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NORTH-SOUTH TYPE DYNAMICS OF RELATIVE ATOROIDAL AUTOMORPHISMS OF FREE GROUPS ON A RELATIVE SPACE OF CURRENTS

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Abstract. This paper, which is the second of a series of three papers, studies dynamical properties of elements of $\mathrm{Out}(F_{\mathtt{n}})$, the outer automorphism group of a nonabelian free group $F_{\mathtt{n}}$. We prove that, for every exponentially growing outer automorphism of $F_{\mathtt{n}}$, there exists a preferred compact topological space, the space of currents relative to a malnormal subgroup system, on which ϕ acts by homeomorphism with a North-South dynamics behavior.

1. Introduction

Let $\mathtt{n} \geqslant 2$. This paper is the second of a sequence of three papers where we study the growth of the conjugacy classes of elements of $F_{\mathtt{n}}$ under iterations of elements of $\mathrm{Out}(F_{\mathtt{n}})$, the outer automorphism group of a nonabelian free group of rank $\mathtt{n}$. An outer automorphism $\phi \in \mathrm{Out}(F_{\mathtt{n}})$ is *exponentially growing* if there exist $g \in F_{\mathtt{n}}$, a free basis $\mathfrak{B}$ of $F_{\mathtt{n}}$ and a constant $K > 0$ such that, for every $m \in \mathbb{N}^*$, we have

$$\ell_{\mathfrak{B}}(\phi^m([g])) \geqslant e^{Km},$$

where $\ell_{\mathfrak{B}}(\phi^m([g]))$ denotes the length of a cyclically reduced representative of $\phi^m([g])$ in the basis $\mathfrak{B}$. Such an element g is said to be *exponentially growing under iteration of* ϕ and the set of elements of $F_{\mathtt{n}}$ which have exponential growth under iteration of ϕ is the *pure exponential part of* ϕ. It is known, using for instance the train track technology of Bestvina and Handel (see [BH]), that every element g of $F_{\mathtt{n}}$ which is not exponentially growing under iteration of ϕ is *polynomially growing under iteration of* ϕ, that is, there exists an integer $K \in \mathbb{N}$ such that, for every $m \in \mathbb{N}^*$, we have

$$\ell_{\mathfrak{B}}(\phi^m([g])) \leqslant (m+1)^K.$$

Initiated by Švarc, Milnor and Wolf, and particularly developed by Guivarc'h, Gromov and Grigorchuk, growth problems in groups are a major field of study in geometric and dynamical group theory, see for instance [LS, Man, Hel]. Many works study the subfield of the element growths under iteration of group automorphisms (see for instance [BFH1, Lev, CU]), for instance in the context of hyperbolic groups. See in particular [Cou] for examples of intermediate growth rates. As another example, Dahmani and Krishna [DS] found a sufficient condition for the suspension of an automorphism of a hyperbolic group to be relatively hyperbolic, and this condition is linked with the structure of the set of all elements of the hyperbolic group which

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have polynomial growth under iterations of the considered automorphism. Such exponentially growing outer automorphisms of $F_{\mathtt{n}}$ were already studied in distinct contexts. For instance, Bestvina, Feighn and Handel [BFH1] used them to prove the Tits alternative for $\mathrm{Out}(F_{\mathtt{n}})$.

If $\phi \in \mathrm{Out}(F_{\mathtt{n}})$, we denote by $\mathrm{Poly}(\phi)$ the set of elements g of $F_{\mathtt{n}}$ such that g is polynomially growing under iteration of ϕ. Let $\mathrm{Poly}(H) = \bigcap_{\phi\in H} \mathrm{Poly}(\phi)$. The aim of this series of papers is to prove Theorem 1.1.

Theorem 1.1. *Let* $\mathtt{n} \geqslant 3$ *and let* H *be a subgroup of* $\mathrm{Out}(F_{\mathtt{n}})$*. There exists* $\phi \in H$ *such that* $\mathrm{Poly}(\phi) = \mathrm{Poly}(H)$*.*

Informally, Theorem 1.1 shows that the exponential growth of a subgroup H of $\mathrm{Out}(F_{\mathtt{n}})$ is encaptured by the exponential growth of a single element of H. Indeed, if $g \in F_{\mathtt{n}}$ has exponential growth for some element $\psi \in H$, then g has exponential growth for an element $\phi \in H$ given by Theorem 1.1. The proof relies on dynamical properties of the action of outer automorphisms on some preferred topological space. In this article, we study the dynamical properties of the elements of the subgroup H of $F_{\mathtt{n}}$ that will be used in [Gue2] in order to construct an element $\phi \in H$ given by Theorem 1.1.

Let $\phi \in \mathrm{Out}(F_{\mathtt{n}})$ be an exponentially growing outer automorphism. In this article, we construct natural (compact, metrizable) topological spaces X on which a subgroup of $\mathrm{Out}(F_{\mathtt{n}})$ containing ϕ acts by homeomorphisms with the additional property that ϕ acts with *North-South dynamics*: there exist two proper disjoint closed subsets of X such that every point of X which is not contained in these subsets converges to one of the two subsets under positive or negative iteration of ϕ. North-South dynamics are preferred tools to apply ping-pong arguments similar to the ones of Tits [Tit] and are used to obtain structural properties of some groups.

The topological space X that we use in the proof of Theorem 1.1 is constructed in such a way that it allows us to create a dictionary between dynamical properties of the action of ϕ on X and growth properties of elements of $F_{\mathtt{n}}$ under iteration of ϕ. In order to construct X, we first need to detect all the elements g of $F_{\mathtt{n}}$ such that the length of $[g]$ with respect to any basis of $F_{\mathtt{n}}$ grows at most polynomially fast under iteration of ϕ. Levitt [Lev] proved that there exist finitely many finitely generated subgroups $H_1, \ldots, H_k$ of $F_{\mathtt{n}}$ such that the conjugacy class of an element g of $F_{\mathtt{n}}$ is not exponentially growing under iteration of ϕ if and only if g is contained in a conjugate of some H_i for $i \in \{1, \ldots, k\}$. Moreover, the set $\mathcal{A}(\phi) = \{[H_1], \ldots, [H_k]\}$ is a *malnormal subgroup system*: for every $i \in \{1, \ldots, k\}$, the group H_i is a malnormal subgroup of $F_{\mathtt{n}}$ and for all distinct subgroups A and B such that $[A], [B] \in \mathcal{A}(\phi)$, we have $A \cap B = \{e\}$. Every element of $F_{\mathtt{n}}$ which is contained in a conjugate of some H_i with $i \in \{1, \ldots, k\}$ has polynomial growth under iteration of ϕ. Moreover, we have $\mathrm{Poly}(\phi) = \bigcup_{i=1}^{r} \bigcup_{g\in F_{\mathtt{n}}} gH_ig^{-1}$.

In [Gue1], we constructed a compact, metrizable space, called the space of projectivised currents relative to $\mathcal{A}(\phi)$, denoted by $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi))$, which is the space of projectivised Radon measures on the double boundary of $F_{\mathtt{n}}$ relative to $\mathcal{A}(\phi)$, equipped with the weak-$*$ topology (see Section 2.4 for precise definitions). In [Gue1], we proved that the set of currents associated with $\mathcal{A}(\phi)$-*nonperipheral* conjugacy classes of elements of g of $F_{\mathtt{n}}$, that is, such that g is not contained in the conjugacy class of some H_i with $i \in \{1, \ldots, k\}$, is dense in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi))$. Thus, the set of conjugacy classes of elements of $F_{\mathtt{n}}$ whose length grows exponentially fast under iteration of ϕ is dense in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi))$. If we denote by

$\mathrm{Out}(F_{\mathtt{n}}, \mathcal{A}(\phi))$ the subgroup of $\mathrm{Out}(F_{\mathtt{n}})$ consisting of every element $\psi \in \mathrm{Out}(F_{\mathtt{n}})$ such that $\psi(\mathcal{A}(\phi)) = \mathcal{A}(\phi)$, the group $\mathrm{Out}(F_{\mathtt{n}}, \mathcal{A}(\phi))$ acts by homeomorphisms on $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi))$ by pushing forward the measures. In this article, we prove Theorem 1.2.

Theorem 1.2 (See Theorem 5.1)**.** *Let* $\mathtt{n} \geqslant 3$ *and let* ϕ *be an exponentially growing outer automorphism. The outer automorphism* ϕ *acts with North-South dynamics on the space* $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi))$*.*

In fact, we prove a slightly stronger result since we prove a *uniform North-South dynamics result*, that is, the convergence in the North-South dynamics statement can be made uniform on compact subsets of $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi))$. As explained above, North-South dynamics results given by Theorem 1.2 will be a key point in the proof of Theorem 1.1.

Such dynamical results already appear in similar contexts. For instance, Tits proved in [Tit] its alternative for linear groups using North-South dynamics and ping-pong arguments. In the context of the mapping class group $\mathrm{Mod}(S)$ of a compact connected orientable surface S of genus at least 2, pseudo-Anosov elements act with North-South dynamics on the space of projectivised measured foliations ([Thu], see also the work of Ivanov [Iva]) or the curve complex [MM]. Using this North-South dynamics, Ivanov [Iva] (see also the work of McCarthy [McC]) later proved a Tits alternative for subgroups of $\mathrm{Mod}(S)$. Similarly, North-South dynamics results were obtained for certain classes of outer automorphisms of $F_{\mathtt{n}}$. For instance, *fully irreducible outer automorphisms* act on the compactified Outer space [LL] or the space of projectivised currents ([Mar], see also the work of Uyanik [Uya1]) with a North-South dynamics and *atoroidal outer automorphisms* act on the space of projectivised currents with a North-South dynamics [LU2, Uya2]. Clay and Uyanik [CU] applied this result in the proof of the fact that, for every subgroup H of $\mathrm{Out}(F_{\mathtt{n}})$, either H contains an atoroidal outer automorphism or there exists a nontrivial element g of $F_{\mathtt{n}}$ such that, for every element $\phi \in H$, there exists $k \in \mathbb{N}^*$ such that we have $\phi^k([g]) = [g]$. Such dynamical results were later extended to relative contexts by Gupta [Gup1, Gup2]. We note that if $\mathcal{F}$ is a nonsporadic free factor system and if $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ is fully irreducible and atoroidal relative to $\mathcal{F}$, then Theorem 5.1 implies [Gup1, Theorem A]. Moreover, the North-South dynamics result proved by Gupta is not sufficient to prove Theorem 1.2 since we also need to deal with *sporadic* free factor systems.

In order to prove Theorem 1.1, we will need a slightly stronger result than Theorem 1.2. Indeed, let $\phi \in \mathrm{Out}(F_{\mathtt{n}})$ and let $\mathcal{A}(\phi) = \{[H_1], \dots, [H_k]\}$. Suppose that ϕ preserves the conjugacy class of a corank one free factor A of $F_{\mathtt{n}}$. Let $\mathcal{A}(\phi) \wedge A$ be the malnormal subgroup system consisting in the conjugacy classes of the intersection of the conjugates of the subgroups H_i with $i \in \{1, \dots, k\}$ with A. By Theorem 1.2, there exist closed disjoint subsets $\Delta_{\pm}(\phi|_A)$ such that the outer automorphism $\phi|_A \in \mathrm{Out}(A, \mathcal{A}(\phi) \wedge A)$ acts with North-South dynamics on $\mathbb{P}\mathrm{Curr}(A, \mathcal{A}(\phi) \wedge A)$ with respect to $\Delta_{\pm}(\phi|_A)$. There is a canonical embedding $\mathbb{P}\mathrm{Curr}(A, \mathcal{A}(\phi) \wedge A) \hookrightarrow \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi) \wedge A)$, and we denote by $\Delta_{\pm}(\phi)$ the image of $\Delta_{\pm}(\phi|_A)$ in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi) \wedge A)$. We will need to understand the dynamics of ϕ on the space $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi) \wedge A)$. As there might exist elements in $F_{\mathtt{n}}$ which have polynomial growth under iterations of ϕ and which are not contained in a conjugate of A, one cannot apply Theorem 1.2 to obtain a North-South dynamics result. However, we obtain the following result.

Theorem 1.3 (See Theorem 6.4). *Let $\mathtt{n} \geqslant 3$ and let $\phi \in \mathrm{Out}(F_{\mathtt{n}})$ be an exponentially growing outer automorphism which preserves a corank one free factor A. There exist two closed compact subsets $\widehat{\Delta}_{\pm}(\phi)$ of $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi) \wedge A)$ such that the following holds. Let $U_{\pm}$ be open neighborhoods of $\Delta_{\pm}(\phi)$ in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi) \wedge A)$ and $\widehat{V}_{\pm}$ be open neighborhoods of $\widehat{\Delta}_{\pm}(\phi)$ in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi) \wedge A)$. There exists $M \in \mathbb{N}^*$ such that for every $n \geqslant M$, we have*

$$\phi^{\pm n}(\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi) \wedge A) - \widehat{V}_{\mp}) \subseteq U_{\pm}.$$

In [CU, Theorem 4.15], Clay and Uyanik proved an analogue of Theorem 1.3 in the context of atoroidal outer automorphisms of $F_{\mathtt{n}}$. In Theorem 1.3, the two closed subsets $\widehat{\Delta}_{\pm}(\phi)$ have nonempty intersection, so that Theorem 1.3 is not a North-South dynamics result as defined above. However, Theorem 1.3 gives a sufficiently precise description of the dynamics of ϕ for our considerations. The intersection $\widehat{\Delta}_{+}(\phi) \cap \widehat{\Delta}_{-}(\phi)$ corresponds informally to the polynomial growth part of ϕ. This intersection, denoted by K_{PG} in the rest of the article, is the closure in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi) \wedge A)$ of the $(\mathcal{A}(\phi) \wedge A)$-nonperipheral elements of $F_{\mathtt{n}}$ which have polynomial growth under iteration of ϕ. In Section 3.3, we present a complete study of the subspace K_{PG} in a more general context.

In fact, Section 3 is devoted to the study of the polynomial growth of an exponentially growing outer automorphism. Following the works of Bestvina, Feighn and Handel [BFH1, BFH2], of Feighn and Handel [FH] and of Handel and Mosher [HM], we use appropriate relative train track representatives of a power of an exponentially growing outer automorphism ϕ in order to describe $\mathcal{A}(\phi)$ geometrically. It gives rise to a (not necessarily connected) topological graph G^* such that the fundamental group of every connected component G_c^* of G^* injects into $F_{\mathtt{n}}$ and such that the set $\{[\pi_1(G_c^*)]\}_{G_c^* \in \pi_0(G^*)}$ where $\pi_1(G_c^*)$ is viewed as a subgroup of $F_{\mathtt{n}}$ is equal to $\mathcal{A}(\phi)$ (see Proposition 3.14). We then use this characterization of $\mathcal{A}(\phi)$ in Section 3.3 in order to describe the subset K_{PG}.

We now sketch a proof of Theorem 1.2. The proofs of Theorem 1.2 and Theorem 1.3 given in this paper are long and quite technical, this is why we postpone the proof of Theorem 1.1 in [Gue2]. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}})$ be exponentially growing. The first step is to construct the closed subsets $\Delta_{\pm}(\phi)$ associated with ϕ as defined in Theorem 1.2. This is done in Section 4. In order to construct them, we use as inspiration the construction given by Lustig and Uyanik in [LU2] (see also [Uya2, Gup1]). We choose an appropriate relative train track representative $f\colon G \to G$ of a power of ϕ, where G is a graph whose fundamental group is isomorphic to $F_{\mathtt{n}}$. A current of $\Delta_{+}(\phi)$ is then constructed by considering occurrences of paths in $\lim_{m\to\infty} f^m(e)$, where e is an edge in G whose length grows exponentially fast under iteration of f (see Proposition 4.4). Currents of $\Delta_{-}(\phi)$ are then defined similarly using a representative of a power of ϕ^{-1}. We then prove Theorem 1.2 in Section 5. Let $[\mu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi)) - \Delta_{\pm}(\phi)$ be the current associated with a $\mathcal{A}(\phi)$-nonperipheral conjugacy class $[w] \in F_{\mathtt{n}}$. Then $[w]$ is represented by a circuit γ_w in the graph G. In order to show that we have $\lim_{m\to\infty} \phi^m([\mu]) \in \Delta_{+}(\phi)$, we prove that the proportion of the path $f^m(\gamma_w)$ which grows exponentially fast under iteration of f tends to 1 as m goes to infinity. This fact is sufficient to prove that

$$\lim_{m\to\infty} \phi^m([\mu]) \in \Delta_{+}(\phi)$$

(see Lemma 5.20). We then conclude the proof using the density of currents associated with nonperipheral elements in $F_{\mathtt{n}}$ proved in [Gue1]. Theorem 1.3 is then proved in Section 6 using a combination of Theorem 1.2 and the description of the space K_{PG}.

2. Preliminaries

2.1. **Malnormal subgroup systems of $F_{\mathtt{n}}$.** Let $\mathtt{n}$ be an integer greater than 1 and let $F_{\mathtt{n}}$ be a free group of rank $\mathtt{n}$. A *subgroup system of* $F_{\mathtt{n}}$ is a finite (possibly empty) set $\mathcal{A}$ whose elements are conjugacy classes of nontrivial (that is distinct from $\{1\}$) finite rank subgroups of $F_{\mathtt{n}}$. There exists a partial order on the set of subgroup systems of $F_{\mathtt{n}}$, where $\mathcal{A}_1 \leqslant \mathcal{A}_2$ if for every subgroup A_1 of $F_{\mathtt{n}}$ such that $[A_1] \in \mathcal{A}_1$, there exists a subgroup A_2 of $F_{\mathtt{n}}$ such that $[A_2] \in \mathcal{A}_2$ and A_1 is a subgroup of A_2. The *stabilizer in* $\mathrm{Out}(F_{\mathtt{n}})$ *of a subgroup system* $\mathcal{A}$, denoted by $\mathrm{Out}(F_{\mathtt{n}}, \mathcal{A})$, is the subgroup of $\mathrm{Out}(F_{\mathtt{n}})$ consisting of all elements $\phi \in \mathrm{Out}(F_{\mathtt{n}})$ such that $\phi(\mathcal{A}) = \mathcal{A}$.

Recall that a subgroup A of $F_{\mathtt{n}}$ is *malnormal* if for every element $x \in F_{\mathtt{n}} - A$, we have $xAx^{-1} \cap A = \{e\}$. A subgroup system $\mathcal{A}$ is said to be *malnormal* if every subgroup A of $F_{\mathtt{n}}$ such that $[A] \in \mathcal{A}$ is malnormal and, for all subgroups A_1, A_2 of $F_{\mathtt{n}}$ such that $[A_1], [A_2] \in \mathcal{A}$, if $A_1 \cap A_2$ is nontrivial then $A_1 = A_2$. An element $g \in F_{\mathtt{n}}$ is $\mathcal{A}$*-peripheral* (or simply *peripheral* if there is no ambiguity) if it is trivial or conjugate into one of the subgroups of $\mathcal{A}$, and $\mathcal{A}$*-nonperipheral* otherwise.

An important class of examples of malnormal subgroup systems is given by the *free factor systems*. A *free factor system of* $F_{\mathtt{n}}$ is a (possibly empty) set $\mathcal{F}$ of conjugacy classes $\{[A_1], \ldots, [A_r]\}$ of nontrivial subgroups $A_1, \ldots, A_r$ of $F_{\mathtt{n}}$ such that there exists an integer $k \in \mathbb{N}$ with $F_{\mathtt{n}} = A_1 * \ldots * A_r * F_k$. The free factor system $\mathcal{F}$ is *sporadic* if $(k+r, k) \leqslant (2,1)$ for the lexicographic order, and is *nonsporadic* otherwise. Therefore, the sporadic free factor systems are those of the form $\{[C]\}$ where C has rank at least equal to $n-1$ and those of the form $\{[A], [B]\}$ with $F_{\mathtt{n}} = A * B$. An ascending sequence of free factor systems $\mathcal{F}_1 \leqslant \ldots \leqslant \mathcal{F}_i = \{[F_{\mathtt{n}}]\}$ of $F_{\mathtt{n}}$ is called a *filtration of* $F_{\mathtt{n}}$.

Given a free factor system $\mathcal{F}$ of $F_{\mathtt{n}}$, a *free factor of* $(F_{\mathtt{n}}, \mathcal{F})$ is a subgroup A of $F_{\mathtt{n}}$ such that there exists a free factor system $\mathcal{F}'$ of $F_{\mathtt{n}}$ with $[A] \in \mathcal{F}'$ and $\mathcal{F} \leqslant \mathcal{F}'$. When $\mathcal{F} = \varnothing$, we say that A is a *free factor of* $F_{\mathtt{n}}$. A free factor of $(F_{\mathtt{n}}, \mathcal{F})$ is *proper* if it is nontrivial, not equal to $\{[F_{\mathtt{n}}]\}$ and if its conjugacy class does not belong to $\mathcal{F}$.

Another class of examples of malnormal subgroup systems is the following one. An outer automorphism $\phi \in \mathrm{Out}(F_{\mathtt{n}})$ is *exponentially growing* if there exists $g \in F_{\mathtt{n}}$ such that the length of the conjugacy class $[g]$ of g in $F_{\mathtt{n}}$ with respect to some basis of $F_{\mathtt{n}}$ grows exponentially fast under iteration of ϕ. If $\phi \in \mathrm{Out}(F_{\mathtt{n}})$ is not exponentially growing, then ϕ is *polynomially growing.* For an automorphism $\alpha \in \mathrm{Aut}(F_{\mathtt{n}})$, we say that α is exponentially growing if there exists $g \in F_{\mathtt{n}}$ such that the length of g grows exponentially fast under iteration of α. Otherwise, α is polynomially growing.

Let $\phi \in \mathrm{Out}(F_{\mathtt{n}})$ be exponentially growing. A subgroup P of $F_{\mathtt{n}}$ is a *polynomial subgroup* of ϕ if there exist $k \in \mathbb{N}^*$ and a representative α of ϕ^k such that $\alpha(P) = P$ and $\alpha|_P$ is polynomially growing.

By [Lev, Proposition 1.4], there exist finitely many conjugacy classes $[H_1]$, …, $[H_k]$ of maximal polynomial subgroups of ϕ. Moreover, the proof of [Lev, Proposition 1.4] implies that the set $\mathcal{H} = \{[H_1], \dots, [H_k]\}$ is a malnormal subgroup system. Indeed, Levitt shows that there exists a nontrivial $\mathbb{R}$-tree T in the boundary of Culler and Vogtmann Outer space [CV] on which $F_{\mathtt{n}}$ acts with trivial arc stabilizers, such that ϕ preserves the homothety class of T and such that the groups $H_1 \dots, H_k$ are elliptic in T. If two distinct subgroups A, B of $F_{\mathtt{n}}$ such that $[A], [B] \in \mathcal{H}$ fix distinct points in T, then their intersection is trivial. If A and B fix the same point x in T, then, up to taking a power of ϕ, the element ϕ preserves $[\mathrm{Stab}(x)]$ and an inductive argument on the rank using $\phi|_{\mathrm{Stab}(x)}$ (the rank of $\mathrm{Stab}(x)$ is less than $\mathtt{n}$ by a result of Gaboriau-Levitt [GL]) shows that the intersection of A and B is trivial. We denote this malnormal subgroup system by $\mathcal{A}(\phi)$.

Note that if H is a subgroup of $F_{\mathtt{n}}$ such that $[H] \in \mathcal{A}(\phi)$, there exists a representative Φ^{-1} of ϕ^{-1} such that $\Phi^{-1}(H) = H$ and $\Phi^{-1}|_H$ is polynomially growing. Hence we have $\mathcal{A}(\phi) \leqslant \mathcal{A}(\phi^{-1})$. By symmetry, we have

$$\mathcal{A}(\phi) = \mathcal{A}(\phi^{-1}). \tag{1}$$

Let $\mathcal{A}$ be a malnormal subgroup system and let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{A})$ be a relative outer automorphism. We say that ϕ is *atoroidal relative to* $\mathcal{A}$ if, for every $k \in \mathbb{N}^*$, the element ϕ^k does not preserve the conjugacy class of any $\mathcal{A}$-nonperipheral element. We say that ϕ is *expanding relative to* $\mathcal{A}$ if $\mathcal{A}(\phi) \leqslant \mathcal{A}$. Note that an expanding outer automorphism relative to $\mathcal{A}$ is in particular atoroidal relative to $\mathcal{A}$. When $\mathcal{A} = \varnothing$, then the outer automorphism ϕ is expanding relative to $\mathcal{A}$ if and only if for every nontrivial element $g \in F_{\mathtt{n}}$, the length of the conjugacy class $[g]$ of g in $F_{\mathtt{n}}$ with respect to some basis of $F_{\mathtt{n}}$ grows exponentially fast under iteration of ϕ. Therefore, by a result of Levitt [Lev, Corollary 1.6], the outer automorphism ϕ is expanding relative to $\mathcal{A} = \varnothing$ if and only if ϕ is atoroidal relative to $\mathcal{A} = \varnothing$.

Let $\mathcal{A} = \{[A_1], \dots, [A_r]\}$ be a malnormal subgroup system and let $\mathcal{F}$ be a free factor system. Let $i \in \{1, \dots, r\}$. By [SW, Theorem 3.14] for the action of A_i on one of its Cayley graphs, there exist finitely many subgroups $A_i^{(1)}, \dots, A_i^{(k_i)}$ of A_i such that:

(1) for every $j \in \{1, \dots, k_i\}$, there exists a subgroup B of $F_{\mathtt{n}}$ such that $[B] \in \mathcal{F}$ and $A_i^{(j)} = B \cap A_i$;
(2) for every subgroup B of $F_{\mathtt{n}}$ such that $[B] \in \mathcal{F}$ and $B \cap A_i \neq \{e\}$, there exists $j \in \{1, \dots, k_i\}$ such that $A_i^{(j)} = B \cap A_i$;
(3) the subgroup $A_i^{(1)} * \dots * A_i^{(k_i)}$ is a free factor of A_i.

Thus, one can define a new subgroup system as

$$\mathcal{F} \wedge \mathcal{A} = \bigcup_{i=1}^{r} \{[A_i^{(1)}], \dots, [A_i^{(k_i)}]\}.$$

Since $\mathcal{A}$ is malnormal, and since, for every $i \in \{1, \dots, r\}$, the group $A_i^{(1)} * \dots * A_i^{(k_i)}$ is a free factor of A_i, it follows that the subgroup system $\mathcal{F} \wedge \mathcal{A}$ is a malnormal subgroup system of $F_{\mathtt{n}}$. We call it the *meet of* $\mathcal{F}$ *and* $\mathcal{A}$.

2.2. **Graphs, markings and filtrations.** Let $\mathtt{n} \geqslant 2$. A *marked graph* is a pointed (at a vertex $*$), connected, finite graph G (in the sense of [Ser]) whose fundamental group is isomorphic to $F_{\mathtt{n}}$ which is equipped with a *marking*, that is an isomorphism $\rho\colon F_{\mathtt{n}} \to \pi_1(G, *)$.

We denote by VG (resp. $\vec{E}G$) the set of vertices (resp. edges) of G. Given an edge e of G, we denote by $o(e)$ the *origin* of e, by $t(e)$ the *terminal point* of e and by e^{-1} the edge of G such that $o(e^{-1}) = t(e)$ and $t(e^{-1}) = o(e)$. An *edge path γ of length m* is a concatenation of m edges $\gamma = e_1e_2\dots e_m$ such that for every $i \in \{1,\dots,m-1\}$, we have $t(e_i) = o(e_{i+1})$. The length of γ is denoted by $\ell(\gamma)$. The edge path γ is *reduced* if for every $i \in \{1,\dots,m-1\}$, we have $e_i \neq e_{i+1}^{-1}$. A reduced edge path is *cyclically reduced* if $t(e_m) = o(e_1)$ and $e_m \neq e_1^{-1}$. A cyclically reduced edge path is also called a *circuit.* For any edge path γ, there exists a unique reduced edge path homotopic to γ relatively to endpoints, we denote it by $[\gamma]$.

Let G and G' be two marked graphs. A *graph map* is a pointed homotopy equivalence $f\colon G \to G'$ such that $f(VG) \subseteq VG'$ and such that the restriction of f to the interior of an edge is an immersion. Thus, for every edge $e \in \vec{E}G$, the image $f(e)$ determines a reduced edge path $[f(e)]$. Given $\phi \in \mathrm{Out}(F_{\mathtt{n}})$ and (G,ρ) a marked graph, a *topological representative* of ϕ is a graph map $f\colon G \to G$ such that the outer automorphism class of $\rho^{-1} \circ f_* \circ \rho \in \mathrm{Aut}(F_{\mathtt{n}})$ is ϕ.

Let $f\colon G \to G$ be a topological representative. Let $w \in F_{\mathtt{n}}$. We denote by γ_w the unique circuit in G which represents the conjugacy class of w.

A *filtration* for G is an increasing sequence of f-invariant (not necessarily connected) subgraphs $\varnothing = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_k = G$. Let $r \in \{1,\dots,k\}$. The *r-th stratum* in this filtration, denoted by H_r, is the (not necessarily connected) closure of $G_r - G_{r-1}$. For every $r \in \{1,\dots,k\}$, there exists a square matrix M_r associated with the stratum H_r called the *transition matrix* of H_r. The rows and columns of M_r are indexed by the undirected edges in H_r and the entry associated with the pair of undirected edges defined by $(e,e') \in (EH_r)^2$ is the number of occurrences of e' and e'^{-1} in $[f(e)]$.

Recall that a nonnegative square matrix $M = (M_{i,j})_{i,j}$ is *irreducible* if for every (i,j), there exists $p = p(i,j)$ such that $M_{i,j}^p > 0$ and that M is *primitive* if there exists $p \in \mathbb{N}^*$ such that every entry of M^p is positive. For $r \in \{1,\dots,k\}$, we say that the stratum H_r is *irreducible* if its associated matrix is irreducible and we say that H_r is *primitive* if its associated matrix is primitive. Let $r \in \{1,\dots,k\}$ and suppose that M_r is irreducible. Then it has a unique real eigenvalue $\lambda_r \geqslant 1$ called the *Perron-Frobenius* eigenvalue. Let H_r be an irreducible stratum. Then H_r is *exponentially growing (EG)* if $\lambda_r > 1$ and is *nonexponentially growing (NEG)* otherwise. Finally, if the matrix associated with the stratum H_r is the zero matrix, then H_r is called a *zero stratum.*

Let G be a marked graph of $F_{\mathtt{n}}$ and let K be a (possibly disconnected) subgraph of G. The subgraph K determines a free factor system $\mathcal{F}(K)$ of $F_{\mathtt{n}}$ as follows. Let $C_1,\dots,C_k$ be the noncontractible connected components of K. Then, for every $i \in \{1,\dots,k\}$, the connected component C_i determines the conjugacy class $[A_i]$ of a subgroup A_i of $\pi_1(G)$. Then the set $\{[A_1],\dots,[A_k]\}$ is a free factor system $\mathcal{F}(K)$ of $F_{\mathtt{n}}$.

Let $\mathcal{F}_1 \leqslant \dots \leqslant \mathcal{F}_i = \{[F_{\mathtt{n}}]\}$ be a filtration of $F_{\mathtt{n}}$. A *geometric realization of the filtration* is a marked graph G equipped with an increasing sequence

$$\varnothing = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_j = G$$

of subgraphs of G such that for every $k \in \{1,\dots,i\}$ there exists $\ell \in \{1,\dots,j\}$ such that $\mathcal{F}_k = \mathcal{F}(G_\ell)$.

2.3. **Train tracks and CTs.** In this section we introduce the technology of *train tracks*. Train tracks are a type of graph maps introduced by Bestvina and Handel [BH]. Even though there exist outer automorphisms of $F_{\mathtt{n}}$ which do not have a topological representative which is a train track, every outer automorphism has a power which has a topological representative called a *completely split train track map* (CT). CT maps were introduced by Feighn and Handel [FH]. The definition of a CT map being quite technical, we will only state the relevant properties needed for the rest of the article. First we need some preliminary definitions.

Let G be a marked graph of $F_{\mathtt{n}}$ and let $f\colon G \to G$ be a graph map. The map f induces a *derivative map* $Df\colon \vec{E}G \to \vec{E}G$ on the set of edges as follows. For every $e \in \vec{E}G$, the map $Df(e)$ is equal to the first edge of the edge path $f(e)$. A *turn* in G is an unordered pair $\{e_1, e_2\}$ of edges in G with $o(e_1) = o(e_2)$. A turn $\{e_1, e_2\}$ is *degenerate* if $e_1 = e_2$, and is *nondegenerate* otherwise. A turn $\{e_1, e_2\}$ is *illegal* if there exists $k \in \mathbb{N}^*$ such that $\{(Df)^k(e_1), (Df)^k(e_2)\}$ is degenerate, and is *legal* otherwise. An edge path $\gamma = e_1e_2\dots e_i$ is *legal* if for every $j \in \{1, \dots, i-1\}$, the turn $\{e_j^{-1}, e_{j+1}\}$ is legal.

In order to deal with relative outer automorphisms, we also need a notion of relative legal paths. Let $\varnothing = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_j = G$ be the geometric realization of some filtration of $F_{\mathtt{n}}$ which is f-invariant and let $r \in \{1, \dots, j\}$. We say that a turn $\{e_1, e_2\}$ is contained in the stratum H_r if $\{e_1, e_2\} \subseteq \vec{E}H_r$. An edge path γ of G is *r-legal* if every turn in γ that is contained in H_r is legal. A *connecting path* for H_r is a nontrivial reduced path γ in G_{r-1} whose endpoints are in $G_{r-1} \cap H_r$. A path γ in G is *r-taken* (or *taken* if γ is r-taken for some r) if it is contained in the reduced image of an iterate of an edge $e \in \vec{E}H_r$, where H_r is an irreducible stratum. The *height of a path* γ is the maximal r such that γ contains an edge of H_r. We can now define the notion of a *relative train track map* due to Bestvina and Handel [BH].

Definition 2.1. Let $\mathtt{n} \geqslant 3$. Let G be a marked graph and let $f\colon G \to G$ be a graph map equipped with an f-invariant filtration $\varnothing = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_j = G$. The map f is a *relative train track map* if, for each exponentially growing stratum H_r, the following holds:

(1) for every edge $e \in \vec{E}H_r$ and every $k \in \mathbb{N}^*$, we have $(Df)^k(e) \in \vec{E}H_r$;
(2) for every connecting path γ for H_r, the reduced path $[f(\gamma)]$ is also a connecting path for H_r;
(3) if γ is a height r reduced edge path which is r-legal, then so is $[f(\gamma)]$.

In order to explain the properties of CT maps that we will use in this paper, we will need some further definitions regarding edge paths in a graph.

Definition 2.2. Let $\mathtt{n} \geqslant 3$ and let G be a marked graph of $F_{\mathtt{n}}$ equipped with an f-invariant filtration $\varnothing = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_j = G$. Let γ be an edge path of G.

(1) The path γ is a *periodic Nielsen path* if there exists $k \in \mathbb{N}^*$ such that $[f^k(\gamma)] = \gamma$. The minimal such k is the *period*, and if $k = 1$, then γ is a *Nielsen path*.
(2) A *(periodic) indivisible Nielsen path ((p)INP)* is a (periodic) Nielsen path that cannot be written as a nontrivial concatenation of (periodic) Nielsen paths.
(3) The path γ is an *exceptional path* if there exist a cyclically reduced Nielsen path w, edges $e_1, e_2 \in \vec{E}G$ and integers $d_1, d_2, p \in \mathbb{Z}^*$ such that for every

$i \in \{1, 2\}$, we have $f(e_i) = e_i w^{d_i}$ and $\gamma = e_1 w^p e_2^{-1}$. The value $|p|$ is called the *width of* γ.

Definition 2.3. Let $n \geqslant 3$, let G be a marked graph of $F_{\mathtt{n}}$ and let $f : G \to G$ be a relative train track map equipped with a filtration $\varnothing = G_0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_j = G$. Let γ be a reduced edge path or a circuit of G.

(1) A *splitting* of γ is a decomposition of γ into edge subpaths $\gamma = \gamma_1\gamma_2 \ldots \gamma_i$ such that for every $k \in \mathbb{N}^*$, we have
$$[f^k(\gamma)] = [f^k(\gamma_1)] \ldots [f^k(\gamma_i)],$$
that is one can tighten the image of $f^k(\gamma)$ by tightening the image of every $f^k(\gamma_j)$ (where $o(\gamma)$ is the base point in the case where γ is a circuit).

(2) Let γ be a circuit. A *circuital splitting* is a splitting $\gamma = \gamma_1 \ldots \gamma_i$ of γ such that for every $k \in \mathbb{N}^*$, the concatenation $[f^k(\gamma_1)] \ldots [f^k(\gamma_i)]$ defines a path whose initial and terminal directions are distinct.

(3) Let $\gamma = \gamma_1\gamma_2 \ldots \gamma_i$ be a splitting of γ. The splitting is *complete* if for every $j \in \{1, \ldots, i\}$, the subpath γ_j is one of the following:
- an edge in an irreducible stratum;
- an INP;
- an exceptional path;
- a connecting path in a zero stratum that is both maximal (for the inclusion in γ) and taken.

Let $\mathtt{n} \geqslant 2$, let G be a marked graph of $F_{\mathtt{n}}$ and let $f : G \to G$ be a relative train track map with respect to a filtration $\varnothing = G_0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_j = G$. Let γ be an edge path of G. Such paths in the above list are called *splitting units*. When γ has a complete splitting, we say that γ is *completely split*.

Definition 2.4 ([HM, Fact 2.16])**.** Let $p \in \{0, \ldots, j\}$. Let $\gamma = \gamma_1\gamma_2 \ldots \gamma_i$ be a splitting of γ. This splitting is *complete relatively to* G_p, or *relatively complete* if there is no ambiguity, if for every $j \in \{1, \ldots, i\}$, the subpath γ_j is one of the following:
- a splitting unit of height at least equal to $p + 1$;
- a subpath in G_p.

We now describe some properties of CT maps whose complete definition can be found in [FH, Definition 4.7].

Proposition 2.5. *Let* $\mathtt{n} \geqslant 3$ *and let* G *be a marked graph of* $F_{\mathtt{n}}$. *Let* $f : G \to G$ *be a completely split train track (CT) map. Then* f *satisfies the following properties.*

(1) *The map* f *is a relative train track map and every stratum in* G *is either irreducible or a zero stratum* [FH, Definition 4.7].

(2) *If* H_r *is an NEG stratum, then* H_r *consists of a single edge* e_r. *Moreover, either* e_r *is fixed by* f *or* $f(e_r) = e_r u_r$ *where* u_r *is a nontrivial completely split circuit in* G_{r-1}. *The terminal endpoint of each NEG stratum is fixed* [FH, Lemma 4.21].

(3) *For every filtration element* G_r, *the stratum* H_r *is a zero stratum if and only if* H_r *is a contractible component of* G_r [FH, Lemma 4.15].

(4) *For every zero stratum* H_r, *there exists a unique* $\ell > r$ *such that* H_ℓ *is an EG stratum and, for every vertex* $v \in VH_r$, *we have* $v \in VH_r \cap VH_\ell$ *and the link of* v *is contained in* $VH_r \cup VH_\ell$ [FH, Definition 4.7].

(5) *Every periodic Nielsen path has period one* [FH, Lemma 4.13].

(6) *For every edge e in an irreducible stratum, the reduced path $f(e)$ is completely split. For every taken connecting path γ in a zero stratum, $[f(\gamma)]$ is completely split* [FH, Definition 4.7].

(7) *Every completely split path or circuit has a unique complete splitting (see* [FH, Lemma 4.11]*).*

(8) *If γ is an edge path, there exists $k_0 \in \mathbb{N}^*$ such that for every $k \geqslant k_0$, the reduced path $[f^k(\gamma)]$ is completely split* [FH, Lemma 4.25].

(9) *If H_r is an EG stratum, there is at most one INP ρ_r of height r. The initial edges of ρ_r and ρ_r^{-1} are distinct oriented edges in H_r* [FH, Corollary 4.19].

(10) *If H_r is a zero stratum, no Nielsen path intersects H_r in at least one edge* [HM, Fact I.1.43].

(11) *Let H_r be an NEG stratum such that $H_r = \{e_r\}$, such that $f(e_r) = e_r u_r$ and such that u_r is not trivial. There exists an INP σ which intersects H_r nontrivially if and only if u_r is a Nielsen path and there exists $s \in \mathbb{Z}$ such that $\sigma = e_r u_r^s e_r^{-1}$* [FH, Definition 4.7].

Definition 2.6. Let $\mathtt{n} \geqslant 2$ and let G be a marked graph of $F_{\mathtt{n}}$. Let $f\colon G \to G$ be a CT map. Let H_r be an NEG stratum and let e_r be the edge of H_r. Let u_r be such that $f(e_r) = e_r u_r$. The edge e_r is called a *fixed edge* if u_r is trivial, a *linear edge* if u_r is a Nielsen path and a *superlinear edge* otherwise.

Lemma 2.7 ([HM, Fact 1.39])**.** *Let $\mathtt{n} \geqslant 2$ and let G be a marked graph of $F_{\mathtt{n}}$. Let $f\colon G \to G$ be a CT map. Let γ be a Nielsen path. Then γ is completely split, and all terms in the complete splitting of γ are fixed edges and INPs.*

Lemma 2.8 ([HM, Fact 1.41])**.** *Let $\mathtt{n} \geqslant 2$ and let G be a marked graph of $F_{\mathtt{n}}$. Let $f\colon G \to G$ be a CT map.*

(1) *Let H_r be a zero stratum and let H_ℓ be the EG stratum given by Proposition* 2.5(4)*. There does not exist an INP of height ℓ.*

(2) *Let H_r be an EG stratum and let ρ_r be an INP of height r. Then ρ_r has a decomposition $\rho_r = a_0 b_1 a_1 \dots b_k a_k$ where, for every $i \in \{0, \dots, k\}$, the subpath a_i is a nontrivial path contained in H_r and for every $i \in \{1, \dots, k\}$, the subpath b_i is a Nielsen path contained in G_{r-1}.*

An INP is an *EG INP* if the maximal stratum it intersects is an EG stratum and is an *NEG INP* otherwise. Note that, by Proposition 2.5(9), there exist only finitely many EG INPs.

Lemma 2.9. *Let $\mathtt{n} \geqslant 2$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}})$. Suppose that there exists a CT map $f\colon G \to G$ representing a power of ϕ. Let γ' be a nontrivial path in a zero stratum. There does not exist a reduced edge path $\gamma = \alpha\gamma'$ where α is either an INP or a fixed edge.*

Proof. Suppose towards a contradiction that such a path $\gamma = \alpha\gamma'$ exists. Let H_r be the zero stratum containing γ'. Note that, by Proposition 2.5(10), the path α does not contain edges in H_r. By Proposition 2.5(4), there exists $\ell > r$ such that H_ℓ is an EG stratum and such that any edge adjacent to a vertex in H_r and not contained in H_r is in H_ℓ. Hence α has height at least ℓ. Since H_ℓ is an EG stratum, the path α is not a fixed edge. Hence α is an INP. By Lemma 2.8(1), the height of α is not equal to ℓ. Let $j > \ell$ be the height of α. We distinguish between three cases according to the nature of the stratum H_j. By Proposition 2.5(10), the stratum H_j is not a zero stratum. Hence, by Proposition 2.5(1), the stratum H_j is irreducible. By Proposition 2.5(11), if H_j is an NEG stratum, then α is of the form $\alpha = e_j w^k e_j^{-1}$, where $e_j \in H_j$, k is an integer and w is a closed Nielsen path in G_{j-1}. But then e_j^{-1} is adjacent to a vertex in H_r. This contradicts Proposition 2.5(4) since $j > \ell$. If H_j is an EG stratum, then by Lemma 2.8(2), the path α is the concatenation of subpaths in H_j and Nielsen paths of height at most $j-1$, and α ends with an edge in H_j. By Proposition 2.5(4), we see that $j = \ell$. This contradicts Lemma 2.8(1). □

Theorem 2.10 due to Feighn and Handel is the main existence theorem of the CT maps.

