

SYMMETRIES OF JULIA SETS FOR RATIONAL MAPS

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ABSTRACT. Since the 1980s, much progress has been done in completely determining which functions share a Julia set. The polynomial case was completely solved in 1995, and it was shown that the symmetries of the Julia set play a central role in answering this question. The rational case remains open, but it was already shown to be much more complex than the polynomial one. Here, we offer partial extensions to Beardon’s results on the symmetry group of Julia sets, and discuss them in the context of singularly perturbed maps.

1. INTRODUCTION

In complex dynamics, the problem of finding maps with the same Julia set goes back to Julia himself [12]. During the 1980s and early 1990s, a complete description for the polynomial case was obtained through the efforts of Baker, Eremenko, Beardon, Steinmetz and others [1–3, 20]. The culmination of this work is the following theorem: given any Julia set J for a non-exceptional polynomial, there exists a polynomial P such that the set of all polynomials with Julia set J is given by

$$(1.1) \quad \mathfrak{P}(J) = \{\sigma \circ P^n : n \geq 1 \text{ and } \sigma \in \Sigma_J\},$$

where Σ_J denotes the set of symmetries of J – that is, the set of all complex-analytic Euclidean isometries of \mathbb{C} preserving J . A rational function is exceptional if it is conformally conjugate to a power map, a Chebyshev polynomial or a Lattès map – any other rational function is *non-exceptional*. This result highlights the importance of the group of symmetries for the Julia set of polynomials; it completely determines which polynomials share that Julia set.

More specifically, let P be a polynomial, and let $J(P)$ denote its Julia set. We are interested in the set

$$\Sigma(P) := \{z \mapsto \sigma(z) = az + b : |a| = 1, b \in \mathbb{C}, \text{ and } \sigma[J(P)] = J(P)\}$$

of complex-analytic Euclidean isometries of \mathbb{C} preserving $J(P)$, the *symmetries* of $J(P)$, for it tells us – via (1.1) – whether there are any “interesting” polynomials with the same Julia set. It is fortunate, then, that Beardon [2] had already given a complete characterisation of $\Sigma(P)$. His results are twofold: first, he described all possible structures for $\Sigma(P)$ (it is either trivial, or a finite group of rotations about the same point, or the group S^1 of all rotations about a point), and then he gave criteria to determine the structure of $\Sigma(P)$ for any particular polynomial P .

The generalisation to rational maps, however, is not completely understood yet. Levin and Przytycki proved in 1997 that – for a large class of rational functions

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– having the same Julia set is equivalent to having the same measure of maximal entropy [14]. However, Ye [21] gives the following example:

$$(1.2) \quad \begin{aligned} R(z) &= a(z^3 - 3z) + \frac{1}{a(z^3 - 3z)} \\ \text{and } S(z) &= a\omega(z^3 - 3z) + \frac{1}{a\omega(z^3 - 3z)}, \quad a \in \mathbb{C}, \omega^3 = 1, \end{aligned}$$

which are non-exceptional rational functions with the same measure of maximal entropy but that cannot be written in the form $R = \sigma \circ S^n$ (or vice versa) for any integer n and Möbius transformation σ , showing that the classification given by (1.1) fails in general. Here, we prove some partial extensions to Beardon's results on the symmetry group of Julia sets. We apply these results to obtain a complete description of the symmetries for maps of the form $z \mapsto z^m + \lambda/z^d$, previously studied by McMullen, Devaney and others [6, 16].

2. RESULTS

Since rational functions are not holomorphic throughout all of \mathbb{C} , it is natural to consider them as analytic endomorphisms of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Therefore, as opposed to Beardon's set $\mathcal{I}(\mathbb{C}) = \{z \mapsto az + b : |a| = 1\}$ of isometries of \mathbb{C} , our set of possible symmetries shall be the set of holomorphic isometries of $\widehat{\mathbb{C}}$

$$\mathcal{I}(\widehat{\mathbb{C}}) = \left\{ z \mapsto \frac{az - \bar{b}}{bz + \bar{a}} : |a|^2 + |b|^2 = 1 \right\}$$

for the spherical metric on the Riemann sphere. This set is isomorphic as a Lie group to $\text{SO}(3)$ – which means that there is a diffeomorphism $\Phi : \mathcal{I}(\widehat{\mathbb{C}}) \rightarrow \text{SO}(3)$ that respects the group operations – and as such it is compact and connected, but not simply connected. Given a rational function R , this allows us to put our first restriction on the structure of

$$\Sigma(R) = \left\{ \sigma \in \mathcal{I}(\widehat{\mathbb{C}}) : \sigma[J(R)] = J(R) \right\},$$

the symmetries of its Julia set – and the rational equivalent of Beardon's $\Sigma(P)$.

Lemma 2.1. *$\Sigma(R)$ is a closed set.*

Proof. If $J(R) = \widehat{\mathbb{C}}$, then every isometry of the Riemann sphere preserves $J(R)$. Hence, $\Sigma(R)$ is $\mathcal{I}(\widehat{\mathbb{C}})$, which is a closed group, and the conclusion follows.

If $J(R) \neq \widehat{\mathbb{C}}$, we cannot have $\Sigma(R) = \mathcal{I}(\widehat{\mathbb{C}})$ – since $\mathcal{I}(\widehat{\mathbb{C}})$ acts transitively on the sphere, it would always be possible to map a point in the Julia set to some point outside the Julia set! Thus, we take $\sigma \in \mathcal{I}(\widehat{\mathbb{C}}) \setminus \Sigma(R)$. We know that there exists some $z \in J(R)$ such that $\sigma(z) \in F(R) = \widehat{\mathbb{C}} \setminus J(R)$, which is an open set. Therefore, there must be a neighbourhood U of $\sigma(z)$ that does not intersect $J(R)$. Since $\widehat{\mathbb{C}}$ is a homogeneous space, this yields a neighbourhood $V \subset \mathcal{I}(\widehat{\mathbb{C}})$ of the identity such that $\mu\sigma \notin \Sigma(R)$ for every $\mu \in V$. The construction of this neighbourhood V is as follows: for every $w \in U$, the homogeneity of $\widehat{\mathbb{C}}$ implies the existence of some $\gamma \in \mathcal{I}(\widehat{\mathbb{C}})$ such that $\gamma[\sigma(z)] = w$. Since the action of $\mathcal{I}(\widehat{\mathbb{C}})$ on $\widehat{\mathbb{C}}$ is smooth, the collection of such γ for every $w \in U$ yields a neighbourhood of the identity – which is also in this collection for $w = \sigma(z)$. By continuity of the group operations, $V\sigma$ is a neighbourhood of σ which does not intersect $\Sigma(R)$, and thus $\mathcal{I}(\widehat{\mathbb{C}}) \setminus \Sigma(R)$ is open. \square

Though simple, this result has crucial consequences. Firstly, as a closed subset of a compact set, we get that $\Sigma(R)$ is compact; secondly, by Cartan’s closed subgroup theorem, it follows that $\Sigma(R)$ is a Lie subgroup of $\mathcal{I}(\widehat{\mathbb{C}})$ – which means that it is an embedded submanifold of $\mathcal{I}(\widehat{\mathbb{C}})$. Hence, we obtain our first serious restriction on $\Sigma(R)$.

