages for ∞, which are 0 and ∞. Then there are two convex polygons of 2ν sides in the Speiser graph of R_{G_2} and one is in the interior of the other; the vertices of these two polygons are labeled by $\circ$ and $\times$ interchangeably; by connecting the $\circ$ (resp. $\times$) vertices of the inside polygon to the $\times$ (resp. $\circ$) vertices of the outside polygon without intersections we obtain the Speiser graph of R_{G_2}. The right one in Figure 1 is the Speiser graph for R_{G_2} when $\nu = 2$.

Using the symmetry groups of a regular tetrahedron, a regular octahedron and a regular icosahedron, we obtain the Speiser graphs of R_{G_3}, R_{G_4} and R_{G_5} in Figures 4 and 5 respectively.

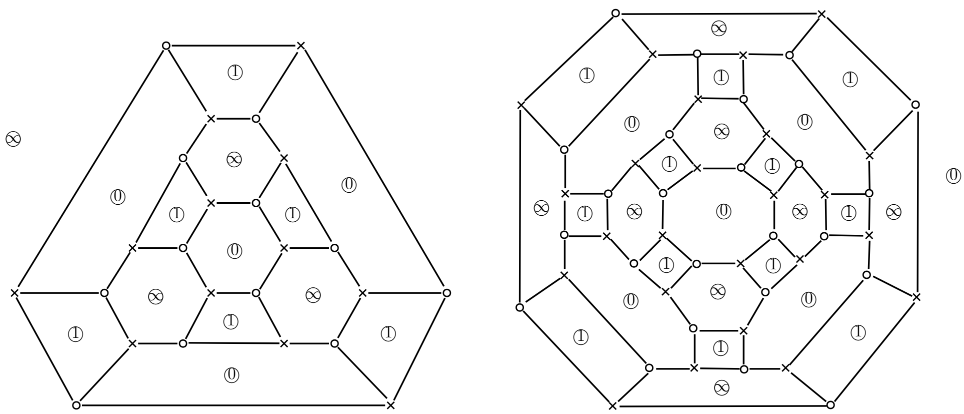

FIGURE 4. Speiser graphs of R_{G_3} (left) and R_{G_4} (right)

Now we start to prove Theorem 1.1.

Proof. Let R be a regularly ramified rational map and d be the degree of R. Assume that R has n distinct critical values denoted by w_j, $j = 1, 2, \cdots, n$. Denote by k_j the local degree of R at the preimages of w_j, where $1 \le j \le n$. By counting the number of the critical points of R, we obtain

$$\sum_{j=1}^{n}(k_j - 1)\frac{d}{k_j} = 2d - 2.$$

Then

$$\sum_{j=1}^{n}\frac{1}{k_j} = n + \frac{2}{d} - 2.$$

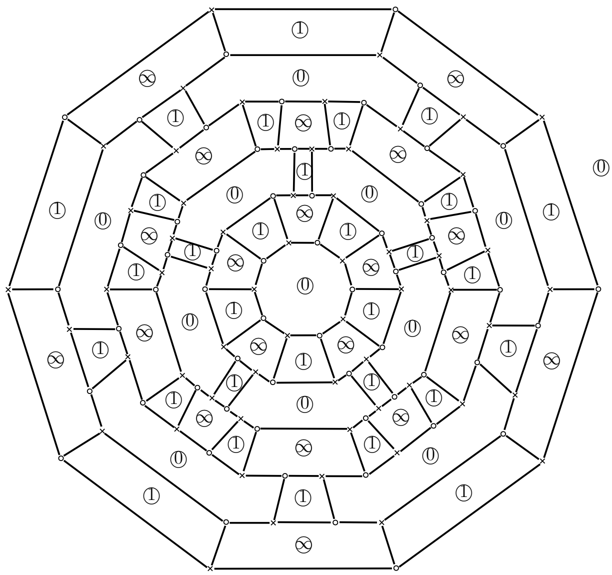

FIGURE 5. Speiser's graph of R_{G_5}

Clearly, $2 \le k_j \le d$ implies $\frac{1}{d} \le \frac{1}{k_j} \le \frac{1}{2}$ and $\frac{n}{d} \le \sum_{j=1}^{n} \frac{1}{k_j} \le \frac{n}{2}$. Thus,

$$\frac{n}{d} \le n + \frac{2}{d} - 2 \le \frac{n}{2}.$$

Therefore, $2 \le n < 4$ and then $n = 2$ or 3.

Case 1. $n = 2$. Then

$$\frac{1}{k_1} + \frac{1}{k_2} = \frac{2}{d}.$$

Since $k_1 \le d$ and $k_2 \le d$, it follows from the previous equation that $k_1 = k_2 = d$. Thus, if a regularly ramified rational map R has two distinct critical values, then $R(z) = z^d$ up to pre- and post-composition by Möbius transformations.

Case 2. $n = 3$. Then

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 1 + \frac{2}{d}. \tag{4.1}$$

Since $2 \le k_1, k_2, k_3 \le d$, this equation first implies that at least one of k_1, k_2 and k_3 is equal to 2. Without loss of generality, we assume that $k_1 = 2$. Then

$$\frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{2} + \frac{2}{d}. \tag{4.2}$$

To search for all positive integer solutions to the equation (4.2), we divide our discussion into two subcases.

Subcase 1. Assume that at least one of k_2 and k_3 is equal to 2. Without loss of generality, we may assume that $k_2 = 2$. Then $k_3 = \nu$ and $d = 2\nu$, where $\nu \ge 2$. It follows that R is a rational map of degree 2ν with three distinct critical values w_1, w_2 and w_3, and the local degree at the preimages of w_1 (resp. w_2 and w_3) is 2 (resp. 2 and ν). The Speiser graph of R contains two convex polygons of 2ν sides with vertices labeled with $\circ$ and $\times$ interchangeably. Since the local degree of the graph is 3, it follows that the Speiser graph of R is as the same as the Speiser graph of R_{G_2} when the order of G_2 is 2ν. Therefore, $R(z) = R_{G_2}(z)$ up to pre- and post-composition by Möbius transformations.