Theorem 2.10 ([FH, Theorem 4.28, Lemma 4.42])**.** *Let* $\mathtt{n} \geqslant 3$. *There exists a uniform constant* $M = M(n) \geqslant 1$ *such that for every* $\phi \in \mathrm{Out}(F_{\mathtt{n}})$ *and every* ϕ^M*-invariant filtration* $\mathcal{C}$ *of* $F_{\mathtt{n}}$*, there exists a CT map* $f \colon G \to G$ *that represents* ϕ^M *and realizes* $\mathcal{C}$*.*

2.4. **Relative currents.** In this section, we define the notion of *currents of* $F_{\mathtt{n}}$ *relative to a malnormal subgroup system.* The section follows [Gue1] (see the work of Gupta [Gup1] for the particular case of free factor systems and Guirardel and Horbez [GH] in the context of free products of groups). It is closely related to the notion of conjugacy classes of $\mathcal{A}$-nonperipheral elements of $F_{\mathtt{n}}$.

Let $\partial_\infty F_{\mathtt{n}}$ be the Gromov boundary of $F_{\mathtt{n}}$. The *double boundary of* $F_{\mathtt{n}}$ is the quotient topological space

$$\partial^2 F_{\mathtt{n}} = (\partial_\infty F_{\mathtt{n}} \times \partial_\infty F_{\mathtt{n}} \backslash \Delta) \,/\, \sim,$$

where $\sim$ is the equivalence relation generated by the flip relation $(x,y) \sim (y,x)$ and Δ is the diagonal, endowed with the diagonal action of $F_{\mathtt{n}}$. We denote by $\{x,y\}$ the equivalence class of (x,y).

Let T be the Cayley graph of $F_{\mathtt{n}}$ with respect to a free basis $\mathfrak{B}$. The boundary of T is naturally homeomorphic to $\partial_\infty F_{\mathtt{n}}$ and the set $\partial^2 F_{\mathtt{n}}$ is then identified with the set of unoriented bi-infinite geodesics in T. Let γ be a finite geodesic path in T. The path γ determines a subset in $\partial^2 F_{\mathtt{n}}$ called the *cylinder set of* γ, denoted by $C(\gamma)$, which consists of all unoriented bi-infinite geodesics in T that contain γ. Such cylinder sets form a basis for a topology on $\partial^2 F_{\mathtt{n}}$, and in this topology, the cylinder sets are both open and closed, hence compact. The action of $F_{\mathtt{n}}$ on $\partial^2 F_{\mathtt{n}}$ has a dense orbit.

For every nontrivial subgroup A of $F_{\mathtt{n}}$, let T_A be the minimal A-invariant subtree of T. Let $\mathcal{A} = \{[A_1], \dots, [A_r]\}$ be a malnormal subgroup system of $F_{\mathtt{n}}$. By malnormality of $\mathcal{A}$, there exists $L \in \mathbb{N}^*$ such that for all distinct subgroups A, B of $F_{\mathtt{n}}$ such that $[A], [B] \in \mathcal{A}$, the diameter of the intersection $T_A \cap T_B$ is at most L (see for instance [HM, Section I.1.1.2]). Let $i \in \{1, \dots, r\}$. Let Γ_i be the set of subgroups B of $F_{\mathtt{n}}$ such that there exists $g_B \in F_{\mathtt{n}}$ such that $B = g_B A_i g_B^{-1}$ and the tree T_B contains the base point e of T. Note that, by malnormality of $\mathcal{A}$, for every $i \in \{1, \dots, r\}$, the set Γ_i is finite. For an element $w \in F_{\mathtt{n}}$, let $\widehat{\gamma_w}$ be the geodesic path in T starting at e and labeled by w. Let C_i be the set of elements w of $F_{\mathtt{n}}$ such that the length of $\widehat{\gamma_w}$ is equal to $L+2$ and, for every $B \in \Gamma_i$, the path $\widehat{\gamma_w}$ is not contained in T_B. Let $\mathscr{C} = \bigcap_{i=1}^r C_i$. Since we are looking at geodesic paths of length equal to $L+2$, the set $\mathscr{C}$ is finite. Moreover, it only depends on the choice of $\mathcal{A}$, $\mathfrak{B}$ and L.

Lemma 2.11 ([Gue1, Lemma 2.3]). *Let $\mathfrak{B}$, T, $\mathcal{A} = \{[A_1], \dots, [A_r]\}$, $L \in \mathbb{N}^*$, $\Gamma_1, \dots, \Gamma_r$, $\mathscr{C}$ be as above. The finite set $\mathscr{C} = \mathscr{C}(A_1, \dots, A_k)$ is nonempty. Moreover, it satisfies the following properties:*

(1) *every $\mathcal{A}$-nonperipheral cyclically reduced element $g \in F_{\mathtt{n}}$ has a power which contains an element of $\mathscr{C}$ as a subword;*

(2) *for every $\mathcal{A}$-nonperipheral cyclically reduced element $g \in F_{\mathtt{n}}$, if c_g is the geodesic ray in T starting from e obtained by concatenating infinitely many edge paths labeled by g, there exists an edge path in c_g labeled by a word in $\mathscr{C}$ at distance at most $L+2$ from $\bigcup_{i=1}^r \bigcup_{B \in \Gamma_i} T_B$;*

(3) *if γ is a path in T which contains a subpath labeled by an element of $\mathscr{C}$, then for every $i \in \{1, \dots, r\}$ and every $g \in F_{\mathtt{n}}$, the path γ is not contained in $T_{gA_ig^{-1}}$.*

Let A be a nontrivial subgroup of $F_{\mathtt{n}}$ of finite rank. The induced A-equivariant inclusion $\partial_\infty A \hookrightarrow \partial_\infty F_{\mathtt{n}}$ induces an inclusion $\partial^2 A \hookrightarrow \partial^2 F_{\mathtt{n}}$. Let

$$\partial^2 \mathcal{A} = \bigcup_{i=1}^r \bigcup_{g \in F_{\mathtt{n}}} \partial^2 \left(gA_ig^{-1}\right).$$

Let $\partial^2(F_{\mathtt{n}}, \mathcal{A}) = \partial^2 F_{\mathtt{n}} - \partial^2 \mathcal{A}$ be the *double boundary of $F_{\mathtt{n}}$ relative to $\mathcal{A}$*. This subset is invariant under the action of $F_{\mathtt{n}}$ on $\partial^2 F_{\mathtt{n}}$ and inherits the subspace topology of $\partial^2 F_{\mathtt{n}}$.

Lemma 2.12 ([Gue1, Lemma 2.5]). *Let $\mathrm{Cyl}(\mathscr{C})$ be the set of cylinder sets of the form $C(\gamma)$, where the element of $F_{\mathtt{n}}$ determined by the geodesic edge path γ contains an element of $\mathscr{C}$ as a subword. We have*

$$\partial^2(F_{\mathtt{n}}, \mathcal{A}) = \bigcup_{C(\gamma) \in \mathrm{Cyl}(\mathscr{C})} C(\gamma).$$

In particular, the space $\partial^2(F_{\mathtt{n}}, \mathcal{A})$ is an open subset of $\partial^2 F_{\mathtt{n}}$.

Lemma 2.13 ([Gue1, Lemma 2.6, Lemma 2.7]). *Let $\mathtt{n} \geqslant 3$ and let $\mathcal{A}$ be a malnormal subgroup system of $F_{\mathtt{n}}$. The space $\partial^2(F_{\mathtt{n}}, \mathcal{A})$ is locally compact and the action of $F_{\mathtt{n}}$ on $\partial^2(F_{\mathtt{n}}, \mathcal{A})$ has a dense orbit.*

We can now define a *relative current*. Let $\mathtt{n} \geqslant 3$ and let $\mathcal{A}$ be a malnormal subgroup system of $F_{\mathtt{n}}$. A *relative current of* $(F_{\mathtt{n}}, \mathcal{A})$ is a (possibly zero) $F_{\mathtt{n}}$-invariant Radon measure μ on $\partial^2(F_{\mathtt{n}}, \mathcal{A})$. The set $\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A})$ of all relative currents on $(F_{\mathtt{n}}, \mathcal{A})$ is equipped with the weak-$*$ topology: a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathrm{Curr}(F_N, \mathcal{A})^{\mathbb{N}}$ converges to a current $\mu \in \mathrm{Curr}(F_N, \mathcal{A})$ if and only if for every Borel subset $B \subseteq \partial^2(F_N, \mathcal{A})$ such that $\mu(\partial B) = 0$ (where ∂B is the topological boundary of B), the sequence $(\mu_n(B))_{n \in \mathbb{N}}$ converges to $\mu(B)$.

The group $\mathrm{Out}(F_{\mathtt{n}}, \mathcal{A})$ acts on $\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A})$ as follows. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{A})$, let Φ be a representative of ϕ, let $\mu \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A})$ and let C be a Borel subset of $\partial^2(F_{\mathtt{n}}, \mathcal{A})$. Then, since ϕ preserves $\mathcal{A}$, we see that $\Phi^{-1}(C) \in \partial^2(F_{\mathtt{n}}, \mathcal{A})$. Then we set

$$\phi(\mu)(C) = \mu(\Phi^{-1}(C)),$$

which is well-defined since μ is $F_{\mathtt{n}}$-invariant.

Every conjugacy class of nonperipheral element $g \in F_{\mathtt{n}}$ determines a relative current $\eta_{[g]}$ as follows. Suppose first that g is *root-free*, that is g is not a proper power of any element in $F_{\mathtt{n}}$. Let γ be a finite geodesic path in the Cayley graph

T. Then $\eta_{[g]}(C(\gamma))$ is the number of axes in T of conjugates of g that contain the path γ. If $g = h^k$ with $k \geqslant 2$ and h root-free, we set $\eta_{[g]} = k\eta_{[h]}$. Such currents are called *rational currents*.

Let G be a pointed connected graph whose fundamental group is isomorphic to $F_{\mathtt{n}}$. Let $\widetilde{G}$ be the universal cover of G. There exists a (nonunique, but fixed) $F_{\mathtt{n}}$-equivariant quasi-isometry $\widetilde{m}\colon \widetilde{G} \to T$ which extends uniquely to a homeomorphism $\widehat{m}\colon \partial_\infty G \to \partial_\infty F_{\mathtt{n}}$. Therefore, if $\widetilde{\gamma}$ is a reduced edge path in $\widetilde{G}$, we can define the cylinder set in $\partial^2 F_{\mathtt{n}}$ defined by $\widetilde{\gamma}$ as

$$C_{\widetilde{m}}(\widetilde{\gamma}) = C([\widetilde{m}(\widetilde{\gamma})]).$$

Let γ be a reduced edge path in G and let $\widetilde{\gamma}$ be a lift of γ in $\widetilde{G}$. Let $\mu \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A})$. We define the *number of occurrences of* γ *in* μ as

$$\langle \gamma, \mu \rangle_{\widetilde{m}} = \mu(C_{\widetilde{m}}(\widetilde{\gamma})). \tag{2}$$

For every such graph G, we fix once and for all the quasi-isometry $\widetilde{m}\colon \widetilde{G} \to T$. Therefore, when the graph G is fixed, we will generally omit the mention of $\widetilde{m}$. We also define the *simplicial length of* μ as:

$$\|\mu\| = \sum_{e \in \vec{E}G} \langle e, \mu \rangle .$$

For any given reduced edge path γ, the functions $\langle \gamma, . \rangle$ and $\|.\|$ are continuous, linear functions of $\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A})$.

Let $\mu \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A})$. The *support of* μ, denoted by $\mathrm{Supp}(\mu)$, is the support of the Borel measure μ on $\partial^2(F_{\mathtt{n}}, \mathcal{A})$. We recall that $\mathrm{Supp}(\mu)$ is a closed subset of $\partial^2(F_{\mathtt{n}}, \mathcal{A})$.

In the rest of the article, rather than considering the space of relative currents itself, we will consider the set of *projectivised relative currents*:

$$\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}) = (\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}) - \{0\})/\sim,$$

where $\mu \sim \nu$ if there exists $\lambda \in \mathbb{R}_+^*$ such that $\mu = \lambda\nu$. The projective class of a current $\mu \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A})$ will be denoted by $[\mu]$. We have the following properties.

Lemma 2.14 ([Gue1, Lemma 3.3]). *Let* $\mathtt{n} \geqslant 3$ *and let* $\mathcal{A}$ *be a malnormal subgroup system of* $F_{\mathtt{n}}$. *The space* $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A})$ *is compact.*

Proposition 2.15 ([Gue1, Theorem 1.1]). *Let* $\mathtt{n} \geqslant 3$ *and let* $\mathcal{A}$ *be a malnormal subgroup system of* $F_{\mathtt{n}}$. *The set of projectivised rational currents about nonperipheral elements of* $F_{\mathtt{n}}$ *is dense in* $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A})$.

3. The polynomially growing subgraph of a CT map

In this section, let $\mathtt{n} \geqslant 3$ and let $\mathcal{F}$ be a free factor system of $F_{\mathtt{n}}$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$. Let $f\colon G \to G$ be a CT map with filtration $\varnothing = G_0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_k = G$ representing a power of ϕ and such that there exists $p \in \{1, \ldots, k-1\}$ such that $\mathcal{F}(G_p) = \mathcal{F}$.

We construct a subgraph of G, called the *polynomially growing subgraph* of G and denoted by G_{PG}, which encaptures the information regarding polynomial growth in the graph G. We then define a notion of length relative to G_{PG}, called the *exponential length*, which measures the time spent by an edge path outside of G_{PG}. Finally, we construct a subspace of $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F})$ which consists in the currents whose support maps to G_{PG}.

3.1. **Definitions and first properties.** We define in this section the *polynomially growing subgraph* G_{PG} of G and prove some of its properties.

Definition 3.1.

(1) Let G_{PG} be the (not necessarily connected) subgraph of G whose edges are the edges e of G in an NEG stratum such that for every $k \in \mathbb{N}^*$, the path $[f^k(e)]$ does not contain a splitting unit which is an edge in an EG stratum.
(2) Let $\mathcal{N}'_{PG}$ be the set of all Nielsen paths in G.
(3) Let $\mathcal{N}_{PG}$ be the subset of $\mathcal{N}'_{PG}$ consisting in all Nielsen paths which are either EG INPs or concatenations of (at least 2) nonclosed EG INPs.
(4) Let $\mathcal{Z}$ be the subgraph of G whose edges are the edges contained in a zero stratum.

Note that, by Lemma 2.7, every path in $\mathcal{N}'_{PG}$ (and hence every path in $\mathcal{N}_{PG}$) has a complete splitting consisting in fixed edges and INPs. Since a complete splitting is unique by Proposition 2.5(7), if γ is a reduced path in $\mathcal{N}_{PG}$, then the splitting of γ given in Definition 3.1(3) is the complete splitting of γ. Moreover, γ is either an EG INP or the complete splitting of γ has at least two splitting units and all of them are nonclosed EG INPs. In particular, the set $\mathcal{N}_{PG}$ does not contain Nielsen paths such that one of their splitting units is either a fixed edge or an NEG INP. Moreover, a Nielsen path which is a concatenation of at least 2 splitting units and such that one of them is a closed EG INP is not in $\mathcal{N}_{PG}$. Excluding such paths from $\mathcal{N}_{PG}$ ensures a finiteness result for $\mathcal{N}_{PG}$ (see Lemma 3.5(1)). Informally, paths in $\mathcal{N}_{PG}$ play the role of low-dynamics bridges between connected components of G_{PG} (see Figure 1). We will see in Proposition 3.14 that a cycle in G has polynomial growth under iteration of f if and only if it is a concatenation of paths in G_{PG} and paths in $\mathcal{N}_{PG}$.

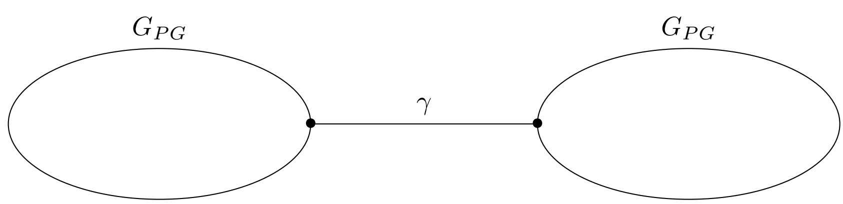

FIGURE 1. A path γ in $\mathcal{N}_{PG}$ between two connected components of G_{PG}

Note that, with p defined at the beginning of Section 3, one can similarly define the *polynomially growing subgraph of* G_p, denoted by $G_{PG,\mathcal{F}}$, which is the subgraph $G_{PG} \cap G_p$. We can also define similarly $\mathcal{N}'_{PG,\mathcal{F}}$, $\mathcal{N}_{PG,\mathcal{F}}$ and $\mathcal{Z}_{\mathcal{F}}$ by considering the paths of $\mathcal{N}'_{PG}$, $\mathcal{N}_{PG}$ and $\mathcal{Z}$ contained in G_p.

We now recall a lemma due to Bestvina and Handel regarding r-legal paths.

Lemma 3.2 ([BH, Lemma 5.8]). *Let $f\colon G \to G$ be a relative train track map. Let H_r be an EG stratum. Suppose that $\sigma = a_1b_1a_2\dots a_\ell b_\ell$ is the decomposition of an r-legal path into subpaths $a_j \subseteq H_r$ and $b_j \subseteq G_{r-1}$ (where a_1 and b_ℓ might be trivial). Then for every $i \in \{1,\dots,\ell\}$, the path $f(a_\ell)$ is a reduced edge path and*

$$[f(\sigma)] = f(a_1)[f(b_1)]f(a_2)\dots f(a_\ell)[f(b_\ell)].$$

Note that if H_r is an EG stratum and if $\sigma = a_1b_1a_2\dots a_\ell b_\ell$ is an r-legal path as in Lemma 3.2, then for every $i \in \{1,\dots,\ell\}$, as $a_i \subseteq H_r$, the path a_i grows exponentially fast under iteration of f. Hence, by Lemma 3.2 the path σ grows exponentially fast under iteration of f. We now prove some results regarding paths in $\mathcal{N}_{PG}$.

Lemma 3.3. *Let σ be an EG INP.*

(1) *There do not exist nontrivial subpaths c, d of σ such that $\sigma = cdc$.*
(2) *Let $\gamma \in \{\sigma^{\pm 1}\}$. There do not exist paths $\gamma_1, \gamma_2, \gamma_3$ such that γ_2 is nontrivial, γ_1 or γ_3 is nontrivial and $\sigma = \gamma_1\gamma_2$ and $\gamma = \gamma_2\gamma_3$.*

Proof. (1) Let r be the height of σ. Suppose towards a contradiction that such a decomposition $\sigma = cdc$ exists. By [BH, Lemma 5.11], there exist two distinct r-legal paths α and β such that $\sigma = \alpha\beta$ and such that the turn $\{Df(\alpha^{-1}), Df(\beta)\}$ is the only height r illegal turn. Moreover, there exists a path τ such that $[f(\alpha)] = \alpha\tau$ and $[f(\beta)] = \tau^{-1}\beta$. Hence c is contained in α and in β and is r-legal. Thus, there exist two paths d_1 and d_2 such that $\alpha = cd_1$ and $\beta = d_2c$.

First we claim that for every $k \in \mathbb{N}^*$, there exists a path τ_k such that $[f^k(\alpha)] = \alpha\tau_k$ and $[f^k(\beta)] = \tau_k^{-1}\beta$. The proof is by induction on k. The base case follows from the existence of τ. Suppose now that τ_{k-1} exists. We have:

$$[f^k(\alpha)] = [f(\alpha\tau_{k-1})] = [f(\alpha)][f(\tau_{k-1})] = \alpha\tau[f(\tau_{k-1})] = \alpha\tau_k,$$

where the second equality comes from the fact that α is r-legal, that α ends with an edge in H_r and from Lemma 3.2. Similarly, we have $[f^k(\beta)] = \tau_k^{-1}\beta$. This proves the claim.

We now claim that, up to taking a power of f, there exists a cycle e such that $[f(c)] = \alpha e\beta$. Indeed, by Proposition 2.5(9), the path σ starts and ends with an edge in H_r. Hence the path c starts and ends with an edge in H_r. Since c is r-legal, we see that the length of $[f^k(c)]$ goes to infinity as k goes to infinity by Lemma 3.2. But, for every $k \in \mathbb{N}^*$, there exists a path τ_k such that $[f^k(\alpha)] = \alpha\tau_k$ and $[f^k(\beta)] = \tau_k^{-1}\beta$. By Lemma 3.2, since c is the initial segment of α and since α is r-legal, there is no identification between $[f(c)]$ and $[f(d_1)]$. Thus, there exists $k_1 \in \mathbb{N}^*$ such that $[f^{k_1}(c)]$ starts with α. Similarly, there exists $k_2 \in \mathbb{N}^*$ such that $[f^{k_2}(c)]$ ends with β. Thus, up to taking a power of f, and since the paths α and β are r-legal, we may suppose that there exists a (reduced) cycle e such that $[f(c)] = \alpha e\beta$.

Finally, we claim that the cycle e is trivial. Indeed, since the paths α and β are r-legal, and since c starts and ends with an edge in H_r, we see that

$$[f(\alpha)] = [f(c)][f(d_1)] = \alpha e\beta[f(d_1)]$$

and

$$[f(\beta)] = [f(d_2)][f(c)] = [f(d_2)]\alpha e\beta.$$

Recall that there exists $k \in \mathbb{N}^*$ such that $[f(\alpha)] = \alpha\tau_k$ and $[f(\beta)] = \tau_k^{-1}\beta$. This implies that $\tau_k = e\beta[f(d_1)]$ and that $\tau_k^{-1} = [f(d_2)]\alpha e$, that is $\tau_k = e^{-1}\alpha^{-1}[f(d_2)]^{-1}$. This shows that $e = e^{-1}$, that is, e is trivial. This proves the claim.

Therefore, we see that $[f(c)] = \alpha\beta = \sigma$. But σ contains a height r illegal turn, whereas c is an r-legal path. This contradicts Proposition 2.5(1) and Definition 2.1(3). This concludes the proof of (1).

(2) Let σ, γ be as in the assertion of the lemma. Suppose towards a contradiction that there exist three paths $\gamma_1, \gamma_2, \gamma_3$ such that γ_2 is nontrivial and $\sigma = \gamma_1\gamma_2$ and $\gamma = \gamma_2\gamma_3$. Suppose first that $\gamma = \sigma$. Then either the two copies of γ_2 in σ overlap or there exists a path γ_4 such that $\sigma = \gamma_2\gamma_4\gamma_2$. The first case is not possible as otherwise σ would contain two illegal turns. This contradicts the fact that σ contains a unique illegal turn (see [BH, Lemma 5.11]). The second case is not possible by Lemma 3.3(1). Suppose now that $\gamma = \sigma^{-1}$. But $\sigma^{-1} = \gamma_2^{-1}\gamma_1^{-1}$. Therefore we see that $\gamma_2^{-1} = \gamma_2$, that is, γ_2 is trivial. This leads to a contradiction. This concludes the proof. $\square$

We now recall a result, due to Feighn and Handel which will be used in the proof of Lemma 3.5.

Lemma 3.4 ([FH, Corollary 4.12]). *Let $f\colon G \to G$ be a CT map and let $\sigma = \sigma_1 \dots \sigma_s$ be the complete splitting of a path σ of G. If τ is an initial segment of σ with terminal endpoint in some σ_j with $j \in \{1, \dots, s\}$, then $\tau = \sigma_1 \dots \sigma_{j-1}\mu_j$ is a splitting of τ, where μ_j is the initial segment of σ_j contained in τ.*

In particular, if τ is a nontrivial Nielsen path, then, for every $i \in \{1, \dots, j\}$, the path σ_i is a Nielsen path and if σ_j is not a single fixed edge then $\mu_j = \sigma_j$.

Lemma 3.5.

(1) *There are only finitely many paths in $\mathcal{N}_{PG}$.*
(2) *Let γ, γ' be paths in $\mathcal{N}_{PG}$. Suppose that γ has a decomposition $\gamma = \gamma_1\gamma_2$ such that γ_2 is an initial segment of γ'. Then $\gamma_1, \gamma_2 \in \mathcal{N}_{PG}$ and $\gamma_1\gamma' \in \mathcal{N}_{PG}$.*
(3) *Let γ, γ' be paths in $\mathcal{N}_{PG}$. Suppose that $\gamma' \subseteq \gamma$. Then one of the following holds:*
   (a) *there exist (possibly trivial) paths $\gamma_1, \gamma_2 \in \mathcal{N}_{PG}$ such that $\gamma = \gamma_1\gamma'\gamma_2$;*
   (b) *there exists an INP σ in the complete splitting of γ such that $\gamma' \subsetneq \sigma$ and γ' is not an initial or a terminal segment of σ.*
(4) *Let γ, γ' be two paths in $\mathcal{N}_{PG}$. Suppose that there exist three paths γ_1, γ_2 and γ_3 such that $\gamma = \gamma_1\gamma_2$, $\gamma' = \gamma_2^{-1}\gamma_3$ and the path $\gamma_1\gamma_3$ is reduced. Then $\gamma_2 \in \mathcal{N}_{PG}$ and $\gamma_1\gamma_3 \in \mathcal{N}_{PG}$.*

Proof. (1) First note that, since there are only finitely many EG strata in G, there are only finitely many EG INPs by Proposition 2.5(9). Let γ be a path in $\mathcal{N}_{PG}$ which is a concatenation of at least 2 nonclosed EG INPs. Let $\gamma = \sigma_1 \dots \sigma_k$ be the complete splitting of γ given by Lemma 2.7. As γ is a concatenation of nonclosed EG INPs, every splitting unit of γ is a nonclosed EG INP.

By Proposition 2.5(9), an INP contained in the complete splitting of γ is entirely determined by its height. For every $i \in \{1, \dots, k\}$, let r_i be the height of σ_i. Let $i \in \{2, \dots, k\}$. Since σ_i is not closed, by [HM, Fact 1.42(1)(a)], one of the endpoints of σ_i is not contained in G_{r_i-1}. Since there exists a unique INP of height r_i by Proposition 2.5(9), either $r_{i-1} < r_i$ or $r_i < r_{i-1}$.

We treat the case $r_1 < r_2$, the case $r_2 < r_1$ being similar. We claim that, for every $i \in \{1, \ldots, k-1\}$, we have $r_{i+1} > r_i$. The proof is by induction on i. The base case is true by hypothesis. Let $i \in \{2, \ldots, k-1\}$. Since $r_{i-1} < r_i$, the origin of σ_i is contained in $G_{r_i - 1}$ and the terminal point of σ_i is not contained in G_{r_i-1}. Thus, the first edge of σ_{i+1} is contained in $\overline{G - G_{r_i-1}}$. Since there exists a unique INP of height r_i we necessarily have $r_i < r_{i+1}$. Thus, the sequence of maximal heights of INPs in γ is (strictly) monotonic. Since there are only finitely many EG strata, there are only finitely many paths in $\mathcal{N}_{PG}$. This concludes the proof of (1).

(2) Let $\gamma, \gamma' \in \mathcal{N}_{PG}$ and let $\gamma = \gamma_1\gamma_2$ be as in the assertion of the lemma. We claim that $\gamma_2 \in \mathcal{N}_{PG}$ and that the splitting units of γ_2 are splitting units of both γ and γ'. This will conclude the proof of Assertion (2) because γ_1 will be a concatenation of splitting units of γ, that is, it will be either an EG INP or a concatenation of nonclosed EG INPs (cf. Definition 3.1(3)). Hence we will have $\gamma_1 \in \mathcal{N}_{PG}$ and $\gamma_1\gamma' \in \mathcal{N}_{PG}$.

We show that γ_2 is a concatenation of INPs which are splitting units of γ'. A similar proof will show that the splitting units of γ_2 will also be splitting units of γ. Indeed, the path γ' has a splitting $\gamma' = \sigma_1'\sigma_2' \ldots \sigma_k'$ which consists in EG INPs. Let r' be the height of σ_1'. By Proposition 2.5(9), there exists a unique unoriented INP of height r' and this INP starts and ends with an edge in $H_{r'}$.

Let σ be the INP of γ which has a decomposition $\sigma = \sigma_1\sigma_2$, where σ_2 is a nontrivial initial segment of γ'. As every splitting unit of γ is an EG INP, so is σ. Let r be the height of σ. Since the first edge of σ_1' is of height r', we cannot have $r' > r$.

If $r = r'$, then by the uniqueness statement in Proposition 2.5(9), we see that $\sigma_1' \in \{\sigma, \sigma^{-1}\}$. Note that if σ_1 is nontrivial, there exist reduced paths τ_1, τ_2 such that $\sigma = \sigma_1\tau_1$ and $\sigma_1' = \tau_1\tau_2$. This contradicts Lemma 3.3(2) applied to σ and σ_1'. Thus, we see that $\sigma = \sigma_1'$ and $\sigma_1' \subseteq \gamma_2$.

If $r' < r$, then by Lemma 2.8(2), the path σ has a decomposition $\sigma = a_1b_1 \ldots b_{k-1}a_k$ such that, for every $i \in \{1, \ldots, k\}$, the path a_i is a path contained in H_r and for every $i \in \{1, \ldots, k-1\}$, the path b_i is a Nielsen path in G_{r-1}. Hence there exists $i \in \{1, \ldots, k-1\}$ such that σ_1' is contained in b_i. Therefore, we see that $\sigma_1' \subseteq \sigma \subseteq \gamma$. As $\sigma_1' \subseteq \gamma'$, we see that $\sigma_1' \subseteq \gamma \cap \gamma' = \gamma_2$. If $\gamma_2 = \sigma_1'$, then we are done. Otherwise, the path γ_2 contains an edge of σ_2'. As σ_2' is an EG INP, the same argument as for σ_1' shows that $\sigma_2' \subseteq \gamma_2$, and an inductive argument shows that γ_2 is a concatenation of INPs in the splitting of γ'. Hence γ_2 is a Nielsen path. Therefore, we see that $\gamma_2 \in \mathcal{N}_{PG}$ and that γ_2 is composed of splitting units of γ'. Similarly, we see that γ_2 is composed of splitting units which are splitting units of both γ and γ'. Hence γ_1 is composed of splitting units of γ. This concludes the proof of (2).

(3) Let γ, γ' be as in the assertion of the lemma. Let $\gamma = \sigma_1 \ldots \sigma_k$ be the complete splitting of γ and let $\gamma' = \sigma_1' \ldots \sigma_m'$ be the complete splitting of γ', which exist by Lemma 2.7. Recall that every splitting unit of both γ and γ' is an EG INP. There exists $i \in \{1, \ldots, k\}$ such that σ_i contains an initial segment of σ_1'. We claim that σ_1' is either equal to σ_i or γ' is strictly

contained in σ_i. Indeed, let r be the height of σ_i and let r' be the height of σ'_1. Since the first edge of σ'_1 is of height r', we cannot have $r' > r$.

Suppose first that $r' < r$. By Lemma 2.8(2), the path σ_i has a decomposition $\sigma_i = a_1 b_1 \dots b_{p-1} a_p$ such that, for every $i \in \{1, \dots, p\}$, the path a_i is a path in H_r and for every $j \in \{1, \dots, p-1\}$, the path b_j is a Nielsen path in G_{r-1}. Hence there exists $j \in \{1, \dots, p-1\}$ such that σ'_1 is contained in b_j.

We claim that, for every $\ell \in \{1, \dots, m\}$, the splitting unit σ'_ℓ is contained in b_j. The proof is by induction on ℓ. For the base case, we already know that $\sigma'_1 \subseteq b_j$. Suppose that for some $\ell \in \{2, \dots, m\}$, the path $\sigma'_{\ell-1}$ is contained in b_j. By Proposition 2.5(9), the path σ_i ends with an edge in H_r. Hence the path a_p is nontrivial. Since $\sigma'_{\ell-1}$ is contained in b_j, the path σ'_ℓ intersects σ_i nontrivially. Let r_ℓ be the height of σ'_ℓ. Recall that σ'_ℓ is an EG INP. By Proposition 2.5(9), the path σ'_ℓ starts with an edge in H_{r_ℓ}. Hence $r_\ell \leqslant r$. Suppose towards a contradiction that $r_\ell = r$. Then, by the uniqueness statement of Proposition 2.5(9), we see that $\sigma'_\ell \in \{\sigma_i^{\pm 1}\}$. As σ_i contains an initial segment of σ'_ℓ, there exist three paths γ_1, γ_2 and γ_3 of G such that γ_2 is nontrivial and $\sigma_i = \gamma_1\gamma_2$ and $\sigma'_\ell = \gamma_2\gamma_3$. Since $\sigma'_{\ell-1}$ is contained in σ_i, the path γ_1 is nontrivial. This contradicts Lemma 3.3(2). Therefore we have $r_\ell < r$. But then σ'_ℓ cannot intersect a_{j+1}. This implies that σ'_ℓ is contained in b_j. This proves the claim and the fact that $\gamma' \subsetneq \sigma_i$ and γ' is not an initial or a terminal segment of σ_i.

Suppose now that $r = r'$. By the uniqueness statement of Proposition 2.5(9), we see that $\sigma'_1 \in \{\sigma_i^{\pm 1}\}$. As σ_i contains an initial segment of σ'_1, there exist three paths γ_1, γ_2 and γ_3 of G such that γ_2 is nontrivial and $\sigma_i = \gamma_1\gamma_2$ and $\sigma'_1 = \gamma_2\gamma_3$. By Lemma 3.3(2), we necessarily have that γ_1 and γ_3 are trivial. Thus, we see that $\sigma_i = \sigma'_1$. Therefore, γ' is an initial segment of $\sigma_i \dots \sigma_k$ and is a Nielsen path. By Lemma 3.4, for every $j \in \{1, \dots, m\}$, we have $\sigma_{i+j-1} = \sigma'_j$. Thus, there exist (possibly trivial) paths $\gamma_1, \gamma_2 \in \mathcal{N}_{PG}$ such that $\gamma = \gamma_1\gamma'\gamma_2$. This concludes the proof of (3).

(4) Let γ, γ', γ_1, γ_2 and γ_3 be as in the assertion of the lemma. Let $\gamma = \alpha_1 \dots \alpha_k$ and $\gamma' = \beta_1 \dots \beta_\ell$ be the complete splittings of γ and γ' given by Lemma 2.7. By definition of $\mathcal{N}_{PG}$, every splitting unit of γ and γ' is an EG INP.

Let $i \in \{1, \dots, k\}$ be such that α_i contains the first edge of γ_2. Let $j \in \{1, \dots, \ell\}$ be such that β_j contains the last edge of γ_2^{-1}. We claim that $\alpha_i \subseteq \gamma_2$ and that $\beta_j \subseteq \gamma_2^{-1}$. By Lemma 3.4 applied to γ_2^{-1} and γ^{-1}, there exists a path δ_i contained in α_i such that the decomposition $\gamma_2 = \delta_i\alpha_{i+1} \dots \alpha_k$ is a splitting of γ_2. Similarly, there exists a path δ'_j in β_j such that $\gamma_2^{-1} = \beta_1 \dots \beta_{j-1}\delta'_j$ is a splitting of γ_2^{-1}. By Proposition 2.5(9), an EG INP starts with an edge of highest height and an EG INP is entirely determined by its height. Hence $\alpha_k = \beta_1^{-1}$. Note that the paths $\delta_i\alpha_{i+1} \dots \alpha_{k-1}$ and $\beta_2 \dots \beta_{j-1}\delta'_j$ satisfy the same hypotheses as $\delta_i\alpha_{i+1} \dots \alpha_k$ and $\beta_1 \dots \beta_{j-1}\delta'_j$. Applying the same arguments, we see that $i = j$ and for every $s \in \{1, \dots, j-1\}$, we have $\beta_s = \alpha_{k-s+1}^{-1}$. Hence we see that $\delta_i = \delta_j'^{-1}$.

Let r be the height of α_i and let r' be the height of β_j. Note that by Proposition 2.5(9) applied to α_i and β_j, the path δ_i ends with an edge in H_r and $\delta_j'^{-1}$ ends with an edge in $H_{r'}$. Therefore, we see that $r = r'$. By uniqueness of EG INPs of height r_i given by Proposition 2.5(9), and since

$\gamma_1\gamma_3$ is reduced, we see that $\alpha_i = \beta_j^{-1}$, that $\alpha_i \subseteq \gamma_2$ and that $\beta_j \subseteq \gamma_2^{-1}$. This shows that γ_2 is a path in $\mathcal{N}_{PG}$. By Assertion (2) applied to γ and γ_2, the path γ_1 is contained in $\mathcal{N}_{PG}$. Similarly, we see that the path γ_3 is contained in $\mathcal{N}_{PG}$. Since the path $\gamma_1\gamma_3$ is reduced, we see that $\gamma_1\gamma_3 \in \mathcal{N}_{PG}$. This concludes the proof. ☐

Lemma 3.6. *Let γ and γ' be two reduced edge paths in G which are concatenations of paths in G_{PG} and $\mathcal{N}_{PG}$. Suppose that there exist three paths γ_1, γ_2 and γ_3 such that $\gamma = \gamma_1\gamma_2$, $\gamma' = \gamma_2^{-1}\gamma_3$ and $\gamma_1\gamma_3$ is reduced. Then γ_2 and $\gamma_1\gamma_3$ are concatenations of paths in G_{PG} and $\mathcal{N}_{PG}$.*

Proof. Let $\gamma = b_0a_1b_1 \dots a_kb_k$ be the decomposition of the path γ such that for every $i \in \{0, \dots, k\}$, the path b_i is in G_{PG} and for every $i \in \{1, \dots, k\}$, the path a_i is a maximal subpath of γ contained in $\mathcal{N}_{PG}$. The existence of the paths a_i follows from Lemma 3.5(2). Let $\gamma' = d_0c_1d_1 \dots c_\ell d_\ell$ be the similar decomposition of γ'. Let e be the initial edge of γ_2.

Claim. There exists $i \in \{0, \dots, k\}$ such that b_i contains e if and only if there exists $j \in \{0, \dots, \ell\}$ such that the edge e^{-1} is contained in d_j.

Proof. The proof of the two directions being similar, we only prove one direction. Suppose that there exists $i \in \{0, \dots, k\}$ such that b_i contains e. Suppose towards a contradiction that there exists $j \in \{1, \dots, \ell\}$ such that e^{-1} is contained in c_j. It follows that there exists an EG INP σ of c_j such that e^{-1} is contained in σ. Let r be the height of σ. Let δ^{-1} be the subpath of σ contained in γ_2^{-1}. Note that, as γ_2^{-1} is an initial segment of γ', the path δ^{-1} is an initial segment of σ. By Proposition 2.5(9), the path δ^{-1} starts with an edge in H_r. As δ is contained in γ, the terminal edge of δ is an edge in an EG stratum. Since every edge in G_{PG} is contained in an NEG stratum, there exists $s \in \{1, \dots, k\}$ such that a_s contains a terminal segment of δ.

Since the initial edge e of γ_2 is not contained in a_s by hypothesis, the path δ contains the initial segment δ' of a_s. Hence the terminal segment δ'^{-1} of a_s^{-1} is the initial segment δ'^{-1} of σ. By Lemma 3.5(2) applied to a_s^{-1} and σ and Lemma 3.4, the path δ'^{-1} is contained in $\mathcal{N}_{PG}$ and is a concatenation of splitting units of σ. As σ contains a unique splitting unit, this implies that $\delta' = \sigma$. As $\delta' \subseteq \delta^{-1} \subseteq \sigma$, we see that $\delta^{-1} = \sigma$.

Note that the edge δ^{-1} ends with e^{-1}. But σ ends with an edge in an EG stratum by Proposition 2.5(9), that is, e^{-1} is an edge in an EG stratum. But every edge in b_i is contained in an NEG stratum by definition of G_{PG}. This contradicts the fact that $e \subseteq b_i$. This concludes the proof of the claim. ☐

Suppose first that there exists $i \in \{1, \dots, k\}$, such that e is contained in b_i. By the above claim, there exists $j \in \{0, \dots, \ell\}$ such that e^{-1} is contained in d_j. Let τ and τ' be such that $\gamma = b_0a_1b_1 \dots a_i\tau\gamma_2$ and $\gamma' = \gamma_2^{-1}\tau'c_{j+1} \dots d_\ell$. Note that $\tau \subseteq b_i$ and $\tau' \subseteq d_j$. Then we have $\gamma_1 = b_0a_1b_1 \dots a_i\tau$ and $\gamma_3 = \tau'c_{j+1} \dots d_\ell$. Since the path $\gamma_1\gamma_3$ is reduced, so is $\tau\tau'$. Moreover the reduced edge path $\tau\tau'$ is contained in G_{PG} and $\gamma_1\gamma_3 = b_0a_1b_1 \dots a_i\tau\tau'c_{j+1} \dots d_\ell$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Let δ'' be the maximal subpath of b_i contained in γ_2. Then $\gamma_2 = \delta''a_{i+1} \dots b_k$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$.

Suppose now that there exists $i \in \{1, \dots, k\}$ such that the initial edge e of γ_2 is contained in a_i. By the above claim, there exists $j \in \{1, \dots, \ell\}$ such that e^{-1}

is contained in c_j. Let δ' be the terminal segment of a_i contained in γ_2. By Proposition 2.5(9), the terminal edge e' of δ' is an edge in an EG stratum. Since G_{PG} does not contain any edge in an EG stratum, there exists $s \leqslant j$ such that c_s contains e'^{-1}.

We claim that $s = j$. Indeed, suppose towards a contradiction that $s < j$. Let δ^{-1} be the terminal segment of c_s whose first edge is e'^{-1}. Then δ is a terminal segment of a_i and δ is an initial segment of c_s^{-1}. By Lemma 3.5(2) applied to a_i and c_s^{-1}, the path δ is a concatenation of splitting units of a_i and c_s^{-1}. If δ is properly contained in δ', there exists an EG INP σ which is a splitting unit of a_i and such that the last edge of σ is the last edge of δ' not contained in δ. But, by Proposition 2.5(9), the terminal edge e_σ of σ is in an EG stratum. However, the first edge of d_s (which is the edge e_σ^{-1}) is in G_{PG}. This leads to a contradiction. Hence $\delta = \delta'$. But δ intersects c_j nontrivially. Hence we have $s = j$.

Therefore, δ'^{-1} is contained in c_j. We claim that δ'^{-1} is an initial segment of c_j. Indeed, otherwise let ϵ' be the initial segment of c_j whose endpoint is the origin of δ'^{-1}. By Proposition 2.5(9), the first edge of ϵ' is an edge in an EG stratum. Hence there exists $p > i$ such that a_p contains the terminal edge of ϵ'^{-1}. Let ϵ^{-1} be the subpath of ϵ'^{-1} contained in a_p. Then ϵ^{-1} is an initial segment of a_p and ϵ is an initial segment of c_j. By Lemma 3.5(2) applied to a_p^{-1} and c_j, the path ϵ is a concatenation of splitting units of a_p^{-1} and c_j. But since ϵ is properly contained in c_j as it does not intersect δ'^{-1}, the path ϵ is adjacent to a splitting unit of c_j. Since an EG INP starts with an edge in an EG stratum by Proposition 2.5(9), the path b_{p-1} ends with an edge in an EG stratum. This contradicts the fact that b_{p-1} is contained in G_{PG}.