Theorem 2.2. *For a rational map R , $\Sigma(R)$ is (isomorphic to) one of:*

- (i) *The trivial group;*
- (ii) *A group of roots of unity;*
- (iii) *A dihedral group generated by a root of unity $z \mapsto e^{2\pi i/k}z$ and an inversion $z \mapsto 1/z$;*
- (iv) *The orientation-preserving symmetries of a regular tetrahedron, octahedron or icosahedron;*
- (v) *$S[O(1) \times O(2)]$ – i.e., the group of isometries of the form $z \mapsto e^{i\theta}z$ and $z \mapsto e^{i\theta}/z$ for any $\theta \in [0, 2\pi)$;*
- (vi) *All isometries of the Riemann sphere.*

Proof. We note that there is little to be done in cases (i) and (vi) from a symmetry point of view. Although their dynamics may be interesting (case (vi), for instance, contains the Lattès maps), we assume now that $\Sigma(R)$ is neither trivial nor all of $\mathcal{I}(\widehat{\mathbb{C}})$.

The first distinction we must make is between a discrete and a continuous symmetry group. In the former, $\Sigma(R)$ must be a discrete Lie group, and – since $\mathcal{I}(\widehat{\mathbb{C}})$ is compact – this implies that it is finite. The classification of finite subgroups of $SO(3)$ in [5, Theorem 4.1] then gives us cases (ii) through (iv). We do remark that we have changed the nomenclature; Carne refers to the roots of unity as the symmetries of a cone on a regular plane polygon, and to the dihedral groups as symmetries of a double cone on a regular plane polygon.

For continuous symmetry groups, we must study the Lie subgroups of $\mathcal{I}(\widehat{\mathbb{C}})$. Take, then, the connected component H of $\Sigma(R)$ containing the identity – which must be a Lie subgroup of $\mathcal{I}(\widehat{\mathbb{C}})$ with an associated Lie subalgebra $\mathfrak{h} \subset \mathfrak{i}(\widehat{\mathbb{C}})$. Since $\mathfrak{i}(\widehat{\mathbb{C}}) \simeq \mathfrak{so}(3) \simeq \mathbb{R}^3$, where the Lie algebra structure is given by the vector product, it follows that the only proper non-trivial Lie subalgebras of $\mathfrak{i}(\widehat{\mathbb{C}})$ are one-dimensional, and thus H is a one-dimensional Lie subgroup of $\mathcal{I}(\widehat{\mathbb{C}})$.

Now, every one-dimensional Lie group admits a parametrisation using the Lie exponential. Therefore, we write $H = \{\exp[tX] : t \in \mathbb{R}\}$ for some X in its Lie algebra $\mathfrak{h} \subset \mathfrak{i}(\widehat{\mathbb{C}})$. The action of $\mathcal{I}(\widehat{\mathbb{C}})$ on the Riemann sphere yields a flow $\phi : \mathbb{R} \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ defined as $\phi(t, z) = \exp[tX](z)$, which has as associated vector field $\vec{X} : T\widehat{\mathbb{C}} \rightarrow T\widehat{\mathbb{C}}$ – called the action field of H – given by

$$\vec{X}(z) = \partial_t \phi(0, z) = \left. \frac{d}{dt} \exp[tX]z \right|_{t=0} .$$

By the hairy ball theorem, there exists $z_0 \in \widehat{\mathbb{C}}$ such that $\vec{X}(z_0) = 0$; this point satisfies $\phi(t, z_0) = z_0$ for all t , and so it is fixed by every $\sigma \in H$. As every isometry of $\widehat{\mathbb{C}}$ has exactly two antipodal fixed points, it follows that z_0 ’s antipode is also fixed by every element of H . Conjugating the Riemann sphere by an isometry such that $z_0 = 0$, we conclude that H is conjugate to $S^1 = \{e^{it} : t \in [0, 2\pi)\}$.

Now, take a $z \in J(R)$ that is not fixed by the action of H . Its orbit must be a circle, and so $J(R)$ is a collection of circles – either a single circle or an uncountable amount of them. In the former case, $\Sigma(R)$ are the symmetries of a circle, so its elements are either of the form $z \mapsto e^{i\theta}z$ or $z \mapsto e^{i\theta}/z$ (furthermore, Eremenko and van Strien [8] showed that either R or R^2 must be a Blaschke product). We show that the latter case is not possible. The smoothness of each connected component of $J(R)$ implies that all multipliers of repelling periodic orbits are real, by a classic argument going back to Fatou (see [17, Corollary 8.11] or [9, Section 46]). Now, as shown by Eremenko and van Strien [8, Theorem 1], this implies that either R is a Lattès map or $J(R)$ is contained in a single circle. We have assumed that $J(R) \neq \widehat{\mathbb{C}}$, and so we must have $J(R)$ contained in a single circle – and therefore equal to a single circle. We fall back to the previous case, and we are done. \square

Remark 2.3. It should be noted that all cases in Theorem 2.2 can actually be realised as the symmetry group of some rational map. Case (i), in fact, is shown in Theorem 2.12 to be the most common. For cases (ii) through (iv), consider the group $\text{Aut}(R) = \{\sigma \in \text{PSL}(2, \mathbb{C}) : R\sigma = \sigma R\}$. By Proposition 2.4, any $\sigma \in \text{Aut}(R)$ that is also an isometry of $\widehat{\mathbb{C}}$ belongs to $\Sigma(R)$. Now, by a theorem of Doyle and McMullen [7], for any finite subgroup $G \in \text{PSL}(2, \mathbb{C})$ there exists a rational map R_G such that $\text{Aut}(R_G) = G$. Thus, by choosing G as one of the groups in (ii) - (iv), we can guarantee the existence of rational functions with any finite symmetry group. For the continuous symmetry groups, case (v) is realised by power maps $z \mapsto z^{\pm d}$ and case (vi), by the Lattès maps.