Subcase 2. Assume that both of k_2 and k_3 are greater than or equal to 3. Then $k_2 \geq 3$ and the equation (4.2) imply $k_3 < 6$. Similarly, we obtain $k_2 < 6$. So $3 \leq k_2, k_3 \leq 5$. Then by checking the value of d through the equation (4.2) for all possible values of k_2 and k_3, we conclude $k_2 = k_3 = 3$ and $d = 12$, $k_2 = 3$, $k_3 = 4$ and $d = 24$, or $k_2 = 3$, $k_3 = 5$ and $d = 60$. Let the critical values $w_1 = 1$, $w_2 = 0$ and $w_3 = \infty$. Consider the Speiser graph of R under each case.

Let $k_1 = 2$, $k_2 = k_3 = 3$ and $d = 12$. We start with a face $F_1(0)$ of 6 sides and containing 0, and the vertices are labeled by $\circ$ and $\times$ interchangeably. Since the local degree of each vertex is 3 and there is no loop, each vertex of $F_1(0)$ can have one new edge connected to a vertex labeled by a different symbol between $\circ$ and $\times$. We use the third condition for being a Speiser graph to label the six faces sharing an edge with $F_1(0)$ by 1 or ∞. Note that this process is uniquely performed. Denote the faces surrounding $F_1(0)$ by $F_1(\infty)$, $F_1(1)$, $F_2(\infty)$, $F_2(1)$, $F_3(\infty)$ and $F_3(1)$ in a counterclockwise order. On the boundary of the union U_1 of these seven faces, there are six vertices of local degree 3 and six vertices of local degree 2. The only way to extend U_1 to a Speiser graph is to add one edge to each of the six vertices of the local degree 2 and label the other vertex accordingly. These six new edges and the boundary of U_1 give a unique way to produce six new faces by adding six more edges without breaking the conditions for a Speiser graph. The six new faces are labeled by $F_2(0)$, $F_4(1)$, $F_3(0)$, $F_5(1)$, $F_4(0)$ and $F_6(1)$ in a counterclockwise order. The boundary of the union U_2 of U_1 with the six new faces is a polygon of 6 sides with vertices labeled by $\circ$ and $\times$ interchangeably. So we label the face outside U_2 by $F_4(\infty)$. Now we can see that the Speiser graph of R is as the same as the Speiser graph of R_{G_3}, shown as the left one in Figure 4. Therefore, $R(z) = R_{G_3}(z)$ up to pre- and post-composition by Möbius transformations.

If $k_1 = 2$, $k_2 = 3$, $k_3 = 4$ and $d = 24$, then we can prove similarly that the Speiser graph of R is as the same as the Speiser graph of R_{G_4}, shown as the right one in Figure 4. Thus, $R(z) = R_{G_4}(z)$ up to pre- and post-composition by Möbius transformations.

If $k_1 = 2$, $k_2 = 3$, $k_3 = 5$ and $d = 60$, then we can prove similarly that the Speiser graph of R is as the same as the Speiser graph of R_{G_5}, shown in Figure 5. Thus, $R(z) = R_{G_5}(z)$ up to pre- and post-composition by Möbius transformations. □

5. Rigidity for degrees ≤ 6

The proof of Theorem 1.2 is combinatorial and reduces to finding the patterns of corresponding Speiser graphs. Let f be a *totally ramified* rational map of degree d. It follows from [9, Proposition 4.3] or Theorem 3.2 that f has at most three critical values. Now we show that f is regularly ramified if $d \leq 6$.

Degree $d = 2$. In this case, f has two critical points and hence two critical values. It follows from Theorem 3.4 that f is regularly ramified. Moreover, f is equal to $z \to z^2$ up to pre- and post-compositions by Möbius transformations.

Degree $d = 3$. In this case, the local degree of f at each critical point is 3. Thus, by definition, f is regularly ramified.

Degree $d = 4$. The local degree of a critical point c is either 2 or 4. If c has local degree 2, then there is exactly one another critical point c' of local degree 2 mapped to the same critical value as c; if c has local degree 4, there is no other critical point mapped to the same critical value as c. In either case, the rational map is regularly ramified.

Degree $d = 5$. In this case, each critical value v of f has two or one preimages. By Theorem 3.5, we know v cannot have two preimages. So each critical value v has only one preimage. By counting the total number of critical points with multiplicities, we conclude that f has two critical values. Then by Theorem 3.4, f is regularly ramified.

Degree $d = 6$. By Theorem 3.5, it suffices to prove f to be a regularly ramified under the assumption that none of the critical values of f has two preimages. Then each critical value v has either three preimages with local degrees all equal to 2 or one preiamge with local degree equal to 6. By Theorem 3.2, f has two or three critical values. Now by counting the number of the critical points mapped to each critical value, the only pattern to have the total number of the critical points equal to 10 is that f has two critical values and each of them has only one preimage. So f is again regularly ramified.

6. Existence of totally but not regularly ramified rational map

In this section, we prove Theorem 1.3. Because of Theorem 3.6, the proof is reduced to find an appropriate Speiser graph of valence 3 and having no 2-gon face for each $d > 6$. We use different procedures to achieve this goal for odd degrees and even degrees. For odd degrees, we first manually work out a Speiser graph for $d = 7$ and then we modify it to a Speiser graph for $d = 9$ and inductively modify the new one to the next with the degree increased by 2. In this step, we apply the fact that every totally ramified rational map of an odd degree with three critical values is not regularly ramified. For an even degree d, in order to make sure that an obtained Speiser graph for degree d is not a Speiser graph of a regularly ramified rational map of an even degree, we modify the Speiser graph of the map R_{G_2}, given in the introduction with $2\nu = q$, to a Speiser graph for a totally but not regularly ramified rational map.

Let us first introduce the following notion.

Definition 6.1 (Signature)**.** Let f be a rational map and v a critical value of f. Then the *signature* of v is a sequel of the local degrees of f at the preimages of v. The signatures of all critical values of f are called the *signature pattern* of f or the signature pattern of the Speiser graph of f, or simply called the signature pattern of a Speiser graph. We also call the degree of f as the degree of the Speiser graph.

