SINGLE AND DOUBLE TORAL BAND FATOU COMPONENTS IN MEROMORPHIC DYNAMICS

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Abstract. We analyze the existence and types of unbounded Fatou components for elliptic functions and other meromorphic functions with doubly periodic Julia sets. We show that apart from Herman rings and Siegel disks, all types of dynamics can occur in these domains, which are called toral bands. We show that toral bands are not necessarily periodic, and we give results about the number of distinct residue classes of critical points in each toral band.

1. Introduction

An elliptic function is a doubly periodic meromorphic map, which, in many respects makes the dynamics under iteration similar to the dynamics of an iterated simply periodic meromorphic function such as \( \tan z \), \( \sin z \), and \( \cos z \). The parallels go further in the sense that there are many “trig identities” that hold for the basic elliptic building blocks, the Weierstrass \( \wp \) function and its derivative \( \wp' \). The dynamics of \( \wp \) and \( \wp' \) also have some connections to iteration of rational maps of the Riemann sphere, as was known to Lattès as early as 1918. The earliest work on iterating these functions with a view to understanding their Julia and Fatou sets seems to have been started in the early 1990s in a series of papers starting with [1], and including extensive results by Kotus and Urbánski [24] and [25], and a plethora of examples and results appearing in [14]–[23]. Iteration of meromorphic functions has been widely studied by many, for example [1]–[4], [6], [9], and [10].

If \( f \) is an elliptic function with period lattice \( \Lambda \), the dynamics of \( \{ f^n \}_{n \in \mathbb{N}} \) are captured by looking at a single fundamental region. However some components extend over infinitely many fundamental regions. We show that for elliptic functions, unbounded Fatou components, called toral bands, can occur concurrently with many types of dynamics. A few closely related definitions of a toral band appear in papers [16] and [15]. We settle on the definition that captures the essence of all of the variations given while describing the most general Fatou component that can be called a toral band.

Definition 1.1. Assume that \( f_{\Lambda} = f \) is a meromorphic function with a doubly periodic Fatou set over \( \Lambda = [\lambda_1, \lambda_2] \). Then

1. a Fatou component \( A_0 \) of \( f_{\Lambda} \) is a toral band if \( A_0 \) contains an open subset \( U \) which is simply connected in \( \mathbb{C} \), but \( U \) projects to a topological band around the torus \( \mathbb{C}/\Lambda \) containing a homotopically nontrivial curve.
(2) A toral band $A_0$ is a double toral band if it projects to a set in the torus $\mathbb{C}/\Lambda$ that contains closed paths that generate the fundamental group $\pi_1(\mathbb{C}/\Lambda)$.

(3) We say $A_0$ is a vertical toral band if there is a line $L \subset A_0$ where $L$ is parallel to the imaginary axis; and $A_0$ is a horizontal toral band if there is a line $L \subset A_0$ where $L$ is parallel to the real axis.

(4) If $A_0$ is a toral band that is not double, it is called a single toral band.

(5) A toral band is nonperiodic if it maps onto a cycle of Fatou components but is not part of the cycle.

There are meromorphic functions of $\mathbb{C}$ that have doubly periodic Julia and Fatou sets but are not elliptic; for example any map of the form

\begin{equation}
G(z) = z + H(z), \quad H \text{ elliptic,}
\end{equation}

has that property. If $H$ is elliptic with period lattice $\Lambda$, generated by $\lambda_1, \lambda_2 \in \mathbb{C}$, then $G$ satisfying (1.1) commutes with the $\mathbb{Z}^2$ action given by: $\phi^{(m,n)}_\Lambda(z) = z + m\lambda_1 + n\lambda_2$. That is,

\begin{equation}
G \circ \phi^{(m,n)}_\Lambda(z) = \phi^{(m,n)}_\Lambda \circ G(z)
\end{equation}

for all $(m, n) \in \mathbb{Z}^2$ since $H \circ \phi^{(m,n)}_\Lambda = H$. We say a meromorphic function $f$ has a doubly periodic Fatou set (Julia set) if there exists some lattice $\Lambda = [\lambda_1, \lambda_2]$ such that

\begin{equation}
\phi^{(m,n)}_\Lambda F(f) = F(f), \quad \text{and} \quad \phi^{(m,n)}_\Lambda J(f) = J(f),
\end{equation}

where $F(f)$ and $J(f)$ denote the Fatou and Julia sets of $f$. The properties in (1.2) and (1.3) are independent of the generators of $\Lambda$ chosen.

Recall that for an elliptic function, all Fatou components are preperiodic ([6, Theorem 12]), though we show that toral bands can occur for functions satisfying (1.1) as well. There are only finitely many critical values for elliptic functions, causing them to exhibit dynamics like those of rational maps under iteration [15,17]. However, their double periodicity and the singularity at $\infty$ create some differences. In particular, when unbounded Fatou components occur, they represent dynamical properties unique to elliptic functions.

In this paper we analyze unbounded Fatou components that can arise for an iterated elliptic function and address the fundamental question of whether or not toral bands represent new dynamics of meromorphic functions, within the known framework set out by e.g., [6] for meromorphic functions.

Let $\hat{\mathbb{C}}$ denote the Riemann sphere. Recall that $J(f)$ is a Cantor Julia set if $J(f)$ is a compact, totally disconnected, perfect subset of $\hat{\mathbb{C}}$. Up to now, double toral bands, which often yield Cantor Julia sets, have been studied the most ([16], [20], [21], [13]), and there are examples of elliptic functions with double toral bands where $J(f)$ is not Cantor [22]. It has also been shown for $\wp_\Lambda$, that if $\Lambda$ is a square lattice or triangular lattice then $J(\wp_\Lambda)$ is connected [8,16]. On the other hand, by adding a constant to $\wp_\Lambda$, just as in the case of quadratic polynomials, the connectivity of $J(\wp_\Lambda + b)$ changes and in this setting toral bands arise [18,19]. A toral band can be part of a Leau petal cycle, for example [19].

The main purpose of this paper is to analyze how toral bands arise in elliptic functions; we also look at some toral bands arising in the Fatou sets of nonelliptic meromorphic functions. In particular, the existence or nonexistence of a toral band is highly dependent on the location of the critical values of the map $\wp$ (or $\wp'$).
within a fundamental domain. This in turn is completely dependent upon some analytic identities for \( \varphi \) and \( \varphi' \) relating to their periodic lattice. One simple way to think of it is that for a fixed lattice \( \Lambda \), the critical points of \( \varphi_\Lambda + b \) or \( 1/\varphi_\Lambda + b \) are independent of the choice of \( b \), but the critical values are greatly impacted by different choices of \( b \). Placing different critical values in the same component in the attracting basin of a cycle creates toral bands.

Our main results and examples include the following: If \( f_\Lambda \) is elliptic,

1. toral bands cannot contain Herman cycles or Siegel disk cycles, but can be part of a Leau petal cycle.
2. We show that if \( f_\Lambda \) has no Herman rings, every toral band must contain at least 2 critical points, and in particular, critical points that are distinct on the torus \( \mathbb{C}/\Lambda \).
3. We show that toral bands need not be forward invariant or periodic.
4. These seem to be the only constraints. There are examples of toral bands that both form a part of an attracting cycle, and map into a (disjoint) attracting cycle, and maps with two disjoint, unrelated toral bands.
5. There exists a nonelliptic meromorphic function \( G \), with a doubly periodic Fatou set over \( \Lambda \), with a toral band that is part of a Baker domain.

There are many examples that can be obtained numerically and illustrate the wide variety of types of toral bands, and we only present a sampling of examples here whose existence can be proved. More estimates can give additional results; for example there are examples on triangular lattices proved to exist using similar techniques. Due to the stability present in many of the settings, examples with non-real lattices occur as well.

1.1. Preliminary definitions. By \( \Lambda = [\lambda_1, \lambda_2] \) we denote the group \( \Lambda = \{ m\lambda_1 + n\lambda_2 : m, n \in \mathbb{Z} \} \subset \mathbb{C} \). Assuming \( \lambda_1, \lambda_2 \in \mathbb{C} \) are non-zero and linearly independent over \( \mathbb{R} \), we call \( \Lambda \) a lattice. A lattice \( \Lambda \) acts on \( \mathbb{C} \) by translation, each \( \omega \in \Lambda \) inducing the transformation of \( \mathbb{C} \):

\[
T_\omega : z \mapsto z + \omega.
\]

We denote the coset of \( \mathbb{C}/\Lambda \) containing \( z \) by \([z]\) and also refer to \([z]\) as the residue class of \( z \). A closed, connected subset \( Q \) of \( \mathbb{C} \) is a fundamental region for \( \Lambda \) if

1. for each \( z \in \mathbb{C} \), \( Q \) contains at least one point in the same \( \Lambda \)-orbit as \( z \);
2. no two points in the interior of \( Q \) are in the same \( \Lambda \)-orbit.

Lattices determine double periods for elliptic functions.

**Definition 1.2.** An elliptic function \( f : \mathbb{C} \to \hat{\mathbb{C}} \) is a meromorphic function in \( \mathbb{C} \) which is periodic with respect to a lattice \( \Lambda \).

For any lattice \( \Lambda \), the Weierstrass elliptic function is defined by

\[
\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right),
\]

\( z \in \mathbb{C} \). The map \( \wp_\Lambda \) is an even elliptic function, periodic with respect to \( \Lambda \), and has order 2.