Hence δ'^{-1} is an initial segment of c_j and δ' is a terminal segment of a_i. Let τ and τ' be two paths such that $a_i = \tau\delta'$ and $c_j = \delta'^{-1}\tau'$. By Lemma 3.5(4) applied to a_i and c_j, the path δ' is in $\mathcal{N}_{PG}$ and the path $\tau\tau'$ is in $\mathcal{N}_{PG}$. Hence $\gamma_2 = \tau b_i a_{i+1} \dots b_k$ and $\gamma_1\gamma_3 = b_0 a_1 b_1 \dots a_i \tau\tau' c_{j+1} \dots d_\ell$ are concatenations of paths in G_{PG} and in $\mathcal{N}_{PG}$. $\square$

Lemma 3.7. *Let γ be a closed Nielsen path of G. Then γ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$.*

Proof. Let γ be a closed Nielsen path of G. We prove the result by induction on the height r of γ. If $r = 0$, there is nothing to prove. Assume that $r \geqslant 1$. By Lemma 2.7, the path γ is completely split, and every splitting unit in its complete splitting is either an INP or a fixed edge. Let $\gamma = \sigma_1 \dots \sigma_k$ be the complete splitting of γ. For every $i \in \{1, \dots, k\}$, let r_i be the height of σ_i. We prove that for every $i \in \{1, \dots, k\}$, the path σ_i is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$.

Let $i \in \{1, \dots, k\}$. If σ_i is a fixed edge, it is contained in G_{PG}. Suppose that σ_i is an NEG INP. By Proposition 2.5(11), there exist an edge $e_{r_i} \in \vec{E}H_{r_i}$, a Nielsen path w in G_{r_i-1} and an integer $s \in \mathbb{Z}^*$ such that $\sigma_i = e_{r_i} w^s e_{r_i}^{-1}$. Moreover, we have $f(e_{r_i}) = e_{r_i} w$. Hence for every $j \in \mathbb{N}^*$, we have $[f^j(e_{r_i})] = e_{r_i} w^j$. Since w is a Nielsen path, by Lemma 2.7, the path w is completely split and its complete splitting consists of fixed edges and INPs. Thus, for every $j \in \mathbb{N}^*$, the complete splitting of $[f^j(e_{r_i})]$ does not contain splitting units which are edges in EG strata. By definition of G_{PG}, we have $e_{r_i} \in \vec{E}G_{PG}$. Moreover, by the induction hypothesis, the path w^s is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Hence σ_i is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Finally, if σ_i is an EG INP, then it is contained in $\mathcal{N}_{PG}$. Hence γ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. □

Lemma 3.8. *Let γ be either an NEG INP or an exceptional path. Then γ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$.*

Proof. We claim that there exist edges e_1, e_2 and a closed Nielsen path w such that $\gamma = e_1 w e_2^{-1}$ and, for every $i \in \{1, 2\}$, we have $f(e_i) = e_i w^{d_i}$ for some $d_i \in \mathbb{Z}^*$. If γ is an exceptional path, it follows from the definition. If γ is an NEG INP, let r be the height of γ. Then H_r is an NEG stratum. As γ is a Nielsen path, we can apply Proposition 2.5(11) to conclude the proof of the claim. Since e_1 and e_2 are linear edges, for every $k \in \mathbb{N}^*$, the paths $[f^k(e_1)]$ and $[f^k(e_1)]$ do not contain splitting units which are edges in EG strata. Thus e_1 and e_2 are contained in G_{PG}. By Lemma 3.7, the path w is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Hence γ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. □

Lemma 3.9. *Let γ be a Nielsen path in G. Then γ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$.*

Proof. By Lemma 2.7, the path γ is completely split, and every splitting unit in its complete splitting is either an INP or a fixed edge. Let $\gamma = \sigma_1 \dots \sigma_k$ be the complete splitting of γ. Let $i \in \{1, \dots, k\}$. If σ_i is a fixed edge, then σ_i is contained in G_{PG}. If σ_i is an NEG INP then, by Lemma 3.8, the path σ_i is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. If σ_i is an EG INP then, by definition, we have $\sigma_i \in \mathcal{N}_{PG}$. Hence γ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. □

Lemma 3.10.

(1) *Let γ be an edge in G_{PG} (resp. an edge in $G_{PG,\mathcal{F}}$). The path $[f(\gamma)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ (resp. a concatenation of paths in $G_{PG,\mathcal{F}}$ and in $\mathcal{N}_{PG,\mathcal{F}}$).*
(2) *Let γ be an edge path contained in G_{PG} (resp. an edge path in $G_{PG,\mathcal{F}}$). The path $[f(\gamma)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ (resp. a concatenation of paths in $G_{PG,\mathcal{F}}$ and in $\mathcal{N}_{PG,\mathcal{F}}$).*
(3) *Let γ be an edge path which is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ (resp. a concatenation of paths in $G_{PG,\mathcal{F}}$ and in $\mathcal{N}_{PG,\mathcal{F}}$). The path $[f(\gamma)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ (resp. a concatenation of paths in $G_{PG,\mathcal{F}}$ and in $\mathcal{N}_{PG,\mathcal{F}}$).*

Proof. We prove Assertions (1), (2), (3) for paths in G_{PG} and in $\mathcal{N}_{PG}$, the proofs for paths in $G_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$ being similar, using the fact that $f(G_p) = G_p$.

(1) Let γ be an edge of G_{PG}. By definition of G_{PG}, the edge γ is an edge in an NEG stratum. By Proposition 2.5(6), the path $[f(\gamma)]$ is completely split. Let $[f(\gamma)] = \gamma_1 \dots \gamma_m$ be the complete splitting of $[f(\gamma)]$. Since γ is an edge in an NEG stratum, by Proposition 2.5(2), we have $\gamma_1 = \gamma$.

    Suppose towards a contradiction that $[f(\gamma)]$ is not a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. It follows that there exists $i \in \{1, \dots, m\}$ and an edge e of γ_i which is not contained in G_{PG} and is not contained in a subpath of $[f(\gamma)]$ contained in $\mathcal{N}_{PG}$. Hence γ_i is not an EG INP nor a fixed edge. By Lemma 3.8, the path γ_i cannot be an NEG INP or an exceptional path. Hence γ_i is either an edge in an irreducible stratum or a maximal taken connecting path in a zero stratum.

Suppose first that γ_i is a maximal taken connecting path in a zero stratum. By Proposition 2.5(4), the path γ_i cannot be adjacent to an edge in an NEG stratum nor an edge in a zero stratum. As $\gamma_1 = \gamma$, we see that $i \geqslant 3$ and that γ_{i-1} ends with an edge in an EG stratum. By Lemma 2.9 (applied to $\gamma = \gamma_{i-1}\gamma_i$), the path γ_{i-1} is not an EG INP. Therefore we see that γ_{i-1} is an edge in an EG stratum. This contradicts the definition of the edges in G_{PG}.

Hence we are reduced to the case where γ_i is an edge in an irreducible stratum. Therefore, we have $\gamma_i = e$. By definition of G_{PG} and as $e \notin \vec{E}G_{PG}$, there exists $k \in \mathbb{N}^*$ such that $[f^k(\gamma_i)]$ contains a splitting unit which is an edge in an EG stratum. This contradicts the fact that γ is contained in G_{PG}. This concludes the proof of (1).

(2) Let γ be a path in G_{PG}. We prove by induction on the length of γ that $[f(\gamma)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. The case where γ is an edge follows from Assertion (1). Suppose now that the length of γ is at least equal to 2. Let e be the last edge of γ and let γ' be an edge path such that $\gamma = \gamma' e$. Hence γ' and e are paths in G_{PG}. By the induction hypothesis, the paths $[f(\gamma')]$ and $[f(e)]$ are concatenations of paths in G_{PG} and in $\mathcal{N}_{PG}$. It remains to show that identifications between $[f(\gamma')]$ and $[f(e)]$ do not create paths which are not concatenations of paths in G_{PG} and in $\mathcal{N}_{PG}$. Let α, β and σ be paths such that $[f(\gamma')] = \alpha\sigma$, $[f(e')] = \sigma^{-1}\beta$ and $\alpha\beta$ is reduced. By Lemma 3.6 applied to $[f(\gamma')]$ and $[f(e')]$, the path $[f(\gamma)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. This concludes the proof of (2).

(3) Let γ be a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Let $\gamma = \gamma_0'\gamma_1\gamma_1' \dots \gamma_k\gamma_k'$ be a decomposition of γ such that for every $i \in \{1, \dots, k\}$, the path γ_i is a maximal subpath of γ in $\mathcal{N}_{PG}$ and for every $i \in \{0, \dots, k\}$, the path γ_i' is a path in G_{PG}. Such a decomposition is possible by Lemma 3.5(2). We prove the result by induction on k. If $k = 0$, the proof follows from Assertion (2). Suppose that the result is true for $k' < k$. Then the paths $\gamma' = \gamma_0'\gamma_1\gamma_1' \dots \gamma_{k-1}\gamma_{k-1}'$ and $\gamma'' = \gamma_k\gamma_k'$ satisfy the induction hypothesis. Hence the paths $[f(\gamma')]$ and $[f(\gamma'')]$ are concatenations of paths in G_{PG} and in $\mathcal{N}_{PG}$. Let α, β and σ be three paths such that $[f(\gamma')] = \alpha\beta$, $[f(\gamma'')] = \beta^{-1}\sigma$ and $\alpha\beta$ is reduced. By Lemma 3.6, the path $[f(\gamma)] = \alpha\sigma$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. This concludes the proof.

□

For Lemma 3.11, we recall a definition due to Bestvina, Feighn and Handel ([BFH1, Section 6], see also [HM, Definition III.1.2]). Let H_{r_+} be the EG stratum of G of maximal height r_+. By Proposition 2.5(9), there exists at most one unoriented INP ρ_{r_+} of height r_+ (we suppose that ρ_{r_+} is a point if such a nontrivial INP does not exist). Following [HM, Definition III.1.2], let Z_{r_+} be the subgraph of G consisting of all edges e' such that for every $m \in \mathbb{N}^*$ and every splitting unit σ of $[f^m(e')]$, the path σ is not an edge in H_{r_+}. Let $\langle Z_{r_+}, \rho_{r_+} \rangle$ be the set consisting of the following paths:

(i) paths in Z_{r_+};
(ii) paths in $\{\rho_{r_+}, \rho_{r_+}^{-1}\}$;
(iii) concatenations of paths in Z_{r_+} and in $\{\rho_{r_+}, \rho_{r_+}^{-1}\}$.

Note that $\langle Z_{r_+}, \rho_{r_+} \rangle$ contains every path in G_{r_+-1}.

Lemma 3.11. *The set $\langle Z_{r_+}, \rho_{r_+} \rangle$ contains every path which is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$.*

Proof. It suffices to prove that $\langle Z_{r_+}, \rho_{r_+} \rangle$ contains every edge of G_{PG} and every EG INP. Let e be an edge in G_{PG}. By definition of G_{PG}, for every $k \in \mathbb{N}^*$, the complete splitting of $[f^k(e)]$ does not contain a splitting unit which is an edge in an EG stratum. In particular, for every $k \in \mathbb{N}^*$, the complete splitting of $[f^k(e)]$ does not contain a splitting unit which is an edge in H_{r_+}. Hence $e \subseteq Z_{r_+}$ and G_{PG} is a subgraph of Z_{r_+}. Let ρ be an EG INP and let r be the height of ρ. By definition of r_+, we have $r \leqslant r_+$. If $r = r_+$, by Proposition 2.5(9), we have $\rho \in \{\rho_{r_+}, \rho_{r_+}^{-1}\}$, hence we have $\rho \in \langle Z_{r_+}, \rho_{r_+} \rangle$. If $r < r_+$, then ρ is contained in G_{r_+-1}. Hence ρ is contained in $\langle Z_{r_+}, \rho_{r_+} \rangle$ by the above remark. $\square$

We now define a graph which will be used in the proof of Lemma 3.13. Let G^* be the finite, not necessarily connected, graph defined as follows:

(a) vertices of G^* are the vertices in G_{PG} and the endpoints of EG INPs in G which are not in G_{PG};
(b) we add one edge between two vertices corresponding to vertices in G_{PG} if there exists an edge in G_{PG} between them;
(c) we add one edge between two vertices corresponding to the endpoints of an EG INP.

Note that we have a natural continuous application $p_{G^*}\colon G^* \to G$ which sends an edge as defined in (b) to the corresponding edge in G_{PG} and which sends an edge as defined in (c) to the corresponding EG INP in G. Let $x \in VG^*$.

Lemma 3.12.

(1) *If γ is a nontrivial reduced path in G^*, so is $p_{G^*}(\gamma)$.*
(2) *The homomorphism*

$$p'_{G^*}\colon \pi_1(G^*, x) \to \pi_1(G, p_{G^*}(x))$$

induced by p_{G^*} is injective.

Proof. (1) Let γ be a reduced path in G^*. Suppose towards a contradiction that $p_{G^*}(\gamma)$ is not a reduced path in G. Thus, there exist an edge $e \in \vec{E}G$ and two paths a and b such that $p_{G^*}(\gamma) = aee^{-1}b$. Let e^* be an arc in γ such that $p_{G^*}(e^*) = ee^{-1}$. Note that, by definition of p_{G^*}, the application p_{G^*} sends edges of G^* to reduced edge paths in G. In particular, the path e^* is not contained in a single edge of G^*. As the image of an edge in G^* by p_{G^*} is either an edge in G or an edge path, we see that the path e^* is contained in at most two edges of G^*.

Let $e_1, e_2 \in G^*$ be such that $e^* \subseteq e_1e_2$. Suppose first that $p_{G^*}(e_1)$ and $p_{G^*}(e_2)$ are edges in G_{PG}. Then $p_{G^*}(e_1) = e$ and $p_{G^*}(e_2) = e^{-1}$. But, as γ is reduced, we have $e_1 \neq e_2^{-1}$. This implies that $p_{G^*}(e_1) \neq p_{G^*}(e_2)^{-1}$.

Suppose now that $p_{G^*}(e_1)$ is an edge in G_{PG} and $p_{G^*}(e_2)$ is an EG INP. By Proposition 2.5(9), the first edge of $p_{G^*}(e_2)$ is an edge in an EG stratum. By definition, every edge in G_{PG} is an edge in an NEG stratum. Hence the turn $\{p_{G^*}(e_1)^{-1}, p_{G^*}(e_2)\}$ is nondegenerate. Therefore, we see that $p_{G^*}(e^*) \neq ee^{-1}$.

Finally, suppose that $p_{G*}(e_1)$ and $p_{G*}(e_2)$ are EG INPs. for every $i \in \{1,2\}$, let r_i be the height of $p_{G*}(e_i)$. By Proposition 2.5(9), the last edge of $p_{G*}(e_1)$ is in H_{r_1} whereas the first edge of $p_{G*}(e_2)$ is in H_{r_2}. Hence if $r_1 \neq r_2$, there is no identification between $p_{G*}(e_1)$ and $p_{G*}(e_2)$. Therefore, we have $p_{G*}(e^*) \neq ee^{-1}$. If $r_1 = r_2$, then by the uniqueness statement in Proposition 2.5(9), we have $p_{G*}(e_2) \in \{p_{G*}(e_1), p_{G*}(e_1)^{-1}\}$. Hence $e_2 \in \{e_1, e_1^{-1}\}$. As γ is a reduced path, we see that $e_2 = e_1$. Hence e_1 is a loop and $p_{G*}(e_1)$ is a closed EG INP. By Proposition 2.5(9), the initial and terminal edges of $p_{G*}(e_1)$ are distinct unoriented edges. Hence the path $p_{G*}(e_1)p_{G*}(e_2)$ is a reduced path and $p_{G*}(e^*) \neq ee^{-1}$. As we have ruled out every case, we see that such a path e^* does not exist. This concludes the proof of Assertion (1).

(2) Let γ be a nontrivial reduced closed path in G^* based at x. By Assertion (1), the path $p_{G*}(\gamma)$ is a nontrivial reduced closed path in G. Therefore, the kernel of p'_{G*} is trivial. □

Lemma 3.13. *The application $[f]$ which sends a circuit α in G to $[f(\alpha)]$ preserves the set of circuits which are concatenations of paths in G_{PG} and in $\mathcal{N}_{PG}$. Moreover, $[f]$ restricts to a bijection on the set of circuits which are concatenations of paths in G_{PG} and in $\mathcal{N}_{PG}$.*

Proof. The first part follows from Lemma 3.10(3). By [HM, Lemma III.1.6 (2), (5)], the application $[f]$ preserves $\left\langle Z_{r_+}, \rho_{r_+}\right\rangle$ and restricts to a bijection on the set of circuits of $\left\langle Z_{r_+}, \rho_{r_+}\right\rangle$. By Lemma 3.11 concatenations of paths in G_{PG} and in $\mathcal{N}_{PG}$ are contained in $\left\langle Z_{r_+}, \rho_{r_+}\right\rangle$. By Lemma 3.10, the application $[f]$ preserves concatenations of paths in G_{PG} and in $\mathcal{N}_{PG}$. In particular, this shows that $[f]$ is injective when restricted to the set of paths which are concatenations of paths in G_{PG} and in $\mathcal{N}_{PG}$.

For surjectivity, let α be a circuit in G which is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ and let x be a vertex in α which is either an endpoint of an edge in G_{PG} or an endpoint of an EG INP contained in α. Note that by Proposition 2.5(2), the endpoint of every edge in G_{PG} is fixed by f. Moreover, the endpoint of every EG INP is fixed by f. Therefore, f fixes x. The circuit α naturally corresponds to a circuit α' in G^*. Let x' be the vertex of α' corresponding to x (which exists by the choices made on x). Since $[f]$ preserves concatenations of paths in G_{PG} and in $\mathcal{N}_{PG}$ by Lemma 3.10, the application $[f]$ induces an application

$$[f]_{G*} : \pi_1(G^*, x') \to \pi_1(G^*, x').$$

Note that, by Lemma 3.12, the group $\pi_1(G^*, x')$ is naturally identified with a subgroup of $\pi_1(G, x)$. By [BFH1, Lemma 6.0.6], the application $[f]_{G*}$ is a bijection. Hence there exists a closed path β' in G^* such that $[f]_{G*}([\beta']) = \alpha'$. Let β be the circuit corresponding to β' in G. Then β is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ and $[f(\beta)] = \alpha$. □

Proposition 3.14. *Let $\mathtt{n} \geqslant 3$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an exponentially growing outer automorphism, let $f \colon G \to G$ be a CT map representing a power of ϕ. Let $w \in F_{\mathtt{n}}$. There exists a subgroup A of $F_{\mathtt{n}}$ such that $[A] \in \mathcal{A}(\phi)$ and $w \in A$ if and only if the circuit γ_w of G associated with w is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$.*

Proof. Suppose first that γ_w is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. We claim that γ_w has polynomial growth under iteration of f. By Proposition 2.5(8), there exists $m \in \mathbb{N}^*$ such that $[f^m(\gamma_w)]$ is completely split. By Lemma 3.10(3), the path $[f^m(\gamma_w)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Hence every splitting unit of $[f^m(\gamma_w)]$ is either an edge of G_{PG} or an INP. Let $[f^m(\gamma_w)] = \gamma_1 \dots \gamma_k$ be the complete splitting of $[f^m(\gamma_w)]$. For every $i \geqslant m$, we have

$$\ell[f^i(\gamma_w)]) = \sum_{j=1}^{k} \ell([f^i(\gamma_j)]).$$

Therefore, it suffices to prove that, for every $j \in \{1, \dots, k\}$, there exists a polynomial $P_j \in \mathbb{Z}[X]$ such that for every $i \in \mathbb{N}^*$, we have

$$\ell([f^i(\gamma_j)]) = \mathrm{O}(P(i)).$$

Claim. There exists a polynomial $P \in \mathbb{Z}[X]$ such that for every edge $e \in \vec{E}G_{PG}$ and every $i \in \mathbb{N}^*$, we have

$$\ell([f^i(e)]) = \mathrm{O}(P(i)).$$

Proof. As there are finitely many edges in G_{PG}, it suffices to prove the claim for a single edge $e \in \vec{E}G_{PG}$. Let $e \in \vec{E}G_{PG}$. By Proposition 2.5(2), there exists a cyclically reduced, completely split circuit w of height less than the one of e and such that $f(e) = ew$. By Lemma 3.10(1), the path w is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$.

We prove the claim by induction on the height of e. Suppose first that e has minimal height in G_{PG}. By minimality of e, the path w does not contain a splitting unit which is an edge in G_{PG}. Hence w is either trivial or a path in $\mathcal{N}_{PG}$, that is, a closed Nielsen path. If w is trivial then e is a fixed edge and $P = 1$ satisfies the claim. Suppose that w is a closed Nielsen path. For every $i \in \mathbb{N}^*$, we have $[f^i(e)] = ew^i$. Hence $\ell([f^i(e)]) \leqslant i\ell(w) + 1$. Then the polynomial $P(i) = i\ell(w) + 1$ satisfies the assertion of the claim. This proves the base case.

Suppose now that e has height r. Let $w = w_1 \dots w_k$ be the complete splitting of w. Recall that, for every reduced path x in G, we have $[f([f(x)])] = [f^2(x)]$. Thus, for every $i \in \mathbb{N}^*$. we have

$$[f^i(e)] = ew_1 \dots w_k[f(w_1)] \dots [f(w_k)] \dots [f^{i-1}(w_1)] \dots [f^{i-1}(w_k)].$$

Hence, for every $i \in \mathbb{N}^*$, we have

$$\ell([f^i(e)]) = 1 + \sum_{\ell=1}^{k} \sum_{j=0}^{i-1} \ell([f^j(w_\ell)]).$$

Hence it suffices, for every $\ell \in \{1, \dots, k\}$, to find a polynomial $P_\ell \in \mathbb{Z}[X]$ such that, for every $i \in \mathbb{N}^*$, we have

$$\ell([f^i(w_\ell)]) = \mathrm{O}(P_\ell(i)).$$

Let $\ell \in \{1, \dots, k\}$. As w is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$, every splitting unit of w is either an edge in G_{PG} or an INP. If w_ℓ is an edge in G_{PG}, the polynomial P_ℓ exists using the induction hypothesis. If w_ℓ is an INP, then the polynomial $P_\ell(i) = \ell(w_\ell)$ satisfies the conclusion of the claim. This proves the existence of the polynomial P. □

Let $j \in \{1, \ldots, k\}$. If γ_k is an edge in G_{PG} which is a splitting unit of $[f^m(\gamma_w)]$, by the above claim, the polynomial P_j exists. If γ_j is an INP, then the polynomial $P_\ell(x) = \ell(\gamma_j)$ satisfies the conclusion. Thus, the path γ_w has polynomial growth under iteration of $[f]$. Therefore, $[w]$ has polynomial growth under iteration of ϕ. By the definition of $\mathcal{A}(\phi)$, there exists a subgroup A of $F_{\mathtt{n}}$ such that $[A] \in \mathcal{A}(\phi)$ and $w \in A$.

Conversely, suppose that there exists a subgroup A of $F_{\mathtt{n}}$ such that $[A] \in \mathcal{A}(\phi)$ and $w \in A$. Let $m \in \mathbb{N}^*$ be such that $[f^m(\gamma_w)]$ is completely split, which exists by Proposition 2.5(7). Since $[w]$ has polynomial growth under iteration of ϕ, there does not exist a splitting unit of $[f^m(\gamma_w)]$ which is an edge in an EG stratum or a superlinear edge with exponential growth.

Suppose towards a contradiction that a splitting unit σ of $[f^m(\gamma_w)]$ is contained in a zero stratum. By Proposition 2.5(3), every zero stratum of G is contractible. As $[f^m(\gamma_w)]$ is a cycle, it is not contained in a zero stratum. By Proposition 2.5(4), every edge adjacent to σ and not contained in the same stratum as σ is in an EG stratum. Thus, there exists a splitting unit σ' of $[f^m(\gamma_w)]$ such that $\sigma\sigma' \subseteq [f^m(\gamma_w)]$ and the first edge of σ' is in an EG stratum. Hence σ' is either an edge in an EG stratum or an INP. But, by Lemma 2.9, the path σ' is not an INP. This shows that σ' is an edge in an EG stratum. This contradicts the fact that $[w]$ has polynomial growth under iteration of ϕ.

Therefore, every splitting unit of $[f^m(\gamma_w)]$ is either an INP, an exceptional path or an edge in an NEG stratum whose iterates by f do not contain splitting units which are edges in EG strata. Edges in the last category are precisely the edges in G_{PG}. By Lemma 3.8 and Lemma 3.9 every INP and every exceptional path is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Thus, the path $[f^m(\gamma_w)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. By Lemma 3.13, the circuit γ_w is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. ☐

Let $\mathcal{F}$ be a nonsporadic free factor system of $F_{\mathtt{n}}$ and let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$. We say that ϕ is *fully irreducible relative to* $\mathcal{F}$ if no power of ϕ preserves a proper free factor system $\mathcal{F}'$ of $F_{\mathtt{n}}$ such that $\mathcal{F} < \mathcal{F}'$. Corollary 3.15 will be used in [Gue2]. It is a well-known result but we did not find a precise statement in the literature.

Corollary 3.15. *Let* $\mathtt{n} \geqslant 3$ *and let* $\mathcal{F}$ *be a nonsporadic free factor system of* $F_{\mathtt{n}}$. *Let* $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ *be a fully irreducible outer automorphism relative to* $\mathcal{F}$. *There exists at most one (up to taking inverse) conjugacy class* $[g]$ *of root-free* $\mathcal{F}$*-nonperipheral element of* $F_{\mathtt{n}}$ *which has polynomial growth under iteration of* ϕ. *Moreover, the conjugacy class* $[g]$ *is* ϕ*-periodic.*

Proof. Let $f \colon G \to G$ be a CT map representing a power of ϕ and let G' be a subgraph of G such that $\mathcal{F}(G') = \mathcal{F}$. Since ϕ is irreducible relative to $\mathcal{F}$ and since $\mathcal{F}$ is nonsporadic, we see that $\overline{G - G'}$ is an EG stratum H_r. Let $[g]$ be the conjugacy class of a root-free $\mathcal{F}$-nonperipheral element g of $F_{\mathtt{n}}$. Then γ_g has height r.

Suppose that $[g]$ has polynomial growth under iteration of ϕ. By Proposition 3.14, the circuit γ_g is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Since γ_g has height r and since H_r is an *EG* stratum, every subpath α of γ_g contained in H_r is contained in a concatenation of INPs of height r. By Proposition 2.5(9), there exists at most one INP σ of height r. Moreover, one of its endpoints is not contained in $G' = G_{r-1}$ (see [HM, I.Fact 1.42]). Hence σ is necessarily a closed

EG INP. Since the endpoint of σ is not in G_{r-1} and since γ_g is a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$, we see that γ_g is an iteration of the closed path σ. Since g is root-free, we have $\gamma_g = \sigma^{\pm 1}$. This concludes the proof. □

3.2. **The exponential length of a CT map.** In this section, we define the *exponential length function* ℓ_{exp}, and its relative version $\ell_{\mathcal{F}}$, of paths in CT maps. We compute its value for some paths in G. Let $G'_{PG} = G_{PG} \cup \mathcal{Z}$ (see Definition 3.1) and let $G'_{PG,\mathcal{F}} = G_{PG,\mathcal{F}} \cup \mathcal{Z}_{\mathcal{F}}$.

Let γ be a reduced edge path in G. By Lemma 3.5(2), every path of $\mathcal{N}_{PG}$ which is contained in γ is contained in a unique maximal subpath of γ contained in $\mathcal{N}_{PG}$. Thus, the path γ has a unique decomposition into edge paths $\gamma = \gamma_0\gamma'_1\gamma_1 \dots \gamma'_k\gamma_k$ where:

(1) for every $i \in \{0, \dots, k\}$, the path γ_i is a maximal path in $\mathcal{N}_{PG}$ contained in γ (where γ_0 and γ_k might be trivial);
(2) for every $\gamma' \in \mathcal{N}_{PG}$ contained in γ, there exists $i \in \{1, \dots, k\}$ such that $\gamma' \subseteq \gamma_i$.

Such a decomposition of γ is called the *exponential decomposition of* γ. Note that the exponential decomposition of γ is not necessarily a splitting of γ. We denote by $\mathcal{N}_{PG}^{\max}(\gamma)$ the set consisting of all paths γ_i, with $i \in \{0, \dots, k\}$. Similarly, γ has a decomposition $\alpha = \alpha_0\alpha'_1\alpha_1 \dots \alpha'_m\alpha_m$, where for every $i \in \{0, \dots, m\}$, the path α_i is a maximal path in $\mathcal{N}_{PG,\mathcal{F}}$ and for every $\gamma' \in \mathcal{N}_{PG,\mathcal{F}}$ contained in γ, there exists $i \in \{1, \dots, k\}$ such that $\gamma' \subseteq \alpha_i$. Such a decomposition is called the $\mathcal{F}$*-exponential decomposition of* γ. We denote by $\mathcal{N}_{PG,\mathcal{F}}^{\max}(\gamma)$ the set consisting of all paths α_i, with $i \in \{0, \dots, m\}$.

Definition 3.16.

(1) Let γ be a reduced edge path in G. The *exponential length of* γ, denoted by $\ell_{exp}(\gamma)$, is:
$$\ell_{exp}(\gamma) = \ell\left(\gamma \cap \overline{G - G'_{PG}}\right) - \sum_{\alpha \in \mathcal{N}_{PG}^{\max}(\gamma)} \ell\left(\alpha \cap \overline{G - G'_{PG}}\right).$$
(2) Let γ be a reduced edge path in G. The $\mathcal{F}$*-exponential length of* γ, denoted by $\ell_{\mathcal{F}}(\gamma)$, is:
$$\ell_{\mathcal{F}}(\gamma) = \ell\left(\gamma \cap \overline{G - G'_{PG,\mathcal{F}}}\right) - \sum_{\alpha \in \mathcal{N}_{PG,\mathcal{F}}^{\max}(\gamma)} \ell\left(\alpha \cap \overline{G - G'_{PG,\mathcal{F}}}\right).$$
(3) Let γ be a reduced edge path in G and let $\gamma = \gamma_0\gamma'_1\gamma_1 \dots \gamma'_k\gamma_k$ be the exponential decomposition of γ. A *PG-relative complete splitting* of the path γ is a splitting $\gamma = \delta_1 \dots \delta_m$ such that for every $i \in \{1, \dots, m\}$, the path δ_i is one of the following paths:
   - a splitting unit of positive exponential length not contained in some γ_i for $i \in \{0, \dots, k\}$;
   - a maximal taken connecting path in a zero stratum;
   - a subpath of γ which is a concatenation of paths in G_{PG} and paths in $\mathcal{N}_{PG}$.

   We call the above paths *PG-relative splitting units.* If γ is a circuit, a *PG-relative circuital complete splitting* of γ is a circuital splitting of γ which is a *PG*-relative complete splitting of γ.

(4) A *factor* of a PG-relative completely split edge path γ is a concatenation of PG-relative splitting units of some given PG-relative complete splitting of γ.

Note that if γ is an edge path of G, then $\ell_{exp}(\gamma) \geqslant 0$. Indeed, two paths γ_1 and γ_2 contained in $\mathcal{N}_{PG}^{\max}(\gamma)$ are either equal or disjoint. Let $\gamma = \gamma_0\gamma_1'\gamma_1 \dots \gamma_k'\gamma_k$ be the exponential decomposition of γ. For every $i \in \{1, \dots, k\}$, we have

$$\ell_{exp}(\gamma_i') = \ell(\gamma_i' \cap \overline{G - G'_{PG}})$$

and

$$\ell_{exp}(\gamma) = \sum_{i=1}^{k} \ell_{exp}(\gamma_i').$$

We prove the existence of PG-relative complete splittings in Lemma 3.20. Note that a PG-relative complete splitting of a reduced edge path γ is not necessarily unique. Indeed, it might be possible that one can split a PG-relative splitting unit of γ which is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ into two PG-relative splitting units which are concatenations of paths in G_{PG} and in $\mathcal{N}_{PG}$.

In the rest of the section, we describe some properties of the exponential length.

Lemma 3.17. *Let γ be a reduced edge path in G and let $\gamma = \gamma_1\gamma_2$ be a decomposition of γ into two edge paths. We have:*

$$\ell_{exp}(\gamma) \leqslant \ell_{exp}(\gamma_1) + \ell_{exp}(\gamma_2).$$

Proof. It is immediate that

$$\ell(\gamma \cap \overline{G - G'_{PG}}) = \ell(\gamma_1 \cap \overline{G - G'_{PG}}) + \ell(\gamma_2 \cap \overline{G - G'_{PG}}).$$

Let $i \in \{1, 2\}$. Let $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma_i)$. Then there exists $\gamma'' \in \mathcal{N}_{PG}^{\max}(\gamma)$ such that $\gamma' \subseteq \gamma''$. In particular, we have

$$\sum_{\gamma'' \in \mathcal{N}_{PG}^{\max}(\gamma)} \ell(\gamma'' \cap \overline{G - G'_{PG}}) \geqslant \sum_{\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma_1)} \ell(\gamma' \cap \overline{G - G'_{PG}}) + \sum_{\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma_2)} \ell(\gamma' \cap \overline{G - G'_{PG}}).$$

By definition of the exponential length, this concludes the proof. □

Note that we do not necessarily have equality in Lemma 3.17. Indeed, let $\gamma = \gamma_1\gamma_2$ be as in Lemma 3.17. Suppose that the endpoint of γ_1 is contained in a path γ' of $\mathcal{N}_{PG}^{\max}(\gamma)$. Then γ' is not necessarily a concatenation of paths in $\mathcal{N}_{PG}^{\max}(\gamma_1)$ and $\mathcal{N}_{PG}^{\max}(\gamma_2)$. Therefore, we might have:

$$\sum_{\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma)} \ell(\gamma' \cap \overline{G - G'_{PG}}) > \sum_{\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma_1)} \ell(\gamma' \cap \overline{G - G'_{PG}}) + \sum_{\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma_2)} \ell(\gamma' \cap \overline{G - G'_{PG}}),$$

and a strict inequality in Lemma 3.17. In particular, a proper subpath of γ might have greater exponential length than γ itself. For instance, if γ is a reduced path in G such that $\ell_{exp}(\gamma) = 0$, it is possible that there exists a proper subpath γ' of γ such that $\ell_{exp}(\gamma') > 0$. However, there exists a bound, depending only on G, on the difference of the exponential length of a subpath of γ and the exponential length of γ (see Lemma 5.6).

If γ is a path in G such that $\ell_{exp}(\gamma) = 0$, we do not necessarily have $\ell_{exp}([f(\gamma)]) = 0$. Indeed, if γ is an edge in a zero stratum such that $[f(\gamma)]$ contains a splitting unit which is an edge in an EG stratum, we have $\ell_{exp}([f(\gamma)]) > 0$. However, Lemma 3.18 describes an important situation where the map f preserves the property of having zero exponential length.

Lemma 3.18. *Let γ be a reduced edge path which is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. For every $n \in \mathbb{N}$, we have $\ell_{exp}([f^n(\gamma)]) = 0$.*

Proof. Since the $[f]$-image of a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ by Lemma 3.10, it suffices to prove the result for $n = 0$. Let γ be a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Let $\gamma = \gamma_0\gamma'_1\gamma_1 \ldots \gamma'_k\gamma_k$ be the exponential decomposition of γ: for every $i \in \{1, \ldots, k\}$, the path γ_i is a maximal subpath of γ in $\mathcal{N}_{PG}$ and for every $i \in \{0, \ldots, k\}$, the path γ'_i is a path in G_{PG}. Note that for every $i \in \{1, \ldots, k\}$, we have $\gamma_i \in \mathcal{N}_{PG}^{\max}(\gamma)$. By definition of the exponential length, we have $\ell_{exp}(\gamma) = \sum_{i=0}^{k} \ell_{exp}(\gamma'_i) = 0$. □

Corollary 3.19. *Let γ be a path of $\mathcal{N}'_{PG}$. Then $\ell_{exp}(\gamma) = 0$. In particular, if γ is either a closed Nielsen path, an NEG INP or an exceptional path, we have $\ell_{exp}(\gamma) = 0$.*

Proof. By Lemma 3.9, the path γ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. By Lemma 3.18, we have $\ell_{exp}(\gamma) = 0$. The second assertion follows from Lemmas 3.7 and 3.8. □

Lemma 3.20. *Let γ be a completely split edge path and let $\gamma = \gamma_1 \ldots \gamma_m$ be its complete splitting. Let $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma)$. Then either γ' is a concatenation of splitting units of γ or there exists $i \in \{1, \ldots, m\}$ such that $\gamma' \subsetneq \gamma_i$. Moreover, the complete splitting of γ is a PG-relative complete splitting of γ.*

Proof. Let e be the first edge of γ' and let $i \in \{1, \ldots, m\}$ be such that e is contained in γ_i. Let σ be the splitting unit of γ' containing e. By Proposition 2.5(9), the edge e is in an EG stratum. Hence γ_i is either an edge in an EG stratum, an exceptional path or an INP. Since γ' is a Nielsen path, and since γ_i is a splitting unit of γ, we see that γ_i is not an edge in an EG stratum. If γ_i is either an NEG INP or an exceptional path, then Proposition 2.5(11) implies that γ_i starts and ends with edges in NEG strata whose height is strictly higher than the one of e. Since the height of e is equal to the height of σ, we see that γ_i contains σ. An inductive argument shows that γ' is contained in γ_i.

Suppose now that γ_i is an EG INP. By Lemma 3.5(2) applied to γ_i and γ', either γ' is contained in γ_i or γ_i is the initial segment of γ'. If γ' is contained in γ_i, by maximality of γ', we see that $\gamma' = \gamma_i$. Suppose that γ' is the initial segment of the completely split edge path $\gamma_i \ldots \gamma_k$. Then Lemma 3.4 implies that γ' is a factor of γ.

The last assertion of the lemma follows from the following observations. Every splitting unit of γ which is either an INP or an exceptional path is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ by Lemma 3.8. Moreover, by the first assertion of the lemma, every splitting unit of γ which is an edge in an irreducible stratum not contained in G_{PG} does not intersect a path in $\mathcal{N}_{PG}^{\max}(\gamma)$. Hence the complete splitting of γ is a PG-relative complete splitting. □

PG-relative completely split edge paths are well-adapted to the computation of the exponential length as explained by Lemma 3.21.

Lemma 3.21. *Let γ be a PG-relative completely split edge path and let $\gamma = \alpha_1 \dots \alpha_\ell$ be a PG-relative complete splitting.*

(1) *For every path $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma)$, there exists a minimal concatenation of PG-relative splitting units δ of γ such that $\gamma' \subseteq \delta$; every PG-relative splitting unit of δ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$; for every PG-relative splitting unit δ' of δ, the intersection $\delta' \cap \gamma'$ is an element of $\mathcal{N}_{PG}^{\max}(\delta')$.*

(2) *We have $\ell_{exp}(\gamma) = \sum_{i=1}^{\ell} \ell_{exp}(\alpha_i)$ and $\ell_{\mathcal{F}}(\gamma) = \sum_{i=1}^{\ell} \ell_{\mathcal{F}}(\alpha_i)$.*

Proof. (1) Let $\gamma = \gamma_0\gamma_1'\gamma_1 \dots \gamma_k'\gamma_k$ be the exponential decomposition of γ where, for every $i \in \{0, \dots, k\}$, we have $\gamma_i \in \mathcal{N}_{PG}^{\max}(\gamma)$. Let $i \in \{0, \dots, k\}$. Let $j \in \{1, \dots, \ell\}$ be such that α_j contains an initial segment of γ_i. By Proposition 2.5(10), the splitting unit α_j is not contained in a zero stratum. Moreover, by definition of the PG-relative splitting units, if α_j is an edge in an irreducible stratum of positive exponential length, it is not contained in γ_i. Hence, by the description of PG-relative splitting units, the path α_j is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$.

By Proposition 2.5(9), the path γ_i starts with an edge in an EG stratum. Hence there exists a path β_j in $\mathcal{N}_{PG}^{\max}(\alpha_j)$ which contains an initial segment of γ_i. By maximality of γ_i, we see that $\beta_j \subseteq \gamma_i$. Suppose first that $\beta_j = \gamma_i$. Then setting $\delta = \alpha_j$ proves the first assertion. Suppose now that $\beta_j \subsetneq \gamma_i$. By Lemma 3.5(2) applied to $\gamma = \gamma_i^{-1}$ and $\gamma' = \beta_j^{-1}$, the path $[\beta_j^{-1}\gamma_i]$ is a path in $\mathcal{N}_{PG}$. Therefore, by Proposition 2.5(9), the path $[\beta_j^{-1}\gamma_i]$ starts with an edge in an EG stratum. Note that, as α_j is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$, if α_j contains the first edge e of $[\beta_j^{-1}\gamma_i]$, then e would be contained in an EG INP contained in α_j. Since β_j is a maximal subpath of α_j in $\mathcal{N}_{PG}$, we see that $[\beta_j^{-1}\gamma_i]$ is contained in $\gamma'' = \alpha_{j+1} \dots \alpha_\ell$ and is in $\mathcal{N}_{PG}^{\max}(\gamma'')$. We can thus apply the same arguments to the paths $[\beta_j^{-1}\gamma_i]$ and γ''. This concludes the proof of (1).

The proof of (2) follows as the exponential length and the $\mathcal{F}$-length are computed by removing paths in G_{PG} and in $\mathcal{N}_{PG}$. As all subpaths in G_{PG} are contained in a splitting unit of γ and as subpaths in $\mathcal{N}_{PG}$ are obtained by concatenating paths in $\amalg_{j=1}^{\ell}\mathcal{N}_{PG}^{\max}(\alpha_j)$, we see that $\ell_{exp}(\gamma) = \sum_{i=1}^{\ell} \ell_{exp}(\alpha_i)$ and $\ell_{\mathcal{F}}(\gamma) = \sum_{i=1}^{\ell} \ell_{\mathcal{F}}(\alpha_i)$. □

The following property of the exponential length allows us to pass, if needed, to a further iterate of the CT map f.

Lemma 3.22. *For every edge e of $\overline{G - G'_{PG}}$, we have*

$$\lim_{n\to\infty} \ell_{exp}([f^n(e)]) = \infty \text{ and } \lim_{n\to\infty} \ell_{\mathcal{F}}([f^n(e)]) = \infty.$$

Moreover, the sequences $(\ell_{exp}([f^n(e)]))_{n\in\mathbb{N}}$ and $(\ell_{\mathcal{F}}([f^n(e)]))_{n\in\mathbb{N}}$ grow exponentially fast.