Now, we need conditions that allow us to choose, among all these possible geometries, which one corresponds to a given rational map R . We offer a sufficient condition and a necessary one; Ye’s example (1.2) suggests that complete characterisations, like for polynomials, are not possible.

Proposition 2.4. *Let R be a rational function and $\sigma \in \mathcal{I}(\widehat{\mathbb{C}})$, and suppose that R is not a Lattès map. If $R\sigma = \sigma^k R$ for some $k \geq 1$, then $\sigma \in \Sigma(R)$.*

Proof. We will show that the Fatou set of R is invariant under σ . Take $z \in F(R)$. By the Arzelà-Ascoli theorem, for any $\epsilon > 0$ there is a neighbourhood U of z satisfying $\text{diam}[R^m(U)] < \epsilon$ for every $m \geq 1$. Now, we consider how R behaves at $\sigma(z)$. By induction, our hypothesis implies that there exists for all $m \geq 1$ some $l \geq 1$ (which depends on m) such that $R^m\sigma = \sigma^l R^m$. Indeed, for the case $m = 1$, l is easily given by k as per our hypothesis. Now, for any m , $R^{m+1}\sigma = R(R^m\sigma)$, and the induction hypothesis gives us $R^{m+1}\sigma = R(\sigma^l R^m)$. By using that $R\sigma = \sigma^k R$, we can shift the σ ’s “one-by-one” to obtain $R^{m+1}\sigma = \sigma^{kl} R^{m+1}$. Therefore, $\text{diam}[R^m\sigma(U)] = \text{diam}[\sigma^l R^m(U)]$; since σ is an isometry of the Riemann sphere, it leaves the diameter of a set unchanged, and thus $\text{diam}[R^m\sigma(U)] = \text{diam}[R^m(U)]$ for every m . Since the terms on the right-hand side are limited by ϵ , this implies (by the Arzelà-Ascoli theorem) that R^m is a normal family at $\sigma(z)$, and thus $\sigma[F(R)] \subset F(R)$. Since σ^{-1} is also an isometry, we can apply the same reasoning to conclude that $\sigma^{-1}[F(R)] \subset F(R)$, and so $F(R)$ – and thus $J(R)$ – is invariant under σ and $\sigma \in \Sigma(R)$. \square

Our necessary condition even allows us to specify a value for k in Proposition 2.4, albeit in a very specific situation. However, we shall need some technical results

concerning potentials in order to prove it. We refer to [19], [13] and [4] for the necessary concepts.

Lemma 2.5. *Let Ω be a domain on the Riemann sphere and z_1, z_2, \dots points in Ω . Suppose $f : \Omega \rightarrow (0, +\infty]$ is a function such that:*

- (i) *f is harmonic on $\Omega \setminus \bigcup_{i \geq 1} \{z_i\}$;*
- (ii) *As $z \rightarrow z_i$, there exists some $m_i > 0$ such that $f(z) = -m_i \log |z - z_i| + O(1)$ for every $i \geq 1$ (we say that f has a logarithmic pole of order m_i at z_i);*
- (iii) *As $z \rightarrow \partial\Omega$, $f(z) \rightarrow 0$, except maybe for a set of points in $\partial\Omega$ of capacity zero.*

Then, f can be decomposed as

$$f(z) = \sum_{i \geq 1} m_i g_\Omega(z, z_i),$$

where $g_\Omega(z, w)$ is the Green's function of Ω with pole at w .

Proof. Let

$$h_n(z) = f(z) - \sum_{i=1}^n m_i g_\Omega(z, z_i).$$

We shall prove that the sequence h_n converges to zero, and this will give us the conclusion. First, the fact that g_Ω is always non-negative implies that $h_1 \geq h_2 \geq h_3 \geq \dots$ - i.e., h_n is a monotonically decreasing sequence. Next, each h_n is harmonic on Ω with the exception of logarithmic poles at z_i for $i \geq n + 1$ and goes to zero as z approaches the boundary of Ω . Therefore, we have that

$$h_1 \geq h_2 \geq h_3 \geq \dots \geq 0,$$

which means that the sequence h_n converges point-wise to some function $h : \Omega \rightarrow \mathbb{R}$. Since it is the limit of a monotonic sequence of harmonic functions, h is itself harmonic throughout Ω , and it also satisfies $h(z) \rightarrow 0$ as $z \rightarrow \partial\Omega$. By the maximum modulus principle, $h \equiv 0$ and we are done. \square

Next, we would like to prove that symmetries of $J(R)$ somehow respect the critical points of R , which are central to its dynamical behaviour. Indeed, let $C(R)$ stand for the set of critical points of R together with their pre-images (we shall call it the pre-critical set):

$$C(R) = \bigcup_{n \geq 0} R^{-n} \{z \in \widehat{\mathbb{C}} : z \text{ is a critical point of } R\}.$$

We shall demonstrate that any symmetry of $J(R)$ must, if R satisfies adequate conditions, preserve $C(R)$.

First, it is well-known that any rational function R admits a unique measure of maximal entropy μ_R (this was proved independently by Lyubich [15] and Freire, Lopes, and Mañé [10] in 1983). Then, Levin and Przytycki showed that, for non-exceptional rational functions without parabolic domains, Siegel discs, or Herman rings (the latter two being called rotation domains), having the same measure of maximal entropy is in fact equivalent to having the same Julia set. We are going to associate a potential to this fundamental measure of rational maps, denoted its ergodic potential, and show that it is continuous and invariant under symmetries of the Julia set. Through this potential, we shall demonstrate the invariance of a certain pluripotential associated to R , and show that its local minima coincide with

critical points of the lift of R to \mathbb{C}^2 . This, in turn, allows us to conclude that $C(R)$ is invariant under symmetries of $J(R)$.

Definition 2.6. Let R be a rational function, and μ_R its unique measure of maximal entropy. We define its ergodic potential to be the function $u_R : \widehat{\mathbb{C}} \rightarrow \mathbb{R}$ given by

$$u_R(z) = \int_{\widehat{\mathbb{C}}} \log \frac{1}{\rho(z, w)} d\mu_R(w),$$

where ρ denotes the spherical metric on the Riemann sphere.