The derivative of the Weierstrass elliptic function is also an elliptic function which is periodic with respect to \( \Lambda \) defined by

\[
\wp'_\Lambda(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z-w)^3}.
\]
The Weierstrass elliptic function and its derivative are related by the differential equation
\[(1.4) \quad \wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,\]
where \(g_2(\Lambda) = 60\sum_{w \in \Lambda \setminus \{0\}} w^{-4}\) and \(g_3(\Lambda) = 140\sum_{w \in \Lambda \setminus \{0\}} w^{-6}\).

The numbers \(g_2(\Lambda)\) and \(g_3(\Lambda)\) satisfy the following: if \(g_2(\Lambda) = g_2(\Lambda')\) and \(g_3(\Lambda) = g_3(\Lambda')\), then \(\Lambda = \Lambda'\). Define \(\Delta = g_2^3 - 27g_3^2\); given any \(g_2\) and \(g_3\) such that \(\Delta \neq 0\), there exists a lattice \(\Lambda\) having \(g_2 = g_2(\Lambda)\) and \(g_3 = g_3(\Lambda)\) as its invariants \([11]\). We sometimes write \(\Lambda = \Lambda(g_2, g_3)\) to show the dependence of the lattice on its invariants \(g_2\) and \(g_3\).

We say \(\Lambda\) is a real lattice if \(\overline{\Lambda} = \Lambda\), and \(\Lambda\) is real rectangular if and only if \(\Delta > 0\) and \(g_2 > 0\).

**Theorem 1.3.** The following are equivalent:

1. \(\wp(\overline{z}) = \wp(z)\);
2. \(\Lambda\) is a real lattice;
3. \(g_2, g_3 \in \mathbb{R}\).

Given a lattice \(\Lambda\), by substitution into the series definitions, the Weierstrass elliptic function and its derivative satisfy the following properties: for \(k \in \mathbb{C} \setminus \{0\}\),
\[(1.5) \quad \wp_{\Lambda}(ku) = \frac{1}{k^2} \wp_{\Lambda}(u), \quad (\text{homogeneity of } \wp_{\Lambda}),
\quad \wp'_{\Lambda}(ku) = \frac{1}{k^3} \wp'(u), \quad (\text{homogeneity of } \wp'_{\Lambda}).\]

If \(\wp'_{\Lambda}(z_0) = 0\), then \(z_0\) is a critical point for \(\wp_{\Lambda}\) and \(\wp_{\Lambda}(z_0)\) is a critical value for \(\wp_{\Lambda}\).

A key identity for \(\wp''_{\Lambda}(z)\), the second derivative of the Weierstrass elliptic function with period lattice \(\Lambda\), is
\[(1.6) \quad \wp''_{\Lambda}(z) = 6(\wp_{\Lambda}(z))^2 - \frac{g_2(\Lambda)}{2}.
\]

We use the notation \(c_1 = \lambda_1/2, c_2 = \lambda_2/2, \) and \(c_3 = \lambda_1/2 + \lambda_2/2\) to denote the half lattice points in a fundamental region \(Q\). The critical points of \(\wp_{\Lambda}\) are \([c_j]\) for \(j = 1, 2, 3\). The critical values of \(\wp_{\Lambda}\) are denoted by \(e_j = \wp_{\Lambda}(c_j)\) for \(j = 1, 2, 3\), all distinct, and they satisfy the relationship
\[(1.7) \quad e_1 + e_2 + e_3 = 0, \quad e_1e_3 + e_2e_3 + e_1e_2 = -\frac{g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}.
\]

For an elliptic function over \(\Lambda\), denoted \(f_{\Lambda}\) (or often just \(f\)), \(z_0\) is a critical point if \(f'_{\Lambda}(z_0) = 0\) and \(f_{\Lambda}(z_0)\) is a critical value. The Fatou set \(F(f)\) is the set of points \(z \in \mathbb{C}\) such that \(\{f^k : k \in \mathbb{N}\}\) is defined and normal in some neighborhood of \(z\). The Julia set \(J(f)\) is the complement of the Fatou set on the sphere. For each lattice \(\Lambda\), every elliptic function with period lattice \(\Lambda\) is of Class \(S\) \([6]\) and does not have wandering domains \([3]\) or Baker domains \([30]\). We summarize the types of Fatou components that can occur.

**Theorem 1.4.** If \(f_{\Lambda}\) is an elliptic function with period lattice, \(\Lambda\), then every component of \(F(f_{\Lambda})\) is preperiodic, and each forward invariant Fatou component contains one of the following:

1. a linearizing neighborhood of an attracting periodic point;
2. a Böttcher neighborhood of a superattracting periodic point;
(3) an attracting Leau petal for a periodic parabolic point. The periodic point is in \( J(f_\Lambda) \);

(4) a periodic Siegel disk containing an irrationally neutral periodic point;

(5) a Herman ring.

Lemma 1.5. If \( f_\Lambda \) is an elliptic function, then \( J(f_\Lambda)+\Lambda = J(f_\Lambda) \), and \( F(f_\Lambda)+\Lambda = F(f_\Lambda) \).

The proof is given for \( \wp_\Lambda \) in [15], and follows from the double periodicity.

2. Results on toral bands

If \( f \) is elliptic, the only singularities of \( f^{-1} \) that can occur are critical values [6].

A domain always refers to an open connected subset of \( \mathbb{C} \).

Theorem 2.1 ([6]). Suppose \( f \) is a transcendental meromorphic function and \( C = \{ U_1, U_2, \ldots, U_{p-1} \} \) is a cycle of Fatou components of period \( p \). If \( C \) is a cycle of Siegel disks or Herman rings then for every \( j = 1, \ldots, p-1, \)

\[ \partial U_j \subset \cup_{n \geq 1} f^n(\text{sing}(f^{-1})). \]

Corollary 2.2. An elliptic function \( f_\Lambda = f \) cannot have a toral band that is a Siegel disk or a Herman ring.

Proof. Let \( U \) be both a Siegel disk and a toral band. Then \( \partial U \subset J(f) \) and throughout the entire interior of \( U \) we have that \( f \) is conjugate to an irrational rotation. This is impossible by periodicity of \( f \) and the definition of a toral band, which implies that for every point \( z \in U \), there are infinitely many \( \lambda \in \Lambda \) such that \( z + \lambda \in U \). The contradiction gives the result. The same proof works for a Herman ring. \( \square \)

There are conditions on an elliptic function that lead to a Cantor Julia set. Suppose \( f \) is an elliptic function with period lattice \( \Lambda \). Recall that a parabolic fixed point \( z_0 \) of \( f \) has multiplicity 2 if and only if \( f'(z_0) = 1 \) and \( f''(z_0) \neq 0 \) (so that there is only one petal in the Leau domain). The proof of the next result comes from ([27, Lemma 8.1 and Appendix E]), and for the setting of elliptic functions, see also ([16, Section 3]).

Proposition 2.3. Let \( f \) be an elliptic function with period lattice \( \Lambda \). Assume that the Julia set \( J(f) \) contains no critical point. Then \( J(f) \) is totally disconnected if and only if all critical values of \( f \) lie in the same component \( F_o \subset F(f) \). If so, then \( F_o \) is completely invariant and corresponds to an attracting fixed point or parabolic fixed point of multiplicity 2.

Corollary 2.4. Under the hypotheses of Proposition 2.3, \( F(f) \) has a double toral band.

Remark 2.5.

(1) There are results showing that certain lattices and elliptic functions preclude the hypotheses of Proposition 2.3 from holding. For the Weierstrass \( \wp \) function \( \wp_\Lambda \), if \( \Lambda \) is a square lattice [8,15] or triangular lattice ([16, Cor. 3.3]) then \( J(\wp_\Lambda) \) is connected and is therefore not a Cantor set.

(2) By the symmetry of square and triangular lattices, there cannot be a single toral band for \( \wp_\Lambda \) without a double toral band, so neither can exist; this follows from Theorem 2.6.
Theorem 2.7. For a lattice $\Lambda = [\lambda_1, \lambda_2]$ the following hold for every even elliptic function $f = f_\Lambda$ with period lattice $\Lambda$:

1. $(-1)J(f) = J(f)$ and $(-1)F(f) = F(f)$.
2. If $\Lambda$ is square, then $e^{\pi i/2}J(f) = J(f)$ and $e^{\pi i/2}F(f) = F(f)$.
3. If $\Lambda$ is triangular, then $e^{2\pi i/3}J(f) = J(f)$ and $e^{2\pi i/3}F(f) = F(f)$.
4. There is symmetry of $F(f)$ and $J(f)$ about each half lattice point; i.e., for any $j = 1, 2, 3$,

$$(2.1) \quad f(c_j + tc_k) = f(c_j - tc_k), \text{ and } f(c_j + z) = f(c_j - z)$$

for $t \in \mathbb{R}$, $z \in \mathbb{C}$.

It follows as a corollary to Theorem 2.6 (2.1), that if $\Lambda$ is real, then $f$ has symmetry about all half lattice lines parallel to the axes.

The next result is related to Proposition 2.3 and shows how single toral bands can occur for Weierstrass $\wp$ function.

Theorem 2.7. If $\Lambda$ is a rectangular lattice, not necessarily real, then $F(\wp_\Lambda)$ cannot have a component containing all 3 critical values.

Before giving the proof, we mention a corollary. The authors showed earlier that there are many rectangular lattices for which $\wp_\Lambda$ has a toral band $[15][16]$. The next result gives limits to what type of toral bands can occur in the rectangular lattice setting for $\wp_\Lambda$; namely, the typical toral band is a single toral band, or, in the case of rectangular square lattices, no toral bands at all.