Proof. We prove the result concerning ℓ_{exp}, the proof of the result concerning $\ell_{\mathcal{F}}$ follows from the fact that for every reduced edge path γ in G, we have $\ell_{exp}(\gamma) \leqslant \ell_{\mathcal{F}}(\gamma)$. Let e be an edge of $\overline{G - G'_{PG}}$. Since every iterate of e is completely split by Proposition 2.5(6) and since there exists an iterate of e which contains a splitting unit which is an edge in an EG stratum, we may suppose that e is an edge in an EG stratum H_r. Since H_r is an EG stratum, the number of edges in $[f^n(e)] \cap H_r$ grows exponentially fast as n goes to infinity. Therefore the number of splitting units of $[f^n(e)]$ which are edges of H_r grows exponentially fast and $\lim_{n\to\infty} \ell_{exp}([f^n(e)]) = \infty$. □

Lemma 3.23. *There exists $n_0 \in \mathbb{N}^*$ such that for every $k \geqslant n_0$ and every PG-relative completely split edge path γ, we have $\ell_{exp}([f^k(\gamma)]) \geqslant \ell_{exp}(\gamma)$.*

Proof. Let $\gamma = \gamma_1 \dots \gamma_k$ be a PG-relative complete splitting of γ. By Lemma 3.21, it suffices to prove the assertion for every subpath γ_i, with $i \in \{1, \dots, k\}$. Let $i \in \{1, \dots, k\}$. If γ_i is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$, then $\ell_{exp}([f(\gamma_i)]) = \ell_{exp}(\gamma_i) = 0$ by Lemma 3.18. If γ_i is a maximal taken connecting path in a zero stratum, we have $\ell_{exp}(\gamma_i) = 0$. Hence $\ell_{exp}([f(\gamma_i)]) \geqslant \ell_{exp}(\gamma_i)$. In the other cases, γ_i is an edge in an irreducible stratum which is not contained in G_{PG}. By Lemma 3.22, we have $\lim_{n\to\infty} \ell_{exp}([f^n(\gamma_i)]) = \infty$. Hence there exists $n_0 \in \mathbb{N}^*$ such that, for every $k \geqslant n_0$, we have $\ell_{exp}([f^k(\gamma_i)]) \geqslant \ell_{exp}(\gamma_i)$. Since there exist only finitely many edges in irreducible strata, the integer n_0 may be chosen to be independent of γ_i with $i \in \{1, \dots, k\}$. □

Lemma 3.24 in this section shows that the exponential length of a PG-relative completely split edge path encaptures the splitting units which are edges with exponential growth under iteration of f.

Lemma 3.24. *Let γ be a PG-relative completely split edge path, let $\gamma = \gamma_1 \dots \gamma_k$ be a PG-relative complete splitting and let $i \in \{1, \dots, k\}$. Then $\ell_{exp}(\gamma_i) > 0$ if and only if γ_i is an edge in an irreducible stratum not contained in G_{PG}. In particular, the value $\ell_{exp}(\gamma)$ is the number of splitting units which are edges in $\overline{G - G'_{PG}}$.*

Proof. Suppose first that γ_i is either a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ or a maximal taken connecting path in a zero stratum. By Lemma 3.18, we have $\ell_{exp}(\gamma_i) = 0$. Suppose that γ_i is an edge in an irreducible stratum which is not contained in G_{PG}. Since there does not exist an EG INP of length 1, by definition of the exponential length, we have $\ell_{exp}(\gamma_i) = 1 > 0$. This concludes the proof of the first part of the lemma. The computation of $\ell_{exp}(\gamma)$ follows from Lemma 3.21(2). □

3.3. **The space of polynomially growing currents.** In this section, let $\mathcal{F}$ be a free factor system and let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an exponentially growing outer automorphism. Recall the definition of $\mathcal{A}(\phi)$ and $\mathcal{F} \wedge \mathcal{A}(\phi)$ from Section 2.1. We define a subspace of $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$, called the *space of polynomially growing currents*. It consists of the currents whose support is contained in $\partial^2\mathcal{A}(\phi)$ (see Lemma 3.28). In order to define it, we first need to show that the exponential length extends to a continuous function $\Psi\colon \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \to \mathbb{R}$. The space of polynomially growing currents will then be defined as a level set of Ψ.

We first need some preliminary results concerning paths in $\mathcal{N}_{PG}$. For a path $\gamma \in \mathcal{N}_{PG}$, let $\mathcal{N}_{PG}^{++}(\gamma)$ be the subset of $\mathcal{N}_{PG}$ which consists of all paths $\gamma' \in \mathcal{N}_{PG}$ such that $\gamma \subsetneq \gamma'$ and γ' is minimal for this property. Let $\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)$. By Lemma 3.5(3), either γ is properly contained in an INP σ of the complete splitting of γ' or there exist (possibly trivial) paths $\gamma_1, \gamma_2 \in \mathcal{N}_{PG}$ such that $\gamma' = \gamma_1\gamma\gamma_2$. By minimality, either γ_1 or γ_2 is trivial. Moreover, Lemma 3.4 shows that, in this case, splitting units of the complete splittings of γ_1, γ_2 and γ are splitting units of γ'.

Thus the set $\mathcal{N}_{PG}^{++}(\gamma)$ can be partitioned into three disjoint subsets:

$$\mathcal{N}_{PG}^{++}(\gamma) = \mathcal{N}_{PG,INP}^{++}(\gamma) \amalg \mathcal{N}_{PG,left}^{++}(\gamma) \amalg \mathcal{N}_{PG,right}^{++}(\gamma),$$

where $\mathcal{N}_{PG,INP}^{++}(\gamma)$ is the set of paths in $\mathcal{N}_{PG}^{++}(\gamma)$ such that one of their splitting units properly contains γ, $\mathcal{N}_{PG,left}^{++}(\gamma)$ is the set of paths $\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)$ such that $\gamma' = \gamma_1\gamma$ and $\mathcal{N}_{PG,right}^{++}(\gamma)$ is the set of paths $\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)$ such that $\gamma' = \gamma\gamma_2$. One can also define similarly the three sets $\mathcal{N}_{PG,INP,\mathcal{F}}^{++}(\gamma)$, $\mathcal{N}_{PG,left,\mathcal{F}}^{++}(\gamma)$ and $\mathcal{N}_{PG,right,\mathcal{F}}^{++}(\gamma)$ as the restriction to the paths in $\mathcal{N}_{PG,INP}^{++}(\gamma)$, $\mathcal{N}_{PG,left}^{++}(\gamma)$ and $\mathcal{N}_{PG,right}^{++}(\gamma)$ contained in G_p. We emphasize on the fact that a path in $\mathcal{N}_{PG,INP}^{++}(\gamma)$ might contain several occurrences of the path γ. However, a path in $\mathcal{N}_{PG,left}^{++}(\gamma)$ or in $\mathcal{N}_{PG,right}^{++}(\gamma)$ contains a unique occurrence of γ. Indeed, let $\gamma' \in \mathcal{N}_{PG,left}^{++}(\gamma)$ (the proof for $\mathcal{N}_{PG,right}^{++}(\gamma)$ being similar). Then $\gamma' = \gamma_1\gamma_2$ with $\gamma_1 \in \mathcal{N}_{PG}$ and $\gamma_2 = \gamma$. Let γ_3 be an occurrence of γ which contains an edge of γ_1. By Lemma 3.3(2), the path γ_3 cannot intersect γ_2 nontrivially. Hence $\gamma_3 \subseteq \gamma_1$. Hence $\gamma_1 \in \mathcal{N}_{PG}$ and γ_1 contains an occurrence of γ. This contradicts the minimality of γ'.

Lemma 3.25. *Let γ be a path in $\mathcal{N}_{PG}$. Let γ_1, γ_2 be two distinct paths in $\mathcal{N}_{PG}^{++}(\gamma)$. Suppose that there exist three paths μ_1, μ_2, μ_3 such that $\gamma_1 = \mu_1\mu_2$, $\gamma_2 = \mu_2\mu_3$ and γ is contained in μ_2. Then $\gamma_1 \in \mathcal{N}_{PG,left}^{++}(\gamma)$, $\gamma_2 \in \mathcal{N}_{PG,right}^{++}(\gamma)$ and $\mu_2 = \gamma$.*

Proof. By Lemma 3.5(2), the path μ_2 belongs to $\mathcal{N}_{PG}$ and contains γ. Since γ_1 and γ_2 are minimal paths of $\mathcal{N}_{PG}$ for the property of properly containing γ, we have $\mu_2 = \gamma$. Therefore, we see that $\gamma_1 = \mu_1\gamma$ and $\gamma_2 = \gamma\mu_3$. This shows that $\gamma_1 \in \mathcal{N}_{PG,left}^{++}(\gamma)$ and that $\gamma_2 \in \mathcal{N}_{PG,right}^{++}(\gamma)$. □

Lemma 3.25 implies that an occurrence of γ in the intersection of paths in $\mathcal{N}_{PG}^{++}(\gamma)$ is well-controlled. Following Lemma 3.25, we then define $\mathcal{N}_{PG,lr}^{++}(\gamma)$ to be the set of paths of the form $\gamma_1\gamma\gamma_2$, where $\gamma_1\gamma \in \mathcal{N}_{PG,left}^{++}(\gamma)$ and $\gamma\gamma_2 \in \mathcal{N}_{PG,right}^{++}(\gamma)$. We define similarly the set $\mathcal{N}_{PG,lr,\mathcal{F}}^{++}(\gamma)$ to be the set of all paths in $\mathcal{N}_{PG,lr}^{++}(\gamma)$ contained in G_p. As for $\mathcal{N}_{PG,left}^{++}(\gamma)$ and $\mathcal{N}_{PG,right}^{++}(\gamma)$, a path in $\mathcal{N}_{PG,lr}^{++}(\gamma)$ contains a unique occurrence of γ.

Given two paths γ and γ' of G let $N(\gamma', \gamma)$ be the number of occurrences of γ and γ^{-1} in γ'. Let $e \in \vec{E}(\overline{G - G'_{PG}})$. Using the finiteness of $\mathcal{N}_{PG}$ (see Lemma 3.5(1)), let

$$\Psi'_e \colon \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \to \mathbb{R}$$

be the continuous function sending ν to

$$\sum_{\gamma \in \mathcal{N}_{PG}, e \subseteq \gamma} \left(\langle \gamma, \nu \rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \left\langle \gamma', \nu \right\rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG,lr}^{++}(\gamma)} \left\langle \gamma', \nu \right\rangle \right) \ell \left(\gamma \cap \overline{G - G'_{PG}} \right).$$

Let

$$\Psi'_0 \colon \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \to \mathbb{R}$$

be the continuous function

$$\Psi'_0(\nu) = \sum_{e \in \vec{E}(\overline{G - G'_{PG}})} \Psi'_e,$$

and let $\Psi_0\colon \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \to \mathbb{R}$ be the continuous linear function

$$\begin{aligned}\Psi_0(\nu) &= \tfrac{1}{2}\Big(\textstyle\sum_{e \in \vec{E}(\overline{G-G'_{PG}})} \langle e, \nu\rangle - \Psi'_e(\nu)\Big)\\ &= \tfrac{1}{2}\Big(\textstyle\sum_{e \in \vec{E}(\overline{G-G'_{PG}})} \langle e, \nu\rangle\Big) - \tfrac{1}{2}\Psi'_0(\nu).\end{aligned}$$

Definition 3.26. The space of polynomially growing currents, denoted by $K_{PG}(f)$, is the compact subset of $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ consisting of all projective classes of currents $[\nu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ such that:

$$\Psi_0(\nu) = 0.$$

Finally, we define the $\mathcal{F}$-*simplicial length* function $\|.\|_{\mathcal{F}}\colon \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \to \mathbb{R}$ as

$$\begin{aligned}\|\nu\|_{\mathcal{F}} = \frac{1}{2}\Big(&\sum_{e \in \vec{E}(\overline{G-G'_{PG,\mathcal{F}}})} \langle e, \nu\rangle \\ &- \sum_{\gamma \in \mathcal{N}_{PG,\mathcal{F}}, e \subseteq \gamma} \Big(\langle \gamma, \nu\rangle - \sum_{\gamma' \in \mathcal{N}^{++}_{PG,\mathcal{F}}(\gamma)} \langle \gamma', \nu\rangle N(\gamma', \gamma) \\ &+ \sum_{\gamma' \in \mathcal{N}^{++}_{PG,lr,\mathcal{F}}(\gamma)} \langle \gamma', \nu\rangle \Big) \ell\left(\gamma \cap \overline{G - G'_{PG,\mathcal{F}}}\right)\Big).\end{aligned}$$

Lemma 3.27. *Let $w \in F_{\mathtt{n}}$ be a nonperipheral element with conjugacy class $[w]$, associated rational current $\eta_{[w]}$ and associated reduced edge path γ_w in G. Then*

$$\begin{aligned}\Psi_0(\eta_{[w]}) &= \ell_{exp}(\gamma_w);\\ \|\eta_{[w]}\|_{\mathcal{F}} &= \ell_{\mathcal{F}}(\gamma_w).\end{aligned}$$

Therefore $\eta_{[w]} \in K_{PG}(f)$ if and only if

$$\ell_{exp}(\gamma_w) = 0.$$

In particular, there exist a basis $\mathfrak{B}$ of $F_{\mathtt{n}}$ and a constant $C > 0$ such that, for every $\mathcal{F} \wedge \mathcal{A}(\phi)$-nonperipheral element $g \in F_{\mathtt{n}}$, we have $\|\eta_{[g]}\|_{\mathcal{F}} \in \mathbb{N}^*$ and

$$\ell_{\mathfrak{B}}([g]) \geqslant C\|\eta_{[g]}\|_{\mathcal{F}}.$$

Proof. We prove the result for Ψ_0, the proof for $\|\eta_{[w]}\|_{\mathcal{F}}$ being similar. First note that

$$\sum_{e \in \vec{E}(\overline{G-G'_{PG}})} \langle e, \eta_{[w]}\rangle = 2\ell(\gamma_w \cap \overline{G - G'_{PG}}),$$

where the factor 2 follows from the fact that the sum on the left hand side is over oriented edges. Therefore, it remains to prove that

$$\Psi'_0(\eta_{[w]}) = 2 \sum_{\gamma \in \mathcal{N}^{\max}_{PG}(\gamma_w)} \ell\left(\gamma \cap \overline{G - G'_{PG}}\right). \tag{3}$$

Let $\gamma \in \mathcal{N}_{PG}$. Then the value

$$\langle \gamma, \eta_{[w]}\rangle - \sum_{\gamma' \in \mathcal{N}^{++}_{PG}(\gamma)} \langle \gamma', \eta_{[w]}\rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}^{++}_{PG,lr}(\gamma)} \langle \gamma', \eta_{[w]}\rangle$$

measures the number of occurrences of γ or γ^{-1} in γ_w which are not induced by an occurrence of a path $\gamma' \in \mathcal{N}_{PG}$ containing properly γ or γ^{-1} and contained in γ_w. Indeed, an occurrence of γ in a path $\gamma' \in \mathcal{N}_{PG}$ containing properly γ will be counted in $\sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle N(\gamma', \gamma)$. Moreover, if an occurrence of γ is contained in two distinct paths $\gamma_1, \gamma_2 \in \mathcal{N}_{PG}^{++}(\gamma)$, Lemma 3.25 ensures that this occurrence is contained in a path $\gamma_3 \in \mathcal{N}_{PG,lr}^{++}(\gamma)$. Therefore, the value

$$-\sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG,lr}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle$$

measures an occurrence of γ or γ^{-1} in a larger path, and each such occurrence will be counted exactly once. Therefore, the equation below Equation (3) measures the number of occurrences of γ and γ^{-1} in $\mathcal{N}_{PG}^{\max}(\gamma_w)$. Since the sum in the definition of Ψ_0' is over oriented edges, the value $\Psi_0'(\eta_{[w]})$ is exactly twice the number of occurrences of γ and γ^{-1} in $\mathcal{N}_{PG}^{\max}(\gamma_w)$. Thus, Equality (3) holds. The last assertions of Lemma 3.27 then follow by definitions of $K_{PG}(f)$ and of $\ell_{\mathcal{F}}$. □

Note that the proof of Lemma 3.27 also shows that, for every edge

$$e \in \vec{E}(\overline{G - G'_{PG}})$$

and every nonperipheral element $w \in F_{\mathtt{n}}$, the value:

$$\langle e, \eta_{[w]} \rangle - \sum_{\gamma \in \mathcal{N}_{PG}, e \subseteq \gamma} \Big(\langle \gamma, \eta_{[w]} \rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG,lr}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle \Big) N(\gamma, e)$$

measures the number of occurrences of e in γ_w which are not contained in a path of $\mathcal{N}_{PG}^{\max}(\gamma_w)$. Thus, for every nonperipheral element and every edge $e \in \vec{E}(\overline{G - G'_{PG}})$, we have:

$$\langle e, \eta_{[w]} \rangle - \sum_{\gamma \in \mathcal{N}_{PG}, e \subseteq \gamma} \Big(\langle \gamma, \eta_{[w]} \rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG,lr}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle \Big) N(\gamma, e) \geqslant 0.$$

The density of rational currents given by Proposition 2.15 and the continuity of $\langle e, . \rangle$ then show that for every current $\nu \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ and every edge $e \in \vec{E}(\overline{G - G'_{PG}})$, we have :

$$\langle e, \nu \rangle - \sum_{\gamma \in \mathcal{N}_{PG}, e \subseteq \gamma} \Big(\langle \gamma, \nu \rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \nu \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG,lr}^{++}(\gamma)} \langle \gamma', \nu \rangle \Big) N(\gamma, e) \geqslant 0.$$

Lemma 3.28. *Let* $\mathtt{n} \geqslant 3$ *and let* $\mathcal{F}$ *be a free factor system. Let* $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ *be an exponentially growing outer automorphism. Let* $f \colon G \to G$ *be a CT map representing a power of* ϕ*.*

(1) *If* $[\nu] \in K_{PG}(f)$*, then* $\mathrm{Supp}(\nu) \subseteq \partial^2(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \cap \partial^2 \mathcal{A}(\phi)$*. In particular, if* ϕ *is expanding relative to* $\mathcal{F}$*, then* $K_{PG}(f) = \varnothing$*.*

(2) *Conversely, if* $\nu \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ *is such that the support* $\mathrm{Supp}(\nu)$ *of* ν *is contained in* $\partial^2(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \cap \partial^2\mathcal{A}(\phi)$*, then* $[\nu] \in K_{PG}(f)$*. Thus we have*

$$K_{PG}(f) = \{[\mu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}) \mid \mathrm{Supp}(\mu) \subseteq \partial^2(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \cap \partial^2\mathcal{A}(\phi)\}.$$

(3) *If* $\nu \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$*, we have* $\|\nu\|_{\mathcal{F}} = 0$ *if and only if* $\nu = 0$*.*

Proof. The proof of (3) being identical to the proof of (1) and (2) replacing G'_{PG} and $\mathcal{N}_{PG}$ by $G'_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$, we only prove (1) and (2). For the proof of both (1) and (2), let $\mathcal{B}$ be a free basis of $F_{\mathtt{n}}$ and let T be the Cayley graph of $F_{\mathtt{n}}$ associated with $\mathcal{B}$. Let $\mathscr{C}(\mathcal{A}(\phi))$ be the set of elements of $F_{\mathtt{n}}$ associated with $\mathcal{A}(\phi)$ given by Lemma 2.11. Recall that $\mathrm{Cyl}(\mathscr{C}(\mathcal{A}(\phi)))$ is the set of cylinder subsets of the form $C(\gamma)$, where γ is a geodesic edge path in T starting at the base point whose associated element $w \in F_{\mathtt{n}}$ contains a word of $\mathscr{C}(\mathcal{A}(\phi))$ as a subword.

(1) Let $\nu \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ nonzero be such that $\mathrm{Supp}(\nu)$ is not contained in $\partial^2(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \cap \partial^2\mathcal{A}(\phi)$. Then $\mathrm{Supp}(\nu) \cap \partial^2(F_{\mathtt{n}}, \mathcal{A}(\phi)) \neq \varnothing$. Hence the restriction of ν to $\partial^2(F_{\mathtt{n}}, \mathcal{A}(\phi))$ induces a nonzero current $\nu' \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi))$. By Lemma 2.12 applied to $\mathcal{A} = \mathcal{A}(\phi)$ and ν', there exists $C(\gamma) \in \mathscr{C}(\mathcal{A}(\phi))$ such that $\nu(C(\gamma)) > 0$. Let w be the element of $F_{\mathtt{n}}$ associated with γ, and let γ'_w be the reduced circuit in G associated with the conjugacy class of w. Up to taking a larger geodesic edge path $\gamma'' \supseteq \gamma$ in T such that $\nu(C(\gamma'')) > 0$ (which exists by additivity of ν), we may suppose that w is cyclically reduced.

By Lemma 2.11(3), the path γ is not contained in any tree T_A with $[A] \in \mathcal{A}(\phi)$. As w is cyclically reduced, the translation axis in T of w contains γ. This shows that $\{w^{+\infty}, w^{-\infty}\} \notin \partial^2\mathcal{A}(\phi)$ and that w is not contained in any subgroup A with $[A] \in \mathcal{A}(\phi)$. By Proposition 3.14, the circuit γ'_w is not a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Therefore, there exists an edge $e \in \vec{E}(\overline{G - G'_{PG}})$ (contained in γ'_w) such that

$$\langle e, \nu\rangle - \sum_{\gamma \in \mathcal{N}_{PG}, e \subseteq \gamma} \Big(\langle \gamma, \nu\rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \nu\rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG,lr}^{++}(\gamma)} \langle \gamma', \nu\rangle\Big) N(\gamma, e) > 0.$$

Thus, we see that $\Psi_0(\nu) > 0$ and that $[\nu] \notin K_{PG}(f)$. The second part of (1) follows from the fact that if ϕ is expanding relative to $\mathcal{F}$, then $\partial^2\mathcal{A}(\phi) \subseteq \partial^2\mathcal{F}$. This proves (1).

(2) Let $\nu \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ be such that $\mathrm{Supp}(\nu) \subseteq \partial^2(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \cap \partial^2\mathcal{A}(\phi)$. Let e be an edge such that $\langle e, \nu\rangle > 0$. By Lemma 3.5(1), there exists a constant $C_1 > 0$ such that, for every path $\gamma' \in \mathcal{N}_{PG}$, we have $\ell(\gamma') \leqslant C_1$. Recall the definition of the graph G^* and the application $p_{G^*}\colon G^* \to G$. from Lemma 3.12. Let C_2 be the length of a maximal path in a maximal forest of $p_{G^*}(G^*)$. Let $C = \max\{2C_1, C_2\}$.

Claim. Let γ, δ_1 and δ_2 be reduced paths such that $\gamma = \delta_1 e \delta_2$, $\ell(\delta_1), \ell(\delta_2) \geqslant 2C$ and $\langle \gamma, \nu\rangle > 0$. Let $\gamma = \gamma_0\gamma'_1\gamma_1 \dots \gamma'_k\gamma_k$ be the exponential decomposition of γ (where, for every $i \in \{0, \dots, k\}$, the path γ_i is contained in $\mathcal{N}_{PG}$). Either $e \in \vec{E}G'_{PG}$ or e is contained in an EG stratum and there exists $i \in \{0, \dots, k\}$ such that $e \subseteq \gamma_i$.

Proof. Since $\mathrm{Supp}(\nu) \subseteq \partial^2(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \cap \partial^2\mathcal{A}(\phi)$, there exist a subgroup A of $F_{\mathtt{n}}$ such that $[A] \in \mathcal{A}(\phi)$, and two elements a and b of A such that the geodesic path in $\widetilde{G}$ representing $\{a^{+\infty}, b^{+\infty}\} \in \partial^2 A$ contains a lift of γ. If $b = a^{-1}$, then γ is contained in an iterate of a and, by Proposition 3.14, γ is contained in a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$. The claim follows in this case. So we may assume that $b \neq a^{-1}$. Suppose first that the axes $\mathrm{Ax}(a)$ and $\mathrm{Ax}(b)$ of a and b are disjoint. Then there exist $k, \ell \in \mathbb{N}^*$ such that γ is contained in the axis of $a^{-k}b^{\ell}$. Thus, by Proposition 3.14, γ is contained in a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$ and the claim follows in this case.

Suppose now that $\mathrm{Ax}(a) \cap \mathrm{Ax}(b) \neq \varnothing$. Let γ_a' and γ_b' be the reduced circuit in G associated with a and b. Then γ is contained in the union of $\gamma_a' \cup \gamma_b'$. Recall that, by Proposition 3.14, the paths γ_a' and γ_b' are concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$. Hence there exist reduced circuits α and β in G^* and reduced arcs τ, τ_e in G^* such that $p_{G^*}(\alpha) = \gamma_a'$ and $p^*(\beta) = \gamma_b'$ and such that $p_{G^*}(\tau) = \gamma$ and $p_{G^*}(\tau_e) = e$. By the choice of C, and as $\ell(\delta_1), \ell(\delta_2) \geqslant 2C$, one can remove an initial and a terminal segment of τ so that the resulting path τ' is nontrivial, is contained in a subgraph of G^* with no leaf and is such that $\ell(p_{G^*}(\tau')) \geqslant 2C+1$. Thus, there exist subpaths τ_1', τ_1'', τ_2', τ_2'' of τ and a reduced circuit δ of G^* such that:

(i) $\ell(p_{G^*}(\tau_1')), \ell(p_{G^*}(\tau_2')) \geqslant C$,
(ii) $\tau = \tau_1''\tau_1'\tau_e\tau_2'\tau_2''$,
(iii) $\tau' = \tau_1'\tau_e\tau_2' \subseteq \delta$.

By Lemma 3.12(1), the path $p_{G^*}(\delta)$ is a reduced circuit which contains e. Since $\ell(p_{G^*}(\tau_1')), \ell(p_{G^*}(\tau_2')) \geqslant C \geqslant 2C_1$, if $\gamma' \in \mathcal{N}_{PG}^{\max}(p_{G^*}(\delta))$ is such that $e \subseteq \gamma'$, then $\gamma' \subseteq \tau_1' e \tau_2'$. Hence it suffices to prove the claim for $\gamma = p_{G^*}(\delta)$. As δ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$, the claim follows. ☐

Suppose towards a contradiction that there exists an edge $e \in \overline{G - G_{PG}'}$ such that:

$$\langle e, \nu \rangle - \sum_{\gamma \in \mathcal{N}_{PG}, e \subseteq \gamma} \Big(\langle \gamma, \nu \rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \nu \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG,lr}^{++}(\gamma)} \langle \gamma', \nu \rangle \Big) N(\gamma, e) > 0.$$

By additivity of ν, there exists a reduced path γ_0 of length $4C+1$ such that the path γ_0 has a decomposition $\gamma_0 = \gamma_1 e \gamma_2$, where for every $i \in \{1, 2\}$, the path γ_i has length equal to $2C$ and we have $\nu(C(\gamma_0)) > 0$. By the above equation, we can choose γ_0 such that if $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma_0)$, then γ' does not contain e. Thus we have $e \notin G_{PG}'$ and e is not contained in a subpath of $\mathcal{N}_{PG}^{\max}(\gamma_0)$. This contradicts the above claim and this concludes the proof. ☐

Let $\mathcal{F}$ be a free factor system and let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an exponentially growing outer automorphism. Note that, by Lemma 3.28 and since for every $k \in \mathbb{N}^*$, we have $\mathcal{A}(\phi) = \mathcal{A}(\phi^k)$, the space $K_{PG}(f)$ does not depend on the CT map f and does not depend on the chosen power of ϕ. Therefore, we will simply write $K_{PG}(\phi)$ instead. Moreover, since $\mathcal{A}(\phi) = \mathcal{A}(\phi^{-1})$, we see that $K_{PG}(\phi) = K_{PG}(\phi^{-1})$.

For Lemma 3.29, let $C_1 > 0$ be a constant such that for every $\gamma \in \mathcal{N}_{PG}$, we have $\ell(\gamma) \leqslant C_1$. It exists since $\mathcal{N}_{PG}$ is finite by Lemma 3.5(1). Let L be the malnormality constant associated with $\mathcal{A}(\phi)$ as defined above Lemma 2.11 and let

$C_0 = \max\{C_1, L\}$. Let $\mathscr{C}$ be the set of elements of $F_{\mathtt{n}}$ associated with $\mathcal{F} \wedge \mathcal{A}(\phi)$ given above Lemma 2.11. Let $\mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ be the set of reduced paths γ in G such that $C(\gamma) \in \mathrm{Cyl}(\mathscr{C})$, $\ell(\gamma) > C_0$ and γ is not contained in a concatenation of paths in $G_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$.

Lemma 3.29. *Let* $\mathtt{n} \geqslant 3$*, let* $\mathcal{F}$ *be a free factor system of* $F_{\mathtt{n}}$ *and let* $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ *be an exponentially growing outer automorphism. We have*

$$\partial^2(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) = \bigcup_{\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))} C(\gamma).$$

Proof. Let $A_1, \ldots, A_r$ be subgroups of $F_{\mathtt{n}}$ such that $\mathcal{F} \wedge \mathcal{A}(\phi) = \{[A_1], \ldots, [A_r]\}$ and $\mathscr{C} = \mathscr{C}(A_1, \ldots, A_r)$. By Lemma 2.12, we have

$$\partial^2(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) = \bigcup_{C(\gamma) \in \mathrm{Cyl}(\mathscr{C})} C(\gamma).$$

Note that, for every path $\gamma \subseteq G$, we have

$$C(\gamma) = \bigcup_{e \in \vec{E}G, \ell(\gamma e) > \ell(\gamma)} C(\gamma e).$$

Hence we have

$$\partial^2(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) = \bigcup_{C(\gamma) \in \mathrm{Cyl}(\mathscr{C}), \ell(\gamma) > C_0} C(\gamma).$$

So it suffices to prove that we can restrict our considerations to paths γ which are not contained in a concatenation of paths in $G_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$. Let γ be a path such that $C(\gamma) \in \mathrm{Cyl}(\mathscr{C})$ and $\ell(\gamma) > C_0$. By Lemma 2.11(3), the path γ is not contained in any tree $T_{gA_ig^{-1}}$ with $g \in F_{\mathtt{n}}$ and $i \in \{1, \ldots, r\}$. Thus, by Proposition 3.14, there does not exist a circuit in G_p which contains γ and which is a concatenation of paths in $G_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$. Moreover, it is not contained in any path of $\mathcal{N}_{PG}$ since $\ell(\gamma) > C_1$.

Suppose that γ is contained in a concatenation of paths in $G_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$ (which is not a circuit by the above). Recall the definition of G^* and p_{G^*} from Lemma 3.12 and let $G^*_{\mathcal{F}} = p_{G^*}^{-1}(G_p)$. By the above paragraph, either there does not exist an immersed path (not necessarily an edge path) γ^* in $G^*_{\mathcal{F}}$ such that $p_{G^*}(\gamma^*) = \gamma$ or there exists an immersed path γ^* in $G^*_{\mathcal{F}}$ such that $p_{G^*}(\gamma^*) = \gamma$ and γ^* is not contained in a circuit of $G^*_{\mathcal{F}}$ (recall that $G^*_{\mathcal{F}}$ might contain univalent vertices). In the first case, we have $\ell_{\mathcal{F}}(\gamma) > 0$. In the second case, since G^* is finite, by Lemma 3.12, up to considering γ^{-1}, there exists $d \in \mathbb{N}^*$ such that for every path of γ' such that $\gamma\gamma'$ is a reduced path in G and $\ell(\gamma\gamma') = \ell(\gamma) + d$, the path $\gamma\gamma'$ is not the image by p_{G^*} of an immersed path in $G^*_{\mathcal{F}}$. Thus we have $\ell_{\mathcal{F}}(\gamma\gamma') > 0$. Using the fact that

$$C(\gamma) = \bigcup_{e \in \vec{E}G, \ell(\gamma e) > \ell(\gamma)} C(\gamma e),$$

we can replace γ by paths γ'' such that $\gamma \subseteq \gamma''$ and γ'' is not contained in a concatenation of paths in $G_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$. This concludes the proof. □

Let ν be a nonzero current in $\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$. By Lemma 3.28(3), we have $\|\nu\|_{\mathcal{F}} \neq 0$. The following result characterizes limits in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$. The result is due to Kapovich [Kap, Lemma 3.5] for a nonrelative context.

Lemma 3.30. *Let* $\mathtt{n} \geqslant 3$ *and let* $\mathcal{F}$ *be a free factor system of* $F_{\mathtt{n}}$. *Let* $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ *be an exponentially growing outer automorphism. Let* $([\mu_n])_{n\in\mathbb{N}}$ *be a sequence of projective relative currents in* $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ *and let* $[\mu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$. *Let* G *be a graph whose fundamental group is isomorphic to* $F_{\mathtt{n}}$ *and such that there exists a subgraph* G_p *of* G *such that* $\mathcal{F}(G_p) = \mathcal{F}$. *Then* $\lim_{n\to\infty}[\mu_n] = [\mu]$ *if and only if, for every reduced edge path* $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$, *we have*

$$\lim_{n\to\infty} \frac{\langle \gamma, \mu_n \rangle}{\|\mu_n\|_{\mathcal{F}}} = \frac{\langle \gamma, \mu \rangle}{\|\mu\|_{\mathcal{F}}}. \tag{4}$$

Proof. Suppose first that $\lim_{n\to\infty}[\mu_n] = [\mu]$. Thus there exists a sequence $(\lambda_n)_{n\in\mathbb{N}^*}$ of positive real numbers such that $\lim_{n\to\infty} \lambda_n\mu_n = \mu$. By continuity of $\|.\|_{\mathcal{F}}$, we have $\lim_{n\to\infty}\|\lambda_n\mu_n\|_{\mathcal{F}} = \|\mu\|_{\mathcal{F}}$. By linearity of $\|.\|_{\mathcal{F}}$ and $\langle .,. \rangle$ in the second variable, for every reduced edge path $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$, we have

$$\lim_{n\to\infty} \frac{\langle \gamma, \lambda_n\mu_n \rangle}{\|\lambda_n\mu_n\|_{\mathcal{F}}} = \lim_{n\to\infty} \frac{\langle \gamma, \mu_n \rangle}{\|\mu_n\|_{\mathcal{F}}} = \frac{\langle \gamma, \mu \rangle}{\|\mu\|_{\mathcal{F}}}.$$

Suppose now that for every reduced edge path $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$, Equation (4) holds. By Lemma 3.29, for every Borel subset B of $\partial^2(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ such that $\mu(\partial B) = 0$ (where ∂B is the topological boundary of B), we have

$$\lim_{n\to\infty} \frac{\mu_n(B)}{\|\mu_n\|_{\mathcal{F}}} = \frac{\mu(B)}{\|\mu\|_{\mathcal{F}}}.$$

Hence we have $\lim_{n\to\infty}[\mu_n] = [\mu]$. □

4. Stable and unstable currents for relative atoroidal outer automorphisms

Let $\mathtt{n} \geqslant 3$ and let $\mathcal{F}$ be a free factor system of $F_{\mathtt{n}}$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an atoroidal outer automorphism relative to $\mathcal{F}$. In this section, under additional hypotheses on ϕ, we construct two ϕ-invariant convex subsets of $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F})$. We will then show in the following section that, with respect to these convex subsets, the outer automorphism ϕ acts with *generalized north-south dynamics.*

In order to define the extremal points of these simplices, we need some results regarding substitution dynamics.

4.1. Substitution dynamics.

Let A be a finite set with cardinality at least equal to 2. Let ζ be a *substitution on* A, that is, a map from A to the set of nonempty finite words on A. The substitution ζ induces a map on the set of all finite words on A by concatenation, which we still denote by ζ. We can therefore iterate the substitution ζ. For a word w on A, we will denote by $|w|$ the length of w on the alphabet A.

To the substitution ζ one can associate its *transition matrix* M, which is a square matrix whose rows and columns are indexed by letters in A and, for all $a, b \in A$, the value $M(a, b)$ is the number of occurrences of a in $\zeta(b)$. Likewise, for $n \geqslant 1$, the matrix M^n is the transition matrix for ζ^n. We say that a substitution ζ is *irreducible* if its transition matrix is irreducible, and that the substitution is *primitive* if its transition matrix is.

Let $\ell \in \mathbb{N}^*$ and let A_ℓ be the set of words on A of length ℓ. As defined in [Que, Section 5.4.1], the substitution ζ induces a substitution ζ_ℓ on A_ℓ as follows. Consider $w = x_1 \dots x_\ell \in A_\ell$. Then $\zeta_\ell(w) = w_1 w_2 \dots w_{|\zeta(x_1)|}$, where, for every $i \in \{1, \dots, |\zeta(x_1)|\}$, the word w_i is the subword of $\zeta(w)$ of length ℓ starting at the i^{th} position of $\zeta(x_1)$. Therefore, ζ_ℓ is the concatenation of the $|\zeta(x_1)|$ first subwords of $\zeta(w)$ of length ℓ. Note that the number of $i \in \{1, \dots, |\zeta(x_1)|\}$ such that w_i is not contained in $\zeta(x_1)$ is bounded by $\ell - 1$. Let $|\cdot|_\ell$ be the length of words on A_ℓ. Then $|\zeta_\ell(w)|_\ell = |\zeta(x_1)|$. Denote by M_ℓ the transition matrix of ζ_ℓ. Note that, for every $n, \ell \geqslant 1$, we have $(\zeta^n)_\ell = (\zeta_\ell)^n$ as applications on the set of words on A_ℓ and thus $(M^n)_\ell = (M_\ell)^n$.

Consider now a partition of the alphabet $A = \coprod_{i=0}^k B_i$. Suppose that the transition matrix associated with the substitution ζ is lower block triangular with respect to this partition. Therefore, for every $i \in \{0, \dots, k\}$, for every $x \in B_i$ and for every $j < i$, the word $\zeta(x)$ does not contain letters in B_j. In the remainder of the article, for every $i \in \{0, \dots, k\}$ the diagonal block in M corresponding to the block B_i will be denoted by M_{B_i}.

The partition of A induces a partition of A_ℓ as follows. For every $i \in \{0, \dots, k\}$, let $\widetilde{B}_i \subseteq A_\ell$ be the set of all words on A of length ℓ which start with a letter in B_i and which, for every $j < i$, do not contain a letter in B_j. Let $\overline{B}_i$ be the set of all words w on A of length ℓ which start with a letter in B_i and such that there exists $j < i$ such that w contains a letter in B_j (note that $\overline{B}_0$ is empty). Then $\widetilde{B}_i \cup \overline{B}_i$ is the set of all words on A of length ℓ which start with a letter in B_i. The hypothesis on the substitution ζ implies that the transition matrix M_ℓ is lower block triangular with respect to the partition

$$\widetilde{B}_0 \amalg \overline{B}_1 \amalg \widetilde{B}_1 \amalg \dots \amalg \overline{B}_k \amalg \widetilde{B}_k$$

of A_ℓ. As before, for every $i \in \{0, \dots, k\}$, we will denote by $M_{\ell, \overline{B}_i}$ the diagonal block in M_ℓ corresponding to $\overline{B}_i$ and by $M_{\ell, \widetilde{B}_i}$ the diagonal block in M_ℓ corresponding to $\widetilde{B}_i$.

Lemma 4.1 ([Gup1, Lemma 8.8])**.** *Let A be a finite alphabet equipped with a partition $A = \amalg_{i=0}^k B_i$. Let ζ be a substitution and let M be its transition matrix. Let $\ell \in \mathbb{N}^*$.*

(1) *The eigenvalues of $M_{\ell, \widetilde{B}_i}$ are those of M_{B_i} with possibly additional eigenvalues of absolute value at most equal to* 1.
(2) *The eigenvalues of $M_{\ell, \overline{B}_i}$ have absolute value at most equal to* 1.

Fix an integer $p \in \{0, \dots, k\}$. For every $i \geqslant p$, let $\overline{B}_i^{(p)}$ be the subset of $\overline{B}_i$ consisting of all words w of length ℓ which start with a letter in B_i and such that there exists $j < p$ such that w contains a letter in B_j. Then, for every $i \geqslant p$, the block $M_{\ell, \overline{B}_i}$ decomposes into a lower triangular block matrix where the columns and rows corresponding to $\overline{B}_i^{(p)}$ are on the top left. Let $M_{\ell, \overline{B}_i^{(p)}}$ be the corresponding block matrix. By Lemma 4.1(2), the eigenvalues of $M_{\ell, \overline{B}_i^{(p)}}$ have absolute value at most 1. Moreover, for every $i, j \geqslant p$, for every word w contained in $\widetilde{B}_j \cup \overline{B}_j - \overline{B}_j^{(p)}$, the word $\zeta_\ell(w)$ considered as a word on A_ℓ does not contain any word of $\overline{B}_i^{(p)}$. Let $M_\ell(p)$ be the matrix obtained from M_ℓ by deleting, for every $i \geqslant p$, all rows and columns corresponding to elements in $\widetilde{B}_i$, and all rows and columns corresponding to elements of $\overline{B}_i$ which do not belong to $\overline{B}_i^{(p)}$. Note that, by Lemma 4.1(1), the eigenvalues of $M_\ell(p)$ are those of every block M_{B_j} with $j < p$ with possibly additional eigenvalues of absolute value at most 1.

We can now prove a result concerning the number of occurrences of words in iterates of a letter. For words w, v on A, we denote by (w, v) the number of occurrences of w in v, so that $M = ((a, \zeta(b))_{a,b\in A}$. For a word w on A, we denote by $||w||_{(p)}$ the number of letters in w which are contained in some B_j for $j < p$.

Proposition 4.2. *Let A be an alphabet equipped with a partition $A = \amalg_{i=0}^k B_i$. Let ζ be a substitution on A and let M be its transition matrix. Suppose that M is lower triangular by block with respect to the partition of A. Let $p \in \mathbb{N}^*$. Let $a \in \bigcup_{t<p} B_t$ be such that $\zeta(a)$ starts with a. Suppose that there exists $j < p$ such that M_{B_j} is a primitive block whose Perron-Frobenius eigenvalue is greater than 1 and such that there exists $n_j \geqslant 1$ such that $\zeta^{n_j}(a)$ contains a letter of B_j. Let w be a word such that w contains a letter in B_k for some $k < p$. Then*

$$\lim_{n\to\infty} \frac{(w, \zeta^n(a))}{||\zeta^n(a)||_{(p)}}$$

exists and is finite. Furthermore there exists a word w containing a letter in some B_k with $k < p$ such that this limit is positive.

Proof. The proof follows [Gup1, Lemma 8.9] (see also [LU1] for similar statements). First, up to replacing A by the smallest ζ-invariant subalphabet of A containing a (which still satisfies the hypotheses of Proposition 4.2), we may suppose that, for every letter $x \in A$, there exists $n_x \geqslant 1$ such that $\zeta^{n_x}(a)$ contains the letter x. Let α be a word on A with length $\ell \geqslant 1$ that starts with a. Note that, since $a \in \cup_{t<p} B_t$, the word α defines a column and a row in $M_\ell(p)$. Recall that for every n the number of occurrences of a word w in $\zeta^n(a)$ differs from the number of occurrences of the letter $w \in A_\ell$ in $\zeta_\ell^n(\alpha)$ by at most $\ell - 1$. Moreover, we have $(w, \zeta_\ell^n(\alpha)) = M_\ell^n(p)(w, \alpha)$.

Let S be the set of all $s < p$ such that M_{B_s} is a primitive block with associated Perron-Frobenius eigenvalue greater than 1. By assumption, the set S is a nonempty finite set. Let S' be the subset of S consisting of all such B_s such that the associated Perron-Frobenius eigenvalue is maximal. Call this eigenvalue λ. By Lemma 4.1, the eigenvalue λ is also the maximal eigenvalue of the matrix $M_\ell(p)$. Let d_λ be the size of the maximal Jordan block of $M_\ell(p)$ associated with λ. Then the growth under iterates of the maximal Jordan block of $\frac{M_\ell(p)}{\lambda}$ is polynomial of degree d_λ. Therefore, we have

$$\lim_{n\to\infty} \frac{(w, \zeta^n(a))}{\lambda^n n^{d_\lambda}} = \lim_{n\to\infty} \frac{(w, \zeta_\ell^n(\alpha))}{\lambda^n n^{d_\lambda}} = \lim_{n\to\infty} \frac{M_\ell^n(p)(w, \alpha)}{\lambda^n n^{d_\lambda}} = d_{w,a},$$

where $d_{w,a}$ is a real number. Moreover, the limit does not depend on the choice of α since, for any n, and for any two columns of $M_\ell^n(p)$ corresponding to words starting with the same letter, the sum of the values of each column differ by at most $\ell - 1$ (see [Gup1, Lemma 8.6]). Moreover, there exists a word w such that the limit is positive since we quotiented by the growth of the iterates of the Jordan block with maximal eigenvalue.