For instance, the signature of 0 or ∞ for the function $f(z) = z^d$ is (d). The signatures of 0 and ∞ for $f(z) = \frac{1}{2}(z^4 + z^{-4})$ are both $(4, 4)$ and the ones of the critical values 1 and -1 are both $(2, 2, 2, 2)$.

We are concerned with totally ramified rational maps with three critical values. In this case, there is no need to specify which critical value a signature is for since they are equivalent in the sense of Definition 2.2.

Let us first show how to work out a Speiser graph Γ_7 for a totally ramified rational map f of degree 7. Using Theorem 3.5, we know each critical value of f has three preimages and hence the signature of each critical value is $(2, 2, 3)$. Thus, Γ has three hexagon faces labeled by 0, 1 and ∞ respectively, two quadrilateral faces labeled by 0, two by 1 and two by ∞. Using the three conditions of a Speiser

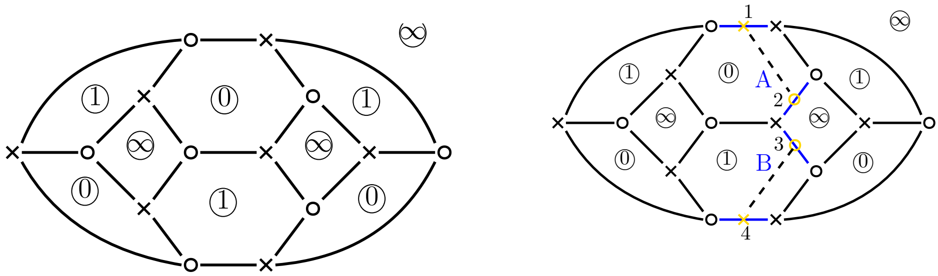

FIGURE 6. Left: Speiser graph of a totally but not regularly ramified rational map of degree 7. Right: Surgeries performed on the left one to obtain a Speiser graph of degree 9. The resulting Speiser graph is shown as the left one in Figure 8.

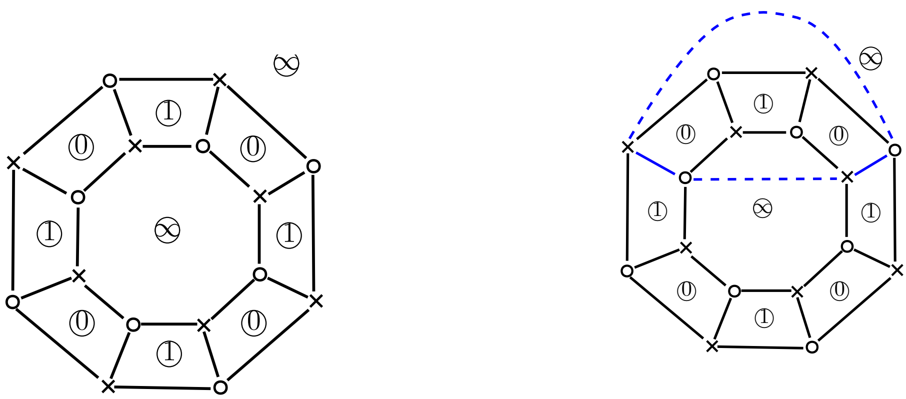

FIGURE 7. Left: The Speiser graph of $f(z) = z^4 + \frac{1}{z^4}$, a regularly ramified rational map of degree 8. Right: Surgery performed on the Speiser graph of $f(z) = z^4 + \frac{1}{z^4}$. The resulting Speiser graph is shown as the right one in Figure 9.

graph and the no 2-gon face condition, we work out such a Speiser graph shown as the left one in Figure 6.

Now we use Γ_7 to construct a Speiser graph which corresponds to a totally ramified rational map of degree 9. Consider the left Speiser graph in Figure 6, on which some surgeries are performed as shown on the right figure in Figure 6. Two hexagons are marked as A and B. Two edges on the boundary of A are denoted by 1 and 2 and two edges on the boundaries of B are denoted by 3 and 4. To obtain a Speiser graph of degree 9 with no 2-gon face, we put exactly one vertex on each edge of 1, 2, 3, 4. The types of the four new added vertices are determined by the vertex on the left-hand side. This means that we need to put $\times$-vertices on the edges 1 and 4, $\circ$-vertices on the edges 2 and 3. After this, we change the types of vertices on the right part of the graph accordingly. The last step is to connect the two new vertices on 1 and 2 by an edge in A, and connect the two new vertices on

3 and 4 by an edge in B. The resulting graph Γ_9 is the Speiser graph of a totally ramified rational map of degree 9.

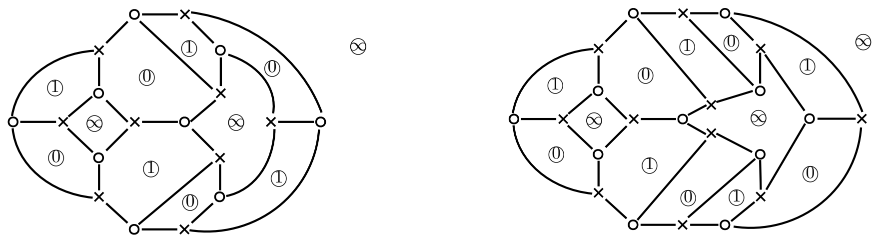

FIGURE 8. From a Speiser graph of degree 9 to a Speiser graph of degree 11

To obtain the Speiser graph of a totally ramified rational map of degree 11, we do the same surgery to Γ_9 as the one on Γ_7, which is shown as the right-hand one on Figure 8. Inductively, we obtain the Speiser graph Γ_d of a totally ramified rational map of each odd degree $d \geq 7$.