Corollary 2.8. Suppose $\Lambda$ is a rectangular lattice and $J(\wp_\Lambda)$ does not contain a critical point; then $F(\wp_\Lambda)$ is not a Cantor set.

We now turn to the proof of the theorem. We write $\wp$ for $\wp_\Lambda$.

Proof of Theorem 2.7. Assume first that $\Lambda = [\lambda_1, \lambda_2]$ is a real lattice with $\lambda_1, \lambda_2 > 0$; then $e_1 = \ell > 0$, $e_2 = -k < 0$, and $e_3 = -\ell + k$. Consider first the case when $e_3 > 0$ and $e_3 < e_1$ (so that $k \in (\ell, 2\ell)$).

Assume that $e_1$ and $e_3$ lie in the same Fatou component; call it $A_0$. We claim that $e_2 \notin A_0$. Suppose so; then all critical values lie in $A_0$, and no critical points lie in $J(f)$. By Proposition 2.3 and $\Lambda$ being real, there is a (real) attracting or parabolic fixed point $p_0$ and an interval in $\mathbb{R}$ about $p_0$ containing either $e_1$, $e_2$, or $e_3$; also $F(\wp)$ consists of the immediate attracting basin of $p_0$.

By our assumption there is a curve $\Gamma \subset A_0 = F(\wp)$ such that $e_1, e_2, e_3$ lie on $\Gamma$. Since $\Gamma$ cannot pass through the origin, (a pole), we can choose it to be a piecewise smooth curve lying completely in $F(\wp) \cap \{z : \text{Im } z \geq 0\}$ from $e_2$ to $e_3$, and then from $e_3$ to $e_1$. Since a preimage of $e_1$ is $c_1$ and a preimage of $e_3$ is $c_3$, there is a component of $\wp^{-1}(\Gamma)$ connecting $c_1$ to $c_3$ and by symmetry about critical points, $c_1$ to $-c_1$. This produces a toral band contained in $A_0$ since both $c_1$ and $c_1 - \lambda_1$ lie in $A_0$.

We make a loop that encloses the origin by taking $\Gamma_0 = \Gamma \cup \overline{\Gamma} \subset F(\wp)$ and contains all the critical values. By construction, $\Gamma_0$ crosses the imaginary axis twice, and the real axis at least 3 times at the $e_j$s. We denote the imaginary axis
by \( i\mathbb{R} = \{ iy : y \in \mathbb{R} \} \). The image curve, \( \varphi_\Lambda(\Gamma_0) \) contains both positive and negative real numbers since \( \varphi_\Lambda(i\mathbb{R}) = (-\infty, e_2) \); also \( \varphi_\Lambda(\mathbb{R}) \subset (e_1, \infty) \). However the image of a connected set is connected, so \( \varphi_\Lambda(\Gamma_0) \) must cross the imaginary axis again to get from points less than \( e_2 < 0 \) to points greater than \( e_3 > 0 \) without passing through the origin. By induction, for each \( n \in \mathbb{N} \), \( \varphi^n(\Gamma_0) \cap i\mathbb{R} \neq \emptyset \). Moreover,

\[
\text{diam } \varphi^n(\Gamma_0) = \sup_{z,w \in \varphi^n\Gamma_0} |z - w| > e_2.
\]

Hence \( \varphi^n(\Gamma_0) \), a compact set, does not converge uniformly to \( \rho_0 \). The contradiction establishes the result. The case when \( e_2, e_3 \) lie in the same Fatou component proceeds similarly; up to relabelling, these are the only possibilities for \( \Lambda \) real.

An arbitrary rectangular lattice is generated by: \( \Lambda = [\alpha \lambda_1, \alpha \lambda_2 i] \) with \( \lambda_1, \lambda_2 > 0 \), \( \alpha = e^{2\pi i \theta} \), \( \theta \in [0, 2\pi) \). The critical values are still collinear, lying on \( \alpha^{-2}\mathbb{R} \) by (1.5), and the proof is the same. If \( \alpha^{-2}\mathbb{R} = i\mathbb{R} \), then we have a real rectangular lattice again. □

Many examples in Section 3 show that the hypotheses of Theorem 2.7 cannot be relaxed too much; also the proof does not work for all elliptic functions since a pole does not always lie between collinear critical values.

2.1. Critical points and toral bands for \( f \). By definition, if \( T \) is a toral band for an elliptic function \( f \), then there exists a point \( z_0 \in F(f) \), a lattice point \( \lambda \in \Lambda \), and a path \( \gamma \subset T \) connecting \( z_0 \) to \( z_0 + \lambda \). (In fact every point in \( T \) has that property.) The next theorem is stated slightly differently in ([16, Theorem 3.1]) and uses the assumption that there are no Herman rings. The proof is the same for an arbitrary elliptic function \( f \) with no Herman rings since \( f \) contains only finitely many critical values and all poles are in \( J(f) \).

**Theorem 2.9.** Suppose \( f \) is an elliptic function with period lattice \( \Lambda \) and \( F(f) \) has no Herman rings. If each critical value of \( f \) that lies in the Fatou set is the only critical value in that component, then

1. each component of \( F(f) \) is simply connected.
2. \( J(f) \) is connected.

**Remark 2.10.** Many elliptic functions satisfy the hypotheses of Theorem 2.9 such as elliptic functions of order \( d = 2 \), and any elliptic function of order \( d \) with poles of order \( d \), to name a few. This is proved by Moreno Rocha in [28] where an elliptic function with a Herman ring is shown to exist.

Consider a fundamental region

\[
Q = \{ s\lambda_1 + t\lambda_2 : 0 \leq s, t \leq 1 \}
\]

for some set of lattice generators \( \lambda_1, \lambda_2 \in \Lambda \).

**Lemma 2.11.** If \( f \) is an elliptic function with period lattice \( \Lambda \), a toral band \( T \) for \( f \) is either simply connected or infinitely connected as a region in \( \hat{\mathbb{C}} \).

**Proof.** A toral band \( T \) is an unbounded Fatou component, and there exists \( \lambda \in \Lambda \) such that \( T = \bigcup_{n \in \mathbb{Z}} (T \cap (Q + n\lambda)) \). Finite (but not simple) connectivity in a region implies infinite connectivity, otherwise \( T \) is simply connected as a Fatou component in \( \hat{\mathbb{C}} \). □
Lemma 2.12. The map \( \varphi_\Lambda : T^2 \cong \mathbb{C}/\Lambda \to \hat{\mathbb{C}} \) is a proper map of \( T^2 \cong \mathbb{C}/\Lambda \) onto \( \hat{\mathbb{C}} \) of topological degree 2. It is a ramified cover with 4 branch points on \( \hat{\mathbb{C}} \): \( e_1, e_2, e_3 \) and \( \infty \).

Theorem 2.13 (Riemann Hurwitz Formula, [5, Thm 5.4.1] and [31, p. 7]). Suppose \( U, V \) are domains and \( f : U \to V \) is analytic. Assume \( U \) is a component of \( f^{-1}(V) \) and there are no critical values on \( \partial(V) \). Let \( \chi(U), \chi(V) \) denote the Euler characteristics of \( U \) and \( V \). Then there exists an integer \( k \) such that number of critical points in \( U \) is equal to \( k\chi(V) - \chi(U) \), where \( k \) is the degree of the map from \( U \) to \( V \).

Remark 2.14.

(1) If \( U \subset \mathbb{C} \) is doubly connected, then \( \chi(U) = 0 \).

(2) If \( V \subset \hat{\mathbb{C}} \), then \( \chi(V) = 1 \) if and only if \( V \) is simply connected.

The authors showed in ([16, Theorem 3.4]) that if a Fatou component contains critical points from 2 or more residue classes for \( \varphi \), then \( F(\varphi) \) has a toral band because it contains at least 2 critical values. We have a partial converse here. Given a lattice \( \Lambda \subset \mathbb{C} \), let \( \Pi : \mathbb{C} \to \mathbb{C}/\Lambda \) denote the covering (or quotient) map of the torus. If \( 0 \neq \lambda \in \Lambda \), then by \( \Pi \lambda \) we denote the quotient map from \( \mathbb{C} \) onto \( \mathbb{C}/[\lambda] \).

Theorem 2.15. Suppose \( f \) is an elliptic function over \( \Lambda \) and has no Herman rings. Assume every Fatou component for \( f \) contains either no critical point or critical points from at most one residue class. Then there are no toral bands in \( F(f) \).

Proof. Suppose \( f \) satisfies the hypotheses and \( T \subset F(f) \) is a toral band. Then each Fatou component, including \( T \) and \( f(T) \), contains either 0 or 1 critical value. Then by Theorem 2.9, \( T \) is simply connected in \( \hat{\mathbb{C}} \) and \( J(f) \) is connected. This implies that \( T \) is a single toral band by [16]. Then there exists some \( \lambda \in \Lambda \) such that \( T \) projects using \( \Pi_\lambda \) to a set \( T_\lambda \) biholomorphically equivalent to an annulus on the cylinder \( \mathbb{C}/[\lambda] \). (The existence of \( \lambda \in \Lambda \) follows from the proof of Lemma 2.11.) The map \( f \) is well-defined and meromorphic on \( \mathbb{C}/[\lambda] \). The set \( T_\lambda \) in turn is biholomorphically equivalent (via a map we label \( \phi \)) to a domain \( U \subset \mathbb{C} \setminus \{0\} \subset \mathbb{C} \) with \( \chi(U) = 0 \). We have that \( f \) induces a well-defined map \( \hat{f} \) as follows: for all \( z \in T \), if \( w = \phi \circ \Pi_\lambda(z) \), set \( \hat{f}(w) = f(z) \).