Let $\|\cdot\|$ be the L_1-norm on $\mathbb{R}^{|A_\ell|}$. By [LU1, Remark 4.1], since $\lim_{n\to\infty}\frac{M_\ell^n(p)(w,\alpha)}{\lambda^n n^{d_\lambda}}$ exists, so does

$$\lim_{n\to\infty}\frac{M_\ell^n(p)(w,\alpha)}{\|M_\ell^n(p)(\alpha)\|},$$

where $\|M_\ell^n(p)(\alpha)\|$ is the norm of the column of $M_\ell^n(p)$ corresponding to α.

Claim. Suppose that there exists $C \geqslant 1$ such that for every $n \in \mathbb{N}$, we have

$$||\zeta^n(a)||_{(p)} \leqslant ||M_\ell^n(p)(\alpha)|| \leqslant C||\zeta^n(a)||_{(p)}.$$

Then

$$\lim_{n\to\infty}\frac{(w,\zeta^n(a))}{||\zeta^n(a)||_{(p)}}$$

exists for all words w on A and is positive for some word w.

Proof. Recall that two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ with values in $\mathbb{R}$ are equivalent if there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ tending to zero such that $u_n = (1+\epsilon_n)v_n$. Recall that there exists $C' > 0$ such that the sequence $(\|M_\ell^n(p)(\alpha)\|)_{n\in\mathbb{N}}$ is equivalent to $(C'\lambda^n n^{d_\lambda})_{n\in\mathbb{N}}$. Recall also that for every n, the value of $\|\zeta^n(a)\|_{(p)}$ is the norm of $M^n(p)(v_a)$, where v_a is the vector whose coordinate is 1 on the coordinate associated with a and 0 otherwise. Hence, since the matrix $M^n(p)$ is nonnegative and not the zero matrix, there exist $C_a, \lambda_a \in \mathbb{R}_+^*$ and $d_a \in \mathbb{N}$ such that the sequence $(\|\zeta^n(a)\|_{(p)})_{n\in\mathbb{N}}$ is equivalent to $(C_a\lambda_a^n n^{d_a})_{n\in\mathbb{N}}$. Thus, by the assumption of the claim, since the limit

$$\lim_{n\to\infty}\frac{M_\ell^n(p)(w,\alpha)}{||M_\ell^n(p)(\alpha)||}$$

exists, and is not equal to zero for some w, the same is true for

$$\lim_{n\to\infty}\frac{(w,\zeta^n(a))}{||\zeta^n(a)||_{(p)}}.$$

This proves the claim. □

Therefore, in order to conclude the proof of the proposition, it remains to prove that the hypothesis of the claim is true in our context. Let $\zeta^n(a) = x_1\dots x_{|\zeta^n(a)|}$ and let

$$\zeta_\ell^n(\alpha) = w_1\dots w_{|\zeta^n(a)|}.$$

Let $X^n(a)$ be the list $x_1,\dots,x_{|\zeta^n(a)|}$ and let $X_{<p}^n(a)$ be the sublist of $X^n(a)$ consisting of all letters in $\cup_{i=1}^{p-1}B_i$. Let $X^{(\ell,n)}(\alpha)$ be the list $w_1,\dots,w_{|\zeta^n(a)|}$ and let $X_{<p}^{(\ell,n)}(\alpha)$ be the sublist of $X^{(\ell,n)}(\alpha)$ which consists of all elements of $X^{(\ell,n)}(\alpha)$ that do not belong to $\cup_{i\leqslant p}\widetilde{B}_i \cup \overline{B}_i - \overline{B}_i^{(p)}$. Note that $|X_{<p}^{(\ell,n)}(\alpha)| = ||M_\ell^n(p)(\alpha)||$ and that $|X_{<p}^n(a)| = ||\zeta^n(a)||_{(p)}$. The fact that $||\zeta^n(a)||_{(p)} \leqslant ||M_\ell^n(p)(\alpha)||$ follows from the fact that we have an injection from $X_{<p}^n(a)$ to $X_{<p}^{(\ell,n)}(\alpha)$ by sending the letter $x_i \in X_{<p}^n(a)$ to $w_i \in X_{<p}^{(\ell,n)}(\alpha)$. Since every word of length ℓ contained in $X_{<p}^{(\ell,n)}(\alpha)$ contains a letter in $X_{<p}^n(a)$, we have an application from $X_{<p}^{(\ell,n)}(\alpha)$ to $X_{<p}^n(a)$ defined as follows. Let $w \in X_{<p}^{(\ell,n)}(\alpha)$ and let $j_w \in \{1,\dots,|\zeta^n(a)|\}$ be the minimal integer such that $x_{j_w} \in X_{<p}^n(a)$ and x_{j_w} is a letter in w. Then the application sends w to x_{j_w}. By construction, the cardinal of the preimage of any $x \in X_{<p}^n(a)$ is at most equal to ℓ. Therefore, we have

$$||\zeta^n(a)||_{(p)} \leqslant ||M_\ell^n(p)(\alpha)|| \leqslant \ell||\zeta^n(a)||_{(p)}.$$

This concludes the proof. □

4.2. **Construction of the attractive and repulsive currents for relative almost atoroidal automorphisms.** Let $\mathtt{n} \geqslant 3$ and let $\mathcal{F} = \{[A_1], \ldots, [A_k]\}$ be a free factor system of $F_{\mathtt{n}}$. We first define a class of outer automorphisms of $F_{\mathtt{n}}$ which we will study in the rest of the article. If $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ and ϕ preserves the conjugacy class of every A_i with $i \in \{1, \ldots, k\}$, we denote by $\phi|_{\mathcal{F}}$ the element $([\phi_1|_{A_1}], \ldots, [\phi_k|_{A_k}])$, where, for every $i \in \{1, \ldots, k\}$, the element ϕ_i is a representative of ϕ such that $\phi_i(A_i) = A_i$ and $[\phi_i|_{\mathcal{A}_i}]$ is an element of $\mathrm{Out}(A_i)$. Note that the outer class of $\phi_i|_{A_i}$ in $\mathrm{Out}(A_i)$ does not depend on the choice of ϕ_i.

Definition 4.3. Let $\mathtt{n} \geqslant 3$ and let $\mathcal{F} = \{[A_1], \ldots, [A_k]\}$ be a free factor system of $F_{\mathtt{n}}$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be exponentially growing. The outer automorphism ϕ is *almost atoroidal relative to* $\mathcal{F}$ if ϕ preserves the conjugacy class of every A_i with $i \in \{1, \ldots, k\}$ and if ϕ preserves a sequence of free factor systems $\mathcal{F} \leqslant \mathcal{F}_1 \leqslant \{F_{\mathtt{n}}\}$ with $\mathcal{F}_1 = \{[B_1], \ldots, [B_\ell]\}$ such that:

(a) $\mathcal{F}_1 \leqslant \{F_{\mathtt{n}}\}$ is sporadic,
(b) for every $i \in \{1, \ldots, \ell\}$, ϕ preserves the conjugacy class of B_i, the element $[\phi_i|_{B_i}]$ is an expanding outer automorphism relative to $\mathcal{F} \wedge \{[B_i]\}$ and ϕ is not expanding relative to $\mathcal{F}$ ($\mathcal{F}$ might be equal to $\mathcal{F}_1$).

The main example of an almost atoroidal automorphism is the following. Suppose that $\mathcal{F}_1 = [A]$ and let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be such that $\phi([A]) = [A]$. Then ϕ is almost atoroidal if $\phi|_{[A]}$ is expanding relative to $\mathcal{F}$. Almost atoroidality allows us to deal with sporadic extensions.

Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an atoroidal or an almost atoroidal outer automorphism relative to $\mathcal{F}$. In this section, we construct a nontrivial convex compact subset in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ associated with ϕ. We follow the construction of [Uya2] in the context of atoroidal automorphisms.

By Theorem 2.10, there exists $M \geqslant 1$ such that ϕ^M is represented by a CT map $f\colon G \to G$ with filtration $\varnothing = G_0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_k = G$ and such that there exists $p \in \{1, \ldots, k\}$ such that $\mathcal{F}(G_p) = \mathcal{F}$.

For a splitting unit σ in G, we say that σ is *expanding* if $\lim_{m\to\infty} \ell_{exp}([f^m(\sigma)]) = +\infty$. Note that, by Lemma 3.24, this is equivalent to saying that there exists $N \in \mathbb{N}^*$ such that $[f^N(\sigma)]$ contains a splitting unit which is an edge in an EG stratum. Moreover, a splitting unit σ which is an expanding splitting unit is either an edge in $\overline{G - G'_{PG}}$ or a maximal taken connecting path in a zero stratum such that a reduced iterate of σ contains an edge in $\overline{G - G'_{PG}}$ as a splitting unit. In particular, there are finitely many expanding splitting units by Proposition 2.5(3).

Let γ and γ' be two finite reduced subpaths of G. We denote by $\#(\gamma, \gamma')$ the number of occurrences of γ in γ' and by $\langle \gamma, \gamma' \rangle$ the sum

$$\langle \gamma, \gamma' \rangle = \#(\gamma, \gamma') + \#(\gamma^{-1}, \gamma'). \tag{5}$$

Proposition 4.4 shows the existence of relative currents associated with relative atoroidal outer automorphisms. Once we have constructed these currents for relative atoroidal outer automorphisms, we will also be able to construct attractive and repulsive simplices for every almost atoroidal outer automorphism relative to $\mathcal{F}$. Proposition 4.4 and its proof are inspired by the same result in the absolute context due to Uyanik [Uya2, Proposition 3.3] and by the proof due to Gupta in the relative fully irreducible context [Gup1, Proposition 8.13]. Recall the definition of $\mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ before Lemma 3.29 and $\mathscr{C}$ before Lemma 2.11.

Proposition 4.4. *Let $\mathtt{n} \geqslant 3$ and let $\mathcal{F}$ be a free factor system of $F_{\mathtt{n}}$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an atoroidal outer automorphism relative to $\mathcal{F}$. Let $f\colon G \to G$ be a CT map that represents a power of ϕ with filtration $\varnothing = G_0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_k = G$ and such that there exists $p \in \{1, \ldots, k\}$ such that $\mathcal{F}(G_p) = \mathcal{F}$. Let $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ and let σ be an expanding splitting unit with fixed initial direction.*

(1) *The limit*
$$\sigma_\gamma = \lim_{m \to \infty} \frac{\langle \gamma, [f^m(\sigma)] \rangle}{\ell_{\mathcal{F}}([f^m(\sigma)])}$$
exists and is finite.

(2) *There exists a unique current $\eta_\sigma \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ such that, for every finite reduced edge path $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$, we have:*
$$\eta_\sigma(C(\gamma)) = \sigma_\gamma.$$

Proof. (1) We may suppose that γ occurs in a reduced iterate of σ as otherwise $\sigma_\gamma = 0$. Note that, since the initial direction of σ is fixed, the splitting unit σ is not contained in a zero stratum. Thus, we see that σ is an expanding splitting unit which is an edge in an irreducible stratum. Let r be the height of σ.

In order to prove the proposition in this case, we want to apply Proposition 4.2 to the CT map f seen as a substitution on the set of splitting units contained in iterates of σ. However, the set of splitting units might be infinite since exceptional paths and INPs may have arbitrarily large lengths.

Instead, we construct a finite alphabet A_γ *depending on* γ. The alphabet is constructed as follows by associating a letter to every splitting unit occurring in a reduced iterate of σ. However some letters will correspond to infinitely many splitting units.

(a) We add one letter for each of the finitely many edges in irreducible strata that are contained in a reduced iterate of σ.

(b) We add one letter for each reduced maximal taken connecting path in a zero stratum contained in a reduced iterate of σ.

(c) We add one letter for each INP contained in a reduced iterate of σ and such that the stratum of maximal height it intersects is an EG stratum.

(d) Let δ be an INP such that the stratum of maximal height it intersects is an NEG stratum and such that it appears in a reduced iterate of σ. By Proposition 2.5(11), there exist an edge e, an integer $s \in \mathbb{Z}$ and a closed Nielsen path w such that $\delta = ew^s e^{-1}$. Note that γ is not contained in w^s since $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ and w^s is a concatenation of paths in $G_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$ by Lemma 3.8 and the fact that ϕ is atoroidal relative to $\mathcal{F}$. Hence if γ is contained in δ, it is either an initial or a terminal segment of δ. Let M_1 be the maximal integer $|d|$ such that γ contains an INP of the form $ew^d e^{-1}$. Let M_2 be the minimal integer $|d|$ such that $\gamma \cap (ew^d e^{-1})$ is either a proper initial or a proper terminal segment of $ew^d e^{-1}$. Let M_3 be the maximal integer $|d|$ such that $ew^d e^{-1}$ is contained in $[f(\sigma')]$ with σ' a splitting unit which is either an edge in an irreducible stratum or a maximal taken connecting path in a zero stratum. Let $M = \max\{M_1, M_2, M_3\}$. We add one letter for each $ew^d e^{-1}$ with $|d| \leqslant M+1$. We add exactly one letter representing every $ew^d e^{-1}$ with $|d| > M+1$.

(e) Let δ be an exceptional path appearing in a reduced iterate of σ. There exist edges e_1, e_2, a nonzero integer s and a closed Nielsen path w such that $\delta = e_1 w^s e_2^{-1}$. Note that γ is not contained in w^s since $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ and w^s is a concatenation of paths in $G_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$ by Lemma 3.8 and the fact that ϕ is atoroidal relative to $\mathcal{F}$. Let M_4 be the maximal integer $|d|$ such that γ contains an exceptional path of the form $e_1 w^d e_2^{-1}$. Let M_5 be the minimal integer $|d|$ such that $\gamma \cap e_1 w^d e_2^{-1}$ is either a proper initial or terminal segment of $e_1 w^d e_2^{-1}$. Let M_6 be the maximal integer $|d|$ such that $e_1 w^d e_2^{-1}$ is contained in $[f(\sigma')]$ with σ' a splitting unit which is either an edge in an irreducible stratum or a maximal taken connecting path in a zero stratum. Let $M' = \max\{M_4, M_5, M_6\}$. We add one letter for each $e_1 w^d e_2^{-1}$ with $|d| \leqslant M'+1$. We add one letter representing every $e_1 w^d e_2^{-1}$ with $|d| > M'+1$.

We claim that the alphabet A_γ is finite. Indeed, since the graph G is finite, so is the number of letters in the first category. By Proposition 2.5(3), the zero strata of G_{r-1} are exactly the contractible components of G_{r-1}. Hence the number of letters in the second category is finite. The number of letters in the third category is finite by Proposition 2.5(9). The remaining letters of A_γ are finite by definition.

Let ζ be the following substitution on A_γ. If $a \in A_\gamma$ represents a unique path in G, we set $\zeta(a) = [f(a)]$. If $a \in A_\gamma$ represents several paths in G, we set $\zeta(a) = a$.

We claim that ζ is a well-defined substitution. Indeed, by Proposition 2.5(6), if a is a letter in A_γ which represents a unique path in G, then $[f(a)]$ is completely split and every splitting unit in $[f(a)]$ is represented by a unique letter by the construction of letters in the fourth and fifth category. Moreover, if $a \in A_\gamma$ represents several paths, then the definition of ζ does not depend on the choice of a representative of a. Hence ζ is a well-defined substitution.

We claim that if $a \in A_\gamma$ represents several paths in G, then, for every representative α of a, the path $[f(\alpha)]$ is represented by a. Indeed, the claim is immediate when a represents several INPs, so we focus on the case where a represents several exceptional paths.

Let e_1, e_2 be edges in G, let w be a closed Nielsen path in G and let $d \in \mathbb{Z}$ be such that $e_1 w^d e_2^{-1}$ is represented by the letter a. There exist a splitting unit σ' of a reduced iterate of σ by $[f]$, an integer $N \in \mathbb{N}^*$ and an integer $d_1 \in \mathbb{Z}$ such that $e_1 w^{d_1} e_2^{-1}$ is a subpath of $[f^N(\sigma')]$. Thus, using the constants given in (e), we have $|d_1| \leqslant M_6 \leqslant M$. By the construction of the alphabet A_γ, there exists a letter a' in A_γ corresponding to the path $e_1 w^{d_1} e_2^{-1}$ and a' represents a unique path. For every $n \in \mathbb{N}$, let $d_n \in \mathbb{Z}$ be such that $[f^n(e_1 w^{d_1} e_2^{-1})] = e_1 w^{d_n} e_2^{-1}$. Then the sequence $(d_n)_{n \in \mathbb{N}}$ is monotonic. Let m_0 be the minimal integer such that the path $e_1 w^{d_{m_0}} e_2^{-1}$ is represented by a. Note that $m_0 > 1$ as a' represents a unique path. By monotonicity, $d_{m_0} \neq d_1$. Thus, if $d_{m_0} > d_1$, then for every $m \geqslant m_0$, we have $d_m \geqslant d_{m_0}$ and if $d_{m_0} < d_1$, then for every $m \geqslant m_0$, we have $d_m \leqslant d_{m_0}$. Hence for every $m \geqslant m_0$, the path $e_1 w^{d_{m+1}} e_2^{-1}$ is represented by a. This shows that if $\alpha \in a$ then $[f(\alpha)] \in a$. This concludes the proof of the claim. Hence ζ only depends on the function $[f(.)]$.

By reordering columns and rows, we may suppose that if M is the matrix associated with ζ, then columns and rows of M with index greater than p are precisely the letters in A_γ representing splitting units which are concatenations of paths in $G_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$. By Lemma 3.10, iterates by ζ of letters of A_γ representing concatenations of paths in $G_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$ are words on A_γ whose letters represent concatenations of paths in $G_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$. Thus, the matrix M is a lower block triangular matrix, where every block of index at most p corresponds to either edges in a common stratum or the 0 matrix when the associated letter is a maximal taken connecting path in a zero stratum.

Since σ is expanding, it has a reduced iterate which contains splitting units which are edges in EG strata. Hence if a_σ is the letter in A_γ corresponding to σ, the iterates $\zeta^n(a_\sigma)$ contain letters of A_γ in a Perron-Frobenius block with eigenvalue greater than 1. Since the initial direction of σ is fixed, by Proposition 4.2, for every word w in the alphabet A_γ, the limit

$$\lim_{m\to\infty} \frac{(w, [\zeta^m(\sigma)])}{||\zeta^m(\sigma)||_{(p)}}$$

exists and is finite. Hence the limit

$$\lim_{m\to\infty} \frac{\langle w, [\zeta^m(\sigma)]\rangle}{||\zeta^m(\sigma)||_{(p)}}$$

exists and is finite.

Claim. There exists a matrix M' obtained from M by multiplying rows and columns by positive scalars and such that, for every $m \in \mathbb{N}^*$, we have $\ell_{\mathcal{F}}([f^m(\sigma)]) = \|M'^m(\sigma)\|_{(p)}$.

Proof. Remark that if $e_1 w^s e_2^{-1}$ is an exceptional path, and if $e_1 w^d e_2^{-1}$ is an exceptional path with distinct width, then their $\mathcal{F}$-lengths are equal and at most equal to 2. Indeed, since ϕ is an atoroidal outer automorphism relative to $\mathcal{F}$, every closed Nielsen path of G is contained in G_p. Since w is a closed Nielsen path, we see that w is a concatenation of paths in $G_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$ by Lemma 3.7. Hence we have

$$\ell_{\mathcal{F}}(e_1 w^s e_2^{-1}) = \ell_{\mathcal{F}}(e_1) + \ell_{\mathcal{F}}(e_2) \leqslant 2.$$

Similarly, if $ew^s e^{-1}$ and $ew^d e^{-1}$ are INP intersecting the same maximal NEG stratum, then their $\mathcal{F}$-lengths are equal and at most equal to 2. Let M' be the matrix obtained from M by multiplying every row corresponding to either an exceptional path not contained in G_p, an INP not contained in G_p, a collection of exceptional paths not contained in G_p, a collection of INPs not contained in G_p or a maximal taken connecting path not contained in G_p, by the corresponding $\mathcal{F}$-length. Note that, by the above remarks, this does not depend on the choice of a representative when the letter corresponds to a collection of paths. Then for every $m \in \mathbb{N}^*$, the value $\|M'^m(\sigma)\|_{(p)}$ corresponds to the sum of the $\mathcal{F}$-length of every splitting unit in $[f^m(\sigma)]$ not contained in G_p. By Lemma 3.20, complete splittings are PG-relative complete splittings. By Lemma 3.21(2), we have $\ell_{\mathcal{F}}([f^m(\sigma)]) = \|M'^m(\sigma)\|_{(p)}$. This proves the claim. □

By the claim, we see that for every $m \in \mathbb{N}^*$, there exists a constant K such that we have

$$\frac{1}{K}||\zeta^m(\sigma)||_{(p)} \leqslant \ell_{\mathcal{F}}([f^m(\sigma)]) \leqslant K||\zeta^m(\sigma)||_{(p)}.$$

Using the claim in the proof of Proposition 4.2 (replacing $\|M_\ell^n(p)(\alpha)\|$ by $\ell_{\mathcal{F}}([f^n(\sigma)])$ which is possible since $\ell_{\mathcal{F}}([f^n(\sigma)])$ is the norm of a matrix by the claim), the limit

$$\lim_{m\to\infty} \frac{\langle w, [f^m(\sigma)]\rangle}{\ell_{\mathcal{F}}([f^m(\sigma)])}$$

exists and is finite.

We now construct a finite set of words $W(\gamma)$ in the alphabet A_γ such that for every $m \in \mathbb{N}^*$, there exists a bijection between occurrences of γ in $[f^m(\sigma)]$ and occurrences of a word $w \in W(\gamma)$ in $[\zeta^m(\sigma)]$. This will conclude the proof of Assertion (1).

Let $W(\gamma)$ be the set of words in A_γ which have a representative consisting of a path contained in a reduced iterate $[f^N(\sigma)]$ of σ which contains γ, which is a concatenation of splitting units of $[f^N(\sigma)]$ and which is minimal for these properties. By construction, every occurrence of γ in a reduced iterate of σ is contained in a word in $W(\gamma)$. The set $W(\gamma)$ is finite since γ is a finite path, since A_γ is finite and since every path representing a letter of a word $w \in W(\gamma)$ must contain an edge of γ by minimality of w.

For every $w \in W(\gamma)$, let m_w be the number of occurrences of γ in w. Since γ is not contained in G_p, the value m_w does not depend on the choice of a representative of w if w represents a collection of paths. Therefore, for every $m \in \mathbb{N}^*$, we have

$$\langle \gamma, f^m(\sigma)\rangle = \sum_{w\in W(\gamma)} m_w \langle w, f^m(\sigma)\rangle .$$

This shows that the limit

$$\sigma_\gamma = \lim_{m\to\infty} \frac{\langle \gamma, f^m(\sigma)\rangle}{\ell_{\mathcal{F}}(f^m(\sigma))}$$

exists and is finite. This concludes the proof of Assertion (1).

(2) Let us prove that for every element $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$, we have:
 (i) $0 \leqslant \sigma_\gamma < \infty$;
 (ii) $\sigma_\gamma = \sigma_{\gamma^{-1}}$;
 (iii) $\sigma_\gamma = \sum_{e\in E} \sigma_{\gamma e}$, where E is the subset of $\vec{E}G$ consisting of all edges that are incident to the endpoints of γ and distinct from the inverse of the last edge of γ.

The point (i) follows from Assertion (1). The second point follows from the definition of $\langle \gamma, f^m(\sigma)\rangle$. In order to prove the third point, remark that $\langle \gamma, f^m(\sigma)\rangle$ and $\sum_{e\in E}\langle \gamma e, f^n(\sigma)\rangle$ differ only when $[f^m(\sigma)]$ ends with γ or γ^{-1}. Therefore the difference between $\langle \gamma, f^m(\sigma)\rangle$ and $\sum_{e\in E}\langle \gamma e, f^m(\sigma)\rangle$ is at most 2. This implies that

$$\left| \frac{\langle \gamma, f^m(\sigma)\rangle}{\ell_{\mathcal{F}}(f^m(\sigma))} - \sum_{e\in E} \frac{\langle \gamma e, f^m(\sigma)\rangle}{\ell_{\mathcal{F}}(f^m(\sigma))} \right| \to 0 \text{ as } n \to \infty.$$

This proves the third point. By [Gue1, Lemma 3.2], since the map $\gamma \mapsto \sigma_\gamma$ satisfies the conditions (i)–(iii), it determines a projective relative current $[n_\sigma] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F})$. This current is unique since a relative current is entirely determined by its set of values on cylinders of finite paths $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ by Lemma 3.29. This concludes the proof. ☐

Definition 4.5. Let $\mathtt{n} \geqslant 3$ and let $\mathcal{F}$ be a free factor system of $F_{\mathtt{n}}$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an atoroidal or an almost atoroidal outer automorphism relative to

$\mathcal{F}$ and let $\mathcal{F}_1$ be a free factor system such that $\mathcal{F} \leqslant \mathcal{F}_1$ and such that the extension $\mathcal{F}_1 \leqslant \{F_{\mathtt{n}}\}$ is sporadic and such that $\phi|_{\mathcal{F}_1}$ is atoroidal relative to $\mathcal{F}$. In the case that ϕ is atoroidal relative to $\mathcal{F}$, we assume that $\mathcal{F}_1 = \{[F_{\mathtt{n}}]\}$. Let $f : G \to G$ be a CT map representing a power of ϕ with filtration

$$\varnothing = G_0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_k = G,$$

such that there exists $i \in \{1, \ldots, k-1\}$ with $\mathcal{F}(G_i) = \mathcal{F}_1$. We define the *simplex of attraction of* ϕ, denoted by $\Delta_+(\phi)$, as the set of projective classes of nonnegative linear combinations of currents μ_σ obtained from Proposition 4.4 applied to $\phi|_{\mathcal{F}_1}$ and f and which correspond to splitting units σ whose exponential length grows exponentially fast under iteration of f. The *simplex of repulsion* of ϕ, denoted by $\Delta_-(\phi)$, is $\Delta_+(\phi^{-1})$.

Remark 4.6. The definitions of attractive and repulsive currents given in Definition 4.5 rely on the choice of CT maps representing powers of the (almost) atoroidal outer automorphisms ϕ and ϕ^{-1}. However, it will be a consequence of Proposition 4.12 and Proposition 5.24 that the attractive and repulsive currents depend only on ϕ.

We now prove properties of the subsets $\Delta_\pm(\phi)$. As explained above Proposition 4.4, there are only finitely many expanding splitting units. Hence the subsets $\Delta_\pm(\phi)$ are closed. Since $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ is a Hausdorff, compact space by Lemma 2.14 and since $\Delta_\pm(\phi)$ are closed subsets, we have the following.

Lemma 4.7. *Let $\mathtt{n} \geqslant 3$ and let $\mathcal{F}$ be a free factor system of $F_{\mathtt{n}}$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an (almost) atoroidal outer automorphism relative to $\mathcal{F}$. The subsets $\Delta_\pm(\phi)$ are compact and contain finitely many extremal points.*

Note that one computes $\|\mu_\sigma\|_{\mathcal{F}}$ by counting the number of occurrences of every PG-relative splitting unit of positive $\mathcal{F}$-length in a reduced iterate of σ and taking the limit. This is precisely the limit of the $\mathcal{F}$-length of reduced iterates of σ by Lemma 3.21. Hence we have the following result.

Lemma 4.8. *Let $\mathtt{n} \geqslant 3$ and let $\mathcal{F}$ be a free factor system of $F_{\mathtt{n}}$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an (almost) atoroidal outer automorphism relative to $\mathcal{F}$. We have $\|\mu_\sigma\|_{\mathcal{F}} = 1$.*

We now prove that the subsets $\Delta_\pm(\phi)$ are ϕ-invariant. We first recall some lemmas.

Lemma 4.9 ([Coo, Bounded Cancellation])**.** *Let $\mathtt{n} \geqslant 2$ and let G be a marked graph of $F_{\mathtt{n}}$. Let $f : G \to G$ be a graph map. There exists a constant C_f such that for any reduced path $\rho = \rho_1\rho_2$ in G we have*

$$\ell([f(\rho)]) \geqslant \ell([f(\rho_1)]) + \ell([f(\rho_2)]) - 2C_f.$$

Lemma 4.10 ([LU2, Lemma 5.7])**.** *For any graph G without valence* 1 *vertices there exists a constant $K \geqslant 0$ such that for any finite reduced edge path γ in G there exists an edge path γ' of length at most K such that the concatenation $\gamma\gamma'$ exists and is a reduced circuit.*

Lemma 4.11. *Let $f : G \to G$ be as in Proposition* 4.4*. Let $K_1 \geqslant 0$ be any constant, let σ be an expanding splitting unit and let η_σ be the current associated with σ given by Proposition* 4.4(2)*. Let $m \in \mathbb{N}$ and let γ'_m be a reduced edge path of length at most K_1. Let $\gamma_m = [f^m(\sigma)]^*\gamma'_m$, where $[f^m(\sigma)]^*$ is obtained from $[f^m(\sigma)]$ by erasing an* *initial and a terminal subpath of length K_1. For every element $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$, we have*

$$\lim_{m\to\infty} \frac{\langle \gamma, \gamma_m \rangle}{\ell_{\mathcal{F}}(\gamma_m)} = \langle \gamma, \eta_\sigma \rangle .$$

Proof. The proof follows [LU2, Lemma 5.8]. Note that $\ell(\gamma'_m) \leqslant K_1$ and that

$$\ell_{\mathcal{F}}([f^m(\sigma)]^*) \geqslant \ell_{\mathcal{F}}([f^m(\sigma)]) - 2K_1.$$

Since σ is expanding, we have $\lim_{m\to\infty} \ell_{\mathcal{F}}([f^m(\sigma)]) = +\infty$. Combining all these facts, we see that

$$\lim_{m\to\infty} \frac{\langle \gamma, \gamma_m \rangle}{\langle \gamma, [f^m(\sigma)] \rangle} = 1$$

and

$$\lim_{m\to\infty} \frac{\ell_{\mathcal{F}}(\gamma_m)}{\ell_{\mathcal{F}}([f^m(\sigma)])} = 1.$$

Hence the result follows from Proposition 4.4(1). □

Proposition 4.12. *Let $\mathtt{n} \geqslant 3$ and let $\mathcal{F}$ be a free factor system of $F_{\mathtt{n}}$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an atoroidal or an almost atoroidal outer automorphism relative to $\mathcal{F}$. Let $f\colon G \to G$ be as in Proposition 4.4. Let σ be an expanding splitting unit and let η_σ be the current associated with σ given by Proposition 4.4(2). There exists $\lambda_\sigma > 1$ such that*

$$\phi(\eta_\sigma) = \lambda_\sigma \eta_\sigma.$$

Proof. The proof follows [LU2, Proposition 5.9]. Let $K \geqslant 0$ be the constant associated with G given by Lemma 4.10. Let $m \in \mathbb{N}$, and let γ'_m be the path of length at most K given by Lemma 4.10 such that $\gamma_m = [f^m(\sigma)]\gamma'_m$ is a reduced circuit. Since $\lim_{t\to\infty} \ell_{exp}([f^t(\sigma)]) = +\infty$, for large values of m, we have $\ell_{exp}(\gamma_m) > 0$. Let w_m be an element of $F_{\mathtt{n}}$ whose conjugacy class is represented by γ_m. Note that, by Lemma 3.27, we have $\ell_{\mathcal{F}}(\gamma_m) = \|\eta_{w_m}\|_{\mathcal{F}}$. By Proposition 3.14, since $\ell_{exp}(\gamma_m) > 0$, we see that w_m is $\mathcal{F} \wedge \mathcal{A}(\phi)$-nonperipheral, hence w_m defines a current $\eta_{[w_m]} \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$.

Let $\alpha_m = [f^{m+1}(\sigma)][f(\gamma'_m)]$. Note that since $\ell(\gamma'_m) \leqslant K$, the value $\ell([f(\gamma'_m)])$ is bounded by a constant K_0 which only depends on K. Let C' be the constant given by Lemma 4.9 and let $K_1 = \max\{K_0, C'\}$. Then, with the notations of Lemma 4.11, the reduced circuit $\gamma''_m = [\alpha_m]$ can be written as a product $\gamma''_m = [f^{m+1}(\sigma)]^*\beta_m$ where $\ell(\beta_m) \leqslant K_1$ and $\ell_{\mathcal{F}}([f^{m+1}(\sigma)]^*) \geqslant \ell_{\mathcal{F}}([f^{m+1}(\sigma)]) - 2K_1$. Applying Lemma 4.11 twice, we see that, for every element $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$, we have

$$\lim_{m\to\infty} \frac{\langle \gamma, \gamma_m \rangle}{\ell_{\mathcal{F}}(\gamma_m)} = \langle \gamma, \eta_\sigma \rangle$$

and

$$\lim_{m\to\infty} \frac{\langle \gamma, \gamma''_m \rangle}{\ell_{\mathcal{F}}(\gamma''_m)} = \langle \gamma, \eta_\sigma \rangle .$$

By Lemma 3.30, we have

$$\lim_{m\to\infty} \frac{\eta_{[w_m]}}{\|\eta_{[w_m]}\|_{\mathcal{F}}} = \eta_\sigma.$$

From the continuity of the $\mathrm{Out}(F_{\mathtt{n}})$-action on $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ and from the fact that $\phi(\eta_{[w_m]}) = \eta_{\phi([w_m])}$, we see that

$$\lim_{m\to\infty} \frac{\eta_{\phi([w_m])}}{\|\eta_{[w_m]}\|_{\mathcal{F}}} = \phi(\eta_\sigma).$$

Since the reduced circuit γ''_m represents the conjugacy class $\phi([w_m])$, the second of the above equalities implies that

$$\lim_{m\to\infty} \frac{\eta_{\phi([w_m])}}{\|\eta_{\phi([w_m])}\|_{\mathcal{F}}} = \eta_\sigma.$$

Recall that $\lim_{m\to\infty} \frac{\ell_{\mathcal{F}}(\gamma_m)}{\ell_{\mathcal{F}}([f^m(\sigma)])} = 1$ and that $\lim_{m\to\infty} \frac{\ell_{\mathcal{F}}(\gamma''_m)}{\ell_{\mathcal{F}}([f^{m+1}(\sigma)])} = 1$. By Lemma 3.27, we have $\ell_{\mathcal{F}}(\gamma_m) = \|\eta_{[w_m]}\|_{\mathcal{F}}$ and $\ell_{\mathcal{F}}(\gamma''_m) = \|\eta_{\phi([w_m])}\|_{\mathcal{F}}$. Recall from the claim in the proof of Proposition 4.4 that $\ell_{\mathcal{F}}([f(\sigma)])$ is the norm of a matrix. The conclusion of Proposition 4.12 then follows from the fact (see [LU1, Remark 3.3]) that there exists $\lambda_\sigma > 1$ such that

$$\lim_{m\to\infty} \frac{\ell_{\mathcal{F}}([f^{m+1}(\sigma)])}{\ell_{\mathcal{F}}([f^m(\sigma)])} = \lambda_\sigma. \qquad \square$$

We now prove a lemma which will be used in [Gue2].

Lemma 4.13. *Let* $\mathtt{n} \geqslant 3$ *and let* $\mathcal{F}$ *be a free factor system of* $F_{\mathtt{n}}$. *Let* $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ *be an expanding outer automorphism relative to* $\mathcal{F}$. *Let* $f\colon G \to G$ *be as in Proposition* 4.4. *Let* σ *be an expanding splitting unit and let* η_σ *be the current associated with* σ *given by Proposition* 4.4(2).

(1) *There exists a projective current* $[\eta] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ *whose support is contained in the support of* η_σ *and such that* $\mathrm{Supp}(\eta)$ *is uniquely ergodic. In particular, the support of every extremal current of* $\Delta_\pm(\phi)$ *contains a closed subset which is uniquely ergodic.*
(2) *There exist only finitely many projective currents* $[\eta] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ *whose support is contained in the support of* η_σ *and such that* $\mathrm{Supp}(\eta)$ *is uniquely ergodic.*

Proof. (1) Note that, since ϕ is expanding relative to $\mathcal{F}$, we have $\mathcal{F} \wedge \mathcal{A}(\phi) = \mathcal{A}(\phi)$. Let $r \in \mathbb{N}$ be the minimal integer such that H_r is an EG stratum and a reduced iterate of σ contains a splitting unit which is an edge of H_r. Such a stratum H_r exists since σ is expanding. Let e be an edge of H_r with fixed initial direction and let η_e be the current in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{A}(\phi))$ associated with e given by Proposition 4.4(2).

Claim. The support of η_e is uniquely ergodic.

Proof. Let G' be the minimal subgraph of G which contains every reduced iterate of e and let A be a subgroup of $F_{\mathtt{n}}$ such that $\pi_1(G')$ is a conjugate of A when $\pi_1(G)$ is identified with $F_{\mathtt{n}}$. Then G' is f-invariant and hence $[A]$ is ϕ-invariant. Let $G'_1, \ldots, G'_k$ be the connected component of $\overline{G' - H_r}$ and let $\mathcal{F}'$ be the free factor system of $F_{\mathtt{n}}$ determined by $G'_1, \ldots, G'_k$. Let $\Phi \in \phi$ be such that $\Phi(A) = A$. Note that $[\Phi|_A] \in \mathrm{Out}(A)$ is fully irreducible relative to $\mathcal{F}'$.

By Proposition 3.14 and Proposition 2.5(9), if γ is a cyclically reduced circuit of G' of height r whose growth under iteration of f is polynomial, then γ contains (up to taking inverse) the only height r EG INP σ_r. As one of the endpoints of σ_r is not contained in G_{r-1} by [HM, Fact I.1.42], we see that either σ_r is not closed and γ does not exist or σ_r is closed and γ is an iterate of σ_r or σ_r^{-1}. Let $b \in F_{\mathtt{n}}$ be the (possibly trivial) element associated with σ_r.

Let $\mathbb{P}\mathrm{Curr}(\mathrm{Supp}(\eta_e))$ be the set of projective currents in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ whose support is contained in $\mathrm{Supp}(\eta_e)$. By minimality of r, there does not exist a splitting unit contained in a reduced iterate of e which is an edge in an EG stratum of height less than r. Thus, every maximal subpath of $G' \cap G_{r-1}$ which is contained in a reduced iterate of σ is a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$. In particular, we see that

$$\mathrm{Supp}(\eta_e) \subseteq \bigcup_{g \in F_{\mathtt{n}}} g\partial^2(A, \mathcal{F}').$$

We now construct an injective application

$$\Theta\colon \mathbb{P}\mathrm{Curr}(\mathrm{Supp}(\eta_e)) \to \mathbb{P}\mathrm{Curr}(A, \mathcal{F}')$$

such that for every projective current $[\mu] \in \mathbb{P}\mathrm{Curr}(\mathrm{Supp}(\eta_e))$ we have

$$\mathrm{Supp}(\Theta([\mu])) = \mathrm{Supp}([\mu]) \cap \partial^2 A.$$

Let $\mathcal{C}(\mathcal{F}')$ be the set of paths in G defined by Lemma 2.12 associated with the free factor system $\mathcal{F}'$. Let $\mathcal{C}_A(\mathcal{F}')$ be the set of paths in $\mathcal{C}(\mathcal{F}')$ contained in G'. Note that no path of $\mathcal{C}_A(\mathcal{F}')$ is contained in $G' \cap G_{r-1}$. Moreover, a path in $\mathcal{C}_A(\mathcal{F}')$ is contained in a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$ if and only if it is contained in the circuit representing a power of b. Thus, up to restricting $\mathcal{C}_A(\mathcal{F}')$ to longer paths (which does not change the fact that the cylinders associated with paths in $\mathcal{C}_A(\mathcal{F}')$ cover $\partial^2(A, \mathcal{F}')$), we may suppose that, for every $\gamma \in \mathcal{C}_A(\mathcal{F}')$, either γ contains σ_r and is contained in a power of σ_r or that γ is not contained in a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$.

Since cylinders associated with paths in $\mathcal{C}_A(\mathcal{F}')$ cover the relative double boundary $\partial^2(A, \mathcal{F}')$, by [Gue1, Lemma 3.2], it suffices to prove that for every projective current $\eta \in \mathbb{P}\mathrm{Curr}(\mathrm{Supp}(\eta_e))$, we can associate a function $\widetilde{\eta}\colon \mathcal{C}_A(\mathcal{F}') \to \mathbb{R}$ such that for every $\gamma \in \mathcal{P}_A(\mathcal{F}')$, we have

(i) $0 \leqslant \widetilde{\eta}(\gamma) < \infty$;
(ii) $\widetilde{\eta}(\gamma) = \sigma_{\gamma^{-1}}$;
(iii) $\widetilde{\eta}(\gamma) = \sum_{e \in E} \sigma_{\gamma e}$, where E is the subset of $\vec{E}G'$ consisting of all edges that are incident to the endpoints of γ and distinct from the inverse of the last edge of γ.

Let $\eta \in \mathbb{P}\mathrm{Curr}(\mathrm{Supp}(\eta_e))$. If $\gamma \in \mathcal{C}_A(\mathcal{F}')$ is not contained in the axis of a conjugate of b, we may set $\widetilde{\eta}(\gamma) = \eta(C(\gamma))$. Since σ_e is r-legal, a reduced iterate of σ_e cannot contain the only height r EG INP. Thus, we may set, for every path $\gamma \in \mathcal{P}_A(\mathcal{F}')$ contained in the axis of a conjugate of b: $\widetilde{\eta}(\gamma) = 0$.

The function $\widetilde{\eta}$ satisfies Conditions (i)–(iii) as η is a relative current whose support is contained in $\bigcup_{g \in F_{\mathtt{n}}} g\partial^2(A, \mathcal{F}')$. Hence it defines a unique current in $\mathbb{P}\mathrm{Curr}(A, \mathcal{F}')$, which we still denote by $\widetilde{\eta}$. Note that for every element $\gamma \in \mathcal{C}_A(\mathcal{F}')$, we have

$$\widetilde{\eta}(C(\gamma) \cap \partial^2 A \cap \partial^2(F_{\mathtt{n}}, \mathcal{A}(\phi))) = \eta(C(\gamma) \cap \partial^2 A \cap \partial^2(F_{\mathtt{n}}, \mathcal{A}(\phi))).$$

Therefore, we have $\mathrm{Supp}(\widetilde{\eta}) = \mathrm{Supp}(\eta) \cap \partial^2 A$. Since $\mathrm{Supp}(\eta_e) \subseteq \bigcup_{g \in F_{\mathtt{n}}} g\partial^2(A, \mathcal{F}')$, the application $\mathbb{P}\mathrm{Curr}(\mathrm{Supp}(\eta_e)) \to \mathbb{P}\mathrm{Curr}(A, \mathcal{F}')$ is injective.

Let $\widetilde{\eta_e} \in \mathbb{P}\mathrm{Curr}(A, \mathcal{F}')$ be the relative current of A associated with η_e. This current coincides with the attractive projective current associated with $[\Phi|_A]$ defined by Gupta in [Gup1, Proposition 8.12]. By [Gup2, Lemma 4.17], the support of $\widetilde{\eta_e}$ is uniquely ergodic. Thus the support of η_e is uniquely ergodic. □

By the claim, it remains to prove that $\mathrm{Supp}(\eta_e) \subseteq \mathrm{Supp}(\eta_\sigma)$. Recall the definition of $\mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ above Lemma 3.29. Note that an element $\beta \in \partial^2(F_{\mathtt{n}}, \mathcal{A}(\phi))$ is contained in the support of η_σ if for every element $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ such that

$\beta \in C(\gamma)$, we have $\eta_\sigma(C(\gamma)) > 0$. Then the support of η_σ contains all the cylinder sets of the form $C(\gamma)$ where $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ and γ is contained in a reduced iterate of σ. In particular, since e is contained in a reduced iterate of σ, we have $\mathrm{Supp}(\eta_e) \subseteq \mathrm{Supp}(\eta_\sigma)$. This proves Assertion (1).