In potential theory, this would be known as the elliptic potential associated to the measure μ_R . Since the logarithm is a subharmonic function, u_R is subharmonic in $\widehat{\mathbb{C}} \setminus J(R)$, and Okuyama [18] proved that it is actually continuous. He also proved that it satisfies $dd^c u_R = \omega - \mu_R$, where ω is the standard area form on the Riemann sphere and d and d^c are the differential operators given by $d = \partial + \bar{\partial}$ and $d^c = (i/(2\pi))(\bar{\partial} - \partial)$.

Proposition 2.7. *For a non-exceptional rational map R without any parabolic or rotation domains and with $\sigma \in \Sigma(R)$, μ_R and u_R are invariant under σ .*

To prove Proposition 2.7, we shall need this technical lemma.

Lemma 2.8. *Let R and S be rational maps of degree ≥ 2 . Then, $J(R) = J(S) \Leftrightarrow J(R)$ is completely invariant under S and $J(S)$ is completely invariant under R . In particular, if $\sigma \in \Sigma(R)$, then $J(R) = J(\sigma R)$.*

Proof. If $J(R) = J(S)$, the complete invariance of the Julia set follows immediately from its definition. To prove the converse, we recall that $J(S)$ is characterised as the minimal closed set with more than three points which is completely invariant under S , hence $J(R) \subset J(S)$. By the symmetry of the hypothesis, we also get that $J(S) \subset J(R)$, and so they are equal.

Now, consider $\sigma \in \Sigma(R)$. In order to conclude that $J(R) = J(\sigma R)$, we shall prove that $J(R)$ is invariant under σR , and vice versa. Firstly, since σ is a symmetry of $J(R)$, we have by definition that $\sigma[J(R)] = J(R)$ and so it is clear that $\sigma R[J(R)] = J(R)$. Also, by the minimality of the Julia set, this implies that $J(\sigma R) \subset J(R)$.

All that is left is to prove that $R[J(\sigma R)] = R^{-1}[J(\sigma R)] = J(\sigma R)$, and we shall do it by contradiction. Suppose, then, that $J(\sigma R)$ is not backward invariant under R (since R is surjective, this is actually equivalent to assuming that $J(\sigma R)$ is not completely invariant). Thus, we can take $z \in J(\sigma R)$ such that $R^{-1}(z)$ contains at least one point – which, by an abuse of notation, we shall denote by $R^{-1}(z)$ – that is not in $J(\sigma R)$. In other words, $R^{-1}(z) \in F(\sigma R)$. However, we already know that $J(\sigma R)$ is a subset of $J(R)$, which is completely invariant under R , and therefore $R^{-1}(z) \in J(R)$. This will be the basis for obtaining a contradiction.

Since $R^{-1}(z)$ is in the Fatou set of σR , it follows from the Arzelà-Ascoli theorem that $\{(\sigma R)^k\}_{k \geq 1}$ is equicontinuous there. Thus, for any $\epsilon > 0$, we can take a neighbourhood U of z such that

$$\text{diam} [(\sigma R)^k R^{-1}(U)] < \epsilon \quad \text{for every } k \geq 1.$$

By taking a term from the family of $(\sigma R)^k$, this becomes

$$\text{diam} [(\sigma R)^{k-1} \sigma(U)] < \epsilon, \quad k \geq 1.$$

Next, we consider what the sequence of mappings $(\sigma R)^{k-1}\sigma$ means for the diameter. σ is an isometry of the Riemann sphere; therefore, none of the σ terms in this expression have any effect on diameter. This means that the end result of $\text{diam} [(\sigma R)^{k-1}\sigma(U)]$ is ultimately determined by the iterations of R . Also, since $z \in J(R)$ and $J(R)$ is completely invariant under both σ and R , this means that U is always mapped to a neighbourhood of a point in $J(R)$. Since R eventually expands all neighbourhoods of points in its Julia set, we can conclude that $\text{diam} [(\sigma R)^{k-1}\sigma(U)]$ should eventually grow larger than any small value of ϵ , and so we have reached a contradiction. \square

Proof of Proposition 2.7. By Lemma 2.8, $\sigma \in \Sigma(R)$ implies that $J(\sigma R) = J(R)$. This means that $\mu_{\sigma R} = \mu_R$, and thus, since the measure of maximal entropy is invariant, we get

$$\mu_R = (\sigma R)_*\mu_R = \sigma_*R_*\mu_R = \sigma_*\mu_R.$$

In other words, μ_R is invariant under symmetries of the Julia set. Now, the expression for $u_R \circ \sigma$ becomes

$$u_R \circ \sigma(z) = \int_{\widehat{\mathbb{C}}} \log \frac{1}{\rho[\sigma(z), w]} d\mu_R(w)$$

and, as σ is an isometry of the metric ρ ,

$$u_R \circ \sigma(z) = \int_{\widehat{\mathbb{C}}} \log \frac{1}{\rho[z, \sigma^{-1}(w)]} d\mu_R(w) = \int_{\widehat{\mathbb{C}}} \log \frac{1}{\rho(z, w)} d(\sigma^{-1})_*\mu_R(w).$$

Since $(\sigma^{-1})_*\mu_R = \mu_R$ (for σ^{-1} is also a symmetry of $J(R)$), we recover the original expression for u_R from the right-hand side of the equality above, and thus $(\sigma^*u_R)(z) := u_R \circ \sigma(z) = u_R(z)$. \square

Although u_R is not trivial to compute, its invariance is a powerful tool for proving the invariance of $C(R)$ under symmetries. However, the proof also relies heavily on another subharmonic function related to R – the lift of its Green’s function, G_R .

Theorem 2.9. *For any non-exceptional rational map R without parabolic or rotation domains and $\sigma \in \Sigma(R)$, σ preserves the pre-critical set.*

Proof. In order to introduce G_R , we shall appeal to the polynomial lift of R in \mathbb{C}^2 . Let R be written in the form $R(z) = P(z)/Q(z)$, where P and Q are co-prime polynomials. If $d \geq 2$ is the degree of R , then the function

$$\widehat{R}(z_1, z_2) = z_2^d (P(z_1 z_2^{-1}), Q(z_1 z_2^{-1}))$$

is a homogeneous polynomial of degree d in \mathbb{C}^2 . Furthermore, if $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$ is the standard projection from \mathbb{C}^2 onto the Riemann sphere (given by $\pi(z_1, z_2) = z_1/z_2$), it is readily seen that $R \circ \pi = \pi \circ \widehat{R}$. Take, then, the escape rate function of \widehat{R} , defined as

$$G_R(z_1, z_2) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \|\widehat{R}^n(z_1, z_2)\|.$$