If \( T \) contains no critical points, then \( U \) contains none as well, and \( V = \hat{f}(U) \) has no critical values. If the valency of \( f \) on points in \( T \) is 1 everywhere, this is true for the valency of \( \hat{f} \) on \( U \). Therefore by the Riemann Hurwitz Formula 2.13, \( \chi(V) = 0 \) and \( V \) is doubly connected. However this is impossible since
\[ V = \hat{f}(U) = f(T) \subset F(f), \]
and each component of \( F(f) \) is simply connected.

If \( T_\lambda \) contains one critical point, then Theorem 2.15 gives that \( \chi(V) = 1/k \), which is also impossible unless \( k = 1 \) and \( V \) is simply connected. However \( f \) is \( k \)-to-one at each critical point, with \( k \geq 2 \) so this is also impossible.

We assemble the above results to obtain the main theorem of this section.

Theorem 2.16. Suppose \( f \) is an elliptic function with period lattice \( \Lambda \), with critical points \( a_1, a_2, \ldots, a_K \), with disjoint residue classes, and \( F(f) \) has no Herman rings. Then \( F(f) \) contains a toral band if and only if there is a component \( T \subset F(f) \subset \hat{\mathbb{C}} \), which projects to a subset \( \hat{T} \subset \mathbb{C}/\Lambda \) with the following properties:
(3.1) Proposition 3.1. We show the following properties and estimates (see Figure 1 and also [22]).

(i) \( \tilde{T} \) contains a generator of \( \pi_1(\mathbb{C}/\Lambda) \), the fundamental group of the torus, and
(ii) \( \tilde{T} \) contains at least 2 distinct residue classes of critical points \([a_i]\) and \([a_j]\).

Proof. \((\Rightarrow)\): Assume \( F(f) \) contains a toral band \( T \). Then \( \Pi(T) = \tilde{T} \subset \mathbb{C}/\Lambda \) satisfies (i). By Theorem 2.15 (ii) holds as well since every toral band must contain critical points from at least two distinct residue classes.

\( T \) is a toral band since it contains both \( \alpha \) and \( \alpha + \lambda \) for some \( \lambda \in \Lambda \).

\((\Leftarrow)\): Assume there is a Fatou component \( T \subset F(f) \) that projects to \( \tilde{T} \subset \mathbb{C}/\Lambda \) satisfying (i) and (ii). By (i) and Definition 1.1 then \( T \) is a toral band. \( \square \)

3. Occurrence of toral band Fatou components

Elliptic functions with double toral bands and Cantor Julia sets have been well-studied [13,16,18,20,21]; we begin by showing double toral bands do not need to be invariant so need not be associated to Cantor Julia sets.

3.1. Noninvariant periodic toral bands. We show the existence of an elliptic function with a double toral band that is periodic but not fixed. The example is similar to one that appears in [22]. The proofs given below are rigorous, but make use of the leminscate constant. We use the classical estimate that appears in [26]: \( \gamma \) is the side length of the square lattice \( \Lambda(\pm 4, 0) \):

\[
\gamma = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}} \approx 2.62206.
\]

\( \Lambda \) denotes a real rhombic square lattice with side length \( \sqrt{2} \), which yields (by the homogeneity equations) that \( g_2 = -\gamma^4 \), \( g_3 = 0 \); \( \Lambda = [1 + i, 1 - i] \), with half lattice points (critical points of \( \wp \)) at \((c_1, c_2, c_3) = ((1 + i)/2, (1 - i)/2, 1)\).

Define the elliptic function

\[
f(z) = f_\Lambda(z) = \frac{1}{(\wp(z)^2 + 1)} - 1.
\]

We show the following properties and estimates (see Figure 1 and also [22]).

Proposition 3.1.

1. The critical points of \( f \) are \([0], [1], [(1 \pm i)/2]\).
2. \( f(0) = -1 \) and \( f(-1) = 0 \) so \([0, -1] \) is a superattracting 2-cycle.
3. The critical values of \( f \) are \( V = \{0, -1, -g_2/(4 + g_2)\} \subset \mathbb{R} \),

\[
\text{with } -1.09593 < -g_2/(4 + g_2) < -1.08827.
\]
4. The range of \( f \) along \( \mathbb{R} \) is \([-1, 0]\), and \( f \) decreases from 0 to \(-1\) on \([-1, 0]\); on \([0, 1]\) \( f \) increases from \(-1\) to 0.
5. There is a unique fixed point for \( f \) in \([-1, 0]\); it is repelling. There are no fixed points in \([0, 1]\).
6. \( f \) is hyperbolic.

Proof. (1) and (2) are straightforward.

To prove (3), we note that \( e_1 = \sqrt{g_2}/2 \), and \( g_2 = -\gamma^4 < -2.6^4 \). Therefore \( e_1 \), the critical value of \( \wp \), satisfies: \( e_1 = i\gamma^2/2 \) and the critical value \( v_1 \) of \( f \) corresponding to \((1 \pm i)/2 \) is \( v_1 = 4/(4 - \gamma^4) - 1 \). Applying (3.1), \( 2.6 < \gamma < 2.65 \), yields

\[
-1.09593 < v_1 < -1.08827.
\]
To show (4) we observe that the sign of \( f'(z) \) is the opposite sign to that of \( \wp'_{\Lambda}(z) \) except at \( z = 0 \), a pole for \( \wp_{\Lambda} \) but \( f'(0) = 0 \). To prove (5), using (4), \( f'(t) \leq 0 \) on \([-1, 0]\) so the map given by

\[
(3.3) \quad h(t) = f(t) - t
\]

is decreasing. Since \( h(-1) = 1 > 0 \), and \( h(0) = -1 < 0 \), there is a unique 0 for \( h \), hence fixed point for \( f \), in that interval. Since \( f(t) < 0 \) on the interval \((0, 1)\), and \( f(1) = 0 \), there are no fixed points in \([0, 1]\). The fixed point must be repelling and separate the basins of attraction along \( \mathbb{R} \) for the fixed points of \( f \circ f = f^{\circ 2} \) at 0 and \(-1\).

(6) To show \( f \) is hyperbolic, it suffices to show that \( v_1 \) is in the attracting basin of the superattracting 2-cycle at 0 and \(-1\). Using quarter lattice value identities, \( \wp_{\Lambda}(-1/2) = \wp_{\Lambda}(1/2) = |e_1| \), so \( h(-1/2) < 0 \). We use the next 2 identities to show that the interval \( J = [-1.25, -0.75] \) lies in the immediate basin of attraction of the cycle \((-1, 0)\) (or of \(-1\) viewed as a fixed point of \( f^{\circ 2} \)). We use the Laurent series expansion of \( \wp \) about 0, and the fact that 1 is a critical point of \( \wp \) to obtain the following:

\[
(3.4) \quad \wp(u) = \frac{1}{u^2} + \frac{g_2}{20} u^2 + \frac{g_2^2}{1200} u^6 + \ldots
\]

\[
\wp(u \pm 1) = \frac{|e_1|^2}{\wp(u)}.
\]

Applying (3.4) to \( u = 1/4 \) and substituting the estimates on \( \gamma \) into (3.2) shows that \( h(-0.75) > 0 \) so the unique fixed point on \( \mathbb{R} \) is in \((-0.75, -0.5)\). Therefore \( v_1 \) and hence all critical values lie in the basin of attraction of the superattracting cycle and \( f \) is hyperbolic. \( \square \)
Corollary 3.2. The Fatou set of \( f \) contains a double toral band, but \( J(f) \) is not a Cantor set.

Proof. Since we have an attracting 2-cycle, the Julia set cannot be totally disconnected. There must be at least 2 disjoint components in \( F(f) \).

To show we have a double toral band, it is enough to show that the boundary curves of a fundamental region of a period square bounded by line segments of the form: \( s_1 = \{ t + ti : t \in [0, 1] \} \), \( s_2 = \{ (1 + t) + (1 - t)i : t \in [0, 1] \} \), \( s_3 = \{ (2 - t) - ti : t \in [0, 1] \} \), \( s_4 = \{ (1 - t) - (1 - t)i : t \in [0, 1] \} \) lie in the attracting basin of \(-1\) as viewed as a superattracting fixed point of \( f^2 \). By \cite{11} we see that \( f(s_1) = [v_1, 0] \subset \mathbb{R} \), and we showed that \( v_1 < -1.09593, -1.08827 \) is in that basin, using the proof of Proposition \( 3.1 \). The other boundary curves lie in the attracting basin by symmetry.

In general, maps of the form
\[
g(z) = \frac{1}{(\varphi_\Lambda(z)^2 + b)} - \beta, \ b, \beta \in \mathbb{C}
\]
on a square lattice \( \Lambda \) cannot have a single toral band.

Lemma 3.3. For maps of the form (3.5) with a square period lattice, if a toral band exists, it is a double toral band.

Proof. We show that if \( T \) is a toral band, then \( iT = T \), or equivalently, \( z \in T \) if and only if \( iz \in T \). If so, then both \( z + \lambda \) and \( z + i\lambda \) are in \( T \) for each \( z \in T \), which will give a double toral band. We use the following:
\[
i\Lambda = \Lambda \Rightarrow \varphi_i\Lambda(iz) = -\varphi_\Lambda(z) = \varphi_i\Lambda(iz)^2 = \varphi_\Lambda(z)^2 \Rightarrow g(iz) = g(z).
\]

Example 3.4. Using \( (g_2, g_3) = (-4, 0) \), \( b = 1 \), and \( \beta = 0 \), we obtain a map that has 0 as a superattracting fixed point, and all points on the real line are attracted to 0 under iteration. Therefore there is a horizontal toral band containing \( [0] \) and \( c_3 \), the real half lattice point for \( \Lambda \). The other two critical points in each fundamental region map to \( \pm t \) under \( \varphi_\Lambda \), so are poles for the map \( g(z) \). By Lemma 3.3, we have a double toral band \( T \) as the imaginary and real axes are in \( T \). It is an example of a double toral band containing exactly 2 residue classes of critical points and not more. We show the Julia and Fatou sets in Figure 2.