(2) Suppose towards a contradiction that there exist infinitely many pairwise distinct projective currents $([\eta_m])_{m\in\mathbb{N}} \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ whose support is contained in the support of η_σ and such that for every $m \in \mathbb{N}$, the support $\mathrm{Supp}(\eta_m)$ is uniquely ergodic. By compactness of $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ (see Lemma 2.14) up to passing to a subsequence, there exists a projective current $[\eta] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ such that $\lim_{m\to\infty}[\eta_m] = [\eta]$. Let $K \in \mathbb{N}^*$ be such that $\mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ contains reduced edge paths of length equal to K. By additivity of η, there exists $\gamma_1, \ldots, \gamma_t \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ of length equal to K such that the support $\mathrm{Supp}(\eta)$ is contained in $\bigcup_{j=1}^t C(\gamma_j)$ and for every $j \in \{1, \ldots, m\}$, we have $\eta(C(\gamma_j)) > 0$. Then, there exists $N \in \mathbb{N}^*$ such that, for every $m \geqslant N$ and every $j \in \{1, \ldots, t\}$, we have $\eta_m(C(\gamma_j)) > 0$. Hence for every $m \geqslant N$, we have

$$\mathrm{Supp}(\eta) \subseteq \bigcup_{j=1}^{t} C(\gamma_j) \subseteq \mathrm{Supp}(\eta_m).$$

By unique ergodicity, for every $m \geqslant N$, we have $[\eta] = [\eta_m]$, a contradiction. $\square$

5. North-South dynamics for expanding relative outer automorphisms

Let X be a compact metric space and let G be a group acting on X by homeomorphisms. We say that an element $g \in G$ acts on X with *generalized north-south dynamics* if the action of g on X has two invariant disjoint closed subsets Δ_- and Δ_+ such that, for every open neighborhood $U_\pm$ of $\Delta_\pm$ and every compact set $K_\pm \subseteq X - \Delta_\mp$, there exists $M > 0$ such that, for every $n \geqslant M$, we have

$$g^{\pm n} K_\pm \subseteq U_\pm.$$

In this section we prove Theorem 5.1. Recall that a relative expanding outer automorphism is in particular relative almost atoroidal (with $\mathcal{F}_1 = \{[F_{\mathtt{n}}]\}$).

Theorem 5.1. *Let $\mathtt{n} \geqslant 3$ and let $\mathcal{F}$ be a free factor system of $F_{\mathtt{n}}$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be a relative expanding outer automorphism. Let $\Delta_+(\phi)$ and $\Delta_-(\phi)$ be the simplexes of attraction and repulsion of ϕ. Then ϕ acts on $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ with generalized north-south dynamics with respect to $\Delta_+(\phi)$ and $\Delta_-(\phi)$.*

Theorem 1.2 in Section 1 follows from Theorem 5.1 since every exponentially growing element of $\mathrm{Out}(F_{\mathtt{n}})$ is expanding relative to its polynomial part.

5.1. **Relative exponential length and goodness.** Let $\mathtt{n} \geqslant 3$ and let $\mathcal{F}$ be a free factor system of $F_{\mathtt{n}}$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an atoroidal or an almost atoroidal outer automorphism relative to $\mathcal{F}$. In this section we define and prove the properties of the objects needed in order to prove Theorem 5.1. Let $f\colon G \to G$ be a CT map representing a power of ϕ with filtration $\varnothing = G_0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_k = G$ and let $p \in \{1, \ldots, k\}$ be such that $\mathcal{F}(G_p) = \mathcal{F}$. The proof of Theorem 5.1 relies on the study of PG-relative completely split edge paths. More precisely, given a reduced circuit γ of G, we study the proportion of subpaths of γ which have PG-relative complete splittings. This proportion will be measured using the exponential length. However, the lack of equality in Lemma 3.17 shows that the exponential length is not well-adapted to study the exponential length of a path by comparing it with the exponential length of its subpaths. Instead, we define a notion of *exponential length of a subpath relative to* γ. We first need some preliminary results regarding splittings of edge paths.

Definition 5.2. Let γ be a reduced edge path in G and let $\gamma = \gamma_0\gamma_1'\gamma_1\dots\gamma_k'\gamma_k$ be the exponential decomposition of γ (see the beginning of Section 3.2). Let α be a subpath of γ. The *exponential length of* α *relative to* γ, denoted by $\ell_{exp}^{\gamma}(\alpha)$, is:

$$\ell_{exp}^{\gamma}(\alpha) = \sum_{i=1}^{k} \ell_{exp}(\alpha \cap \gamma_k').$$

We define the $\mathcal{F}$*-length of* α *relative to* γ similarly replacing ℓ_{exp} by $\ell_{\mathcal{F}}$ and the exponential decomposition by the $\mathcal{F}$-exponential decomposition.

Note that, for every reduced edge path γ of G, we have $\ell_{exp}^{\gamma}(\gamma) = \ell_{exp}(\gamma)$. The exponential length relative to a path γ is well-adapted to compute the exponential length of γ using its subpaths, as shown by Lemma 5.3.

Lemma 5.3. *Let* γ *be a reduced edge path and let* $\gamma' = \alpha\beta \subseteq \gamma$ *be a subpath of* γ*. Then*

$$\ell_{exp}^{\gamma}(\gamma') = \ell_{exp}^{\gamma}(\alpha) + \ell_{exp}^{\gamma}(\beta).$$

In particular, when $\gamma' = \gamma$*, we have*

$$\ell_{exp}(\gamma) = \ell_{exp}^{\gamma}(\alpha) + \ell_{exp}^{\gamma}(\beta).$$

The same statement is true replacing ℓ_{exp}^{γ} *by* $\ell_{\mathcal{F}}^{\gamma}$*.*

Proof. The proof is similar for both ℓ_{exp}^{γ} and $\ell_{\mathcal{F}}^{\gamma}$, so we only do the proof for ℓ_{exp}^{γ}. Let $\gamma = \gamma_0\gamma_1'\gamma_1\dots\gamma_k'\gamma_k$ be the exponential decomposition of γ. Then, for every $i \in \{1,\dots,k\}$, the paths $\alpha \cap \gamma_i'$ and $\beta \cap \gamma_i'$ do not contain a subpath of a path in $\mathcal{N}_{PG}^{\max}(\gamma)$. In particular, for every $i \in \{1,\dots,k\}$, one computes $\ell_{exp}(\alpha \cap \gamma_i')$ and $\ell_{exp}(\beta \cap \gamma_i')$ by removing edges from G_{PG}'. Since $\ell_{exp}^{\gamma}(\gamma')$ is computed by removing edges in G_{PG}' from every γ_i' with $i \in \{1,\dots,k\}$, the proof follows. ☐

In Lemma 5.6, we will show that if γ is a reduced edge path in G and that α is a subpath of γ, then $\ell_{exp}(\alpha)$ and $\ell_{exp}^{\gamma}(\alpha)$ differ by a uniform additive constant. This will allow us to compute directly $\ell_{exp}(\alpha)$ rather than $\ell_{exp}^{\gamma}(\alpha)$.

Let γ be a reduced edge path in G and let $\gamma = \gamma_1\dots\gamma_m$ be a splitting of γ. Let $J_{CS,PG} \subseteq \{\gamma_1,\dots,\gamma_m\}$ be the subset consisting of all subpaths which have a PG-relative complete splitting. If $\ell_{exp}(\gamma) > 0$, let

$$\mathfrak{g}_{CT,PG}(\gamma,\gamma_1,\dots,\gamma_m) = \frac{\sum_{\gamma_i \in J_{CS,PG}} \ell_{exp}^{\gamma}(\gamma_i)}{\ell_{exp}(\gamma)}.$$

The *goodness of* γ, denoted by $\mathfrak{g}(\gamma)$, is the least upperbound of $\mathfrak{g}_{CT,PG}(\gamma)$ over all splittings of γ if $\ell_{exp}(\gamma) > 0$, and is equal to 0 otherwise. When γ is a circuit, the value $\mathfrak{g}_{CT,PG}(\gamma)$ is defined using only circuital splittings.

Since there are only finitely many decompositions of a finite edge path into subpaths, the value $\mathfrak{g}(\gamma)$ is realized for some splitting of γ. A splitting for which $\mathfrak{g}(\gamma)$ is realized is called an *optimal splitting* of γ, and an *optimal circuital splitting* when γ is a circuit.

A subpath of γ which is the concatenation of consecutive splitting units of an optimal splitting of γ is called a *factor* of γ. When $\ell_{exp}(\gamma) = 0$, we use the convention that the only factor of γ is γ itself. The factors of γ that admit a PG-relative complete splitting are called *complete factors*. The factors in an optimal splitting which do not admit PG-relative complete splittings are said to be *incomplete*. Remark that, by Proposition 2.5 (6), (8) and by Lemma 3.10, the $[f]$-image of a PG-relative complete path is PG-relative complete, and the reduced iterates of an incomplete factor are eventually PG-relative complete.

Using Lemma 5.3, we have the following result.

Lemma 5.4. *Let γ be a reduced edge path and let $\gamma = \gamma'_0\gamma_1\gamma'_1\ldots\gamma_m\gamma'_m$ be an optimal splitting of γ, where, for every $i \in \{0,\ldots,m\}$, the path γ'_i is an incomplete factor of γ and, for every $i \in \{1,\ldots,m\}$, the path γ_i is complete. Then*

$$\mathfrak{g}(\gamma) = \frac{\sum_{i=1}^m \ell^{\gamma}_{exp}(\gamma_i)}{\sum_{i=1}^m \ell^{\gamma}_{exp}(\gamma_i) + \sum_{j=0}^m \ell^{\gamma}_{exp}(\gamma'_i)}.$$

Definition 5.5. Let $\mathtt{n}, \mathcal{F}, \phi, f, p$ be as in the beginning of this section. Let $K \geqslant 1$. The CT map f is *$3K$-expanding* if for every edge e of $\overline{G - G'_{PG}}$, we have

$$\ell_{exp}([f(e)]) \geqslant 3K.$$

Note that, by Lemma 3.22, for every $K \geqslant 1$, the CT map f has a power which is $3K$-expanding. Note that, since ϕ is exponentially growing, we have $G \neq G'_{PG}$, so that the definition of $3K$-expanding is not empty.

In the rest of the section, let $K \geqslant 1$ be a constant such that, for every reduced edge path σ which is either in $\mathcal{N}_{PG}$ or a path in a zero stratum, we have $\ell(\sigma) \leqslant \frac{K}{2}$. Such a K exists since $\mathcal{N}_{PG}$ is finite by Lemma 3.5(1) and since every zero stratum is contractible by Proposition 2.5(3). We fix a constant C_f given by Lemma 4.9. Let

$$C = \max\{K, C_f\}. \tag{6}$$

Recall that if σ is a PG-relative splitting unit, σ is either an edge in an irreducible stratum, a path in a zero stratum or a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Thus, the choice of K implies that for every PG-relative splitting unit σ, we have $\ell_{exp}(\sigma) \leqslant \frac{K}{2}$.

Lemma 5.6. *Let γ be a reduced edge path in G and let γ' be a subpath of γ. Let $\gamma = \gamma_0\gamma'_1\gamma_1\ldots\gamma'_k\gamma_k$ be the exponential decomposition of γ. There exist three (possibly empty) subpaths δ_1, δ_2 and τ of γ such that for every $i \in \{1,2\}$, the path δ_i is a proper subpath of a splitting unit of some γ_j, we have $\ell_{exp}(\tau) = \ell^{\gamma}_{exp}(\tau) = \ell^{\gamma}_{exp}(\gamma')$ and $\gamma' = \delta_1\tau\delta_2$. In particular, we have*

$$\ell^{\gamma}_{exp}(\gamma') \leqslant \ell_{exp}(\gamma') \leqslant \ell^{\gamma}_{exp}(\gamma') + 2C \leqslant \ell_{exp}(\gamma) + 2C.$$

The same statement is true replacing ℓ_{exp} by $\ell_{\mathcal{F}}$ and ℓ^{γ}_{exp} by $\ell^{\gamma}_{\mathcal{F}}$.

Proof. The proof is similar for both ℓ_{exp} and $\ell_{\mathcal{F}}$, so we only do the proof for ℓ_{exp}. Since γ' is a subpath of γ, there exist three (possibly trivial) paths δ'_1, τ' and δ'_2 such that:

(a) for every $i \in \{1,2\}$, there exists $k_i \in \{0,\ldots,k\}$ such that the path δ'_i is a subpath of some γ_{k_i};
(b) for every $j \in \{0,\ldots,k\}$, either γ_j is contained in τ' or γ_j does not contain edges of τ';
(c) we have $\gamma' = \delta'_1\tau'\delta'_2$.

The path δ_1' has a decomposition $\delta_1' = \delta_1 f_1$, where f_1 is a (possibly trivial) factor of γ_{k_1} and δ_1 is properly contained in a splitting unit of γ_{k_1} for some fixed choice of optimal splitting of γ_{k_1}. Similarly, the path δ_2' has a decomposition $\delta_2' = f_2\delta_2$, where f_2 is a (possibly trivial) factor of γ_{k_2} and δ_2 is properly contained in a splitting unit of γ_{k_2} for some fixed choice of optimal splitting of γ_{k_2}. Let $\tau = f_1\tau' f_2$. Then $\gamma' = \delta_1\tau\delta_2$. It remains to show that $\ell_{exp}(\tau) = \ell_{exp}^{\gamma}(\tau) = \ell_{exp}^{\gamma}(\gamma')$. Since for every $i \in \{1,2\}$, the path f_i is a path in $\mathcal{N}_{PG}$, we have $\ell_{exp}(\tau) = \ell_{exp}(\tau')$. By (b), one obtains $\ell_{exp}(\gamma')$ by deleting edges in G'_{PG} and every path of $\mathcal{N}_{PG}^{\max}(\gamma)$ contained in τ'. Hence we have

$$\ell_{exp}^{\gamma}(\tau') = \sum_{i=1}^{k} \ell_{exp}(\tau' \cap \gamma_k') = \sum_{i=1}^{k} \ell_{exp}(\tau \cap \gamma_k') = \ell_{exp}^{\gamma}(\tau).$$

Since δ_1 and δ_2 are contained in paths of $\mathcal{N}_{PG}^{\max}(\gamma)$, we have $\ell_{exp}^{\gamma}(\gamma') = \ell_{exp}^{\gamma}(\tau)$, that is, the second equality holds.

We now prove the final inequalities in the lemma. The first inequality follows from the fact that every path in $\mathcal{N}_{PG}^{\max}(\gamma')$ is a subpath of some γ_i for $i \in \{0,\ldots,k\}$. Thus, we have $\ell_{exp}^{\gamma}(\gamma') \leqslant \ell_{exp}(\gamma')$. By Lemma 3.17, we have

$$\ell_{exp}(\gamma') \leqslant \ell_{exp}(\delta_1) + \ell_{exp}(\tau) + \ell_{exp}(\delta_2) \leqslant \ell_{exp}^{\gamma}(\gamma') + \ell(\delta_1) + \ell(\delta_2).$$

By definition of the constant K and the fact that $K \leqslant C$, we have:

$$\ell_{exp}^{\gamma}(\gamma') + \ell(\delta_1) + \ell(\delta_2) \leqslant \ell_{exp}^{\gamma}(\gamma') + 2C \leqslant \ell_{exp}(\gamma) + 2C,$$

where the last inequality follows from Lemma 5.3. □

Lemma 5.7. *Let $f\colon G \to G$ be a $3K$-expanding CT map. Let γ be a PG-relative completely split edge path of positive exponential length. Then*

$$\ell_{exp}([f(\gamma)]) \geqslant 3\ell_{exp}(\gamma).$$

Proof. Consider a PG-relative complete splitting $\gamma = \gamma_0'\gamma_1\gamma_1'\ldots\gamma_m\gamma_m'$ of γ, where, for every $i \in \{0,\ldots,m\}$, the path γ_i' is either a (possibly trivial) concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ or a (possibly trivial) reduced maximal taken connecting path in a zero stratum and, for every $i \in \{1,\ldots,m\}$, the path γ_i is an edge in an irreducible stratum of positive exponential length. By Lemma 3.24, we have

$$\ell_{exp}(\gamma) = \sum_{i=1}^{m} \ell_{exp}(\gamma_i).$$

Since f is $3K$-expanding, for every $i \in \{1,\ldots,m\}$, we have

$$\ell_{exp}([f(\gamma_i)]) \geqslant 3K\ell_{exp}(\gamma_i).$$

Since the reduced image of a PG-relative complete splitting is a PG-relative complete splitting by Lemma 3.10, by Lemma 3.21(2), we see that

$$\ell_{exp}([f(\gamma)]) \geqslant \sum_{i=1}^{m} \ell_{exp}([f(\gamma_i)]) \geqslant \sum_{i=1}^{m} 3K\ell_{exp}(\gamma_i) \geqslant 3\ell_{exp}(\gamma).$$

This concludes the proof. □

Lemma 5.8. *Let $f\colon G \to G$ be a $3K$-expanding CT map. Let $\gamma = \gamma_1\gamma_2$ be a (not necessarily reduced) edge path of positive exponential length, where γ_1 and γ_2 are reduced edge paths. Let $\gamma_1 = a_1b_1\ldots a_kb_k$ be an optimal splitting of γ_1 where for every $i \in \{1,\ldots,k\}$, the path a_i is an incomplete factor and for every $i \in \{1,\ldots,k\}$*

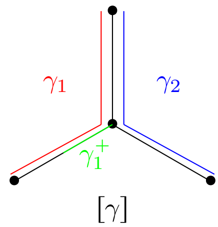

FIGURE 2. Illustration of Lemma 5.8. If a complete factor of γ_1 contained in $[\gamma]$ is not contained in γ_1^+, then it is a complete factor of $[\gamma]$.

the path b_i is complete. For every $i \in \{1,2\}$, let γ_i' be the subpath of γ_i contained in $[\gamma]$. Let $\gamma_1' = \gamma_1^- \gamma_1^+$ be a decomposition of γ_1' into two subpaths where γ_1^+ is the maximal terminal segment of γ_1' such that $\sum_{i=1}^k \ell_{exp}(\gamma_1^+ \cap b_i) = 2C$. Then every PG-relative complete factor b' of γ_1 contained in γ_1^- (for the given optimal splitting) is also a PG-relative complete factor of $[\gamma]$.

Remark 5.9.

(1) We emphasize that, in the statement of Lemma 5.8, if the path γ_1 is PG-relative completely split, the path γ_1' is not necessarily PG-relative completely split. Indeed, there might be some identification with the path γ_2 that might create incomplete factors in γ_1'.
(2) Lemma 5.8 also implies that if γ_1 is PG-relative completely split, the intersection of an incomplete factor of $[\gamma]$ with γ_1' is contained in a terminal segment of γ_1' of exponential length at most equal to $2C$ (see Figure 2). Indeed, the claim in the proof of Lemma 5.8 shows that the path γ_1^- is a complete factor of γ_1, hence a complete factor of $[\gamma]$ by Lemma 5.8. Moreover, we have $k = 1$, a_1 is trivial and $\ell_{exp}(\gamma_1^+) = \ell_{exp}(\gamma_1^+ \cap b_1)$.

Proof. Let $t \in \{1, \dots, k\}$ be the minimal integer such that γ_1^- is contained in $\delta' = a_1 b_1 \dots a_t b_t$. Let $b_t = \delta_1 \dots \delta_{s'}$ be a PG-relative complete splitting of b_t. Let $s \in \{1, \dots, s'\}$ be the minimal integer such that γ_1^- is contained in $\delta = a_1 b_1 \dots a_t \delta_1 \dots \delta_s$. The integer s exists since, by maximality of γ_1^+, for every $i \in \{1, \dots, k\}$, either $\gamma_1^+ \cap a_i = a_i$ or $\gamma_1^+ \cap a_i = \varnothing$.

Claim. We have $\delta = \gamma_1^-$.

Proof. By minimality of t and s, the path γ_1^- contains an edge of δ_s. We claim that δ_s is contained in γ_1'. Indeed, it is clear if δ_s is an edge. Suppose towards a contradiction that δ_s is not contained in γ_1'. Then the concatenation point of γ_1' and γ_2' is contained in δ_s.

If δ_s is a maximal taken connecting path in a zero stratum, then, by the choice of K, we have $\ell(\delta_s) \leqslant \frac{K}{2} \leqslant \frac{C}{2}$. Since $\ell(\gamma_1^+) \geqslant 2C$, the path $\delta_s \cap \gamma_1'$ would be contained in γ_1^+, contradicting the fact that γ_1^- contains the first edge of δ_s.

Suppose that δ_s is a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$. Then $\delta_s \cap \gamma_1'$ has a decomposition $\delta_s \cap \gamma_1' = \beta_1^{(s)} \alpha_1^{(s)} \beta_1^{(s)} \dots \alpha_{k_s-1}^{(s)} \beta_{k_s}^{(s)} \alpha_{k_s}^{(s)}$, where for every $i \in \{1, \dots, k_s\}$, the path $\beta_i^{(s)}$ is contained in G_{PG}, for every $i \in \{1, \dots, k_s - 1\}$, the path $\alpha_i^{(s)}$ is contained in $\mathcal{N}_{PG}^{\max}(\delta_s)$ and $\alpha_{k_s}^{(s)}$ is a subpath of a path in $\mathcal{N}_{PG}^{\max}(\delta_s)$. By the choice of K, we have $\ell_{exp}(\delta_s) \leqslant \ell(\alpha_{k_s}) \leqslant \frac{K}{2} \leqslant \frac{C}{2}$. Since $\ell_{exp}(\gamma_1^+) \geqslant 2C$, the path $\delta_s \cap \gamma_1'$ would be contained in γ_1^+, contradicting the fact that γ_1^- contains the first edge of δ_s.

Hence, in every case, the path δ_s is contained in γ_1'. Note that, since γ_1^+ is the maximal subpath of γ_1' for the property that $\sum_{i=1}^k \ell_{exp}(\gamma_1^+ \cap b_i) = 2C$, the PG-relative splitting unit δ_s is not a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ or a maximal taken connecting path in a zero stratum. Indeed, otherwise it is properly contained in γ_1^+, contradicting the fact that γ_1^- intersects δ_s. Hence δ_s is an edge contained in γ_1^- and $\delta = \gamma_1^-$. □

By the claim, we see that $\gamma_1^- = a_1 b_1 \ldots a_t \delta_1 \ldots \delta_s$ is an optimal splitting of γ_1^-. Let $r \in \{1, \ldots, k\}$ be the minimal integer such that γ_1' is contained in $a_1 b_1 \ldots a_r b_r$. The last edge of γ_1' is either contained in a_r or in b_r. In the first case, for every $i \in \{1, \ldots, k\}$, either b_i is contained in γ_1' or $b_i \cap \gamma_1'$ is at most a point. In the second case, it is possible that $b_r \cap \gamma_1' \neq b_r$ and that $b_r \cap \gamma_1'$ contains an edge. Let α' be the (possibly trivial) terminal segment of γ_1^+ which is properly contained in a splitting unit σ of b_r.

If σ is a maximal taken connecting path in a zero stratum, then, by the choice of K, we have $\ell_{exp}(\alpha') \leqslant \ell(\alpha') \leqslant \ell(\sigma) \leqslant \frac{K}{2} \leqslant \frac{C}{2}$.

Suppose that σ is a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$. Then α' has a decomposition $\alpha' = \beta_1 \alpha_1 \beta_1 \ldots \alpha_{\ell-1} \beta_\ell \alpha_\ell$, where for every $i \in \{1, \ldots, \ell\}$, the path β_i is contained in G_{PG}, for every $i \in \{1, \ldots, \ell - 1\}$, the path α_i is contained in $\mathcal{N}_{PG}^{\max}(\sigma)$ and α_ℓ is a subpath of a path in $\mathcal{N}_{PG}^{\max}(\sigma)$. By the choice of K, we have $\ell_{exp}(\alpha') \leqslant \ell(\alpha_\ell) \leqslant \frac{K}{2} \leqslant \frac{C}{2}$.

Thus, in all cases, we have $\ell_{exp}(\alpha') \leqslant \frac{C}{2}$. Since $\ell_{exp}(\gamma_1^+) \geqslant 2C$, there exists a PG-relative complete factor α_0 of b_r such that $\gamma_1^+ = \delta_{s+1} \ldots \delta_{s'} a_{t+1} b_{t+1} \ldots a_r \alpha_0 \alpha' = \alpha\alpha'$ and

$$\sum_{i=1}^{k} \ell_{exp}(\alpha \cap b_i) \geqslant C.$$

We now prove that every PG-relative complete factor of γ_1 contained in γ_1^- is a PG-relative complete factor of γ. Note that the decomposition $\gamma_1^- \alpha$ is a splitting. Thus, it suffices to prove that, for every $k \in \mathbb{N}^*$, the path $[f^k(\gamma_1^-)]$ is contained in $[f^k(\gamma)]$ as any identification in order to obtain $[f^k(\gamma)]$ which involves a path in $f^k(\gamma_1^-)$ will be induced by an identification in order to obtain $[f^k(\gamma_1^-)]$ from $f^k(\gamma_1^-)$.

By Lemma 5.7 applied to $\delta_{s+1}, \ldots, \delta_{s'}$, to the paths b_i with $i \in \{1, \ldots, k\}$ such that $b_i \subseteq \alpha$ and to α_0, we have

$$\begin{aligned}\sum_{i=1}^{k} \ell_{exp}([f(\alpha)] \cap [f(b_i)]) &\geqslant \sum_{i=s+1}^{s'} \ell_{exp}([f(\delta_i)]) + \sum_{i=t+1}^{r-1} \ell_{exp}([f(b_i)]) + \ell_{exp}([f(\alpha_0)]) \\ &\geqslant 3 \sum_{i=1}^{k} \ell_{exp}(\alpha \cap b_i) \geqslant 3C,\end{aligned}$$

where the first inequality follows from the fact that the decomposition

$$\alpha = \delta_{s+1} \ldots \delta_{s'} a_{t+1} b_{t+1} \ldots a_r \alpha_0$$

is an optimal splitting of α.

Note that, since the decomposition $\gamma_1^-\alpha$ is a splitting, for every $k \in \mathbb{N}^*$, the path $[f^k(\alpha)]$ is contained in $[f^k(\gamma_1^-\alpha)]$. Remark that Lemma 4.9 implies that the segment of $[f(\gamma_1^-\alpha)]$ which is C away from the concatenation point between $[f(\gamma_1^-\alpha)]$ and $[f(\alpha'\gamma_2')]$ remains in $[f([\gamma])]$. In particular, the edges of $[f(\gamma_1^-\alpha)]$ which are cancelled with edges of $[f(\alpha'\gamma_2')]$ are contained in $[f(\alpha)]$. Recall that $\sum_{i=1}^k \ell_{exp}([f(\alpha)] \cap [f(b_i)]) \geqslant 3C$ and that the subpath of $[f(\alpha)]$ which is contained in $[f([\gamma])]$ is obtained by the concatenation of at most C edges of $[f(\alpha)]$. Thus, we see that the sum over i of the exponential length of the subpaths of $[f(\alpha)] \cap [f(b_i)]$ which are contained in $[f([\gamma])]$ is at least equal to $2C$. Hence the path $[f(\gamma_1^-)]$ is a subpath of $[f([\gamma])]$ and $\sum_{i=1}^k \ell_{exp}([f(\gamma_1^+)] \cap [f(b_i)] \cap [f([\gamma])]) \geqslant 2C$.

Thus, we can apply the same arguments to show that for every $k \geqslant 1$, the path $[f^k(\gamma_1^-)]$ is contained in $[f^k([\gamma])]$ and the exponential length of the subpath of $[f^k(\alpha)]$ contained in $[f^k([\gamma])]$ is at least equal to $2C$. Hence every PG-relative complete factor of the path γ_1 contained in γ_1^- is a complete factor of an optimal splitting of $[\gamma]$. □

Lemma 5.10.

(1) *Let $\gamma = \alpha\beta$ be a reduced path. Let $N \in \mathbb{N}^*$ be such that $[f^N(\alpha)]$ has a PG-relative complete splitting and that $[f^N(\beta)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. For every $m \geqslant N$, let α_m, β_m and σ_m be paths such that $[f^m(\alpha)] = \alpha_m\sigma_m$ and $[f^m(\beta)] = \sigma_m^{-1}\beta_m$.*

For every $m \geqslant N$, we have $\ell_{exp}(\sigma_m) \leqslant C$, $\ell_{exp}(\alpha_m) \geqslant \ell_{exp}([f^m(\alpha)]) - C$ and $\ell_{exp}(\beta_m) \leqslant C$.

(2) *Let $\gamma = \beta^{(1)}\alpha\beta^{(2)}$ be a reduced path. Let $N \in \mathbb{N}^*$ be such that $[f^N(\alpha)]$ has a PG-relative complete splitting and, for every $i \in \{1, 2\}$, the path $[f^N(\beta^{(i)})]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. For every $m \geqslant N$, let α_m, $\beta_m^{(1)}$, $\beta_m^{(2)}$, and $\sigma_m^{(1)}$, $\sigma_m^{(2)}$ be paths such that $[f^m(\alpha)] = \sigma_m^{(1)}\alpha_m\sigma_m^{(2)}$, $[f^m(\beta^{(1)})] = \beta_m^{(1)}\sigma_m^{(1)-1}$ and $[f^m(\beta^{(2)})] = \sigma_m^{(2)-1}\beta_m$.*

For every $m \geqslant N$, either $\ell_{exp}(\alpha_m) \leqslant 2C$ or we have $\ell_{exp}(\sigma_m^{(1)}), \ell_{exp}(\sigma_m^{(2)}) \leqslant C$, $\ell_{exp}(\alpha_m) \geqslant \ell_{exp}([f^m(\alpha)]) - 2C$ and $\ell_{exp}(\beta_m^{(1)}), \ell_{exp}(\beta_m^{(2)}) \leqslant C$.

Proof. Assertion (2) follows from Assertion (1) by applying Assertion (1) twice: one with $\gamma = \alpha\beta^{(2)}$ and one with $\gamma = \alpha^{-1}\beta^{(1)}$. If for some $m \in \mathbb{N}^*$, $\ell_{exp}(\alpha_m) \geqslant 2C$, there is no identification between $[f^m(\beta^{(1)})]$ and $[f^m(\beta^{(2)})]$ by Lemma 4.9, so Assertion (2) follows from Assertion (1). Therefore, we focus on the proof of Assertion (1).

Let $m \geqslant N$. When σ_m is reduced to a point, we have $\ell_{exp}(\alpha_m) = \ell_{exp}([f^m(\alpha)])$ and $\ell_{exp}(\beta_m) = \ell_{exp}([f^m(\beta)]) = 0$ by Lemma 3.18. This concludes the proof in this case. So we may suppose that σ_m is nontrivial.

Let $[f^m(\alpha)] = a_1 \dots a_k$ be a PG-relative complete splitting of $[f^m(\alpha)]$. Suppose that, for every $i \in \{1, \dots, k\}$ such that a_i is a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$, the path a_i is a maximal subpath of $[f^m(\alpha)]$ for the property of being a factor which is a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$. For every $j \in \{1, \dots, k\}$, let r_j be the height of a_j.

Let $i \in \{1, \dots, k\}$ be such that a_i contains the first edge of σ_m. Let $\sigma' \in \mathcal{N}_{PG}^{\max}(\sigma_m)$. Note that there exists $\sigma'' \in \mathcal{N}_{PG}^{\max}([f^m(\alpha)])$ such that $\sigma' \subseteq \sigma''$. By Lemma 3.21(1) applied to σ'' and $[f^m(\alpha)]$, the path σ'' is contained in a factor which is a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$. By the maximality assumption, there exists $j \in \{1,\dots,k\}$ such that $\sigma' \subseteq \sigma'' \subseteq a_j$. Hence we can compute $\ell_{exp}(\sigma_m)$ by removing, for every $j \in \{1,\dots,k\}$, paths in the intersection $\sigma_m \cap a_j$. Thus, we have

$$\ell_{exp}(\sigma_m) = \sum_{j>i} \ell_{exp}(a_j) + \ell_{exp}(a_i \cap \sigma_m).$$

Note that, by Lemma 3.10, the path $[f^m(\beta)] = \sigma_m^{-1}\beta_m$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Let $j \in \{i,\dots,k\}$.

Claim. If $j > i$, then either a_j is not an edge in an EG stratum and $\ell_{exp}(a_j \cap \sigma_m) = 0$, or $\ell_{exp}((a_i \dots a_j) \cap \sigma_m) \leqslant C$. If $j = i$, then $\ell_{exp}(a_j \cap \sigma_m) \leqslant C$.

Proof. We distinguish several cases, according to the nature of a_j.

(i) Suppose that a_j is maximal taken connecting path in a zero stratum. By definition we have $\ell_{exp}(a_j \cap \sigma_m) = 0$.

(ii) Suppose that a_j is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. If $j > i$, we have $a_j \cap \sigma_m = a_j$. By Lemma 3.18 applied to a_j, we have $\ell_{exp}(a_j \cap \sigma_m) = 0$.

Suppose that $i = j$. Suppose that the first edge of σ_m is not contained in a path in $\mathcal{N}_{PG}^{\max}(a_i)$. Then a_i has a decomposition $a_i = a_i^0 a_i^1 a_i^2$ where a_i^1 is a path contained in G_{PG} such that the first edge of σ_m is contained in a_i^1 and such that, for every path $\delta \in \mathcal{N}_{PG}^{\max}(a_i)$, either $\delta \subseteq a_i^0$ or $\delta \subseteq a_i^2$. Note that a terminal segment of a_i whose first edge is contained in a_i^1 is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. In particular, the path $a_i \cap \sigma_m$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. By Lemma 3.18 applied to $a_i \cap \sigma_m$, we have $\ell_{exp}(a_i \cap \sigma_m) = 0$.

Suppose now that the first edge of σ_m is contained in a path $\delta \in \mathcal{N}_{PG}^{\max}(a_i)$. Then a_i has a decomposition $a_i^1 \delta a_i^2$, where the first edge of σ_m is contained in δ. Note that a_i^2 is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ which is contained in σ_m. By Lemma 3.17 applied to $a_i \cap \sigma_m = (\delta \cap \sigma_m)a_i^2$, by Lemma 3.18 applied to a_i^2 and by definition of the constant K, we have

$$\ell_{exp}(a_i \cap \sigma_m) \leqslant \ell_{exp}(\delta \cap \sigma_m) + \ell_{exp}(a_i^2) = \ell_{exp}(\delta \cap \sigma_m) \leqslant \ell(\delta) \leqslant K \leqslant C.$$

(iii) Suppose that a_j is an edge in an irreducible stratum with positive exponential length. Since $[f^m(\beta)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$, there exists a path $\gamma' \in \mathcal{N}_{PG}^{\max}([f^m(\beta)])$ such that a_j is contained in γ'. By Lemma 3.21(1), every path in $\mathcal{N}_{PG}^{\max}([f^m(\alpha)])$ is contained in a minimal factor of $[f^m(\alpha)]$ consisting in PG-relative splitting units which are concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$. Since a_j is a PG-relative splitting unit of $[f^m(\alpha)]$ which is not a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$, the path a_j is not contained in a path of $\mathcal{N}_{PG}^{\max}([f^m(\alpha)])$. Hence the path γ' is not contained in σ_m as otherwise it would be contained in a path of $\mathcal{N}_{PG}^{\max}([f^m(\alpha)])$. Therefore, we see that $(a_i \dots a_j) \cap \sigma_m \subseteq \gamma'$. Hence, by the choice of K, we have

$$\ell_{exp}((a_i \dots a_j) \cap \sigma_m) \leqslant \ell((a_i \dots a_j) \cap \sigma_m) \leqslant \ell(\gamma') \leqslant C.$$

This proves the claim as we have considered all possible PG-relative splitting units. □

Let $m \in \mathbb{N}^*$. By the claim, either $\ell_{exp}((a_i \dots a_j) \cap \sigma_m) \leqslant C$ or, for every $j > i$, we have $\ell_{exp}(a_j \cap \sigma_m) = 0$. In the second case, we have

$$\ell_{exp}(\sigma_m) = \sum_{j>i} \ell_{exp}(a_j) + \ell_{exp}(a_i \cap \sigma_m) = \ell_{exp}(a_i \cap \sigma_m) \leqslant C,$$

where the last inequality follows from the case $j = i$ of the claim. Hence, for every $m \in \mathbb{N}^*$, we have $\ell_{exp}(\sigma_m) \leqslant C$. Note that, by Lemma 3.17 applied to $[f^m(\alpha)] = \alpha_m \sigma_m$, we have

$$\ell_{exp}(\alpha_m) \geqslant \ell_{exp}([f^m(\alpha)]) - \ell_{exp}(\sigma_m) \geqslant \ell_{exp}([f^m(\alpha)]) - C.$$

It remains to prove that $\ell_{exp}(\beta_m) \leqslant C$. But β_m can be written as $\beta_m = \delta_1\delta_2$ where δ_2 is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ and δ_1 is a (possibly trivial) path contained in a path of $\mathcal{N}_{PG}^{\max}([f^m(\beta)])$. By Lemma 3.18 applied to δ_2 and by the choice of K (since δ_1 is a subpath of a path in $\mathcal{N}_{PG}$), we have

$$\ell_{exp}(\beta_m) \leqslant \ell_{exp}(\delta_1) + \ell_{exp}(\delta_2) = \ell_{exp}(\delta_1) \leqslant \ell(\delta_1) \leqslant C.$$

This concludes the proof. $\square$

Lemma 5.11. *Let $L \geqslant 1$. There exists $n_0 = n_0(L) \in \mathbb{N}^*$ which satisfies the following properties. Let γ be a reduced edge path of G such that $\ell_{exp}(\gamma) \leqslant L$. For every $n \geqslant n_0$ and every optimal splitting of $[f^n(\gamma)]$, either $[f^n(\gamma)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ or the following two assertions hold:*

(a) *the path $[f^n(\gamma)]$ contains a complete factor of exponential length at least equal to $10C$;*
(b) *the exponential length of an incomplete factor of $[f^n(\gamma)]$ is at most equal to $8C$.*

Proof. By Lemma 3.22, there exists an integer $m' \in \mathbb{N}^*$ depending only on f such that for every edge e of $\overline{G - G'_{PG}}$ and every $n \geqslant m'$, we have $\ell_{exp}[f^n(e)] \geqslant 16C+1$. Let $\gamma = \gamma_0\gamma'_1\gamma_1 \dots \gamma'_\ell\gamma_\ell$ be the exponential decomposition of γ. Let

$$\gamma = \beta_0\alpha_1\beta_1 \dots \alpha_k\beta_k$$

be a nontrivial decomposition of γ such that, for every $i \in \{0, \dots, k\}$, the path β_i is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ and for every $i \in \{1, \dots, k\}$, the path α_i is a concatenation of edges in irreducible strata not contained in some γ_j with $j \in \{0, \dots, \ell\}$ and paths in zero strata. The main point of the proof is to show that, up to applying an iterate of $[f]$, there is no cancellation between the subpaths α_i.

For every $i \in \{1, \dots, k\}$, we have $\ell_{exp}(\gamma) = \sum_{i=1}^{k} \ell_{exp}(\alpha_i)$ by definition of the exponential length. Therefore, since $\ell_{exp}(\gamma) \leqslant L$, for every $i \in \{1, \dots, k\}$, we have $\ell_{exp}(\alpha_i) \leqslant L$. Note that, for every $i \in \{1, \dots, k\}$, we have $\ell_{exp}(\alpha_i) = \ell(\alpha_i) - \ell(\alpha_i \cap \mathcal{Z})$ where $\mathcal{Z}$ is the subgraph of G consisting in all zero strata. By the choice of C the length of every path contained in a zero stratum is at most equal to C. Hence for every $i \in \{1, \dots, k\}$, we have $\ell(\alpha_i) \leqslant CL$.

By Proposition 2.5(8) there exists $m'' \in \mathbb{N}^*$ depending only on L such that, for all $i \in \{1, \dots, k\}$ and $m \geqslant m''$, the path $[f^m(\alpha_i)]$ is completely split. Let $m = m' + m''$. By Lemma 3.21(2), for every $n \geqslant m$ and every $i \in \{1, \dots, k\}$, since $[f^{n-m'}(\alpha_i)]$ is completely split, one computes its exponential length by adding the exponential length of all its splitting units. Thus, if $[f^{n-m'}(\alpha_i)]$ contains a splitting unit which is an edge e in $\overline{G - G'_{PG}}$, we have

$$\ell_{exp}([f^n(\alpha_i)]) \geqslant \ell_{exp}([f^{m'}(e)]) \geqslant 16C + 1. \tag{7}$$

Let C_m be a bounded cancellation constant for f^m given by Lemma 4.9. Note that if there exists $i \in \{1, \ldots, k-1\}$ such that $\ell(\beta_i) < C_m$, then there might exist some identifications between $[f^m(\alpha_{i-1})]$ and $[f^m(\alpha_i)]$ when reducing the paths in order to obtain $[f^m(\gamma)]$. This is why we replace the decomposition $\gamma = \beta_0\alpha_1\beta_1 \ldots \alpha_k\beta_k$ of γ by a new one.

The new decomposition is defined as follows. Since every lift of f^m to the universal cover of G is a quasi-isometry, there exists $M_m > 0$ depending only on m such that, for every reduced edge path of length $\ell(\beta) > M_m$, we have $\ell([f^m(\beta)]) \geqslant 2C_m + 1$.

Let $\Gamma_m = \{\beta_i \mid \ell(\beta_i) \leqslant M_m\}$. Note that $|\Gamma_m| \leqslant k+1$. Note that, by Lemma 2.9 and Proposition 2.5(4), for every $i \in \{1, \ldots, k\}$, if β_{i-1} or β_i is not trivial, then α_i is not contained in a zero stratum. In particular, we may suppose that, for every $i \in \{1, \ldots, k\}$, we have $\ell_{exp}(\alpha_i) > 0$. Thus, since $\ell_{exp}(\gamma) = \sum_{i=1}^k \ell_{exp}(\alpha_i) \leqslant L$, and, for every $i \in \{1, \ldots, k\}$, we have $\ell_{exp}(\alpha_i) > 0$, we see that $k \leqslant L$. Hence we have $|\Gamma_m| \leqslant k+1 \leqslant L+1$.

Claim. There exist $m_1 \geqslant m$ depending only on $|\Gamma_m|$ (and hence on L) and a decomposition $\gamma = \beta_0^{(1)}\alpha_1^{(1)}\beta_1^{(1)} \ldots \alpha_{k_1}^{(1)}\beta_{k_1}^{(1)}$ such that:

(a′) for every $i \in \{1, \ldots, k_1\}$, the path $[f^{m_1}(\alpha_i^{(1)})]$ is completely split;

(b′) for every $i \in \{0, \ldots, k_1\}$, the path $\beta_i^{(1)}$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$;

(c′) for every $i \in \{0, \ldots, k_1\}$, the subpath of $[f^{m_1}(\beta_i^{(1)})]$ contained in $[f^{m_1}(\gamma)]$ is not reduced to a point;

(d′) for every $i \in \{1, \ldots, k_1\}$, for every $n \geqslant m'$, if $[f^{n-m'}(\alpha_i^{(1)})]$ contains a splitting unit which is an edge in $\overline{G - G'_{PG}}$ then $\ell_{exp}([f^n(\alpha_i^{(1)})]) \geqslant 16C+1$.