The definition implies that G_R is plurisubharmonic. It also implies that G_R satisfies $G_R(t(z_1, z_2)) = G_R(z_1, z_2) + \log |t|$ for any $t \in \mathbb{C}$, and thus $G_R(z_1, z_2) - \log \|(z_1, z_2)\|$ is constant on sets of the form $\{t(z_1, z_2) : t \in \mathbb{C}\}$. Therefore (see, for instance, [4, pp. 7–8]) there exists a unique function $g : \widehat{\mathbb{C}} \rightarrow \mathbb{R}$ such that

$$G_R - \log \|\cdot\| = g \circ \pi,$$

called the *Green's function* of R . Furthermore (see [4, Theorems 9 and 18]), this function g satisfies $dd^c g + \omega = \mu_R$. Since $\mu_R = \omega - dd^c u_R$ as well, we conclude that $dd^c(g + u_R) = 0$ – or, in other words, $g + u_R$ is harmonic throughout $\widehat{\mathbb{C}}$. Since the only harmonic functions on the Riemann sphere are constant, it follows that $u_R = -g + C$ for some real constant C , which we shall promptly ignore.

Thus, we have that $\log \|\cdot\| - G_R = \pi^* u_R$. By Proposition 2.7, $\sigma^* u_R = u_R$, so that $\log \|\cdot\| - G_R = \pi^*(\sigma^* u_R) = (\sigma\pi)^* u_R$. The symmetry σ , being an isometry of $\widehat{\mathbb{C}}$, lifts to an isometry Σ of \mathbb{C}^2 , and therefore

$$\log \|\cdot\| - G_R = \Sigma^*(\pi^* u_R) = \Sigma^*(\log \|\cdot\| - G_R) = \Sigma^* \log \|\cdot\| - \Sigma^* G_R.$$

The fact that Σ is an isometry implies that the logarithm cancels out, so that $G_R = \Sigma^* G_R$. In particular, local minima of G_R over $F(R)$ are mapped by Σ onto local minima of G_R over $F(R)$; we are going to prove that these points are exactly those with $D\widehat{R} = 0$ (where $D\widehat{R}$ denotes the differential of \widehat{R}). Indeed, consider the sequence

$$G_n := \frac{1}{d^n} \log \|\widehat{R}^n\|.$$

Each of its terms is p.s.h. and smooth, and the sequence converges to G_R uniformly over compact sets. Now, by the chain rule, the points with $D\widehat{R}^n = 0$ are the minima of G_n ; these, in turn, are exactly the n th pre-images of critical points of \widehat{R} – i.e., those with $D\widehat{R} = 0$. And, also by the chain rule, any local minimum of G_n is also a local minimum of G_m for any $m \geq n$. Since the convergence to G_R is uniform over compact sets, this means that G_R also has a local minimum at every such point. Conversely, every local minimum of G_R comes from a converging sequence of local minima of G_n for sufficiently large values of n .

We conclude, thus, that local minima of G_R correspond to $C(\widehat{R})$, the pre-critical set of \widehat{R} . Since we already knew that local minima of G_R are invariant under Σ , this implies that $\Sigma[C(\widehat{R})] = C(\widehat{R})$; finally, we come back to the Riemann sphere – Σ pushes down to σ , $C(\widehat{R})$ pushes down to $C(R)$ and we conclude that $\sigma[C(R)] = C(R)$. \square

We are now ready to prove the necessary condition.

Proposition 2.10. *Suppose R is non-exceptional without parabolic or rotation domains and $\sigma \in \Sigma(R)$ fixes a superattracting fixed point z_0 of R , with local degree $m > 1$. Then, $R\sigma = \sigma^m R$.*

Proof. Consider the function

$$f(z) = - \lim_{n \rightarrow \infty} \frac{1}{m^n} \log |\Phi[R^n(z)]|,$$

where $\Phi : U \rightarrow B(0; r)$ is a biholomorphism conjugating R to $z \mapsto z^m$. It is well-defined throughout the immediate basin of attraction for z_0 , denoted $\mathcal{A}(z_0)$, with poles at pre-images of z_0 with order given by the multiplicity of the pre-image. By Böttcher's theorem, R sends level curves of f onto level curves of f – in fact, $f[R(z)] = mf(z)$. Furthermore, we can apply Lemma 2.5 to f and write

$$f(z) = \sum_{i \geq 0} m_i g_{F_0}(z, z_i),$$

where F_0 is the connected component of $F(R)$ containing z_0 , and we have enumerated the pre-images of z_0 in F_0 as z_0, z_1, \dots (z_0 is a pre-image of itself). Now, we have:

$$f[\sigma(z)] = \sum_{i \geq 0} m_i g_{F_0}[\sigma(z), z_i] = \sum_{j \geq 0} m_j g_{F_0}[\sigma(z), \sigma(z_j)],$$

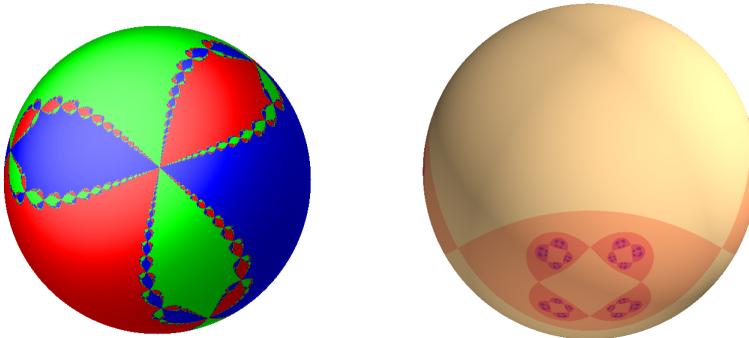
where in the last inequality we have used Theorem 2.9 to ensure that σ permutes the z_i , and permuted the indices accordingly. Next, our hypothesis that σ is a symmetry of $J(R)$ fixing z_0 implies that σ is a conformal mapping of F_0 onto itself, and – since Green’s functions are preserved by conformal mappings – we conclude that $g_{F_0}[\sigma(z), \sigma(z_i)] = g_{F_0}(z, z_i)$ for every i . Therefore, $f\sigma(z) = f(z)$, which means that there is a neighbourhood V of z_0 that is forward-invariant under both R and σ and which is contained in U – one need only define V as any level curve of f that is completely contained in U . Again by Böttcher’s theorem, Φ maps V into a circle $B(0; \delta)$ and conjugates R to z^m .