We can find many other unusual examples of the form given in (3.5) for square lattices. For example, choosing \( \beta = \varphi_\Lambda^{-1}(\sqrt{-b}) \) and \( b \) so that there is an attracting fixed point often gives an example of a double toral band, and 0 is always a prepole. The Hausdorff dimension of the resulting Julia set was estimated in \cite{21}. Since Lemma 3.3 does not hold for maps of the form \( \varphi_\Lambda + b \) or \( 1/\varphi_\Lambda + b \) new techniques are needed.

3.2. Nonperiodic toral bands. In this section, we discuss elliptic functions of the form \( f_{\Lambda,b}(z) = 1/\varphi_\Lambda(z) + b \) on real lattices \( \Lambda \). The function \( f_{\Lambda,b} \) has two distinct residue classes of real critical points and either one or two non-real critical point residue classes, depending on the shape of the lattice. This family of functions was studied in \cite{21} with \( b = 0 \) and \( \Lambda \) real rectangular, where it was shown that whenever the lattice invariants satisfied the inequality \( g_3 > -4g_2 + 256 \), \( f_{\Lambda,0}(z) \) has a double toral band containing the real and imaginary axes and \( J(f_{\Lambda,0}) \) is a Cantor set.
Figure 2. Double toral band for Example 3.4 of the form (3.5). The superattracting fixed point at 0 has a basin containing 2 critical points and the other 2 critical points are poles. The Julia set is black.

Figure 3. The graph of $f_{\Lambda,0}(z)$ on $\mathbb{R}$

We choose a real lattice $\Lambda = [\lambda_1, \lambda_2]$ where $\lambda_1 > 0$.

**Proposition 3.5** ([21]). If $\Lambda$ is a real lattice then

1. If $\Lambda$ is not square then the critical points of $f_{\Lambda,b}$ are $[0], [c_1], [c_2], [c_3]$.
2. If $\Lambda$ is square then the critical points of $f_{\Lambda,b}$ are $[0], [c_1]$, and $[c_2]$, and $[c_3]$ are poles of $f_{\Lambda}$. 
When \( \Lambda \) is a real lattice and \( b \in \mathbb{R} \), the function \( f_{\Lambda,b}(z) \) maps both the real and imaginary axes to the real line. We show the graph of a typical function \( f_{\Lambda,b}(z) \) in Figure 3 for the invariants \( g_2 = 2.7 \) and \( g_3 = -0.8 \) and \( b = 0 \). The next proposition considers \( f_{\Lambda,b} \) as a function from the real line to itself.

**Proposition 3.6.** If \( \Lambda \) is a real rectangular or triangular lattice and \( b \in \mathbb{R} \) then

1. \( f_{\Lambda,b} \) is an even, degree two elliptic function with period lattice \( \Lambda \).
2. \( f_{\Lambda,b}: \mathbb{R} \to [b, 1/e_1 + b] \) is piecewise monotonic and onto.
3. \( f_{\Lambda,b} \) is strictly increasing on \([0, c_1]\) and strictly decreasing on \([c_1, \lambda_1]\).
4. If \( \Lambda \) is real rectangular, then \( f_{\Lambda,b} \) maps the imaginary axis to the real interval \([1/e_2 + b, b]\), where \( 1/e_2 < 0 \).
5. \( f_{\Lambda,b} \) has one inflection point \( d \) in \((0, c_1)\) and is concave up on \((0, d)\) and concave down on \((d, c_1)\).
6. If \( b = 0 \) then 0 is a superattracting fixed point for \( f_{\Lambda,b} \).

**Proof.** Parts (1)–(4) follow immediately from properties of \( \wp_\Lambda \) on real lattices. To prove part (5), we compute the second derivative

\[
f''_{\Lambda,b} =\frac{2(\wp_\Lambda')^2 - \wp_\Lambda \wp_\Lambda''}{(\wp_\Lambda)^3}.
\]

Using Equations 1.4 and 1.6 we simplify to

\[
f''_{\Lambda,b} =\frac{2\wp_\Lambda^3 - 1.5g_2\wp_\Lambda - 2g_3}{(\wp_\Lambda)^3}.
\]

Notice that \( f''_{\Lambda,b}(0) = 2 \), so \( f \) is concave up at the origin.

For \( \Lambda \) real rectangular, notice that the numerator of Equation 3.6 is a real cubic polynomial, which has exactly one real root precisely when

\[
\frac{27}{16}g_2^3 - 27g_3^2 > 0.
\]

This inequality follows immediately from the inequality \( g_2^3 - 27g_3^2 > 0 \) for real rectangular lattices.

When \( \Lambda \) is triangular, we have \( g_2 = 0 \) and the numerator of Equation 3.6 simplifies to \( 2\wp_\Lambda^3 - 2g_3 \), which has exactly one real root. \( \square \)

We can use Proposition 3.6 to find lattices and values of \( b \) for which \( f_{\Lambda,b} \) has a toral band containing the real axis.

**Theorem 3.7.** Let \( \Lambda \) be a real rectangular or triangular lattice and \( b \in \mathbb{R} \). Then

1. If \( f_{\Lambda,b} \) has only one fixed point on \( \mathbb{R} \) which is attracting, then \( f_{\Lambda,b} \) has a toral band containing the real axis.
2. If \( \Lambda \) is real rectangular and \( g_3 > -4g_2 + 256 \), then \( f_{\Lambda,b} \) has a toral band containing the real axis.
3. If \( \Lambda \) is real triangular and \( g_3 > 64 \), then \( f_{\Lambda,b} \) has a toral band containing the real axis.

**Proof.** For (1), every real number must iterate to the attracting fixed point by Proposition 3.6. For (2) and (3), we have that \( 0 \leq |f'_{\Lambda,b}(z)| < 1 \) for all \( z \in \mathbb{R} \) (see [20][21]). Using Proposition 3.6, \( f_{\Lambda,b} \) must have exactly one fixed point on \( \mathbb{R} \) which is attracting. By (1), \( f_{\Lambda,b} \) has a toral band containing the real axis. \( \square \)
Next, we use Proposition 3.6 to prove that a toral band for a real lattice can contain two distinct critical points from non-real residue classes. By choosing an appropriate value of \( b \), we can guarantee that the two real critical points of \( f_{\Lambda,b} \) are part of a superattracting two-cycle. In this case, the real axis is not contained in the toral band.

To begin, we find rectangular lattices for which \( \varphi_{\Lambda} \) has two critical values lying close to one another in Propositions 3.8 and 3.9 leading to many examples of elliptic functions with toral bands.

**Proposition 3.8.** Let \( \varepsilon > 0 \), and set \( g_3 = 4(1 + \varepsilon)(-2 - \varepsilon) \) and \( g_2 = 4 - g_3 \). Then \( \Lambda \) is a real rectangular lattice with \( e_1 = 1, e_2 = -2 - \varepsilon \), and \( e_3 = 1 + \varepsilon \).

**Proof.** Notice that \( \Lambda \) is real rectangular since \( g_3^3 - 27g_3^2 > 0 \) for small \( \varepsilon \). Since \( g_3 < 0 \), we have that \( e_1, e_3 > 0 \) and \( e_2 < 0 \). Equation 1.7 gives \( e_1 = 1, e_2 = -2 - \varepsilon \), and \( e_3 = 1 + \varepsilon \).

**Proposition 3.9.** Given \( \varepsilon > 0 \), choose \( g_2 = 3 + 4\varepsilon^2 \) and \( g_3 = 1 - 4\varepsilon^2 \). Then the three critical values of \( \varphi_{\Lambda} \) are \( e_1 = 1, e_2 = -5 - \varepsilon \), and \( e_3 = -5 + \varepsilon \).

**Proof.** Notice that \( \Lambda \) is real rectangular since \( g_3^3 - 27g_3^2 > 0 \) for small \( \varepsilon \) and \( g_2 + g_3 = 4 \). Since \( g_3 > 0 \), we have that \( e_1 > 0 \) and \( e_2, e_3 < 0 \). Equation 1.7 gives \( e_1 = 1, e_2 = -5 - \varepsilon \), and \( e_3 = -5 + \varepsilon \).

Using Proposition 3.9 we begin by choosing a lattice for which \( e_1 = 1 \), and \( e_2 \) and \( e_3 \) are arbitrarily close to \(-1/2\).

**Theorem 3.10.** Let \( \Lambda \) be a real rectangular lattice, and define

\[
(3.7) \quad f(z) = f_{e_1\Lambda, c_1^2}(z) = \frac{1}{\varphi_{e_1\Lambda}(z)} - c_1^2.
\]

Then

1. \( f \) has critical values at \( 0, -c_1^2, c_1^2/e_2 - c_1^2 < 0 \), and \( c_1^2/e_3 - c_1^2 < 0 \).
2. \( f \) decreases from 0 to \(-c_1^2\) on the interval \([-c_1^2, 0]\).
3. \( f \) has a superattracting two-cycle at \( \{0, -c_1^2\} \).
4. \( f \) has a unique fixed point on \( \mathbb{R} \) which lies in the interval \([-c_1^2, 0]\), and it is repelling.
5. Every real number either belongs to the Julia set or iterates to this two-cycle.