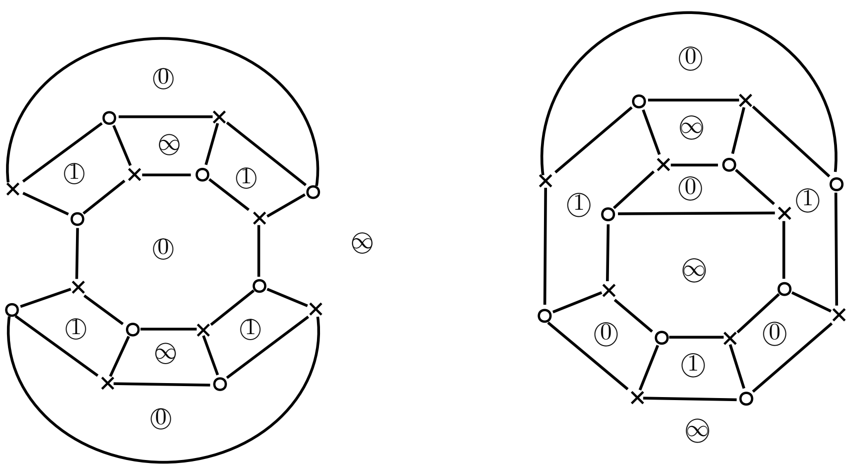

FIGURE 9. Speiser graphs of totally but not regularly ramified rational maps of degree 8

It remains to find Speiser graphs for constructing totally but not regularly ramified rational maps of even degrees greater than or equal to 8. As mentioned at the beginning, in this step we do surgeries on the Speiser graph of regularly ramified rational map R_{G_2} of degree ≥ 8. The surgeries for the degree 8 case are shown on the right graph in Figure 7. We cut off the two blue solid edges and connect their vertices by blue dashed edges. Then we mark their faces accordingly. One can easily see that the resulting new graph (shown as the right one in Figure 9) is a Speiser graph for constructing a totally but not regularly ramified rational map of degree 8. Similarly, we obtain another Speiser graph for constructing a different (in the sense of Definition 2.1) totally but not regularly ramified rational map of degree 8, which is shown as the left one in Figure 9.

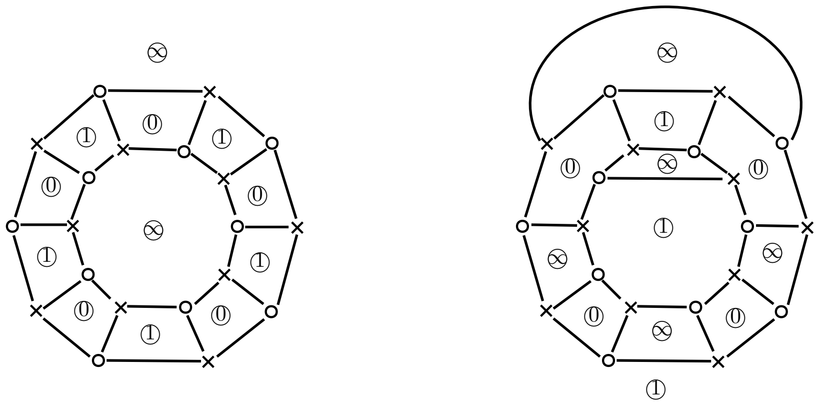

FIGURE 10. Speiser graphs of degree 10. The left one is for $f(z) = z^5 + \frac{1}{z^5}$ with signature pattern $(5,5)$, $(2,2,2,2,2)$, $(2,2,2,2,2)$; the right one is for a degree 10 totally but not regularly ramified rational map with signature pattern $(2,2,3,3)$, $(2,2,2,2,2)$, $(2,4,4)$.

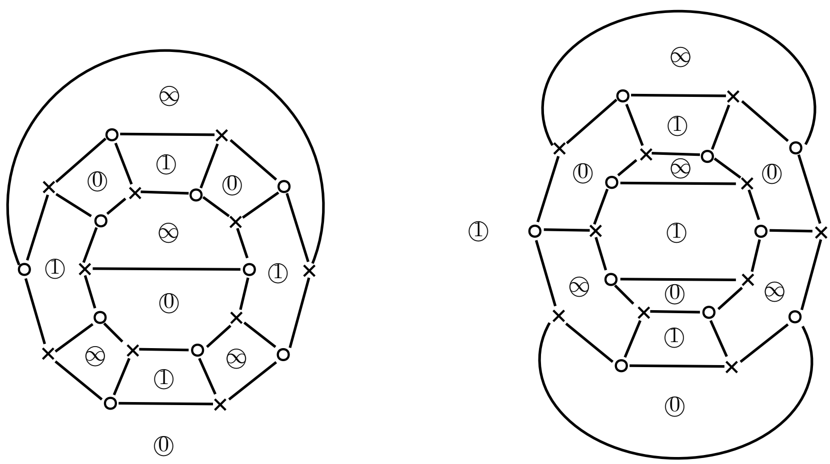

FIGURE 11. Different constructions of a Speiser graphs for a degree 10 totally but not regularly ramified rational map with signature pattern $(2,2,3,3)$, $(2,2,3,3)$, $(2,2,3,3)$

Generally, we start with the Speiser graph of $f_d(z) = z^d + 1/z^d$ with $d \geq 4$. Then we perform similar surgeries as applied to the Speiser graph of f_4 to obtain a Speiser graph for constructing a totally but not regularly ramified rational map of degree $2d$. Shown in Figure 10 is the case for $d = 5$. Note that as the degree increases, there are more choices for performing similar surgeries. For example, by using similar surgeries two different examples of Speiser graphs are obtained for constructing degree 10 totally but not ramified rational maps, which are shown in Figure 11.

7. Some results on counting totally ramified rational maps

In Section 4, we have obtained a complete classification of all regularly ramified rational maps (Theorem 1.1); in Section 5, we have showed that any totally ramified rational map of degree ≤ 6 is regularly ramified (Theorem 1.2); in Section 6, we have proved the existence of a totally ramified but not regularly ramified rational map of degree d for each $d \geq 7$ (Theorem 1.3). In this section, we explore the classification of totally ramified but not regularly ramified rational maps of degree d for $d > 7$. We prove the classification results for $d = 7, 8, 9, 10$, from which we conclude Theorem 1.4. The work of this section is divided into the proofs of the following four propositions. Some open problems are raised at the end of this section.

Let $T_d(n)$ be the collection of the equivalence classes of totally ramified rational maps with n critical values and of degree d, and denote its cardinality by $|T_d(n)|$.