Now, consider the functions $\hat{\sigma} = \Phi\sigma\Phi^{-1} : B(0; \delta) \rightarrow B(0; \delta)$ and $\hat{R} = \Phi R\Phi^{-1} : B(0; \delta) \rightarrow B(0; \delta)$. We already know that $\hat{R}(z) = z^m$; now, notice that $\hat{\sigma}$ is an automorphism of $B(0; \delta)$, and thus it is an isometry of the hyperbolic metric on $B(0; \delta)$. Since it also fixes 0, it follows that it is of the form $\hat{\sigma}(z) = e^{i\theta}z$. Therefore,

$$\hat{R}\hat{\sigma}(z) = (e^{i\theta}z)^m = e^{i(m\theta)}z^m = \hat{\sigma}^m\hat{R}(z),$$

and so $R\sigma = \sigma^m R$. □

In Figure 1, we show examples of Julia sets with finite, non-trivial symmetry groups. Figure 1a is the Julia set for the Newton map of the polynomial $z \mapsto z^3 + 1$, and Figure 1b corresponds to the map $z \mapsto z^2 + 1/z^2$. The latter belongs to a family which shall be discussed in further details in Section 3.



(a) Phase portrait for the Newton map of $z \mapsto z^3 + 1$. Each colour represents the attraction basin of minus a cube root of unity.

(b) Phase portrait for the rational map $z \mapsto z^2 + 1/z^2$. Lighter points belong to the basin of attraction of infinity, and black points are in the Julia set.

FIGURE 1. Examples for rational maps with finite symmetry groups

Finally, a known application of symmetries of a Julia set is in the description of all polynomials that share a Julia set [20]. Although (1.2) provides an example where the simple criterion for polynomials fails for rational maps, we can nevertheless offer a sufficient condition for having the same Julia set that goes beyond commuting.

Proposition 2.11. *Let R and S be rational maps of degree ≥ 2 such that $SR = \sigma RS$ for some $\sigma \in \Sigma(R)$. Then, R and S have the same Julia set.*

Proof. We shall prove that, under the hypotheses, $F(R)$ is completely invariant under S and vice versa. Since both are surjective on $\widehat{\mathbb{C}}$, it suffices to prove backward invariance, i.e., $S^{-1}[F(R)] \subset F(R) \subset S^{-1}[F(R)]$.

Firstly, notice that, for all $k \geq 1$, we obtain by induction – the argument is analogous to the one used in Proposition 2.4 – that $SR^k = (\sigma R)^k S$. Now, let M be a Lipschitz constant for S in the spherical metric. For $z \in F(R)$, the definition of the Fatou set means that $\{R^k\}_{k \geq 1}$ is normal, and therefore equicontinuous by the Arzelà-Ascoli theorem, at z . As such, for any ϵ , there exists a neighbourhood U of z such that $\text{diam}[R^k(U)] < \epsilon/M$ for every $k \geq 1$. By the induction formula above, we have:

$$\text{diam}[(\sigma R)^k S(U)] = \text{diam}[SR^k(U)] \leq M \text{diam}[R^k(U)] < \epsilon.$$

This tells us that $\{(\sigma R)^k\}_{k \geq 1}$ is equicontinuous on $S(U)$, and thus $S(z) \in F(\sigma R) = F(R)$. Therefore, $S[F(R)] \subset F(R)$, and $F(R) \subset S^{-1}[F(R)]$.

Now, let $V = S^{-1}(U)$ for $U \subset F(R)$. Since $F(R) = F(\sigma R)$, we can pick U such that $\text{diam}[(\sigma R)^k(U)] < \epsilon$ for every $k \geq 1$ for an arbitrary choice of $\epsilon > 0$. Then,

$$\begin{aligned} \text{diam}[SR^k(V)] &= \text{diam}\{SR^k[S^{-1}(U)]\} \\ &= \text{diam}\{(\sigma R)^k S[S^{-1}(U)]\} \\ &= \text{diam}[(\sigma R)^k(U)] < \epsilon, \end{aligned}$$

and so $\{SR^k\}_{k \geq 1}$ is equicontinuous on V . Since S is Lipschitz continuous, so is $\{R^k\}_{k \geq 1}$, and by the Arzelà-Ascoli theorem we have that $\{R^k\}_{k \geq 1}$ is normal on $S^{-1}[F(R)]$. It follows that $S^{-1}[F(R)] \subset F(R)$, and so we can conclude that $F(R)$ is backward invariant under S . Finally, this implies that $F(R)$ – and thus $J(R)$ – is completely invariant under S .

For the remaining statement, the complete invariance of $F(S)$ under R , we recall that $\Sigma(R)$ is a group; hence, σ^{-1} is also a symmetry of the Julia set, and it satisfies $RS = \sigma^{-1}SR$. We can apply the same argument as above, concluding that $F(S)$ is completely invariant under R , and by Lemma 2.8 we obtain $J(R) = J(S)$. \square

We end this section with an observation on the prevalence of symmetries for rational maps. In [2], Beardon remarks that most polynomials have trivial Julia sets. Here, as another application of Lemma 2.8, we prove a similar statement – with a couple more technicalities.

Consider a rational map of degree d as

$$R(z) = \frac{P(z)}{Q(z)} = \frac{a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0}{b_d z^d + b_{d-1} z^{d-1} + \cdots + b_0}.$$

It can be naturally identified with its coefficient vector

$$(a_d, a_{d-1}, \dots, a_0, b_d, b_{d-1}, \dots, b_0) \in \mathbb{C}^{2d+2}.$$

However, not every vector in \mathbb{C}^{2d+2} gives rise to a degree d rational function – we must make sure that the polynomials P and Q do not share any roots. Thus, we can consider the set of feasible coefficient vectors to be the complement of the roots for the resultant polynomial $\text{res}(P, Q)$. Since this polynomial is homogeneous and multiples of a coefficient vector lead to the same rational function, we conclude that the space Rat_d of all rational maps of degree d is, in fact, an affine (quasi-projective) variety in $\mathbb{P}^{2d+1}(\mathbb{C})$. The Zariski topology on this variety yields a way to define “general” subsets of Rat_d . We say that a property holds for a *generic* rational map in Rat_d if it holds except for a countable union of Zariski closed sets. In this language, Ye [21, Theorem 1.2] showed that generic rational functions R of degree $d \geq 3$ have a “unique” measure of maximal entropy μ_R , in the sense that any other rational function S with $\mu_S = \mu_R$ must be an iterate of R . In view of that, we have:

Theorem 2.12. *For generic $R \in \text{Rat}_d$, $d \geq 3$, $\Sigma(R)$ is trivial.*

Proof. Suppose that $\sigma \in \Sigma(R)$ is not the identity. Then, by Lemma 2.8, $J(\sigma R) = J(R)$ and so by Levin and Przytycki’s results $\mu_R = \mu_{\sigma R}$ – we can assume that R is non-exceptional without parabolic or rotation domains, because such rational maps are contained in a Zariski F_σ set. However, by Ye [21, Theorem 1.2], for a generic $R \in \text{Rat}_d$ with $d \geq 3$, having the same measure of maximal entropy means that $\sigma R = R^n$ for some natural n . Of course, this is not the case here; hence, R must be contained in a countable union of Zariski closed sets in Rat_d . \square

3. APPLICATIONS

We apply these results to certain families of singularly perturbed maps, also called McMullen maps. These are rational functions of the form

$$R_\lambda(z) = z^m + \frac{\lambda}{z^d}$$

for $m \geq 2$, $d \geq 1$ and $\lambda \in \mathbb{C}$. These maps have been previously studied by McMullen [16], Devaney and others [6], who have already exploited particular Möbius functions that preserve $J(R_\lambda)$. We provide a way to determine all isometries of $\widehat{\mathbb{C}}$ that do so.

Theorem 3.1. *The Julia set of R_λ has the following symmetries:*

- (i) $z \mapsto e^{i\theta} z^{\pm 1}$ for any $\theta \in \mathbb{R}$, if $\lambda = 0$;
- (ii) $z \mapsto \mu z^{\pm 1}$, where $\mu^{m+d} = 1$, if $m = d$ and $|\lambda| = 1$;
- (iii) $z \mapsto \mu z$, where $\mu^{m+d} = 1$, otherwise.

Proof. The case $\lambda = 0$ reduces to $R_0(z) = z^m$, and its Julia set is a circle; hence, (i) follows from Theorem 2.2.

Now, if $\lambda \neq 0$, any symmetry must either fix infinity or map it to another point. We start with the symmetries fixing infinity. These are, of course, a subgroup of $\Sigma(R_\lambda)$ made of symmetries of the form $z \mapsto e^{i\theta} z$ for some values of $\theta \in \mathbb{R}$. Our task here is to ascertain the possible values of θ . Let $\sigma(z) = \mu z$, where $\mu^{m+d} = 1$. We shall prove that $\sigma \in \Sigma(R_\lambda)$. We have:

$$R_\lambda \sigma(z) = \mu^m z^m + \frac{\lambda}{\mu^d z^d} = \mu^m z^m + \mu^m \frac{\lambda}{z^d} = \mu^m R_\lambda(z) = \sigma^m R_\lambda(z),$$

and so, by Proposition 2.4, σ is a symmetry of $J(R_\lambda)$. On the other hand, any symmetry fixing infinity must, by Proposition 2.10, satisfy $R_\lambda\sigma = \sigma^m R_\lambda$. If $\sigma(z) = \nu z$ with $|\nu| = 1$, then

$$R_\lambda\sigma(z) = \nu^m z^m + \nu^{-d} \frac{\lambda}{z^d} = \nu^m z^m + \nu^m \frac{\lambda}{z^d} = \sigma^m R_\lambda(z),$$

and thus $\nu^{-d} = \nu^m$ and $\nu^{m+d} = 1$. This means that the symmetries fixing infinity are a subgroup of $\Sigma(R_\lambda)$ isomorphic to $\mathbb{Z}/(m+d)\mathbb{Z}$.

Now, we consider any remaining symmetries. First, we invoke Theorem 2.2: since $\Sigma(R_\lambda)$ only admits certain structures, any symmetry group that properly contains $\{z \mapsto \nu z : \nu^{m+d} = 1\}$ as a subgroup must also contain a symmetry of the form $\sigma(z) = \mu/z$ with $\mu^{m+d} = 1$. In other words, the only possible structures for $\Sigma(R_\lambda)$ are $\{z \mapsto \mu z : \mu^{m+d} = 1\}$ or $\{z \mapsto \mu z : \mu^{m+d} = 1\} \cup \{z \mapsto \mu/z : \mu^{m+d} = 1\}$. What is left, thus, is to decide when it is one or the other.

Thus, consider that σ has the form $\sigma(z) = \mu/z$. We shall again call upon results from potential theory. Let $g_0(z, 0)$ and $g_\infty(z, \infty)$ denote the Green's functions for the connected components of $F(R_\lambda)$ containing 0 and ∞ , respectively. Since conformal mappings send Green's functions to Green's functions, we have that $g_\infty[\sigma(z), \infty] = g_0(z, 0)$ for z in a neighbourhood of 0. At the same time, R_λ maps this neighbourhood of 0 to a neighbourhood of infinity with multiplicity d , and thus $g_\infty[R_\lambda(z), \infty] = dg_0(z, 0)$ by the uniqueness of the Green's function of a domain. Therefore, $g_\infty[R_\lambda(z), \infty] = dg_\infty[\sigma(z), \infty]$ and $d \log |\Phi\sigma(z)| = \log |\Phi R_\lambda(z)|$, where Φ is the Böttcher function for R_λ ; hence, by applying the series expansion for Φ ,

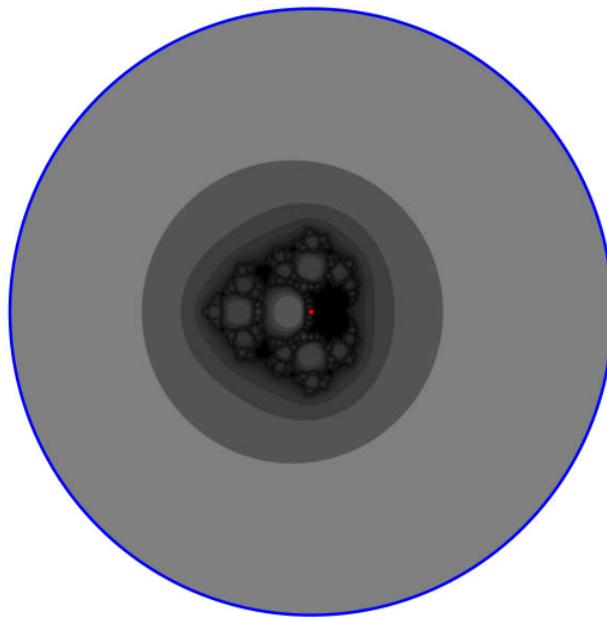
$$\left[\frac{\mu}{z} + a_0 + \dots \right]^d = \alpha \left[z^m + \frac{\lambda}{z^d} + \dots \right],$$

where $|\alpha| = 1$. Comparing the coefficients in the series expansion, we conclude that $m = d$ and, simultaneously, $\mu^d = \alpha\lambda$. Since $|\mu| = |\alpha| = 1$, it follows that $|\lambda| = 1$ and we are done. \square