**Proof.** Using the homogeneity property, we have \( 1/(\varphi_{e_1\Lambda}(-c_1^2)) = c_1^2 \). This gives that \( f(-c_1^2) = 0 \) and \( f(0) = -c_1^2 \). The remaining properties follow from Propositions 3.5 and 3.6.

The next result gives sufficient conditions for the existence of a nonperiodic toral band.

**Proposition 3.11.** Let \( \Lambda \) be a real rectangular lattice and let \( f \) be as in (3.7). If the two negative critical values lie in the same Fatou component, \( f \) has a nonperiodic toral band.

**Proof.** The negative critical values of \( f_{e_1\Lambda, c_1^2}(z) \) are \( v_2 = c_1^2/e_2 - c_1^2 < 0 \), and \( v_3 = c_1^2/e_3 - c_1^2 < 0 \), both of which lie in \( \mathbb{R} \). By assumption, they lie in the same Fatou component \( V \), so at least one critical point in each of \([v_2]\) and \([v_3]\) lies in a Fatou component \( U \) with \( f_{e_1\Lambda, c_1^2}(U) = V \). Then \( U \) is a toral band by Theorem 2.16.
We then use Theorem 3.10 to construct an elliptic function of the form \( f_{c_1, c_1}^{\Lambda} (z) \) for which the two negative critical values lie close to each other. When these two negative critical values lie in the same Fatou component, then we have two nonreal critical points lying in the same toral band by Proposition 3.11. The details are similar to the proof of Proposition 3.1 and we do not produce them here.

**Example 3.12.** As an example, the Fatou set for the parameters \( g_2 = 3.00001 \) and \( g_3 = .99999 \) is shown in Figure 4. Here, \( f_{c_1, c_1}^{\Lambda} (z) \) has a nonperiodic toral band that iterates to a 2-cycle.

### 3.3. Two parallel single toral bands.

For real rectangular lattices and \( b = 0 \), we can use Proposition 3.6 to find elliptic functions \( f_{\Lambda} (z) = 1/\wp_{\Lambda} (z) \) that have a toral band containing the imaginary axis.

**Proposition 3.13.** If \( \Lambda \) is real rectangular and the critical value \( 1/e_2 \) of \( f_{\Lambda} \) is in the immediate attracting basin of the origin, then \( 1/\wp_{\Lambda} \) has a vertical toral band containing the imaginary axis.

**Proof.** Let \( W \) denote the immediate attracting basin of the origin, and suppose \( 1/e_2 \) lies in \( W \). We claim that the entire interval \([1/e_2, 0]\) lies in \( W \). If \( W = \mathbb{R} \), then the claim is obviously true. If \( W \neq \mathbb{R} \), then Proposition 3.6 (3) and (5) imply that there is a fixed point \( p \in [0, 1/e_1] \) with \((1/\wp_{\Lambda}(z))'(p) \geq 1 \) and that the immediate attracting basin \( W = (-p, p) \). Thus \( 1/e_2 \in (-p, p) \), and \([1/e_2, 0] \subset (-p, p) = W \).
Figure 5. \( f_\Lambda \) with parallel fixed toral bands when \( g_2 = 2.7 \) and \( g_3 = -0.8 \), from Remark 3.14

Proposition 3.6(4) implies that the imaginary axis maps to the interval \([1/e_2, 0]\). Since 0 is contained in both the imaginary axis and its image under \(1/\wp_\Lambda(z)\), the imaginary axis lies in \( W \), and \(1/\wp_\Lambda(z)\) has a vertical toral band.

Remark 3.14. Assume that \( f_\Lambda = 1/\wp_\Lambda \), and \( \Lambda \) is real, then 0 is a superattracting fixed point and the critical values are: \( v_j = 1/e_j, j = 1, 2, 3 \) and 0, all real. Certain \( \Lambda(g_2, g_3) \) can give two toral band cycles corresponding to two different attracting fixed points, each cycle containing two critical values and critical points from two distinct residue classes. If \( v_2 < 0 \) lies in the immediate basin of the origin, and \( v_1, v_3 > 0 \) lie in the immediate basin of another real attracting fixed point (see Figure 5), two vertical parallel toral bands can occur, as shown in Figure 5. It is possible to find rectangular lattices \( \Lambda \) so that \( f_\Lambda(z) \) has a fixed toral band containing the origin as well as a second toral band which contains an attracting fixed point or a cycle of higher period.

The point at 0 is a superattracting fixed point for \( f_\Lambda = 1/\wp_\Lambda \), for any lattice. We give a numerical example with an attracting 2-cycle, for a lattice that is not real, outlined in gray, shown in Figure 6.

3.4. Bifurcations occurring simultaneously with toral bands. Here, we examine elliptic functions of the form \( \wp_\Lambda + b \) that have toral bands with interesting properties. We begin with some results about \( \wp_\Lambda + b \) on real rectangular or real triangular lattices. Using calculus terminology on the family of maps \( \wp_\Lambda + b \) with \( b \) real, restricted to the real line, the next proposition describes the graph of \( \wp_\Lambda \) restricted to \( \mathbb{R} \). This allows us to use techniques from one real dimensional maps.
Figure 6. Parallel toral bands, one fixed at 0 and one part of a two-cycle (shown in magenta), when $g_2 = -10.72 - 7.52i, g_3 = -7.24 + 5.54i$

**Proposition 3.15** ([14]). Suppose $\Lambda$ is a real rectangular or real triangular lattice and $b \in \mathbb{R}$. Then for $\varphi_\Lambda + b$ restricted to $\mathbb{R}$,

1. the critical value $e_1 + b$ is the minimum of $\varphi_\Lambda + b$.
2. $\varphi_\Lambda + b$ is strictly decreasing on $(0, c_1]$ and strictly increasing on $[c_1, \lambda_1)$.
3. $\varphi_\Lambda + b$ is symmetric with respect to $c_1$.
4. $\varphi_\Lambda$ is concave up on $(0, \lambda_1)$.

We begin by examining real rectangular lattices. One approach in this case is to find invariants $(g_2, g_3)$ for which two of the critical values lie close together (using (1.7)). When two critical values of $\varphi_\Lambda + b$ lie in the same Fatou component, then $\varphi_\Lambda + b$ has a toral band by Theorem 2.16. For example, when the lattice invariants lie close to the curve $\Delta = 0$, then two critical values lie close to each other.

In general it is impossible to solve explicitly for fixed points of $\varphi_\Lambda$ on a specific lattice $\Lambda$. However, for the map $\varphi + b$ restricted to $\mathbb{R}$, many properties are known about the derivatives and concavity of the function at the fixed point. If a careful
choice of lattice yields two close critical values for the map \( \varphi_\Lambda \) and an attracting fixed point containing them in their basin, then we see period doubling bifurcations in some settings. We outline this procedure below. The procedure is quite flexible as there are many parameters to vary such as lattice, critical values, and even the choice of family of elliptic functions.

Recall that Propositions 3.8 and 3.9 give lattices for which two of the critical values lie arbitrarily close to each other.

**Remark 3.16.** Procedure for obtaining toral bands as part of higher period attracting cycles.

1. Choose a real rectangular lattice with \( e_1 \) and \( e_3 \) positive and close to each other using Proposition 3.8.
2. At \( a_0 = e_1 - c_1 \) the map \( f(x) = \varphi(x) - a_0 \) has a superattracting fixed point at \( c_1 \). The shape of \( \varphi_\Lambda - a_0 \) described in Proposition 3.15 implies that there is a unique repelling fixed point \( p_0 \) in \((c_1, \lambda_1)\) and interval \((-p_0 + \lambda_1, p_0)\), containing both \( e_1 \) and \( e_3 \), is the immediate basin of this superattracting fixed point. In this case, \( \varphi_\Lambda - a_0 \) will have a fixed toral band containing \( c_1 \) and \( c_3 \).
3. We restrict our attention to the map \( f(x) = \varphi(x) - a \), where \( x \in (0, \lambda_1) \) and \( a > a_0 \).
4. As \( a \) increases, there is a unique fixed point, \( p_a = p(a) \in (0, c_1) \). The point \( p_a \) decreases analytically as \( a \) increases, \( f'(p_a) < 0 \), and \( |f'(p_a)| \searrow \infty \) as \( a \) increases.
5. Since \(|p_a - c_1|\) increases as \( a \) increases, as long as \( p_a \) is attracting, the basin of attraction increases and the toral band persists. For some value of \( a \), \( f'(p_a) = -1 \) by the Intermediate Value Theorem.
6. When several technical conditions are satisfied, we then have a period doubling bifurcation ([7, Proposition 7.7.5]). They are typically satisfied, since: (i) when \( f \) can be written as \( M \circ \varphi_\Lambda \), with \( M(z) = (az + b)/(cz + d) \), \( ad - bc \neq 0 \), \( f \) has negative Schwarzian derivative as a map on \( \mathbb{R} \) (see [18], Prop 4.4); (ii) the function \( p(a) = p_a \) is smooth and monotone decreasing in \( a \) (i.e., \( p'(a) < 0 \)). Most of these properties follow directly from Proposition 3.15.
7. The toral band will persist for a while through the bifurcations, until the basins of attraction split up the critical values.