Proposition 7.1 (Classification for degree 7)**.** $|T_7(3)| = 1$.

Proof. By definition, it is clear that possible signatures of critical values are $(2, 5)$, $(3, 4)$ and $(2, 2, 3)$. By Theorem 3.5, the first two cases cannot happen. Thus, the only possible signature pattern for a totally ramified rational map of degree 7 is $(2, 2, 3)$, $(2, 2, 3)$, $(2, 2, 3)$. If such a rational map f exists, then the Speiser graph Γ_f has two 4-gon faces and one 6-gon face labeled with 0, 1 and ∞ respectively. This signature pattern is achieved by the Speiser graph shown as the left one in Figure 9. More delicately, it is unique. This is concluded by manually checking all the nine faces can not form any other Speiser graph. □

Proposition 7.2 (Classification for degree 8)**.** $|T_8(3)| = 3$.

Proof. Let v_j, $j = 1, 2, 3$, be the critical values of f and i_j be the number of the preimages of v_j. Using Theorem 3.1 or the Riemann-Hurwitz formula, we know $i_1 + i_2 + i_3 = d + 2 = 10$. The totally ramified condition implies that $i_j \leq 4$ for $j = 1, 2, 3$. Then each $i_j \geq 2$ and only one of i_1, i_2 and i_3 is less than or equal to 3. Without loss of generality, we assume that $2 \leq i_1 \leq 3$. If $i_1 = 2$, then $i_2 = i_3 = 4$; if $i_1 = 3$, then we may assume $i_2 = 3$ and $i_3 = 4$.

Using Theorem 3.5, we know if $i_1 = 2$, then the signature of v_1 has to be $(4, 4)$ and hence the signatures of v_2 and v_3 are the same $(2, 2, 2, 2)$. For this signature pattern, we have proved in Section 4 that there is a unique a Speiser graph realizing it, which is the signature pattern of R_{G_2} with degree 8. See the left one in Figure 6.

If $i_1 = 3$, there are three possible signature patterns in the sense of Definition 2.1. They are: $(2, 2, 4)$, $(2, 2, 4)$, $(2, 2, 2, 2)$; $(2, 3, 3)$, $(2, 3, 3)$, $(2, 2, 2, 2)$; $(2, 3, 3)$, $(2, 2, 4)$, $(2, 2, 2, 2)$. The first two signature patterns are realized by unique Speiser graphs of valence 3 and having no 2-gon face under the equivalence relation given in Definition 2.4, which are given in Figure 9.

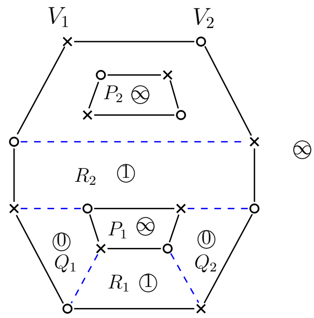

FIGURE 12. No Speiser graph for the signature pattern $(2,3,3)$, $(2,2,4)$, $(2,2,2,2)$

Now we show that the third signature pattern cannot realized by a Speiser graph of valence 3 and having no 2-gon face. Suppose that such a Speiser graph exists. We further assume that the signatures of ∞, 0 and 1 are $(2,2,4)$, $(2,2,2,2)$ and $(2,3,3)$, respectively. Draw an octagon P with the outside labeled by ∞. There are two quadrilaterals P_1 and P_2 inside P labeled by ∞ sharing no vertex with each other and sharing no vertex with P. It follows that all 16 vertices, half of which are labeled by $\times$ and the other half are labeled by $\circ$, are used as the vertices of P, P_1 and P_2 (see Figure 12). So the ending points of other edges of the Speiser graph are among these sixteen vertices. Since the signature of 0 is $(2,2,2,2)$, there should be four quadrilaterals labeled by 0 in the Speiser graph and two of them are adjacent to a pair of the opposite sides of P_1 or P_2. Without loss generality, we assume that these two quadrilaterals are denoted by Q_1 and Q_2 and they are adjacent to P_1. Then at least one of Q_1 and Q_2 shares one edge with P and one edge with P_1, which we denote by Q_1. In the meantime, there are two faces R_1 and R_2 labeled by 1 and adjacent to the other pair of opposite sides of P_1 and at least one of them is a quadrilateral, which we denote by R_1. Then R_1 shares edges with P_1 and P. It follows that Q_2 shares edges with P_1 and P. Since R_2 is a hexagon, it follows that $\tilde{P} = P \setminus (P_1 \cup Q_1 \cup Q_2 \cup R_1 \cup R_2)$ is a quadrilateral containing P_2 in its interior (see Figure 12). Let v_1 and v_2 be the two vertices of P that are not on $P_1 \cup Q_1 \cup Q_2 \cup R_1 \cup R_2$. In order to satisfy the homogeneity and biparity conditions of a Speiser graph, the remaining three edges have endpoints among v_1, v_2 and the vertices of P_2 and each of them is required to connect a $\times$-vertex to a $\circ$-vertex. These conditions force the resulting graph to be a Speiser graph with a 2-gon face. This is a contradiction to Theorem 3.3. Therefore, the signature pattern $(2,3,3)$, $(2,2,4)$, $(2,2,2,2)$ cannot be realized by a Speiser graph of valence 3 and having no 2-gon face. □

Proposition 7.3 (Classification for degree 9). $|T_9(3)| = 3$.

Proof. Using the same notion and the same idea in the proof of the previous proposition, we obtain all possible signature patterns. They are: $(2,2,2,3)$, $(2,2,2,3)$, $(2,3,4)$; $(2,2,2,3)$, $(2,2,2,3)$, $(3,3,3)$; $(2,2,2,3)$, $(2,2,2,3)$, $(2,2,5)$. All of them are uniquely realized by Speiser graphs of valence 3 and having no 2-gon face, which are given as the left one in Figure 8 and the two in Figure 13 respectively. □

Proposition 7.4 (Partial classification for degree 10). $|T_{10}(3)| \geq 7$.