Proof. The proof is by induction on $|\Gamma_m|$. Suppose first that $\Gamma_m = \varnothing$. By the definition of $|\Gamma_m|$ and M_m, for every $i \in \{0, \ldots, k\}$, the path $[f^m(\beta_i)]$ has length at least equal to $2C_m + 1$. By Lemma 4.9, for every $i \in \{0, \ldots, k\}$, the subpath of $[f^m(\beta_i)]$ contained in $[f^m(\gamma)]$ is not reduced to a point. So the integer $m_1 = m$ and the decomposition $\gamma = \beta_0\alpha_1\beta_1 \ldots \alpha_k\beta_k$ satisfy the assertions of the claim (Assertion (d′) follows from Equation (7)).

Suppose now that $\Gamma_m \neq \varnothing$. Then

$$\sum_{i=1}^{k} \ell(\alpha_i) + \sum_{\beta_i \in \Gamma_m} \ell(\beta_i) \leqslant kCL + M_mL \leqslant CL^2 + M_mL.$$

Let $m_2' \geqslant m$ be such that for every path β of length at most equal to $CL^2 + M_mL$ and every $n \geqslant m_2'$, the path $[f^n(\beta)]$ is completely split. Then γ has a decomposition $\gamma = \beta_0^{(2)}\alpha_1^{(2)}\beta_2^{(2)} \ldots \alpha_{k_2}^{(2)}\beta_{k_2}^{(2)}$ such that, for every $i \in \{1, \ldots, k_2\}$, the path $[f^{m_2'}(\alpha_i^{(2)})]$ is completely split and for every $i \in \{0, \ldots, k_2\}$, the path $\beta_i^{(2)}$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ of length greater than M_m. Let $m_2 = m_2' + m'$. Then for every $i \in \{1, \ldots, k_2\}$, the paths $[f^{m_2}(\alpha_i^{(2)})]$ and $[f^{m_2-m'}(\alpha_i^{(2)})]$ are completely split. Moreover, if $[f^{m_2-m'}(\alpha_i^{(2)})]$ contains a splitting unit which is an edge in $\overline{G - G'_{PG}}$, then $\ell_{exp}([f^m(\alpha_i^{(2)})]) \geqslant 16C + 1$ as in Equation (7).

Let C_{m_2} be a bounded cancellation constant associated with f^{m_2} and let $M_{m_2} \geqslant M_m$ be such that, for every reduced edge path of length $\ell(\beta) > M_{m_2}$, we have $\ell([f^{m_1}(\beta)]) \geqslant 2C_{m_2} + 1$. Let $\Gamma_{m_2} = \{\beta_i^{(2)} \mid \ell(\beta_i) \leqslant M_{m_2}\}$. Note that $|\Gamma_{m_2}| <$

$|\Gamma_m|$. Hence we can apply the induction hypothesis to the decomposition $\gamma = \beta_0^{(2)}\alpha_1^{(2)}\beta_2^{(2)}\dots\alpha_{k_2}^{(2)}\beta_{k_2}^{(2)}$ to obtain the desired decomposition of γ. This concludes the proof of the claim. □

Let m_1 and $\gamma = \beta_0^{(1)}\alpha_1^{(1)}\beta_1^{(1)}\dots\alpha_{k_1}^{(1)}\beta_{k_1}^{(1)}$ be as in the assertion of the claim. By Assertion (c′) of the claim, for every $i \in \{1,\dots,k_1\}$, there is no identification between edges of $[f^{m_1}(\alpha_i^{(1)})]$, $[f^{m_1}(\alpha_{i-1}^{(1)})]$ and $[f^{m_1}(\alpha_{i+1}^{(1)})]$ when reducing in order to obtain $[f^{m_1}(\gamma)]$.

For every $i \in \{1,\dots,k_1\}$, since $[f^{m_1}(\alpha_i^{(1)})]$ is PG-relative completely split, we can distinguish three possible cases for $[f^{m_1}(\alpha_i^{(1)})]$:

(i) the path $[f^{m_1}(\alpha_i^{(1)})]$ contains a PG-relative splitting unit which is an edge in $\overline{G - G'_{PG}}$ (by Lemma 3.24 this case happens exactly when $\ell_{exp}([f^{m_1}(\alpha_i^{(1)})]) > 0$);
(ii) $\ell_{exp}([f^{m_1}(\alpha_i^{(1)})]) = 0$ and the path $[f^{m_1}(\alpha_i^{(1)})]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$;
(iii) $\ell_{exp}([f^{m_1}(\alpha_i^{(1)})]) = 0$ and $[f^{m_1}(\alpha_i^{(1)})]$ contains a maximal taken connecting path in a zero stratum.

We claim that if there exists $i \in \{1,\dots,k_1\}$ such that $[f^{m_1}(\alpha_i^{(1)})]$ satisfies (iii), then $[f^{m_1}(\gamma)]$ is contained in a zero stratum. Indeed, suppose that $[f^{m_1}(\alpha_i^{(1)})]$ satisfies (iii). By Lemma 3.24 applied to the PG-relative completely split edge path $[f^{m_1}(\alpha_i^{(1)})]$, since $\ell_{exp}([f^{m_1}(\alpha_i^{(1)})]) = 0$ the path $[f^{m_1}(\alpha_i^{(1)})]$ does not contain an edge in $\overline{G - G'_{PG}}$. Therefore, the path $[f^{m_1}(\alpha_i^{(1)})]$ is a concatenation of paths in G'_{PG} and in $\mathcal{N}_{PG}$. By Proposition 2.5(4) and Lemma 2.9, there is no path in a zero stratum which is adjacent to a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Hence $[f^{m_1}(\alpha_i^{(1)})] = \sigma$, where σ is a maximal taken connecting path in a zero stratum not contained in G_{PG}. But the endpoints of σ are the endpoints of $[f^{m_1}(\beta_{i-1}^{(1)})]$ and $[f^{m_1}(\beta_i^{(1)})]$, which are concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. As above, this implies that $[f^{m_1}(\gamma)] = \sigma$.

Since zero strata are contractible, there exists $m_3 \in \mathbb{N}^*$ such that $[f^{m_3}(\gamma)]$ is PG-relative completely split. Hence Assertion (b) of Lemma 5.11 follows. Applying a further power of $[f]$ (which can be chosen uniformly as there are finitely many reduced edge paths contained in a zero stratum), there exists $m_4 \in \mathbb{N}^*$ such that $[f^{m_4}(\gamma)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ or it satisfies Assertion (a) of Lemma 5.11. This concludes the proof of Lemma 5.11 in case (iii).

Hence we may suppose that for every $i \in \{1,\dots,k_1\}$, the path $[f^{m_1}(\alpha_i^{(1)})]$ satisfies either (i) or (ii). Note that if $i \in \{1,\dots,k_1\}$ is such that the path $[f^{m_1}(\alpha_i^{(1)})]$ satisfies (i), then $[f^{m_1}(\alpha_i^{(1)})]$ also satisfies the hypothesis of Assertion (d′) of the claim. Thus

$$\ell_{exp}([f^{m_1+m'}(\alpha_i^{(1)})]) \geqslant 16C + 1.$$

Let $m'_1 = m_1 + m'$ and let $n' \geqslant m'_1$. Let

$$\Lambda_{exp} = \{\alpha_i^{(1)} \mid \ell_{exp}([f^{n'}(\alpha_i^{(1)})]) \geqslant 16C + 1\}.$$

For every $j \in \{1, \dots, k_1\}$ and every $n \in \mathbb{N}^*$, let $\alpha_j^{(n)}$ be the subpath of $[f^n(\alpha_j^{(1)})]$ contained in $[f^n(\gamma)]$. For every $j \in \{0, \dots, k_1\}$ and every $n \in \mathbb{N}^*$, let $\beta_j^{(n)}$ be the subpath of $[f^n(\beta_j^{(1)})]$ contained in $[f^n(\gamma)]$.

Suppose first that Λ_{exp} is not empty and let $\alpha_i^{(1)} \in \Lambda_{exp}$. By Lemma 5.10(2) applied to $\beta^{(1)} = [f^{n'}(\beta_{i-1}^{(1)})]$, $\alpha = [f^{n'}(\alpha_i^{(1)})]$ and $\beta^{(2)} = [f^{n'}(\beta_i^{(1)})]$, we have

$$\ell_{exp}(\alpha_i^{(n')}) \geqslant 14C + 1.$$

Using Remark 5.9(2) twice (once with $\gamma_1 = [f^{n'}(\alpha_i^{(1)})]$ and $\gamma_2 = [f^{n'}(\beta_i^{(1)} \dots \alpha_{k_1}^{(1)}\beta_{k_1}^{(1)})]$, and once with $\gamma_1 = [f^{n'}(\alpha_i^{(1)})]^{-1}$ and $\gamma_2 = [f^{n'}(\beta_0^{(1)} \dots \alpha_{i-1}^{(1)}\beta_{i-1}^{(1)})]^{-1})$, we see that the path $\alpha_i^{(n')}$ contains a complete factor of $[f^{n'}(\gamma)]$ of exponential length at least equal to $14C + 1 - 4C = 10C + 1$. This proves Assertion (a) of Lemma 5.11 when Λ_{exp} is not empty.

Moreover, Remark 5.9(2) implies that the intersection of an incomplete factor of $[f^{n'}(\gamma)]$ with $\alpha_i^{(n')}$ is contained in the union of an initial and a terminal segment of $\alpha_i^{(n')}$ of exponential lengths at most $2C$. For every $i \in \{1, \dots, k_1\}$ such that $\alpha_i^{(1)} \in \Lambda_{exp}$, let τ_i^1 be the maximal initial segment of $\alpha_i^{(n')}$ of exponential length equal to $2C$ and let τ_i^2 be the maximal terminal segment of $\alpha_i^{(n')}$ of exponential length equal to $2C$.

We now prove Assertion (b) of Lemma 5.11 when Λ_{exp} is not empty. Suppose that there exists $i \in \{1, \dots, k_1\}$ such that $\alpha_i^{(1)} \notin \Lambda_{exp}$, so that in particular $[f^{m_1}(\alpha_i^{(1)})]$ does not satisfy (i). Then $[f^{m_1}(\alpha_i^{(1)})]$ satisfies (ii) and is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. By Lemma 3.10(3), the path $[f^{n'}(\alpha_i^{(1)})]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. By Lemma 3.6, the path $[[f^{n'}(\beta_{i-1}^{(1)})][f^{n'}(\alpha_i^{(1)})]$ $[f^{n'}(\beta_i^{(1)})]]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Thus, the path $\beta_{i-1}^{(n')}\alpha_i^{(n')}\beta_i^{(n')}$ is a subpath of a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Hence $[f^{n'}(\gamma)]$ has a decomposition

$$[f^{n'}(\gamma)] = \epsilon_1\alpha_1^{(n',+)}\epsilon_2 \dots \alpha_{k_2}^{(n',+)}\epsilon_{k_2},$$

where for every $j \in \{1, \dots, k_2\}$, the path $\alpha_j^{(n',+)}$ is the reduced image of a path in Λ_{exp} and for every $j \in \{0, \dots, k_2\}$, the path ϵ_j is contained in a path ι_j which is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Hence, for every $j \in \{0, \dots, k_2\}$, we have $\ell_{exp}(\iota_j) = 0$ by Lemma 3.18 and, by Lemma 5.6, we have $\ell_{exp}(\epsilon_j) \leqslant 2C$.

If γ' is an incomplete factor of $[f^{n'}(\gamma)]$, as explained above, there exists $i \in \{1, \dots, k_2\}$ such that γ' is contained in $\tau_{i-1}^2\epsilon_{i-1}\tau_i^1$. By Lemma 5.6, we have

$$\ell_{exp}(\gamma') \leqslant \ell_{exp}(\tau_{i-1}^2\epsilon_{i-1}\tau_i^1) + 2C.$$

By Lemma 3.17, the exponential length of γ' is at most equal to

$$\ell_{exp}(\tau_{i-1}^2) + \ell_{exp}(\epsilon_{i-1}) + \ell_{exp}(\tau_i^1) + 2C \leqslant 6C + \ell_{exp}(\epsilon_{i-1}) \leqslant 8C.$$

This proves (b) when Λ_{exp} is not empty.

Finally, suppose that Λ_{exp} is empty. For every $j \in \{1, \dots, k_1\}$, the path $[f^{m_1}(\alpha_j^{(1)})]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. By Lemma 3.6, the path $[f^{m_1}(\gamma)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. By Lemma 3.10, for every $n' \geqslant m_1$, the path $[f^{n'}(\gamma)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. $\square$

Lemma 5.12. *Let $f\colon G \to G$ be a $3K$-expanding CT map. There exists $N \in \mathbb{N}^*$ such that for every reduced edge path γ and every $m \geqslant N$, the total exponential length of incomplete factors in any optimal splitting of $[f^m(\gamma)]$ is uniformly bounded by $8C\ell_{exp}(\gamma)$.*

Proof. By Proposition 2.5(8), there exists $N \in \mathbb{N}^*$ such that, for every reduced edge path α of length at most equal to $C+1$, the path $[f^N(\alpha)]$ is completely split. Suppose first that $\ell_{exp}(\gamma) = 0$. Then, by definition of the exponential length, the path γ is a concatenation of paths in G'_{PG} and in $\mathcal{N}_{PG}$. By Proposition 2.5(4), every edge in a zero stratum is adjacent to either an edge in a zero stratum or an edge in an EG stratum. Moreover, by Lemma 2.9, there does not exist a subpath of γ contained in a zero stratum which is adjacent to a Nielsen path. Hence γ is either a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ or a path in a zero stratum.

In the first case, the path γ is PG-relative completely split. In the second case, by the definition of the constant K and Equation (6), we have $\ell(\gamma) \leqslant K \leqslant C$. By the choice of N, for every $m \geqslant N$, the path $[f^m(\gamma)]$ is completely split. By Lemma 3.20, for every $m \geqslant N$, the path $[f^m(\gamma)]$ is PG-relative completely split.

So we may suppose that $\ell_{exp}(\gamma) > 0$. Let $\gamma = \gamma_0\gamma'_1\gamma_1 \dots \gamma'_\ell\gamma_\ell$ be the exponential decomposition of γ (see the beginning of Section 3.2). By Lemma 2.9, there does not exist a subpath of γ contained in a zero stratum which is adjacent to a Nielsen path. Therefore, the path γ has a decomposition $\alpha_0\beta_1\alpha_1 \dots \beta_k\alpha_k$ where, for every $i \in \{0, \dots, k\}$, the path α_i is a (possibly trivial) concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ and, for every $i \in \{1, \dots, k\}$, the path β_i is a concatenation of a (possibly trivial) maximal reduced path in a zero stratum and an edge in an irreducible stratum not contained in G_{PG} or in some γ_i. By construction of K, for every $i \in \{1, \dots, k\}$, we have $\ell(\beta_i) \leqslant C+1$. By the choice of N, for every $m \geqslant N$, the path $[f^m(\beta_i)]$ is completely split.

Note that, for every $i \in \{1, \dots, k\}$, we have $\ell_{exp}(\beta_i) = 1$ and that

$$\ell_{exp}(\gamma) = \sum_{i=1}^{k} \ell_{exp}(\beta_i) = k.$$

By Lemma 3.10, for every $i \in \{0, \dots, k\}$ and every $m \geqslant M$, the path $[f^m(\alpha_i)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. By Lemma 3.18, for every $m \geqslant M$, we have $\ell_{exp}([f^m(\alpha_i)]) = 0$. By Lemma 5.6, the exponential length of the subpath of $[f^m(\alpha_i)]$ contained in $[f^m(\gamma)]$ is at most equal to $2C$.

For every $i \in \{0, \dots, k\}$ (resp. $i \in \{1, \dots, k\}$) and every $m \geqslant N$, let $\alpha_{i,m}$ (resp. $\beta_{i,m}$) be the subpath of $[f^m(\alpha_i)]$ (resp. $[f^m(\beta_i)]$) contained in $[f^m(\gamma)]$. By Remark 5.9(2), for every $i \in \{1, \dots, k\}$ and every $m \geqslant N$, the exponential length of any incomplete factor in $\beta_{i,m}$ is at most equal to $4C$. By Lemma 3.17, for every $m \geqslant N$, the sum of the exponential lengths of the incomplete factors in $[f^m(\gamma)]$ is at most equal to

$$\sum_{i=0}^{k} \ell_{exp}(\alpha_{i,m}) + 4Ck \leqslant 2C(k+1) + 4kC \leqslant 4Ck + 4Ck = 8Ck = 8C\ell_{exp}(\gamma).$$

The conclusion of the lemma follows. □

Lemma 5.13. *Let $f\colon G \to G$ be a $3K$-expanding CT map. Let γ be a reduced edge path in G. Suppose that γ has a splitting $\gamma = b_1ab_2$ where, for every $i \in \{1, 2\}$,* *the (possibly trivial) path b_i is PG-relative completely split. If $\ell^{\gamma}_{exp}(a) = 0$ then $\ell_{exp}(a) = 0$.*

Proof. Let $\gamma = \gamma_0\gamma_1'\gamma_1 \ldots \gamma_k'\gamma_k$ be the exponential decomposition of γ. By Lemma 5.6, there exist three (possibly trivial) paths δ_1, δ_2 and τ such that for every $i \in \{1, 2\}$, the path δ_i is a proper initial or terminal subpath of a splitting unit of some γ_j we have $\ell_{exp}(\tau) = \ell^{\gamma}_{exp}(\tau) = \ell^{\gamma}_{exp}(a)$ and $a = \delta_1\tau\delta_2$. Since $\ell^{\gamma}_{exp}(a) = 0$, we have $\ell_{exp}(\tau) = 0$. Hence τ is a concatenation of paths in G'_{PG} and in $\mathcal{N}_{PG}$.

By Proposition 2.5(4), every edge in a zero stratum is adjacent to either an edge in a zero stratum or an edge in an EG stratum. Moreover, by Lemma 2.9, there does not exist a subpath of γ contained in a zero stratum which is adjacent to a Nielsen path. Hence τ is either a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ or a path in a zero stratum.

If τ is contained in a zero stratum, by Lemma 2.9, we see that δ_1 and δ_2 are trivial, that is, $a = \tau$. Thus, we have $\ell_{exp}(a) = \ell_{exp}(\tau) = 0$.

So we may suppose that τ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Suppose towards a contradiction that there exists $i \in \{1, 2\}$ such that δ_i is not trivial. For every $i \in \{1, 2\}$ such that $\delta_i \neq \varnothing$, let σ_i be the splitting unit of some γ_j containing δ_i and let r_i be the height of σ_i. By [BH, Lemma 5.11], for every $i \in \{1, 2\}$ such that δ_i is not trivial, there exist two distinct r_i-legal paths α_i and β_i such that $\sigma_i = \alpha_i\beta_i$ and such that the turn $\{Df(\alpha_i^{-1}), Df(\beta_i)\}$ is the only height r_i illegal turn. Moreover, there exists a path τ_i' such that $[f(\alpha_i)] = \alpha_i\tau_i'$ and $[f(\beta_i)] = \tau_i'^{-1}\beta_i$. Let $\epsilon_1^{(1)}, \epsilon_1^{(2)}$ be two paths such that $\sigma_1 = \epsilon_1^{(1)}\epsilon_1^{(2)}$, the path $\epsilon_1^{(1)}$ is contained in b_1 and the path $\epsilon_1^{(2)}$ is contained in a. Similarly, let $\epsilon_2^{(1)}, \epsilon_2^{(2)}$ be two paths such that $\sigma_2 = \epsilon_2^{(1)}\epsilon_2^{(2)}$, the path $\epsilon_2^{(2)}$ is contained in b_2 and the path $\epsilon_2^{(1)}$ is contained in a.

Claim.

(1) For every path $b \in \mathcal{N}^{\max}_{PG}(b_1)$ (resp. $b \in \mathcal{N}^{\max}_{PG}(b_2)$), the path b does not contain edges of $\epsilon_1^{(1)}$ (resp. $\epsilon_2^{(2)}$).
(2) The path $\epsilon_1^{(1)}$ is r_1-legal and the path $\epsilon_2^{(2)}$ is r_2-legal.

Proof. We prove the claim for b_1, the proof for b_2 being similar.

(1) Let $b \in \mathcal{N}^{\max}_{PG}(b_1)$. There exists $c \in \mathcal{N}^{\max}_{PG}(\gamma)$ such that $b \subseteq c$. Moreover, by Lemma 3.5(3) applied to $\gamma' = b$ and $\gamma = c$, either b is a concatenation of splitting units of c or b is properly contained in a splitting unit of c and is not an initial or a terminal segment of c. Since b_1 is an initial segment of γ, the second case cannot occur. Hence b is a concatenation of splitting units of c. Since σ_1 is not contained in b_1, the path b cannot contain edges of σ_1. Since $\epsilon_1^{(1)} \subseteq \sigma_1$, the path b cannot contain edges of $\epsilon_1^{(1)}$.
(2) Suppose towards a contradiction that $\epsilon_1^{(1)}$ is not r_1-legal. Then it contains the illegal turn $\{Df(\alpha_1^{-1}), Df(\beta_2)\}$. Recall that the path b_1 is *PG*-relative completely split. By the description of *PG*-relative splitting units, the illegal turn must be contained in a *PG*-relative splitting unit of b_1 which is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Since the last edge of α_1 is an edge in an EG stratum, the last edge of α_1 must be contained in a

path contained in $\mathcal{N}_{PG}$. Hence $\epsilon_1^{(1)}$ intersects a path in $\mathcal{N}_{PG}^{\max}(b_1)$. This contradicts Assertion (1). □

By Assertion (2) of the claim, for every $i \in \{1,2\}$ such that σ_i is not trivial, the path $\epsilon_i^{(i)}$ is r_i-legal. Moreover, by Assertion (1) of the claim an INP contained in b_i cannot intersect the path $\epsilon_i^{(i)}$. Since the paths b_1 and b_2 are PG-relative completely split, the paths b_1 and b_2 split respectively at the origin of $\epsilon_1^{(1)}$ and at the end of $\epsilon_2^{(2)}$. So we may suppose that $b_1 = \epsilon_1^{(1)}$ and $b_2 = \epsilon_2^{(2)}$. Therefore, there exists a (possibly trivial) path τ_1 such that, up to taking a power of f so that the length of $[f(b_1)]$ is greater than α_1, we have $[f(b_1)] = \alpha_1\tau_1$ and $[f(\epsilon_1^{(2)})] = \tau_1^{-1}\beta_1$. Similarly, there exists a path τ_2 such that $[f(\epsilon_2^{(1)})] = \alpha_2\tau_2$ and $[f(b_2)] = \tau_2^{-1}\beta_2$.

Since γ splits at the concatenation points of b_1, a and b_2, the paths τ_1^{-1} and τ_2 contained in $[f(\epsilon_1^{(2)})][f(\tau)][f(\epsilon_2^{(1)})]$ must be identified when passing to $[f(a)]$. Suppose first that $[f(\tau)]$ is a point. Then since the EG INPs σ_1 and σ_2 are uniquely determined by their initial and terminal edges by Proposition 2.5(9), we see that $\sigma_1 = \sigma_2^{-1}$. But then there are some identifications between b_1 and b_2, which contradicts the fact that b_1ab_2 is a splitting.

Thus, we may suppose that $[f(\tau)]$ is nontrivial. By Lemma 3.10, since τ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ so is $[f(\tau)]$. Note that, since an EG INP is completely determined by its initial and terminal edges by Proposition 2.5(9), if $[f(\tau)]$ contains the initial or the terminal edge of an EG INP σ, then σ is contained in $[f(\tau)]$. Note that there are identifications between edges of $[f(\epsilon_1^{(2)})]$ and $[f(\tau)]$ or between edges of $[f(\tau)]$ and $[f(\epsilon_2^{(1)})]$. Therefore, $[f(\tau)]$ starts with σ_1^{-1} or $[f(\tau)]$ ends with σ_2^{-1}. Thus, one of the following holds:

(a) $[f(\tau)] = \sigma_1^{-1}\tau'$ with τ' a (possibly trivial) path which is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ which does not end by σ_2^{-1};
(b) $[f(\tau)] = \tau'\sigma_2^{-1}$ with τ' a (possibly trivial) path which is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ which does not start by σ_1^{-1};
(c) $[f(\tau)] = \sigma_1^{-1}\tau'\sigma_2^{-1}$ with τ' a (possibly trivial) path.

We treat the three cases simultaneously by considering Case (c) and assuming that σ_1^{-1} and σ_2^{-1} might be trivial. Note that $\sigma_1^{-1}\tau'\sigma_2^{-1}$ is reduced since it is equal to $[f(\tau)]$, so that there is no identification between α_1^{-1} and τ' and between τ' and β_2^{-1}.

Let e_{σ_1} be the terminal edge of σ_1 and let e_{σ_2} be the initial edge of σ_2. By Proposition 2.5(9), both e_{σ_1} and e_{σ_2} are edges in EG strata. Since f is $3K$-expanding, for every $i \in \{1,2\}$, the path $[f(e_{\sigma_i})]$ has length at least equal to $3K$. Recall that, for every $i \in \{1,2\}$, by definition of K, we have $\ell(\sigma_i) \leqslant K$, so that $\ell(\alpha_i), \ell(\beta_i) \leqslant K$. Since $[f(\epsilon_1^{(2)})] = \tau_1^{-1}\beta_1$ and $[f(\epsilon_2^{(1)})] = \alpha_2\tau_2$, the path $[f(e_{\sigma_1})]$ contains a nondegenerate terminal segment of τ_1^{-1} and the path $[f(e_{\sigma_2})]$ contains a nondegenerate initial segment of τ_2. As e_{σ_1} is r_1-legal and as f is a relative train track by Proposition 2.5(1), we see that the last edge of τ_1^{-1} is not the last edge of α_1. Similarly, the first edge of τ_2 is not the first edge of β_2. Therefore, we have $[\tau_1^{-1}\beta_1\sigma_1^{-1}] = \tau_1^{-1}\alpha_1^{-1}$ and $[\sigma_2^{-1}\alpha_2\tau_2] = \beta_2^{-1}\tau_2$. Thus we have

$$[[f(\epsilon_1^{(2)})][f(\tau)][f(\epsilon_2^{(1)})]] = [\tau_1^{-1}\beta_1\sigma_1^{-1}\tau'\sigma_2^{-1}\alpha_2\tau_2] = [\tau_1^{-1}\alpha_1^{-1}\tau'\beta_2^{-1}\tau_2],$$

and there is no identification between τ_1^{-1} and α_1^{-1}, α_1^{-1} and τ', τ' and β_2^{-1} and β_2^{-1} and τ_2. Therefore, if τ' is not trivial, then we have a contradiction as τ_1^{-1} and τ_2 are not identified in $[f(a)]$.

Suppose that τ' is trivial. Then the paths τ_1^{-1} and τ_2 are identified in $[f(a)]$ only if a terminal segment of α_1^{-1} is identified with an initial segment of β_2^{-1}. Since EG INP are uniquely determined by their initial and terminal edges by Proposition 2.5(9), we see that $\sigma_1 = \sigma_2^{-1}$. Hence $\alpha_1^{-1} = \beta_2$ and either τ_1^{-1} is an initial segment of τ_2^{-1} or τ_2 is an initial segment of τ_1.

Up to changing the orientation of γ, we may suppose that τ_1^{-1} is an initial segment of τ_2^{-1}. If $\tau_1^{-1} = \tau_2^{-1}$, then $[f(a)]$ is a vertex. Moreover, as $\sigma_1 = \sigma_2^{-1}$, the segment $b_1 = \epsilon_1^{(1)}$ is equal to b_2^{-1}. Therefore, a terminal segment of b_1 is identified with an initial segment of b_2, a contradiction. If τ_1^{-1} is a proper initial segment of τ_2^{-1}, then τ_2 is identified with edges in b_1, a contradiction. As we have considered every case, we see that δ_1 and δ_2 are trivial and $\ell_{exp}(a) = \ell_{exp}(\tau) = 0$. □

Lemma 5.14. *Let $f\colon G \to G$ be a $3K$-expanding CT map. There exists $n_0 \in \mathbb{N}^*$ such that for every $n \geqslant n_0$, and every closed reduced edge path γ of G, we have:*

$$\mathfrak{g}([f^n(\gamma)]) \geqslant \mathfrak{g}(\gamma).$$

Proof. By Lemma 3.23, there exists $N_0 \in \mathbb{N}^*$ such that, for every $n \geqslant N_0$ and every PG-relative splitting unit σ, the exponential length of the path $[f^n(\sigma)]$ is at least equal to the one of σ. By Lemma 5.12, there exists N_1 such that for every $n \geqslant N_1$ and every closed reduced edge path γ of G, the total exponential length of incomplete segments in any optimal splitting of $[f^n(\gamma)]$ is bounded by $8C\ell_{exp}(\gamma)$. Let $N_2 = \lceil \log_3(10C + 16C^2) \rceil \in \mathbb{N}^*$ be such that for every $x, y \geqslant 0$ such that $(x, y) \neq (0, 0)$, we have

$$\frac{(3^{N_2} - 2C)x}{(3^{N_2} - 2C)x + 8C(1 + 2C)y} \geqslant \frac{x}{x + y}.$$

Let $n_0 = \max\{N_0, N_1, N_2\}$.

Let γ be a closed reduced edge path in G. All splittings of γ are circuital splittings in what follows. Let $\gamma = \alpha_0\beta_1\alpha_1 \ldots \beta_k\alpha_k$ be an optimal splitting of γ, where for every $i \in \{0, \ldots, k\}$, the path α_i is an incomplete factor of γ and for every $i \in \{1, \ldots, k\}$, the path β_i is a PG-relative complete factor of γ. First note that, for every $i \in \{1, \ldots, k\}$, and for every $n \geqslant 1$, the path $[f^n(\beta_i)]$ is PG-relative completely split by Proposition 2.5(6) and Lemma 3.10. Therefore, if $n \geqslant n_0 \geqslant N_0$, the total exponential length of such PG-relative complete segments is nondecreasing under $[f^n]$. We now distinguish two cases, according to the growth of the paths β_i.

Suppose first that for every $i \in \{1, \ldots, k\}$, the exponential length of β_i relative to γ is equal to zero. Since the splitting $\gamma = \alpha_0\beta_1\alpha_1 \ldots \beta_k\alpha_k$ is optimal and since for every $i \in \{1, \ldots, k\}$, we have $\ell_{exp}^{\gamma}(\beta_i) = 0$, we have $\mathfrak{g}(\gamma) = 0$. Therefore, for every $n \in \mathbb{N}^*$, we have $\mathfrak{g}([f^n(\gamma)]) \geqslant \mathfrak{g}(\gamma)$.

Suppose now that there exists $i \in \{1, \ldots, k\}$ such that the exponential length of β_i relative to γ is positive. By Lemma 3.22, the sequence $(\ell_{exp}([f^n(\beta_i)]))_{n \in \mathbb{N}^*}$ grows exponentially with n. We can now modify the splitting of γ into the following splitting: $\gamma = \alpha'_0\beta'_1\alpha'_1 \ldots \beta'_m\alpha'_m$ where:

(a) for every $j \in \{0, \ldots, m\}$, the path α'_i is a concatenation of incomplete factors and complete factors of zero exponential length relative to γ of the previous splitting;

(b) for every $j \in \{1,\dots,m\}$, the path β_i' is a complete factor of positive exponential length relative to γ of the previous splitting.

Note that, by definition of the exponential length relative to γ, for every $i \in \{1,\dots,m\}$ and every path $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma)$, the path β_i' is not contained in γ'. Therefore, if there exists $j \in \{0,\dots,m\}$ and $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma)$ such that α_j' intersects γ' nontrivially, then γ' is contained in $\beta_{j-1}'\alpha_j'\beta_j'$. In particular, Lemma 5.13 applies and for every $j \in \{0,\dots,m\}$, if $\ell_{exp}^{\gamma}(\alpha_j') = 0$, then $\ell_{exp}(\alpha_j') = 0$. Let Λ be the subset of $\{0,\dots,m\}$ such that for every $j \in \Lambda$, we have $\ell_{exp}^{\gamma}(\alpha_j') > 0$.

By Lemma 5.6 and Lemma 5.7, for every $j \in \{1,\dots,m\}$ and every $M \in \mathbb{N}^*$, we have

$$\ell_{exp}^{[f^M(\gamma)]}([f^M(\beta_i')]) \geqslant \ell_{exp}([f^M(\beta_i')]) - 2C \geqslant 3^M \ell_{exp}(\beta_i') - 2C \geqslant (3^M - 2C)\ell_{exp}^{\gamma}(\beta_i').$$

By Lemma 5.6, for every $j \in \{0,\dots,m\}$, we have $\ell_{exp}^{\gamma}(\alpha_j') \leqslant \ell_{exp}(\alpha_j')$. Note that, for every $i \in \{1,\dots,m\}$, and every $n \in \mathbb{N}^*$, the path $[f^n(\beta_i')]$ is PG-relative completely split. In particular, for every $n \in \mathbb{N}^*$, any incomplete factor of $[f^n(\gamma)]$ is contained in a reduced iterate of some α_i'. Thus, by Lemma 5.12, for every $n \geqslant n_0 \geqslant N_1$, the total exponential length of incomplete segments in $[f^n(\gamma)]$ is bounded by $8C\sum_{j=1}^{k} \ell_{exp}(\alpha_j') = 8C\sum_{j\in\Lambda} \ell_{exp}(\alpha_j')$. Note that the function

$$x \mapsto \frac{x}{x + 8C\sum_{j\in\Lambda} \ell_{exp}(\alpha_j')}$$

is nondecreasing. Recall that, for every $n \in \mathbb{N}^*$, the goodness function is a supremum over splittings of $[f^n(\gamma)]$. Thus, by Lemma 5.4, for every $n \geqslant n_0$, we have:

$$\mathfrak{g}([f^n(\gamma)]) \geqslant \frac{(3^n - 2C)\sum_{i=1}^{m} \ell_{exp}^{\gamma}(\beta_i')}{(3^n - 2C)\sum_{i=1}^{m} \ell_{exp}^{\gamma}(\beta_i') + 8C\sum_{j\in\Lambda} \ell_{exp}(\alpha_j')}.$$

By Lemma 5.6, we have

$$8C\sum_{j\in\Lambda} \ell_{exp}(\alpha_j') \leqslant 8C\sum_{j\in\Lambda}(\ell_{exp}^{\gamma}(\alpha_j') + 2C) \leqslant 8C(1+2C)\sum_{j\in\Lambda} \ell_{exp}^{\gamma}(\alpha_j'),$$

where the last inequality follows from the fact that, for every $j \in \Lambda$, we have $\ell_{exp}^{\gamma}(\alpha_j') \geqslant 1$. Therefore, since $n_0 \geqslant N_2$, for every $n \geqslant n_0$, we have:

$$\frac{(3^n - 2C)\sum_{j=1}^{m} \ell_{exp}^{\gamma}(\beta_j')}{(3^n - 2C)\sum_{j=1}^{m} \ell_{exp}^{\gamma}(\beta_j') + 8C(1+2C)\sum_{j\in\Lambda} \ell_{exp}^{\gamma}(\alpha_j')} \geqslant \frac{\sum_{j=1}^{m} \ell_{exp}^{\gamma}(\beta_j')}{\sum_{j=1}^{m} \ell_{exp}^{\gamma}(\beta_j') + \sum_{j\in\Lambda} \ell_{exp}^{\gamma}(\alpha_j')}.$$

By Lemma 5.3, we have

$$\ell_{exp}(\gamma) = \sum_{j=1}^{m} \ell_{exp}^{\gamma}(\beta_j') + \sum_{j=0}^{m} \ell_{exp}^{\gamma}(\alpha_j') = \sum_{j=1}^{m} \ell_{exp}^{\gamma}(\beta_j') + \sum_{j\in\Lambda} \ell_{exp}^{\gamma}(\alpha_j').$$

Thus, we see that

$$\frac{\sum_{j=1}^{m} \ell_{exp}^{\gamma}(\beta_j')}{\sum_{j=1}^{m} \ell_{exp}^{\gamma}(\beta_j') + \sum_{j\in\Lambda} \ell_{exp}^{\gamma}(\alpha_j')} = \mathfrak{g}(\gamma),$$

which gives the result. $\square$

Remark 5.15. In the next lemmas, we will adopt the following conventions.
Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an atoroidal or an almost atoroidal outer automorphism relative to $\mathcal{F}$. Let $f \colon G \to G$ be a CT map representing a power of ϕ with filtration

$$\varnothing = G_0 \subsetneq \ldots \subsetneq G_k = G.$$

Let $p \in \{1, \ldots, k-1\}$ be such that $\mathcal{F}(G_p) = \mathcal{F}$. By Lemma 3.22, up to taking a power of f, we may suppose that f is $3K$-expanding. By Lemma 5.14, up to passing to a power of f, we may suppose that for every closed reduced edge path γ of G, we have $\mathfrak{g}([f(\gamma)]) \geqslant \mathfrak{g}(\gamma)$.

Lemma 5.16. *Let $f \colon G \to G$ be as in Remark* 5.15.

(1) *For every $\delta > 0$, there exists $m \in \mathbb{N}^*$ such that for every reduced edge path γ such that $\mathfrak{g}(\gamma) \geqslant \delta$ and every $n \geqslant m$, the total exponential length relative to $[f^n(\gamma)]$ of complete factors in $[f^n(\gamma)]$ denoted by $TEL(n,\gamma)$ is at least*

$$TEL(n,\gamma) \geqslant \mathfrak{g}(\gamma)\ell_{exp}(\gamma)(3^n - 2C).$$

(2) *For every $\delta > 0$ and every $\epsilon > 0$, there exists $m \in \mathbb{N}^*$ such that for every cyclically reduced circuit γ such that $\ell_{exp}(\gamma) > 0$ and $\mathfrak{g}(\gamma) \geqslant \delta$ and every $n \geqslant m$, we have $\mathfrak{g}([f^n(\gamma)]) \geqslant 1 - \epsilon$.*

Proof. Let $\gamma = \alpha_0\beta_1\alpha_1 \ldots \alpha_k\beta_k$ be an optimal splitting, where for every $i \in \{0, \ldots, k\}$, the path α_i is an incomplete factor of γ and for every $i \in \{1, \ldots, k\}$, the path β_i is a PG-relative complete factor of γ. We may assume that $\ell_{exp}(\gamma) > 0$, otherwise $\mathfrak{g}(\gamma) = 0$ and the result is immediate. Note that, since $\mathfrak{g}(\gamma) \geqslant \delta > 0$, there exists $i \in \{1, \ldots, k\}$ such that $\ell_{exp}^{\gamma}(\beta_i) > 0$. Let Λ_γ be the set consisting of all complete factors β_i of γ whose exponential length relative to γ is positive. Let $\ell_{exp}^{\gamma}(\Lambda_\gamma)$ be the sum of the exponential lengths relative to γ of all factors that belong to Λ_γ. Note that

$$\ell_{exp}^{\gamma}(\Lambda_\gamma) = \sum_{\beta_i \in \Lambda_\gamma} \ell_{exp}^{\gamma}(\beta_i) = \mathfrak{g}(\gamma)\ell_{exp}(\gamma).$$

Note that, for every $n \in \mathbb{N}^*$, the value $TEL(n,\gamma)$ is a supremum over all splittings of $[f^n(\gamma)]$. Thus, by Lemma 5.6 and Lemma 5.7, for every $n \in \mathbb{N}^*$, we have:

$$TEL(n,\gamma) \geqslant \sum_{\beta_i \in \Lambda_\gamma} \ell_{exp}^{[f^n(\gamma)]}([f^n(\beta_i)]) \geqslant (3^n - 2C)\ell_{exp}^{\gamma}(\Lambda_\gamma) \geqslant (3^n - 2C)\mathfrak{g}(\gamma)\ell_{exp}(\gamma).$$

This proves (1). We now prove (2). By Lemma 5.12, there exists $n_0 \in \mathbb{N}^*$ such that for every $n \geqslant n_0$, the total exponential length of incomplete segments in $[f^n(\gamma)]$ is bounded by $8C\ell_{exp}(\gamma)$. By Lemma 5.6, the total exponential length relative to γ of incomplete segments in $[f^n(\gamma)]$ is hence bounded by $10C\ell_{exp}(\gamma)$. Note that, for every $n \in \mathbb{N}^*$, the value $\mathfrak{g}([f^n(\gamma)])$ is a supremum over all splittings of $[f^n(\gamma)]$. Thus, by Lemma 5.4, for every $n \geqslant n_0$, we have:

$$\begin{aligned}\mathfrak{g}([f^n(\gamma)]) &\geqslant \frac{\mathfrak{g}(\gamma)\ell_{exp}(\gamma)(3^n - 2C)}{10C\ell_{exp}(\gamma) + \mathfrak{g}(\gamma)\ell_{exp}(\gamma)(3^n - 2C)} \\ &= \frac{\mathfrak{g}(\gamma)(3^n - 2C)}{10C + \mathfrak{g}(\gamma)(3^n - 2C)} \geqslant \frac{\delta(3^n - 2C)}{10C + \delta(3^n - 2C)}.\end{aligned}$$

The last term is independent of γ and converges to 1 as n goes to infinity. Therefore the conclusion of Lemma 5.16 holds for some n large enough which does not depend on γ. This proves (2) and this concludes the proof. $\square$

5.2. **North-South dynamics for a relative atoroidal outer automorphism.** Let $\mathtt{n} \geqslant 3$ and let $\mathcal{F}$ be a free factor system of $F_{\mathtt{n}}$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an atoroidal or an almost atoroidal automorphism relative to $\mathcal{F}$. In this subsection we prove Theorem 5.1. The proof of Theorem 5.1 is inspired by the proof of the same result due to Uyanik [Uya2] in the context of an atoroidal outer automorphism, that is, in the special case when $\mathcal{F} = \varnothing$. The proof relies on the study of splittings of reduced edge paths in the graph associated with a CT map representing a power of ϕ. Indeed, we show that, when a cyclically reduced edge path representing $w \in F_{\mathtt{n}}$ has a splitting which is close to a complete splitting, then some iterate of ϕ sends $[w]$ into an open neighborhood of $\Delta_+(\phi)$ (see Definition 4.5), and this iterate can be chosen uniformly (see Lemma 5.20).

Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an almost atoroidal outer automorphism. Let $\mathcal{F} \leqslant \mathcal{F}_1 \leqslant \mathcal{F}_2 = \{[F_{\mathtt{n}}]\}$ be a sequence of free factor system given in this definition. Let $f\colon G \to G$ be a CT map representing a power of ϕ with filtration $\varnothing = G_0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_k = G$ and such that there exist p and i in $\{1, \ldots, k\}$ with $\mathcal{F}(G_p) = \mathcal{F}$ and $\mathcal{F}(G_i) = \mathcal{F}_1$. We denote by $\mathrm{Curr}(\mathcal{F}_1, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$ the set of currents of $\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$ whose support is contained in $\partial^2\mathcal{F}_1$.

Note that, since the extension $\mathcal{F}_1 \leqslant \{[F_{\mathtt{n}}]\}$ is sporadic, either $\mathcal{F}_1 = \{[H_1], [H_2]\}$ or $\mathcal{F}_1 = \{[H]\}$ for some subgroups H, H_1, H_2 of $F_{\mathtt{n}}$. Up to assuming that H_2 is the trivial group, we may assume that $\mathcal{F}_1 = \{[H_1], [H_2]\}$. Moreover, we have $\mathcal{F}_1 \wedge \mathcal{A}(\phi) = \{[A_1], \ldots, [A_s], [B_1], \ldots, [B_t]\}$ where, for every $j \in \{1, \ldots, s\}$, the group A_j is contained in H_1 and for every $j \in \{1, \ldots, t\}$, the group B_j is contained in H_2. Since $\mathcal{F}_1 \wedge \mathcal{A}(\phi)$ is a malnormal subgroup system, the set $\{[A_1], \ldots, [A_s]\}$ is a malnormal subgroup system of H_1 and the set $\{[B_1], \ldots, [B_t]\}$ is a malnormal subgroup system of H_2.