This method goes over to real triangular lattices with \( g_3 > 0 \) since the Schwarzian derivative of \( \varphi_\Lambda \) (and therefore \( \varphi_\Lambda + b, b \in \mathbb{C} \)) is negative [23]. We can modify the above procedure to obtain nonperiodic horizontal toral bands that iterate to attracting cycles.

**Remark 3.17.**

1. **A procedure for obtaining nonperiodic toral bands that iterate to higher period attracting cycles.** We modify the first step in Remark 3.16 by choosing a real rectangular lattice for which \( e_2 \) and \( e_3 \) are both negative and lie close together by applying Proposition 3.9. Using properties of the Schwarzian derivative described in step (6), whenever \( f(x) = \varphi(x) - a \) has an attracting cycle, the cycle must contain the real critical value \( e_1 \) as described in Remark 3.16. When \( e_2 \) and \( e_3 \) lie in the same Fatou component on the negative real axis, \( f(x) = \varphi(x) - a \) will have a horizontal
Figure 7. A toral band that maps onto a Siegel disk. The fixed point is green and a critical orbit is shown in red, with a close-up on the right. A fundamental region is outlined in black, and the horizontal boundaries lie in the toral band.

...toral band containing the nonreal critical points \( c_2 \) and \( c_3 \) that iterate to the attracting cycle. The bifurcations of the attracting cycle proceed as described in Remark 3.16. Basins of attraction split up the critical values \( e_2 \) and \( e_3 \) and \( f \) no longer has a toral band.

(2) A modification for a nonperiodic toral band that iterates to a Siegel disk. Recall that Corollary 2.2 shows that toral bands cannot be Siegel disks. Moving to consider complex values of \( b \) for \( \wp'_\Lambda + b \), we can construct an example of a nonperiodic toral band that maps to a Siegel disk. We choose a value for \( b \) so that there is a fixed point with derivative whose absolute value is 1, but \( f'(z_0) \) is irrational and satisfies a Diophantine condition. In order to ensure we have a toral band, we (first) choose a lattice as above so that \( e_2 \) and \( e_3 \) are both negative and extremely close to each other. Then for any choice of \( b \), \( b_2 = e_1 + b \) and \( b_3 = e_3 + b \) remain close, and this typically leads to the existence of a toral band. For the example shown in Figure 7, we use \((g_2, g_3) \approx (6, 2.83)\), which determines the rectangular lattice shown, and then setting \( b \approx -2.41 + .023i \) gives the desired fixed point with derivative \( \exp(2\pi i(1/4)^{1/3}) \), using a technique given by the authors in (15, Prop. 11.1). The Julia and Fatou set structure for the map \( f \) are shown in Figure 7.

It is not difficult to find (numerically) examples illustrating the procedure described in Remarks 3.16 and 3.17, but giving precise bounds for the parameters is lengthy.

3.5. Toral bands for elliptic functions involving \( \wp'_\Lambda \). We turn to the other building block of elliptic functions, namely \( \wp'_\Lambda \). In this section we prove the existence of a small sample of toral bands for the family of functions \( \wp'_\Lambda + b \). Unlike the examples studied earlier in the paper, these functions are not even, and, for \( b \neq 0 \),
are not odd. Further, the half lattice points $c_j$ are not critical points. As before, the points $c_j$ refer to the half lattice points, the critical points of $\varphi_\Lambda$.

For any lattice $\Lambda$, $\varphi_\Lambda'$ has four residue classes (distinct cosets) of critical points $w_j$, $j = 1, \ldots, 4$ at points $z$ where $(\varphi_\Lambda(z))^2 = \frac{g_2}{12}$ with four corresponding critical values given by the following formula:

$$u_j = \varphi_\Lambda'(w_j) = \pm \left(-g_3 \pm \left(\frac{g_2}{3}\right)^{3/2}\right)^{1/2}$$

all distinct unless the lattice is triangular.

When the lattice $\Lambda = [\lambda_1, \lambda_2]$, $\lambda_1, \lambda_2 > 0$, is rectangular, $\varphi_\Lambda'$ satisfies the following properties:

1. In the period parallelogram $Q$ given in (2.2), two critical points, $w_1$ and $w_2$, of $\varphi_\Lambda$ lie symmetrically (with respect to $c_3$) on the vertical line through the half lattice point $c_1$, and the other two, $w_3$ and $w_4$, lie symmetrically (with respect to $c_3$) on the horizontal line through the half lattice point $c_2$.

2. $\varphi_\Lambda'$ maps the vertical lines $c_1 + yi + \Lambda, y \in \mathbb{R}$ to the imaginary axis.

3. Along the vertical line segment $c_1 + yi, 0 \leq y \leq \lambda_2$, the value of $\varphi_\Lambda'$ is $0$ at $c_1$. Moving upward along the line, $\varphi_\Lambda'$ increases until reaching the critical point $w_1$, then decreases to $0$ at the point $c_3$, then decreases until reaching the critical point at $w_2$, then increases to $0$ at $c_1 + \lambda_2i$.

For the rest of this section, we use the rectangular lattice $\Lambda$ associated with the invariants $(g_2, g_3) = (11, -7)$. For the Weierstrass elliptic function $\varphi_\Lambda$, these invariants give rise to the critical values $e_1 = 1, e_2 = (-1 - 2\sqrt{2})/2$, and $e_3 = (-1 + 2\sqrt{2})/2$ (see, e.g., [15]).

**Theorem 3.18.** Let $\Lambda$ be the lattice with invariants $(g_2, g_3) = (11, -7)$, and let $k \in \mathbb{Z}$, and $b \in \mathbb{R}$. Then $f_{\Lambda, k, b}(z) = \varphi_\Lambda'(z) + (2k + 1)c_1 + bi$ has a toral band.

**Proof.** Assume first that $k = 0$ and $b = 0$, so

$$f_{\Lambda, 0, 0}(z) \equiv f_{\Lambda}(z) = \varphi_\Lambda'(z) + c_1.$$  

Since $\varphi_\Lambda'(c_1) = 0$, $c_1$ is a fixed point for $f_{\Lambda}$. Using Equation 1.6 $f_{\Lambda}'(c_1) = \varphi_\Lambda''(c_1) = 6(1)^2 - 11/2 = 1/2$, so $c_1$ is attracting.

Let $L_0$ be the vertical line $c_1 + yi$, $y \in \mathbb{R}$. Then $f_{\Lambda} : L_0 \to L_0$ since $\varphi_\Lambda'$ maps $L_0$ to the imaginary axis. More specifically, since the two imaginary critical values for $\varphi_\Lambda'$ are located at $\pm iv_1 = \pm (7 - (11/3)^{3/2})^{1/2} \approx \pm 0.145369i$, we have that $f_{\Lambda}$ maps the line $L_0$ to the line segment on the line $L_0$ lying between $-iv_1$ and $iv_1$.

In addition, for all $z$ on $L_0$, we have $e_3 = (-1 + 2\sqrt{2})/2 \leq \varphi_\Lambda(z) \leq 1 = e_1$, so

$$8 - 6\sqrt{2} \leq f_{\Lambda, k, b}'(z) \leq \frac{1}{2}$$

by Equation 1.6. Therefore, $|f_{\Lambda, k, b}'(z)| \leq 1/2$ for all $z \in L_0$.

Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(y) = \text{Im}(f_{\Lambda}(c_1 + iy))$. Then $g : [-c_2, c_2] \to [-v_1, v_1], g$ is periodic with respect to $\lambda_2$, and $g(0) = 0$. Figure 8 shows the graph of $g$. Using the contraction mapping theorem, 0 is the unique fixed point of $g$. This implies that all points on $L_0$ must iterate under $f_{\Lambda}$ to the attracting fixed point at $c_1$. Thus $f_{\Lambda}$ has a toral band containing the line $L_0$.

For an arbitrary $k \in \mathbb{Z}$, we repeat the same argument on the vertical line $L_k$ described by $(2k + 1)c_1 + yi$, $y \in \mathbb{R}$ since $f_{\Lambda, k, 0} : L_k \to L_k$ and $f_{\Lambda, k, 0}$ has an attracting fixed point at $(2k + 1)c_1$. For an arbitrary $b \in \mathbb{R}$, we have that $f_{\Lambda, k, b}(z)$
maps the line $L_k$ to the segment on the line $L_k$ lying between $-iv_1 + ib$ and $iv_1 + ib$. Using $g(y) = \text{Im}(f_\Lambda((2k+1)c_1 + iy))$, we have a unique attracting fixed point $p$ lying on $L_k$ with $-v_1 + b \leq \text{Im}(p) \leq v_1 + b$ since $|f'_{\Lambda,k,b}(z)| \leq 1/2$ for all $z$ on $L_k$. Therefore, $p$ is an attracting fixed point for $f_{\Lambda,k,b}$, all points on $L_k$ iterate to $p$, and $f_{\Lambda,k,b}$ has a toral band.

We show the Fatou set of $f_{\Lambda,0,0}(z)$ in Figure 9. Critical points from $[w_1]$, $[w_2]$, $[w_3]$, and $[w_4]$ (red), and the attracting fixed point $\lambda_1/2$ (green) are labeled in the figure.
3.5.1. Examples on other lattices. The technique in the proof of Theorem 3.18 is applicable to other lattices. Here we restrict to real rectangular lattices, so \( g_2 > 0 \) and \( \Delta > 0 \). We define a real rectangular lattice \( \Lambda = [\lambda_1, \lambda_2] \), to be a standard lattice if \( \varphi_\Lambda(\lambda_1/2) = \varphi_\Lambda(e_1) = 1 \). Using Equation (3.2) with \( e_1 = 1 \) we obtain
\[
1 + e_2 + e_3 = 0, \quad e_2 e_3 = \frac{g_3}{4}, \quad e_2 + e_3 + e_2 e_3 = -\frac{g_2}{4},
\]
and thus all standard lattices lie on the line segment \( g_3 = -g_2 + 4 \) with \( 3 < g_2 < 12 \) and \( \Delta > 0 \).