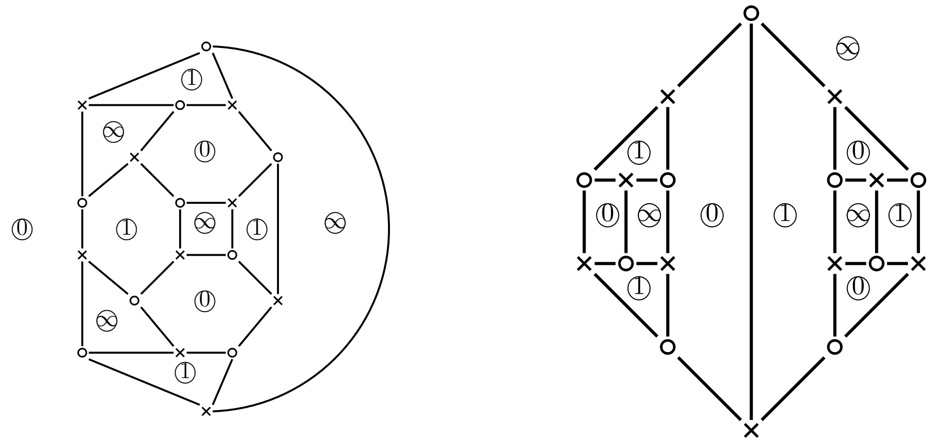

FIGURE 13. The Speiser graphs of degree 9 with signature pattern $(2,2,2,3)$, $(3,3,3)$, $(2,2,2,3)$ (left) and $(5,2,2)$, $(2,2,2,3)$ and $(2,2,2,3)$ (right)

Proof. Using Theorem 3.1 or the Riemann-Hurwitz formula, we can work out all possible signature patterns, which are listed as follows.

$(5,5)$, $(2,2,2,2,2)$, $(2,2,2,2,2)$ (see Figure 10 Left for Speiser graph)

$(2,2,3,3)$, $(2,2,2,2,2)$, $(2,4,4)$ (see Figure 10 Right for Speiser graph)

$(2,2,3,3)$, $(2,2,3,3)$, $(2,2,3,3)$ (see Figure 11 for Speiser graph)

$(2,2,3,3)$, $(2,2,2,2,2)$, $(3,3,4)$ (see Figure 14 Left for Speiser graph)

$(2,2,2,4)$, $(2,2,2,2,2)$, $(2,3,5)$ (see Figure 14 Right for Speiser graph)

$(2,2,3,3)$, $(2,2,2,2,2)$, $(2,2,6)$ (see Figure 15 Left for Speiser graph)

$(2,2,2,4)$, $(2,2,2,4)$, $(2,2,3,3)$ (see Figure 15 Right for Speiser graph)

$(2,2,3,3)$, $(2,2,2,2,2)$, $(2,3,5)$ (no Speiser graph)

$(2,2,2,4)$, $(2,2,2,2,2)$, $(2,4,4)$ (no Speiser graph)

$(2,2,2,4)$, $(2,2,2,2,2)$, $(3,3,4)$ (no Speiser graph)

$(2,2,2,4)$, $(2,2,2,4)$, $(2,2,2,4)$ (no Speiser graph)

$(2,2,3,3)$, $(2,2,3,3)$, $(2,2,2,4)$ (no Speiser graph)

The first seven of them can be realized by Speiser graphs of valence 3 and having no 2-gon face, which are shown in the figures indicated in the parentheses respectively, and the other five cannot be realized by any Speiser graph of valence 3 and having no 2-gon face. Since we don't know if the Speiser graph for each signature pattern is unique, it follows that $|T_{10}(3)| \geq 7$. □

From the work to prove the previous propositions, we can see that using Theorem 3.1 or Riemann-Hurwitz formula, one can find all possible signature patterns, which we call signature pattern candidates. Furthermore, let us call a signature pattern candidate regularly if each sequel in the pattern consists of a constant. In Section 4, we have shown that if a signature pattern candidate is regular, then it can be realized by a unique Speiser graph of valence 3 and having no 2-gon face. On the

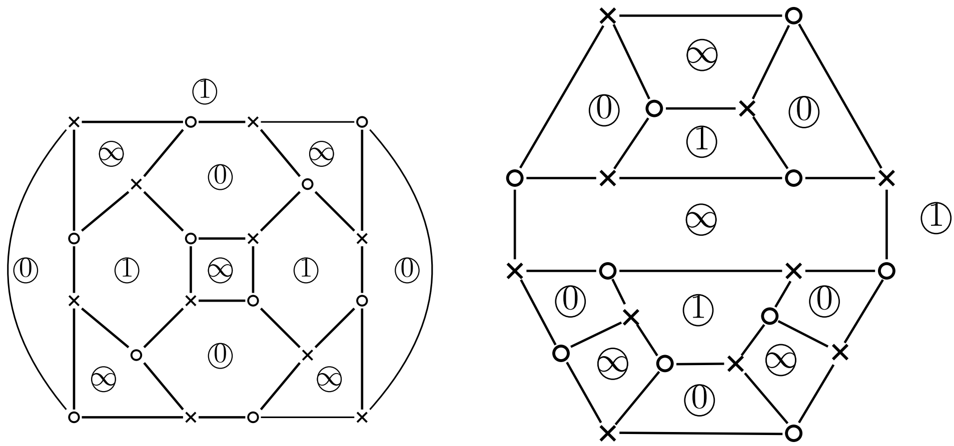

FIGURE 14. Speiser graphs of degree 10 with signature patterns $(2,2,3,3)$, $(2,2,2,2,2)$, $(3,3,4)$ (left) and $(2,2,2,4)$, $(2,2,2,2,2)$, $(2,3,5)$ (right)