Figures 2a and 3a illustrate this theorem. If $m = d = 2$, the isometries $z \mapsto \mu z$, where $\mu^4 = 1$, are always a symmetry of $J(R_\lambda)$ (see Figures 2b and 2d). If, in addition, $\lambda = 0$ (red dot) or $|\lambda| = 1$ (blue circle), then additional symmetries arise: in the former case, $J(R_0)$ is the unit circle, and so has all rotations as its symmetries as well as inversions with respect to the unit circle. In the latter, composing a rotation by a fourth root of unity with an inversion also yields a symmetry of $J(R_\lambda)$ (see Figure 2c). If, on the other hand, $m = 2$ but $d = 1$, the region $|\lambda| = 1$ has nothing special with regards to symmetry. For any $\lambda \in \mathbb{C}^*$, the symmetry group consists of rotations by $2\pi/3$ and $4\pi/3$ radians as in Figures 3b, 3c and 3d. For $\lambda = 0$ (red dot) the Julia set is again the unit circle.

The figures show (in black) the connectedness locus – i.e., the values of λ for which $J(R_\lambda)$ is connected; we remark that, since the families discussed here are very similar, they have similar connectedness loci – in order to emphasise one thing: the structure of the symmetry group has no regards for any topological changes to $J(R_\lambda)$. Indeed, while the structure of the Julia set changes drastically on the boundaries of the black region, the symmetry group “ignores” these changes and instead undergoes change as $|\lambda| = 1$ for $m = d = 2$, as in Figure 2a.

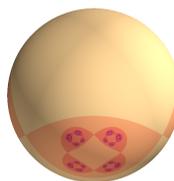
Remark 3.2. This sudden, quickly reversed change at $|\lambda| = 1$ when $m = d$ can, in fact, be understood as the transversal movement of half-symmetries of R_λ through



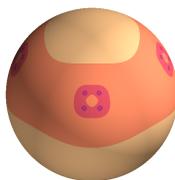
(a) Parameter plane for the family $z \mapsto z^2 + \lambda/z^2$. We highlight the regions where $\Sigma(R_\lambda)$ is different.



(b) $\lambda = 0.1$

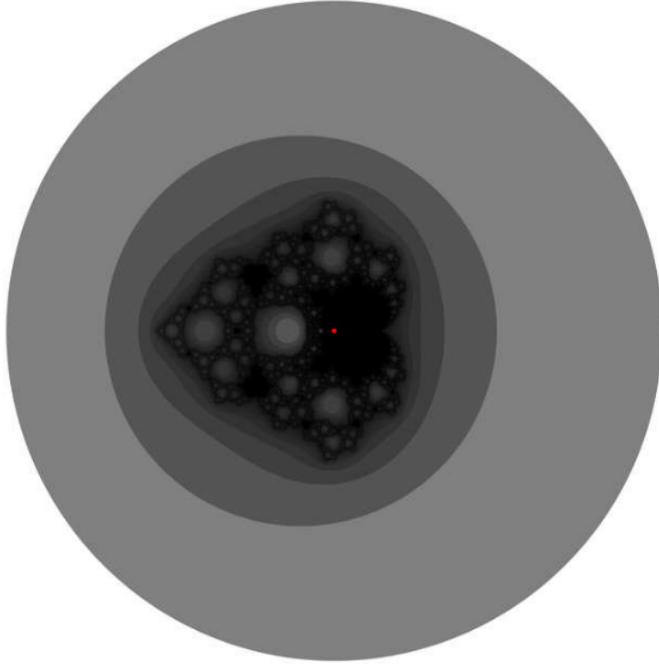


(c) $\lambda = 1$

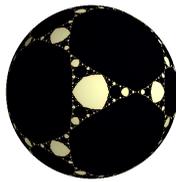


(d) $\lambda = 10$

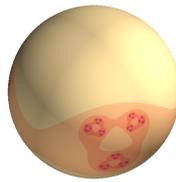
FIGURE 2. (a) The parameter plane for a family of singular perturbations with $m = d = 2$; (b–d) Julia sets for particular values of λ



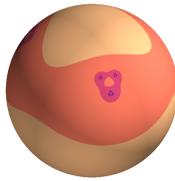
(a) Parameter plane for the family $z \mapsto z^2 + \lambda/z$. We highlight the regions where $\Sigma(R_\lambda)$ is different.



(b) $\lambda = 0.1$



(c) $\lambda = 1$



(d) $\lambda = 10$

FIGURE 3. (a) The parameter plane for a family of singular perturbations with $m = 2$ and $d = 1$; (b–d) Julia sets for particular values of λ

$\mathcal{I}(\widehat{\mathbb{C}})$ in $\mathrm{PSL}(2, \mathbb{C})$. In [11], the half-symmetries of a rational map R were defined as the Möbius transformations $\gamma \in \mathrm{PSL}(2, \mathbb{C})$ satisfying $R\gamma = R$ – i.e., they permute the fibres of R . It is readily seen that these form a group $\mathcal{G}(R) \subset \mathrm{PSL}(2, \mathbb{C})$, and that $\gamma[J(R)] = J(R)$ for any $\gamma \in \mathcal{G}(R)$. It was proved by Hu and his co-workers that $\mathcal{G}(R)$ is always a finite group, and that it is conjugate to a group of isometries (see Theorem 2.2). For R_λ and $m = d$, then, it is easy to check that $\gamma_\lambda(z) = \sqrt[m]{\lambda}/z$ is a half-symmetry depending on the value of λ . In other words, there is a continuous function $\lambda \mapsto \gamma_\lambda$ from \mathbb{C} to $\mathrm{PSL}(2, \mathbb{C})$, and its image intersects $\mathcal{I}(\widehat{\mathbb{C}}) \subset \mathrm{PSL}(2, \mathbb{C})$ precisely as $|\lambda| = 1$ – and then it moves away again, giving us the strange “bifurcation” of $\Sigma(R_\lambda)$ at the blue circle in Figure 2a.

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