**Proposition 3.19.** Let \( \Lambda \) be a standard lattice with \( 10 < g_2(\Lambda) < 12 \), and let \( k \in \mathbb{Z} \) and \( b \in \mathbb{R} \). Then \( f_{\Lambda,k,b}(z) = \varphi_\Lambda'(z) + (2k + 1)c_1 + bi \) has a toral band.

**Proof.** As in the proof of Theorem 3.18 \( \lambda_1/2 \) is an attracting fixed point and \( f_\Lambda : L_k \to L_k \). For a standard lattice, \( e_1 = 1 \). When \( 10 < g_2(\Lambda) < 12 \) and \( z \in L_k \), we have
\[
f_\Lambda'(z) < 6e_1^2 - \frac{g_2}{2} < 1.
\]
Using the first two equations in Equation (3.8) we have \(-e_3 - 1)e_3 = g_3/4\). Since \( 10 < g_2(\Lambda) < 12 \), we have \(-8 < g_3(\Lambda) < -6\), so we solve for the positive root and obtain
\[
e_3 = -\frac{1 + \sqrt{1 - g_3}}{2} = -\frac{1 + \sqrt{1 - (-g_2 + 4)}}{2}.
\]
Thus for \( z \in L_k \) we have
\[
f_\Lambda'(z) > 6 \left( -\frac{1 + \sqrt{1 - (-g_2 + 4)}}{2} \right)^2 - \frac{g_2}{2} > -1
\]
when \( 10 < g_2(\Lambda) < 12 \). Thus \( f_\Lambda(z) \) is a contraction on \( L_k \), and the rest of the proof proceeds as in the proof of Theorem 3.18 \( \square \)

Two lattices \( \Lambda_1 \) and \( \Lambda_2 \) are similar if there exists a nonzero \( m \in \mathbb{C} \) such that \( \Lambda_2 = m\Lambda_1 \). For \( \Lambda \) real rectangular, the lattice \( m\Lambda \) is real when \( m \) is either real or purely imaginary [11].

In Proposition 3.20 we let \( m > 1 \) and consider the analog of the functions considered in Proposition 3.19 using the lattice \( m\Lambda \). Using the homogeneity property for lattice invariants, when \( \Lambda \) has invariants \((g_2(\Lambda),g_3(\Lambda))\) then the lattice \( m\Lambda \) has invariants
\[
(g_2(m\Lambda),g_3(m\Lambda)) = (g_2(\Lambda)/m^4,g_3(\Lambda)/m^6).
\]

**Proposition 3.20.** Let \( \Lambda \) be a standard lattice with \( 10 < g_2(\Lambda) < 12 \), and let \( k \in \mathbb{Z} \), \( b \in \mathbb{R} \), and \( m > 1 \). Then \( f_{m\Lambda,k,b}(z) = \varphi_{m\Lambda}'(z) + (2k + 1)m c_1 + bi \) has a toral band.

**Proof.** Note that since \( m > 1 \), when \( z \in L_k \), \( mz \in L_k \). We have
\[
f_{m\Lambda}'(mz) = 6(\varphi_{m\Lambda}(mz))^2 - \frac{g_2(m\Lambda)}{2} = 6 \left( \frac{1}{m^2} \varphi_\Lambda(z) \right)^2 - \frac{1}{m^4} \frac{g_2(\Lambda)}{2} = \frac{1}{m^4} f_\Lambda'(z).
\]
Since \( m > 1 \), this implies that \( |f_{m\Lambda}'(mz)| < 1 \) and \( f_{m\Lambda}' \) is a contraction on \( L_k \). The rest of the proof proceeds as in the proof of Theorem 3.18 \( \square \)
Figure 10. The black curve shows where $\Delta = 0$. Invariants $(g_2, g_3)$ of lattices that are similar to the real rectangular lattice $\Lambda$ with invariants $(g_2, g_3) = (5, -1)$ lie on the blue curve.

Figure 11. Invariants of lattices for which $f_{\Lambda,k,b}(z)$ has a toral band lie in the gray region.

The invariants $(g_2, g_3)$ used in Proposition 3.20 lie along the similar lattice curve to the left of the line segment on standard lattices when $10 < g_2(\Lambda) < 12$, and lie very close to the bottom curve of the discriminant. We show a zoom of a small section in Figure 11. The black curve is the discriminant $\Delta = 0$. The red line is a section of the standard lattices described in Proposition 3.19, and the grey shaded area represents lattices that satisfy the hypotheses of Proposition 3.20.
4. Baker domains for doubly periodic Julia and Fatou sets

There are many examples of meromorphic and entire functions with Baker domains; our results about toral bands up to now are quite different insofar as toral bands for elliptic functions are unbounded domains in the Fatou set that always are preperiodic with bounded dynamics. For an elliptic function, toral bands map onto, or are in a cycle of, a periodic Fatou component containing a nonrepelling cycle or a rotation domain. Since for every lattice \( \wp \) and \( \wp' \) have only finitely many critical values and no asymptotic values in \( \mathbb{C} \), there are no Baker domains for elliptic functions (see [9] and the references mentioned there).

However we can construct meromorphic functions with doubly periodic Julia and Fatou sets with toral bands that are Baker domains using maps satisfying (1.2) and (1.3). These are a variation of similar simply periodic maps discussed in [16] and the references mentioned there.

Example 4.1. We fix the lattice \( \Lambda = \Lambda(4,0) = [\lambda_1,\lambda_1i], \lambda_1 > 0 \), so that \( e_1 = 1, e_2 = -1 \), and \( e_3 = 0 \). Define

\[
G(z) = z + \wp_{\Lambda}(z) - a, \quad a > 0.
\]

We choose the value of \( a \) shortly. Since

\[
G'(z) = 1 + \wp'_{\Lambda}(z),
\]

we have \( G'(z_0) = 0 \) if and only if \( \wp'_{\Lambda}(z_0) = -1 \); this occurs in each fundamental region since \( \wp'_{\Lambda}(z + \lambda) = \wp'_{\Lambda}(z) \) for each \( \lambda \in \Lambda \), so there are infinitely many distinct critical points.

We also have that (1.2) holds:

\[
G(z + \lambda) = G(z) + \lambda, \quad \text{for each } z \in \mathbb{C}, \lambda \in \Lambda,
\]

which gives us symmetry with respect to \( \Lambda \) in the Julia and Fatou sets.

We choose \( w_0 \in \mathbb{R} \) to be the unique point in \((0,\lambda_1)\) such that \( \wp'_{\Lambda}(w_0) = -1 \). Then using (4.1) with \( z = w_0 \),

\[
v_0 = w_0 + \wp_{\Lambda}(w_0) - a \in \mathbb{R}
\]

is a critical value for \( G \), and so are all the elements of the set: \( V = \{v_0 + \lambda\}_{\lambda \in \Lambda}, \) giving us an unbounded postcritical set (with \( \infty \) an accumulation point).

Choosing \( a = \wp_{\Lambda}(w_0) > 1 \) (because \( e_1 = 1 \) is the minimum value of \( \wp_{\Lambda} \) on \( \mathbb{R} \)) gives that \( \{w_0 + \lambda\}_{\lambda \in \Lambda} \) are distinct superattracting fixed points of \( G \). This holds since

\[
G'(w_0 + \lambda) = 1 + \wp'_{\Lambda}(w_0) = 0.
\]

A repelling fixed point in \((0,\lambda_1)\) can be given explicitly as well. We have \( G(-w_0 + \lambda_1) = -w_0 + \lambda_1 + \wp_{\Lambda}(-w_0 + \lambda_1) - a = -w_0 \) since \( \wp_{\Lambda} \) is even and periodic, but

\[
G'(-w_0 + \lambda_1) = 1 + 1 = 2,
\]

since \( \wp'_{\Lambda} \) is odd and periodic. For each \( m \in \mathbb{Z} \), \( w_m = w_0 + m\lambda_1 \) is a real superattracting fixed point and \( v_m = -w_0 + m\lambda_1 \) is a repelling fixed point.
Figure 12. A meromorphic map $G(z) = z + \varphi_A(z) - a$ that has a Baker domain, using $a \approx 1.107$. The basins of a few of the fixed points are colored; the dashed orange line is contained in a Baker domain for $G$. The red lines outline a fundamental region for $\varphi_A$.

To show $G$ has a Baker domain, consider the horizontal half lattice line

$$L = \{ t + c_2 : t \in \mathbb{R} \}.$$

$G(L) \subset L$ and for any $t \in \mathbb{R}$, and $z = t + c_2 \in L$,

$$G(z) - z = \varphi_A(t + c_2) - a \in \mathbb{R},$$

since $\varphi_A(t + c_2), a \in \mathbb{R}$. In fact $\varphi_A(L) \subset [e_2, e_3] = [-1, 0]$ [18]. We claim

$$\lim_{n \to \infty} G^n(t + c_2) = -\infty,$$

so $L$ is in a Baker domain. To show the claim, if $z = t + c_2 \in L$

$$G(z) - z = \varphi_A(z) - a$$

< 0 \quad \text{for all } z \in L.

Figure 12 shows a Baker domain in $F(G)$ (with an arrow showing the direction of points under iteration), and some attracting basins for the fixed points.

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