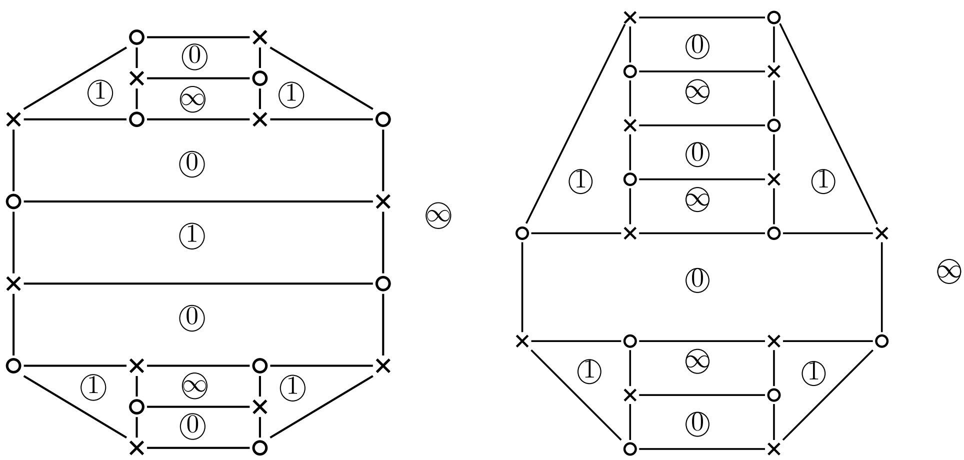

FIGURE 15. Speiser graph of degree 10 with signature pattern $(2,2,3,3)$, $(2,2,2,2,2)$, $(2,2,6)$ (left) and $(2,2,2,4)$, $(2,2,2,4)$, $(2,2,3,3)$ (right)

other hand, the work in this section has shown that non-regular signature pattern candidates cannot be always realized by Speiser graphs of valence 3 and having no 2-gon face. We like to finish this section with the following questions on the signature pattern candidates for constructing totally but not regularly ramified rational maps with three critical points and of degree ≥ 10.

(1) How to determine whether or not a signature pattern candidate can be realized by a Speiser graph of valence 3 and having no 2-gon face?
(2) Is such a Speiser graph unique if exists?
(3) How does $|T_d(3)|$ grow as d goes to ∞?

8. Explicit formulas of totally but not regularly ramified rational maps of degrees 7 or 8

In this section, we show how to find the formulas for totally but not regularly ramified rational maps of degree 7 or 8.

Example 8.1 (Degree 7)**.** The rational map given by

$$f(z) = \frac{2i}{49}\frac{(z+i)^3(z-a)^2(z+\bar{a})^2}{(z^2-b^2)^2}, \text{ where } a = \frac{3}{4}\left(\sqrt{7}+i\right) \text{ and } b = \frac{3}{\sqrt{7}}, \tag{8.1}$$

is totally but not regularly ramified.

In the following, instead of trying to prove that f is totally but not regularly ramified, we present how to find this formula.

Through postcomposition by a Möbius transformation, we may assume that the three critical values of f are arranged at 0, 1 and ∞. Using the Speiser graph Γ_f (the left one in Figure 6) and through precomposition by a Möbius transformation, we may assume that the zeros of f are arranged at $-i$, a and $-\bar{a}$ with multiplicities equal to 3, 2 and 2 respectively and the poles of f are arranged at b, $-b$ and ∞ with multiplicities equal to 2, 2 and 3 respectively. Then

$$f(z) = \lambda\frac{(z+i)^3(z-a)^2(z+\bar{a})^2}{(z^2-b^2)^2},$$

where a, b (real) and λ are to be determined.

The derivative f' of f can be written as

$$f'(z) = \frac{\lambda(z+i)^2(z-a)(z+\bar{a})P(z)}{(z+b)^3(z-b)^3},$$

where $P(z) = c_4z^4 + c_3z^3 + c_2z^2 + c_1z + c_0$ is a degree 4 polynomial with coefficients $c_4 = 3$, $c_3 = -a+\bar{a}$, $c_2 = a\bar{a} + 2ia - 2i\bar{a} - 7b^2$, $c_1 = -5\bar{a}b^2 + 5ab^2 + 4ia\bar{a} - 4ib^2$, $c_0 = -2i\bar{a}b^2 + 3a\bar{a}b^2 + 2iab^2$.

In order to determine the values for a, b and λ, we use the conditions that i is a double root of P, $-a$ and $\bar{a}$ are simple roots of P, and f takes the same value at $-a$, $\bar{a}$ and i.

Using the synthetic division to obtain the remainder of P divided by $z-i$, we have the first equation:

$$\left(5b^2-1\right)i\,(a-\bar{a}) - 6a\bar{a} + \left(18b^2+12\right) = 0. \tag{8.2}$$

Again using the synthetic division to obtain the remainder of $\frac{P(z)}{z-i}$ when divided by $z-i$, we obtain the second equation:

$$\left(7b^2-1\right)i\,(a-\bar{a}) + \left(3b^2-5\right)a\bar{a} + \left(11b^2+3\right) = 0. \tag{8.3}$$

In the mean time, we obtain

$$Q(z) = \frac{P(z)}{(z-i)^2} = 3z^2 + (-a+\bar{a}+6i)\,z + \left(a\bar{a} - 7b^2 - 9\right). \tag{8.4}$$

Note that $-a$ and $\bar{a}$ are roots of Q. Using the relationship between roots and coefficients, we obtain the third and fourth equations:

$$-a+\bar{a} = -\frac{-a+\bar{a}+6i}{3} \tag{8.5}$$

and

$$-a\,\bar{a} = \frac{a\bar{a} - 7b^2 - 9}{3}. \tag{8.6}$$

Using the equations (8.2) and (8.3), we can express

$$i\,(a - \bar{a}) = -\frac{6\left(9b^2 - 7\right)}{15b^2 - 1} \tag{8.7}$$

and

$$a\,\bar{a} = \frac{71b^2 - 9}{15b^2 - 1}. \tag{8.8}$$

Clearly, the equation (8.5) implies $a - \bar{a} = \frac{3}{2}i$. Then the equation (8.7) implies $b^2 = \frac{9}{7}$. Finally, either the equation (8.8) or (8.6) implies $a\bar{a} = 9/2$. Therefore, we can take $a = \frac{3}{4}(\sqrt{7} + i)$ and $b = \frac{3}{\sqrt{7}}$.

It remains to verify that f takes the same value at i, $-a$ and $\bar{a}$. Clearly,

$$\begin{aligned} f(i) &= \lambda \frac{-8i\,[-1 - a\bar{a} - i\,(a - \bar{a})]^2}{(b^2 + 1)^2} \\ &= \lambda \frac{-8i\left[-1 - \frac{9}{2} - i\left(\frac{3}{2}i\right)\right]^2}{\left(\frac{9}{7} + 1\right)^2} = -\frac{49i}{2}\lambda. \end{aligned}$$

Now we compute the value $f(-a)$. At first,

$$f(-a) = \lambda \frac{(-a + i)^3\,(2a)^2\,(-a + \bar{a})^2}{a^2 - b^2},$$

where $(-a+i)^3 = \frac{1}{16}\left(47i - 45\sqrt{7}\right)$, $4a^2 = \frac{9}{4}\left(6 + 2\sqrt{7}i\right)$, $\left(a^2 - b^2\right)^2 = \frac{81}{32(49)}(-87 + 91\sqrt{7}i)$. Thus,

$$f(-a) = \lambda\,\frac{\frac{1}{16}\left(47i - 45\sqrt{7}\right)\frac{9}{4}\left(6 + 2\sqrt{7}i\right)\left(-\frac{9}{4}\right)}{\frac{81}{32(49)}\left(-87 + 91\sqrt{7}i\right)} = -\frac{49i}{2}\lambda.$$

Similarly, we can verify that $f(\bar{a}) = -\frac{49i}{2}\lambda$ as well. By taking $\lambda = \frac{2i}{49}$, we can see that $f(i) = f(-a) = f(\bar{a}) = 1$. Thus, f, given by (8.1), is a totally ramified rational map, which is not regularly ramified because of the nature of the Speiser graph.

As we have known, the quotient map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}/G$ is a regularly ramified rational map of degree 8 if G is a dihedral group of order 8 and its Speiser graph is the right one in Figure 9. We also know that the degree 8 rational maps with Speiser graphs equal to the ones in Figure 9 are totally but not regularly ramified. In the following, we provide formulas for such degree 8 rational maps.

Example 8.2 (Degree 8)**.** Up to pre- and post-composition by Möbius transformations, the degree 8 rational map whose Speiser graph is equal to the left one in Figure 9 is given by

$$g(z) = -\frac{1}{4}\left(\frac{z^2(z^2 + 2)}{z^2 + 1}\right)^2. \tag{8.9}$$

Using the symmetry in the left Speiser graph in Figure 9, we may assume that a rational map g with this Speiser graph has zeros arranged at 0 (of order 4) and $\pm ai$ (of order 2), and poles arranged at ∞ (of order 4) and $\pm bi$ (of order 2), where a and b are two positive real numbers and $a \neq b$. Then g is of the form

$$g(z) = \lambda \frac{z^4(z^2+a^2)^2}{(z^2+b^2)^2},$$

where λ is a complex number. Then

$$g'(z) = 4\lambda \frac{z^3\left(z^2+a^2\right)\left(z^4+2b^2z^2+a^2b^2\right)}{(z^2+b^2)^3}. \tag{8.10}$$

Let z_0 be a root of the equation given by

$$z^4+2b^2z^2+a^2b^2=0. \tag{8.11}$$

Then $\frac{z_0^2+a^2}{z_0^2+b^2} = -\frac{z_0^2}{b^2}$ and $z_0^2 = -b\left(b \pm \sqrt{b^2-a^2}\right)$. Thus,

$$g(z_0) = \lambda\left[\frac{z_0^2\left(z_0^2+a^2\right)}{z_0^2+b^2}\right]^2 = \lambda\left[z_0^2\left(-\frac{z_0^2}{b^2}\right)\right]^2 = \lambda\frac{z_0^8}{b^4} = \lambda\left(b \pm \sqrt{b^2-a^2}\right)^4.$$

The condition for g to be a totally ramified rational map is that g takes the same values at all zeros of the equation (8.11). To satisfy this equation, we let $a^2 = 2b^2$. Then

$$g(z_0) = \lambda(b \pm bi)^4 = -\lambda 4b^4.$$

Take $b = 1$ and $\lambda = -\frac{1}{4}$. Then $g(z_0) = 1$ at any root z_0 of the equation (8.11). Therefore, the rational map g given in (8.9) is a degree 8 totally ramified, but not regularly ramified, rational map.

Example 8.3 (Degree 8)**.** There exists a nonzero real number a such that

$$f(z) = \frac{z^2\left(z^2-a^2\right)^3}{\left(z^2+\frac{1}{a^2}\right)^3} \tag{8.12}$$

has the signature pattern $(2,3,3)$, $(2,3,3)$, $(2,2,2,2)$ and hence the reduced Speiser graph Γ_f is the right one in Figure 9. The numerical approximation of a is $1.6818\cdots$.

The pattern presented in the right Speiser graph in Figure 9 helps us to reach that the map f given by (8.12) may contain a desired rational map.

Clearly, both signatures of 0 and ∞ for f are $(2,3,3)$. We need to find a nonzero real number a such that the other critical points of f are simple and they have the same critical value.

From the equation $f'(z)=0$, we know that the other critical points c of f satisfy

$$z^4 + 2\left(a^2+\frac{2}{a^2}\right)z^2 - 1 = 0. \tag{8.13}$$

Denote by

$$b_1 = -\left(a^2+\frac{2}{a^2}\right) + \sqrt{a^4+\frac{4}{a^4}+5} \text{ and } b_2 = -\left(a^2+\frac{2}{a^2}\right) - \sqrt{a^4+\frac{4}{a^4}+5}.$$

When $a \neq 0$, the other critical points of f are $\pm\sqrt{b_1}$ and $\pm\sqrt{b_2}$, which are pairwise different. It remains to find a nonzero real number a such that f has the same value at these four critical points; that is, a satisfies the following equation

$$\frac{b_1\left(b_1 - a^2\right)^3}{\left(b_1 + \frac{1}{a^2}\right)^3} = \frac{b_2\left(b_2 - a^2\right)^3}{\left(b_2 + \frac{1}{a^2}\right)^3}, \tag{8.14}$$

which leads to the existence and the numerical approximation of a.

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