Let

$$X(\mathcal{F}_1) = \mathrm{Curr}(H_1, \{[A_1], \ldots, [A_s]\}) \times \mathrm{Curr}(H_2, \{[B_1], \ldots, [B_t]\}).$$

Let $\mu \in \mathrm{Curr}(\mathcal{F}_1, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$. We set $\psi_1(\mu) = (\mu|_{\partial^2 H_1}, \mu|_{\partial^2 H_2}) \in X(\mathcal{F}_1)$. Since μ is $F_{\mathtt{n}}$-invariant, $\psi_1(\mu)$ does not depend on the choice of the representatives of the conjugacy classes of H_1 and H_2. Let $(\mu_1, \mu_2) \in X(\mathcal{F}_1)$. Since the subgroup system $\mathcal{F}_1 \wedge \mathcal{A}(\phi)$ is malnormal, for every $j \in \{1, 2\}$, the current μ_j can be extended in a canonical way to a current $\mu_j^* \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$. The current μ_j^* is such that, for every Borel subset B of $\partial^2(F_{\mathtt{n}}, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$, we have

$$\mu_j^*(B) = \mu_j^*(B \cap \partial^2 H_j) = \mu_j(B \cap \partial^2 H_j).$$

We set $\psi_2((\mu_1, \mu_2)) = \mu_1^* + \mu_2^*$. By the property of μ_j^* described above, we see that $\psi_2((\mu_1, \mu_2)) \in \mathrm{Curr}(\mathcal{F}_1, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$. The maps ψ_1 and ψ_2 are clearly continuous.

Lemma 5.17. *The space* $\mathrm{Curr}(\mathcal{F}_1, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$ *is homeomorphic to* $X(\mathcal{F}_1)$.

Proof. We prove that ψ_1 and ψ_2 are inverse from each other. Let $\mu \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$. Then $\psi_2 \circ \psi_1(\mu) = (\mu|_{\partial^2 H_1})^* + (\mu|_{\partial^2 H_2})^*$. Note that μ and $\psi_2 \circ \psi_1(\mu)$ coincide on Borel subsets contained in $\partial^2\mathcal{F}_1$. Since both have supports contained in $\partial^2\mathcal{F}_1$, they are equal. Conversely, let $(\mu_1, \mu_2) \in X(\mathcal{F}_1)$. Then

$$\psi_1 \circ \psi_2((\mu_1, \mu_2)) = ((\mu_1^* + \mu_2^*)|_{\partial^2 H_1}, (\mu_1^* + \mu_2^*)|_{\partial^2 H_2}).$$

But $\mu_2^*|_{\partial^2 H_1} = 0$ and $\mu_1^*|_{\partial^2 H_2} = 0$. Hence we have

$$((\mu_1^* + \mu_2^*)|_{\partial^2 H_1}, (\mu_1^* + \mu_2^*)|_{\partial^2 H_2}) = (\mu_1^*|_{\partial^2 H_1}, \mu_2^*|_{\partial^2 H_2}) = (\mu_1, \mu_2).$$

This concludes the proof. $\square$

Given $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$, we refer to the definition of $\mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ given above Lemma 3.29.

Lemma 5.18. *Let* $\mathtt{n} \geqslant 3$ *and let* $\mathcal{F}$ *be a free factor system of* $F_{\mathtt{n}}$. *Let* $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ *be an almost atoroidal outer automorphism. Let* $\mathcal{F} \leqslant \mathcal{F}_1 \leqslant \mathcal{F}_2 = \{F_{\mathtt{n}}\}$ *be a sequence of free factor systems given in this definition. Let* $f\colon G \to G$ *be a CT map representing a power of* ϕ *with filtration* $\varnothing = G_0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_k = G$ *and such that there exist* p *and* i *in* $\{0, \ldots, k-1\}$ *such that* $\mathcal{F}(G_p) = \mathcal{F}$ *and* $\mathcal{F}(G_i) = \mathcal{F}_1$.

(1) *The graph* $\overline{G - G_i}$ *either is a topological arc whose endpoints are in* G_i *or it retracts onto a circuit* C *and there exists exactly one topological arc that connects* C *and* G_i.
(2) *There does not exist an EG stratum or a zero stratum of height greater than* i. *If* $\overline{G - G_i}$ *is a topological arc, every edge in* $\overline{G - G_i}$ *is contained in* G_{PG}. *Otherwise every edge of the circuit* C *in* $\overline{G - G_i}$ *is contained in* G_{PG}.
(3) *Let* γ *be a path of* G_i *which is not contained in a concatenation of paths of* $G_{PG,\mathcal{F}_1}$ *and* $\mathcal{N}_{PG,\mathcal{F}_1}$. *Then* γ *is not contained in a concatenation of paths in* G_{PG} *and in* $\mathcal{N}_{PG}$.
(4) *We have*
$$\partial^2(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) = \bigcup_{\gamma \in \mathcal{P}(\mathcal{F}_1 \wedge \mathcal{A}(\phi))} C(\gamma).$$
In particular, we have
$$\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) = \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F}_1 \wedge \mathcal{A}(\phi)).$$
(5) *For every edge path* γ *in* G, *the value* $\ell_{\mathcal{F}_1}(\gamma) - \ell_{exp}(\gamma)$ *is the number of edges of* $\overline{G - G_i}$ *contained in* γ. *In particular, for every path* γ *contained in* G_i, *we have*
$$\ell_{\mathcal{F}_1}(\gamma) = \ell_{exp}(\gamma)$$
and for every current $\mu \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ *whose support is contained in* $\partial^2\mathcal{F}_1$, *we have*
$$\Psi_0(\mu) = \|\mu\|_{\mathcal{F}_1}.$$
(6) *Let* γ *be a circuit in* G. *For every* $m \in \mathbb{N}^*$, *we have*
$$\ell_{\mathcal{F}_1}([f^m(\gamma)]) - \ell_{exp}([f^m(\gamma)]) = \ell_{\mathcal{F}_1}(\gamma) - \ell_{exp}(\gamma).$$
(7) *Suppose that* $\mathcal{F} \wedge \mathcal{A}(\phi) = \{[A_1], \ldots, [A_r]\}$. *One of the following holds.*
   - *There exist distinct* $i, j \in \{1, \ldots, r\}$ *such that*
   $$\mathcal{A}(\phi) = (\mathcal{F} \wedge \mathcal{A}(\phi)) - \{[A_i], [A_j]\}) \cup \{[A_i * A_j]\}.$$
   - *There exist* $i \in \{1, \ldots, r\}$ *and an element* $g \in F_{\mathtt{n}}$ *such that*
   $$\mathcal{A}(\phi) = (\mathcal{F} \wedge \mathcal{A}(\phi)) - \{[A_i]\}) \cup \{[A_i * \langle g \rangle]\}.$$
   In that case, there exists a subgroup A *of* $F_{\mathtt{n}}$ *such that* $\mathcal{F}_1 = \{[A]\}$ *and* $F_{\mathtt{n}} = A * \langle g \rangle$.
   - *There exists* $g \in F_{\mathtt{n}}$ *such that* $\mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi) \cup \{[\langle g \rangle]\}$. *In that case, there exists a subgroup* A *of* $F_{\mathtt{n}}$ *such that* $\mathcal{F}_1 = \{[A]\}$ *and* $F_{\mathtt{n}} = A * \langle g \rangle$.

Proof. (1) It is a consequence of [HM, Lemma II.2.5]. Note that, in the terminology of [HM, Lemma II.2.5], the first case is called a *one-edge extension* and the second case is called a *lollipop extension.*

(2) By Proposition 2.5(4), it suffices to show that there does not exist an EG stratum of height greater than i. This follows from [BFH1, Corollary 3.2.2] (where the stratum described in it is the whole graph $\overline{G - G_i}$)

We now prove the second part of Assertion (2). Let w be an element of $F_{\mathtt{n}}$ represented by γ. Then there exists a subgroup A of $F_{\mathtt{n}}$ such that $[A] \in \mathcal{A}(\phi)$ and $w \in A$. Since $\phi|_{\mathcal{F}_1}$ is expanding relative to $\mathcal{F}$ but ϕ is not expanding relative to $\mathcal{F}$ by Definition 4.3(b), there exists a reduced circuit γ in G which is not contained in G_i which has polynomial growth under iterates of f. By Proposition 3.14, the circuit γ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. By the first part of Assertion (2), the intersection $\gamma \cap \overline{G - G_i}$ does not contain EG INPs, hence consists in edges in G_{PG}.

Thus, if $\overline{G - G_i}$ is a lollipop, then the circuit C in $\overline{G - G_i}$ is contained in γ, hence is contained in G_{PG}. If $\overline{G - G_i}$ is a topological arc, the graph $\overline{G - G_i}$ is contained in γ, hence consists in edges in G_{PG}. This proves (2).

(3) Let γ be as in Assertion (3). By Assertion (2), every edge of $\overline{G - G_i}$ is contained in an NEG stratum. In particular, there does not exist an EG INP of height greater than i. Hence $\mathcal{N}_{PG} = \mathcal{N}_{PG,\mathcal{F}_1}$. Since γ is contained in G_i and since $G_{PG} \cap G_i = G_{PG,\mathcal{F}_1}$, the path γ is not contained in a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$.

(4) Since $\phi|_{\mathcal{F}_1}$ is expanding relative to $\mathcal{F}$, we see that $\mathcal{F}_1 \wedge \mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi)$. Thus, we have $\partial^2(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) = \partial^2(F_{\mathtt{n}}, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$. Assertion (4) then follows from Lemma 3.29 applied to $\mathcal{F}_1 \wedge \mathcal{A}(\phi)$.

(5) By Assertion (2), there does not exist an EG INP of height at least $i+1$. Hence $\ell_{\mathcal{F}_1}(\gamma)$ differs from $\ell_{exp}(\gamma)$ by the number of edges in G_{PG} of height at least $i+1$. Since every edge in $\overline{G - G_i}$ is in G_{PG} by Assertion (2), the conclusion of the first claim of Assertion (5) follows. The claim about paths contained in G_i is then a direct consequence.

Let μ be a current in $\mathrm{Curr}(\mathcal{F}_1, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$. By Lemma 5.17, there exists $(\mu_1, \mu_2) \in X(\mathcal{F}_1)$ such that $\mu = \mu_1^* + \mu_2^*$. Since rational currents are dense in $\mathrm{Curr}(H_1, \{[A_1], \ldots, [A_s]\})$ and $\mathrm{Curr}(H_2, \{[B_1], \ldots, [B_t]\})$ by Proposition 2.15, linear combination of rational currents is dense in $\mathrm{Curr}(\mathcal{F}_1, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$. The last claim of Assertion (5) then follows from the linearity and continuity of Ψ_0 and $\|.\|_{\mathcal{F}_1}$.

(6) Let $m \in \mathbb{N}^*$. By Assertion (5), it suffices to prove that the number of edges in $\overline{G - G_i}$ contained in $[f^m(\gamma)]$ is equal to the number of edges in $\overline{G - G_i}$ contained in γ. In the case that $\overline{G - G_i}$ is a lollipop extension and that γ is the circuit C in $\overline{G - G_i}$, then γ is fixed by f by [HM, Definition I.1.29 (3)] (that is the filtration associated with f is *reduced*). Hence $[f^m(\gamma)] = \gamma$ and the claim follows.

Otherwise, if $\overline{G - G_i}$ is either a one-edge extension or a lollipop extension, the circuit γ is not contained in $\overline{G - G_i}$. Moreover, if γ or $[f^m(\gamma)]$ contains an edge in $\overline{G - G_i}$, then it contains $\overline{G - G_i}$. Hence it suffices to count the number of occurrences of $\overline{G - G_i}$ in γ and $[f^m(\gamma)]$. Since f preserves G_i, the result follows from Assertion (1) and [BFH1, Corollary 3.2.2] (where the stratum in it is the graph $\overline{G - G_i}$).

(7) Note that since $\phi|_{\mathcal{F}_1}$ is expanding relative to $\mathcal{F}$, we have $\mathcal{F}_1 \wedge \mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi)$. Recall the definition of the graph G^* and the map $p_{G^*} \colon G^* \to G$ from above Lemma 3.12. By Proposition 3.14 and Lemma 3.12(2), the malnormal

subgroup system $\mathcal{A}(\phi)$ is precisely the subgroup system associated with the fundamental groups of the connected components of G^*. Moreover, the malnormal subgroup system associated with $\mathcal{F}_1 \wedge \mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi)$ is the subgroup system associated with the connected components of $p_{G^*}^{-1}(G_i)$.

By Assertion (1), the graph $\overline{G - G_i}$ is either a topological arc or a lollipop. Suppose first that $\overline{G - G_i}$ is a topological arc. By Assertion (2), the graph $\overline{G - G_i}$ consists in edges in G_{PG}. Thus, the graph G^* is obtained from $p_{G^*}^{-1}(G_i)$ by adding a topological arc τ. If the endpoints of τ are in two distinct connected components of G^*, then the first case of Assertion (7) occurs and otherwise the second case of Assertion (7) occurs. Moreover, if the second case occurs, the extension $\mathcal{F}_1 \leqslant \{[F_{\mathtt{n}}]\}$ is an HNN extension. Thus there exists a subgroup A of $F_{\mathtt{n}}$ such that $\mathcal{F}_1 = \{[A]\}$. By [BFH1, Corollary 3.2.2], one can obtain an element g of $F_{\mathtt{n}}$ such that $F_{\mathtt{n}} = A * \langle g \rangle$ by taking a circuit in the image of p_{G^*} which contains $\overline{G - G_i}$ exactly once.

Suppose now that $\overline{G - G_i}$ is a lollipop extension. By Assertion (2), the circuit C in $\overline{G - G_i}$ consists in edges in G_{PG}. Thus, either G^* is obtained from $p_{G^*}^{-1}(G_i)$ by adding a lollipop extension or G^* is obtained from $p_{G^*}^{-1}(G_i)$ by adding a connected component which is homotopy equivalent to a circle. If G^* is obtained from $p_{G^*}^{-1}(G_i)$ by adding a lollipop extension, the second case of Assertion (7) occurs. If G^* is obtained from $p_{G^*}^{-1}(G_i)$ by adding a connected component which is homotopy equivalent to a circle, the third case of Assertion (7) occurs. The proof of the fact about HNN extension is similar to the proof for the one-edge extension case. This concludes the proof. □

Remark 5.19. By Lemma 5.18(1), $\overline{G - G_i}$ is either a topological arc or it retracts onto a circuit C and there exists exactly one topological arc that connects C and G_i. In the second case, we will adopt the convention that $\overline{G - G_i} = C$, so that, by Lemma 5.18(2), in both cases of Lemma 5.18(1), every edge in $\overline{G - G_i}$ is in G_{PG}.

Lemma 5.20. *Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ and let $f\colon G \to G$ be as in Remark* 5.15.

(1) *Let U be an open neighborhood of $\Delta_+(\phi)$, let V be an open neighborhood of $K_{PG}(\phi)$ (see Definition* 3.26*). There exist $N \in \mathbb{N}^*$ and $\delta \in (0,1)$ such that for every $m \geqslant 1$ and every $w \in F_{\mathtt{n}}$ with $\mathfrak{g}(\gamma_w) > \delta$ and $\eta_{[w]} \notin V$, we have*
$$(\phi^N)^m(\eta_{[w]}) \in U.$$

(2) *Suppose that ϕ is an almost atoroidal outer automorphism relative to $\mathcal{F}$. Let $\mathcal{F} \leqslant \mathcal{F}_1 \leqslant \mathcal{F}_2$ be an associated sequence of free factor systems.*

For every $\epsilon > 0$ and $L > 0$, there exist $\delta \in (0,1)$ and $M > 0$ such that, for every $n \geqslant M$, for every nonperipheral element $w \in F_{\mathtt{n}}$ with $\mathfrak{g}(\gamma_w) > \delta$, there exists $[\mu_w] \in \Delta_+(\phi)$ such that for every reduced edge path $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ of length at most L contained in G_i:
$$\left| \frac{\langle \gamma, [f^n(\gamma_w)] \rangle}{\ell_{exp}([f^n(\gamma_w)])} - \frac{\langle \gamma, [\mu_w]) \rangle}{\|[\mu_w]\|_{\mathcal{F}_1}} \right| < \epsilon.$$

Proof. The proof is similar to the one of [LU2, Lemma 6.1]. By Lemma 5.3 and Lemma 5.16(1), up to passing to a power of f, we may assume that for every $w \in F_{\mathtt{n}}$

such that $\mathfrak{g}(\gamma_w) \geqslant \frac{1}{2}$, and every $n \in \mathbb{N}^*$, we have $\mathfrak{g}([f^n(\gamma_w)]) \geqslant \mathfrak{g}(\gamma_w)$ and

$$\ell_{exp}([f^n(\gamma_w)]) \geqslant TEL(n, \gamma) \geqslant (3^n - 2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w). \tag{8}$$

Let $N \in \mathbb{N}^*$ be such that $3^N > 2C$. Let $\lambda > 0$ be such that, for every edge $e \in \vec{E}G$ and every $n \in \mathbb{N}^*$, we have

$$\ell([f^n(e)]) \leqslant \lambda^n. \tag{9}$$

By Lemma 3.30, a sequence $([\nu_m])_{m\in\mathbb{N}}$ of projective relative currents tends to a projective current $[\nu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ if for every $\epsilon > 0$ and $R > 0$ there exists $M \in \mathbb{N}^*$ such that, for every $m \geqslant M$ and every reduced edge path $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ with $\ell(\gamma) \leqslant R$, we have

$$\left| \frac{\langle \gamma, \nu \rangle}{\|\nu\|_{\mathcal{F}}} - \frac{\langle \gamma, \nu_m \rangle}{\|\nu_m\|_{\mathcal{F}}} \right| < \epsilon. \tag{10}$$

For every $\mathcal{F}$-expanding splitting unit σ, we denote by $\mu(\sigma)$ the corresponding current given by Proposition 4.4. By Lemma 4.8, we have $\|\mu(\sigma)\|_{\mathcal{F}} = 1$. Since $\Delta_+(\phi)$ is compact by Lemma 4.7, there exist $\epsilon, R > 0$ such that for every $m \geqslant M$, if there exists $\nu \in \Delta_+(\phi)$ such that ν_m, ν, R, ϵ satisfy Equation (10), then $\nu_m \in U$. Since there are only finitely many expanding splitting units of positive exponential length and finitely many edge paths $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ such that $\ell(\gamma) \leqslant R$, there exists $M_0 \in \mathbb{N}^*$ such that for every $m \geqslant M_0$, for every expanding splitting unit σ and for every reduced edge path $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ with $\ell(\gamma) \leqslant R$, we have:

$$\left| \frac{\langle \gamma, [f^m(\sigma)] \rangle}{\ell_{\mathcal{F}}([f^m(\sigma)])} - \langle \gamma, \mu(\sigma) \rangle \right| < \frac{\epsilon}{6}.$$

Recall that $\langle \gamma, \mu(\sigma) \rangle$ is equal to $\mu(\sigma)(C(\gamma))$ by definition of the number of occurrences of γ in $\mu(\sigma)$. Let γ' be a reduced edge path in G. By Lemma 5.6, for every reduced edge path σ of G contained in γ', we have $\ell_{\mathcal{F}}(\sigma) \geqslant \ell_{\mathcal{F}}^{\gamma'}(\sigma) \geqslant \ell_{\mathcal{F}}(\sigma) - 2C$. Hence there exists $M_1 \in \mathbb{N}^*$ such that for every $m \geqslant M_1$, for every expanding splitting unit σ, for every edge path γ' containing σ as a splitting unit and for every reduced edge path $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ with $\ell(\gamma) \leqslant R$, we have:

$$\left| \frac{\langle \gamma, [f^m(\sigma)] \rangle}{\ell_{\mathcal{F}}^{[f^m(\gamma')]}([f^m(\sigma)])} - \langle \gamma, \mu(\sigma) \rangle \right| < \frac{\epsilon}{6}. \tag{11}$$

Recall the definition of the continuous function $\Psi_0 \colon \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \to \mathbb{R}$ given above Definition 3.26. Recall that, by Lemma 3.28(3), for every current $\mu \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$, we have $\|\mu\|_{\mathcal{F}} > 0$. Let

$$\begin{array}{rccc} \Psi\colon & \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) & \to & \mathbb{R}, \\ & [\nu] & \mapsto & \frac{\Psi_0(\nu)}{\|\nu\|_{\mathcal{F}}}. \end{array}$$

Since Ψ is continuous and since $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) - V$ is compact, there exists $s > 0$ such that for every $\nu \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) - V$, we have:

$$\Psi([\nu]) \geqslant s.$$

In particular, by Lemma 3.27, for every nonperipheral element $w \in F_{\mathtt{n}}$ such that $\eta_{[w]} \notin V$, we have

$$\frac{\ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}}(\gamma_w)} = \frac{\Psi_0(\eta_{[w]})}{\|\eta_{[w]}\|_{\mathcal{F}}} = \Psi([\eta_{[w]}]) \geqslant s. \tag{12}$$

Now let $w \in F_{\mathtt{n}}$ be a nonperipheral element such that $\mathfrak{g}(\gamma_w) \geqslant \frac{1}{2}$ and $\eta_{[w]} \notin V$. Let $\gamma_w = \alpha_0\beta_1\alpha_1 \dots \alpha_k\beta_k$ be an optimal splitting of γ_w, where for every $i \in \{0, \dots, k\}$, the path α_i is an incomplete factor of γ_w and for every $i \in \{1, \dots, k\}$, the path β_i is a complete factor of γ. Using this optimal splitting, we construct another decomposition of γ_w, which is not necessarily a splitting of γ_w, but is well-adapted for our considerations.

Since concatenations of paths in G_{PG} and in $\mathcal{N}_{PG}$ have zero exponential length by Lemma 3.18, we change the decomposition in such a way that every subpath of γ_w which is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ is in some α_i for $i \in \{1, \dots, k\}$. In particular, for every $i \in \{1, \dots, k\}$, the exponential lengths of β_i and α_i are equal to their exponential lengths relative to γ_w. Let $i \in \{0, \dots, k\}$. The path α_i has a decomposition $\alpha_i = \alpha_i^{(1)}\alpha_i^{(1')} \dots \alpha_i^{(k_i)}\alpha_i^{(k_i')}$ where, for every $j \in \{1, \dots, k_i\}$, the path $\alpha_i^{(j)}$ is a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$ and, for every $j \in \{1, \dots, k_i\}$, the path $\alpha_i^{(j')}$ is a path in $\overline{G - G_{PG}}$ such that every edge of $\alpha_i^{(j')}$ either has positive exponential length relative to γ_w or is in a zero stratum.

Note that, by Proposition 2.5(4), for every $j \in \{1, \dots, k_i\}$ and every maximal subpath τ of $\alpha_i^{(j')}$ contained in some zero stratum, the path τ is adjacent to a path in γ_w of positive exponential length. Suppose that τ is nontrivial. Since no zero path is adjacent to a path which is a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$ by Lemma 2.9 and Proposition 2.5(4), either $\alpha_i = \tau$ or $\ell_{exp}(\alpha_i^{(j')}) > 0$. In the first case, we have $\ell(\tau) \leqslant C$ by definition of C. Thus, there exists $n \in \mathbb{N}^*$ such that $[f^n(\tau)]$ is completely split. Therefore, if the first case occurs, we may suppose, up to taking a power of f, that α_i is completely split and is a splitting unit of some β_j.

Let $i \in \{1, \dots, k\}$. Since β_i does not contain splitting units which are concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$, every splitting unit of β_i is an edge in $\overline{G - G'_{PG}}$ or a maximal taken connecting path in a zero stratum. By Lemma 3.22, every splitting unit of β_i which is an edge in $\overline{G - G'_{PG}}$ is expanding.

Let σ' be a splitting unit of β_i which is a maximal taken connecting path in a zero stratum and which is not expanding. Let $n \in \mathbb{N}^*$ be such that $[f^n(\sigma')]$ is completely split. By Lemma 3.22 and Lemma 3.21, the path $[f^n(\sigma')]$ does not contain splitting units which are edges in $\overline{G - G_{PG}}$. If $[f^n(\sigma')]$ contains a splitting unit which is contained in a zero stratum, then an inductive argument shows that, up to taking a larger n, the path $[f^n(\sigma')]$ is a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$. Thus, the $\mathcal{F}$-length of σ' grows at most polynomially fast under iterates of f.

Combining all the above remarks, we see that γ_w has a decomposition

$$\gamma_w = a_0 b_0 a_1 c_1^{(1)} c_2^{(1)} \dots c_{k_1}^{(1)} a_2 b_2 \dots a_t c_1^{(t)} c_2^{(t)} \dots c_{k_t}^{(t)} a_{t+1} b_{t+1} a_{t+2},$$

where:

(a) for every $i \in \{0, \dots, t+2\}$, the path a_i is either possibly trivial, a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ or a maximal taken connecting path whose $\mathcal{F}$-length grows at most polynomially fast;

(b) for every $i \in \{0, \dots, t+1\}$, the path b_i is a subpath of positive exponential length relative to γ_w of an incomplete path of γ_w such that every edge of b_i either has positive exponential length relative to γ_w or is in a zero stratum;

(c) for every $i \in \{1, \dots, t\}$ and every $j \in \{1, \dots, k_i\}$, the path $c_j^{(i)}$ is a (possibly trivial) expanding splitting unit of a complete factor of γ_w.

Recall that the length of every path in a zero stratum is bounded by C. Thus, for every $i \in \{0, \dots, t+1\}$, we have

$$\ell(b_i) \leqslant C\ell_{exp}(b_i).$$

We claim that the exponential length relative to γ_w of one of the edges at the concatenation point of two consecutive nontrivial paths of the form a_ib_i, b_ia_{i+1}, $a_ic_1^{(i)}$, $c_j^{(i)}c_{j+1}^{(i)}$ or $c_{k_i}^{(i)}a_{i+1}$ is positive. Indeed, for every $i \in \{1, \dots, t\}$ (resp. $i \in \{0, \dots, t+1\}$) and every $j \in \{1, \dots, k_i\}$, the path $c_j^{(i)}$ (resp. b_i) either has positive exponential length relative to γ_w or is contained in a zero stratum. Note that by hypothesis, for every $i \in \{0, \dots, t+1\}$, the path b_i is not contained in a zero stratum. Moreover, if b_i is adjacent to a path a_i, then the first edge of b_i is not in a zero stratum by Proposition 2.5(4), Lemma 2.9 and the fact that the paths in zero strata that we consider in our subdivision are maximal. Hence one of the edges at the concatenation point of every path of the form a_ib_i, b_ia_{i+1} has positive exponential length relative to γ_w.

By maximality of the splitting units contained in zero strata, one of the splitting units in a path $c_j^{(i)}c_{j+1}^{(i)}$ is an edge in $\overline{G - G'_{PG}}$, hence has positive exponential length relative to γ_w. Since paths in zero strata and concatenations of paths in G_{PG} and $\mathcal{N}_{PG}$ cannot be adjacent by Proposition 2.5(4) and Lemma 2.9, paths of the form $a_ic_1^{(i)}$ and $c_{k_i}^{(i)}a_{i+1}$ have positive exponential length since in this case $c_1^{(i)}$ or $c_{k_i}^{(i)}$ is an edge in $\overline{G - G'_{PG}}$. This proves the claim.

Remark that, by construction and the definition of goodness of a reduced path, we have

$$\sum_{i=1}^{t}\sum_{j=1}^{k_i} \ell_{exp}(c_j^{(i)}) = \ell_{exp}(\gamma_w)\mathfrak{g}(\gamma_w).$$

Note that the length of reduced iterates of edges in G_{PG} grows at most polynomially fast, hence the $\mathcal{F}$-length of reduced iterates of edges in G_{PG} grows at most polynomially fast. Let $C' > 0$ and $k \in \mathbb{N}^*$ be such that, for every splitting unit σ' which is either an edge in G_{PG} or a maximal taken connecting path in a zero stratum whose $\mathcal{F}$-length grows at most polynomially fast, and every $m \in \mathbb{N}^*$, we have:

$$\ell_{\mathcal{F}}([f^m(\sigma')]) \leqslant C'm^k\ell_{\mathcal{F}}(\sigma').$$

The constants C' and k exist by the claim in Proposition 3.14.

Let $i \in \{0, \dots, t+2\}$ and let $a_i = \alpha_0 \dots \alpha_{\ell_i}$ be a decomposition of a_i such that, for every $j \in \{0, \dots, \ell_i\}$, the path α_{ℓ_i} is either an edge in G_{PG}, a path in $\mathcal{N}_{PG}^{\max}(a_i)$ or a maximal taken connecting path in a zero stratum whose $\mathcal{F}$-length grows at most polynomially fast. By Lemma 3.17, for every $m \in \mathbb{N}^*$, we have

$$\ell_{\mathcal{F}}([f^m(a_i)]) \leqslant \sum_{j=0}^{\ell_i} \ell_{\mathcal{F}}([f^m(\alpha_j)]) \leqslant C'm^k \sum_{j=1}^{\ell_i} \ell_{\mathcal{F}}(\alpha_j) = C'm^k\ell_{\mathcal{F}}(a_i),$$

where the last equality follows from the fact that a path in $\mathcal{N}_{PG}$ is contained in some subpath α_j by hypothesis. In particular,

$$\sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^m(a_i)]) \leqslant C'm^k \sum_{i=0}^{t+2} \ell_{\mathcal{F}}(a_i) \leqslant C'\ell_{\mathcal{F}}(\gamma_w)m^k, \tag{13}$$

where the last inequality follows from the fact that, by hypothesis, every path in $\mathcal{N}_{PG}^{\max}(\gamma)$ is contained in some a_i. Thus, if $\mathfrak{g}(\gamma_w) \geqslant \frac{1}{2}$, there exists $C'' > 0$ such that, for every $n \geqslant N$, by Equations (8), (13) and (12), we have:

$$\begin{aligned}\frac{\sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^n(a_i)])}{\ell_{exp}([f^n(\gamma_w)])} &\leqslant \frac{C'\ell_{\mathcal{F}}(\gamma_w)n^k}{(3^n-2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w)} \\ &\leqslant \frac{C'\frac{1}{s}\ell_{exp}(\gamma_w)n^k}{(3^n-2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w)} \\ &\leqslant C''\frac{n^k}{(3^n-2C)\mathfrak{g}(\gamma_w)}.\end{aligned}$$

Up to taking a larger $N \in \mathbb{N}^*$, we may suppose that, for every $n \geqslant N$, we have

$$C''\frac{n^k}{(3^n-2C)\mathfrak{g}(\gamma_w)} \leqslant \frac{\epsilon}{48\mathfrak{g}(\gamma_w)R}. \tag{14}$$

Recall that, for every reduced edge path γ of G, we have

$$\ell_{exp}(\gamma) \leqslant \ell_{\mathcal{F}}(\gamma).$$

Thus, for every $n \geqslant N$ and every nonperipheral element $w \in F_{\mathtt{n}}$ such that $\mathfrak{g}(\gamma_w) \geqslant \frac{1}{2}$, by Equation (8), we have

$$\frac{2R\ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}}([f^n(\gamma_w)])} \leqslant \frac{2R\ell_{exp}(\gamma_w)}{(3^n-2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w)} = \frac{2R}{(3^n-2C)\mathfrak{g}(\gamma_w)}.$$

Up to taking a larger N, we may assume that for every $n \geqslant N$ and every $w \in F_{\mathtt{n}}$ such that $\mathfrak{g}(\gamma_w) \geqslant \frac{1}{2}$, we have:

$$\frac{2R\ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}}([f^n(\gamma_w)])} \leqslant \frac{2R}{(3^n-2C)\mathfrak{g}(\gamma_w)} \leqslant \frac{\epsilon}{12\mathfrak{g}(\gamma_w)}. \tag{15}$$

Let

$$\delta = \max\left\{\frac{1}{1+\frac{\epsilon}{6}}, \frac{1}{1+\frac{2RC\epsilon\lambda^N}{(3^N-2C)6}}, \frac{1}{2}\right\}.$$

Thus, in order to prove the first assertion of Lemma 5.20, it suffices to show that for every $m \geqslant N$ and every $w \in F_{\mathtt{n}}$ such that $\mathfrak{g}(\gamma_w) > \delta$ and $\eta_{[w]} \notin V$, the projective current $[\nu_m] = \phi^m([\eta_w])$ is close to an element $[\nu]$ in $\Delta_+(\phi)$ in the sense of Equation (10). Since the goodness function is monotone by Remark 5.15, it suffices to prove it for $m = N$.

Let $w \in F_{\mathtt{n}}$ such that $\mathfrak{g}(\gamma_w) > \delta$ and $\eta_{[w]} \notin V$. By Equation (14) and the fact that $\mathfrak{g}(\gamma_w) \geqslant \delta \geqslant \frac{1}{2}$, we have

$$\begin{aligned}\frac{\sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^N(a_i)])}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} &\leqslant \frac{\sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^N(a_i)])}{\ell_{exp}([f^N(\gamma_w)])} \\ &\leqslant C''\frac{N^k}{(3^N-2C)\mathfrak{g}(\gamma_w)} \leqslant C''\frac{N^k}{(3^N-2C)\delta} \leqslant \frac{\epsilon}{24R}.\end{aligned} \tag{16}$$

Moreover, by Equation (15) and the fact that $\mathfrak{g}(\gamma_w) \geqslant \delta \geqslant \frac{1}{2}$, we have

$$\frac{2R\ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} \leqslant \frac{\epsilon}{6}. \tag{17}$$

Note that, for every $w \in F_{\mathtt{n}}$ such that $\mathfrak{g}(\gamma_w) > \delta$ and $\eta_{[w]} \notin V$, we have:

$$\begin{aligned}\frac{2RC\lambda^N(1-\mathfrak{g}(\gamma_w))\ell_{exp}(\gamma_w)}{(3^N-2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w)} &= 2RC\frac{\lambda^N}{3^N-2C}\left(\frac{1}{\mathfrak{g}(\gamma_w)}-1\right)\\ &\leqslant 2RC\frac{\lambda^N}{3^N-2C}\left(\frac{1}{\delta}-1\right) \leqslant \frac{\epsilon}{6},\end{aligned} \tag{18}$$

where the last inequality follows from the definition of δ.

Let $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ be of length at most R. By the triangle inequality, we have

$$\begin{aligned}&\left|\frac{\left\langle\gamma,[f^N(\gamma_w)]\right\rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} - \frac{\left\langle\gamma,\sum_{i=1}^t\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])\mu(c_j^{(i)})\right\rangle}{\sum_{i=1}^t\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])}\right|\\ &\leqslant \left|\frac{\left\langle\gamma,[f^N(\gamma_w)]\right\rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} - \sum_{i=1}^t\sum_{j=1}^{k_i}\frac{\left\langle\gamma,[f^N(c_j^{(i)})]\right\rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])}\right|\\ &+\left|\sum_{i=1}^t\sum_{j=1}^{k_i}\frac{\left\langle\gamma,[f^N(c_j^{(i)})]\right\rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} - \frac{\sum_{i=1}^t\sum_{j=1}^{k_i}\left\langle\gamma,[f^N(c_j^{(i)})]\right\rangle}{\sum_{i=1}^t\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])}\right|\\ &+\left|\frac{\sum_{i=1}^t\sum_{j=1}^{k_i}\left\langle\gamma,[f^N(c_j^{(i)})]\right\rangle}{\sum_{i=1}^t\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])}\right.\\ &\qquad\left.-\frac{\left\langle\gamma,\sum_{i=1}^t\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])\mu(c_j^{(i)})\right\rangle}{\sum_{i=1}^t\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])}\right|.\end{aligned} \tag{19}$$

Note that an occurrence of γ or γ^{-1} in $[f^N(\gamma_w)]$ might happen either in some $[f^N(c_j^{(i)})]$ or in some $[f^N(a_i)]$ or in some $[f^N(b_i)]$ or it might cross over the concatenation points. Recall that one of the edges at the concatenation point of paths of the form a_ib_i, b_ia_{i+1}, $a_ic_1^{(i)}$, $c_j^{(i)}c_{j+1}^{(i)}$ or $c_{k_i}^{(i)}a_{i+1}$ has positive exponential length relative to γ_w. Recall also that the length of γ is at most equal to R. Thus the number of such crossings is at most $2R\ell_{exp}(\gamma_w)$. Thus:

$$\begin{aligned}&\left|\frac{\left\langle\gamma,[f^N(\gamma_w)]\right\rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} - \sum_{i=1}^t\sum_{j=1}^{k_i}\frac{\left\langle\gamma,[f^N(c_j^{(i)})]\right\rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])}\right|\\ &\qquad\leqslant \frac{2R\ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} + \sum_{i=0}^{t+2}\frac{\left\langle\gamma,[f^N(a_i)]\right\rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} + \sum_{i=0}^{t+1}\frac{\left\langle\gamma,[f^N(b_i)]\right\rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])}.\end{aligned}$$

Since γ is not contained in a concatenation of paths in $G_{PG,\mathcal{F}}$ and $\mathcal{N}_{PG,\mathcal{F}}$, if γ is contained in $[f^N(a_i)]$ for $i \in \{1,\dots,t+1\}$, then γ contains an edge of $[f^N(a_i)]$ of positive $\mathcal{F}$-length relative to $[f^N(a_i)]$. Hence we have $\left\langle\gamma,[f^N(a_i)]\right\rangle \leqslant 2\ell_{\mathcal{F}}([f^N(a_i)])$. By Equations (17) and (16) with $n = N$, we have

$$\frac{2R\ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} + \sum_{i=0}^{t+2}\frac{\left\langle\gamma,[f^N(a_i)]\right\rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} \leqslant \frac{2R\ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} + \frac{2\sum_{i=0}^{t+1}\ell_{\mathcal{F}}([f^N(a_i)])}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} \leqslant \frac{\epsilon}{4}.$$

Moreover, since for every $i \in \{0, \dots, t+1\}$, we have $\ell(b_i) \leqslant C\ell_{exp}(b_i)$ and by Equations (8), (12) and (18), we see that:

$$\begin{aligned}\sum_{i=0}^{t+1} \frac{\langle \gamma, [f^N(b_i)] \rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} &\leqslant \sum_{i=0}^{t+1} \frac{\ell([f^N(b_i)])}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} \leqslant \sum_{i=0}^{t+1} \frac{C\lambda^N \ell_{exp}(b_i)}{(3^N - 2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w)} \\ &\leqslant \frac{C\lambda^N(1-\mathfrak{g}(\gamma_w))\ell_{exp}(\gamma_w)}{(3^N-2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w)} \leqslant \frac{\epsilon}{6}.\end{aligned}$$

For the third term of Inequality (19), note that, since $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$, it is not contained in a concatenation of paths in $G_{PG,\mathcal{F}}$ and in $\mathcal{N}_{PG,\mathcal{F}}$. Therefore, if c is a reduced edge path of $[f^N(\gamma_w)]$, an occurrence of γ always appears with an edge e of c such that $\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}(e) = 1$. Since $\ell(\gamma) \leqslant R$, such an edge e can be crossed by at most R occurrences of γ in c. Thus, for every reduced edge path c in $[f^N(\gamma_w)]$, we have $\langle \gamma, c \rangle \leqslant 2R\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}(c)$.

Hence we have

$$\left| \frac{\sum_{i=1}^t \sum_{j=1}^{k_i} \left\langle \gamma, [f^N(c_j^{(i)})] \right\rangle}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f(c_j^{(i)})])} \right| \leqslant 2R.$$

Since

$$\ell_{\mathcal{F}}([f^N(\gamma_w)]) = \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})]) + \sum_{i=0}^{t+1} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(a_i b_i a_{i+1})]),$$

using Lemma 5.3 and Lemma 5.6 for the last inequality we have:

$$\begin{aligned}&\left| \sum_{i=1}^t \sum_{j=1}^{k_i} \frac{\left\langle \gamma, [f^N(c_j^{(i)})] \right\rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} - \frac{\sum_{i=1}^t \sum_{j=1}^{k_i} \left\langle \gamma, [f^N(c_j^{(i)})] \right\rangle}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])} \right| \\
&= \left| \frac{\left(\sum_{i=1}^t \sum_{j=1}^{k_i} \left\langle \gamma, [f^N(c_j^{(i)})] \right\rangle\right)}{\left(\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f(c_j^{(i)})])\right)} \right. \\
&\qquad \left. \times \frac{\left(\sum_{i=0}^{t+1} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f(a_i b_i a_{i+1})])\right)}{\left(\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})]) + \sum_{i=0}^{t+1} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(a_i b_i a_{i+1})])\right)} \right| \\
&\leqslant \left| \frac{\left(\sum_{i=1}^t \sum_{j=1}^{k_i} \left\langle \gamma, [f^N(c_j^{(i)})] \right\rangle\right) \left(\sum_{i=0}^{t+1} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f(a_i b_i a_{i+1})])\right)}{\left(\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f(c_j^{(i)})])\right) \left(\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])\right)} \right| \\
&\leqslant \left| \frac{\left(\sum_{i=1}^t \sum_{j=1}^{k_i} \left\langle \gamma, [f^N(c_j^{(i)})] \right\rangle\right) \left(\sum_{i=0}^{t+1} \ell_{\mathcal{F}}([f^N(b_i)]) + 2\sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^N(a_i)])\right)}{\left(\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f(c_j^{(i)})])\right) \left(\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])\right)} \right| \\
&\leqslant 2R \left| \frac{\sum_{i=0}^{t+1} \ell_{\mathcal{F}}([f^N(b_i)]) + 2\sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^N(a_i)])}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])} \right|.\end{aligned}$$

Recall that we have

$$\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{exp}(c_j^{(i)}) = \ell_{exp}(\gamma_w)\mathfrak{g}(\gamma_w)$$

and, for every $i \in \{1,\dots,t\}$ and every $j \in \{1,\dots,k_i\}$, we have either $\ell_{exp}(c_j^{(i)}) = 1$ or $\ell_{exp}(c_j^{(i)}) = 0$. Hence, we have:

$$\begin{array}{rcl} \sum_{i=1}^{t}\sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})]) & \geqslant & \sum_{i=1}^{t}\sum_{j=1}^{k_i} (\ell_{\mathcal{F}}([f^N(c_j^{(i)})]) - 2C) \\ & \geqslant & \sum_{i=1}^{t}\sum_{j=1}^{k_i} (3^N - 2C) \\ & \geqslant & (3^N - 2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w), \end{array}$$

where the first inequality follows from Lemma 5.6 and the second inequality follows from the fact that f is $3K$-expanding and $K \geqslant 1$. Thus, we have

$$\begin{array}{l} 2R\left|\frac{\sum_{i=0}^{t+1}\ell_{\mathcal{F}}([f^N(b_i)])+2\sum\limits_{i=0}^{t+2}\ell_{\mathcal{F}}([f^N(a_i)])}{\sum_{i=1}^{t}\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])}\right| \\ \leqslant 2R\left|\frac{\sum_{i=0}^{t+1}\ell_{\mathcal{F}}([f^N(b_i)])}{(3^N-2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w)}\right| + 2R\left|\frac{2\sum_{i=0}^{t+2}\ell_{\mathcal{F}}([f^N(a_i)])}{(3^N-2C)\delta\ell_{exp}(\gamma_w)}\right|. \end{array}$$

By Equation (9), we have

$$\begin{aligned} \sum_{i=0}^{t+1}\ell_{\mathcal{F}}([f^N(b_i)]) \leqslant \sum_{i=0}^{t+1}\ell([f^N(b_i)]) \leqslant \lambda^N\sum_{i=0}^{t+1}\ell(b_i) & \\ \leqslant C\lambda^N\sum_{i=0}^{t+1}\ell_{exp}(b_i) \leqslant C\lambda^N\ell_{exp}(\gamma_w)(1-\mathfrak{g}(\gamma_w)). \end{aligned}$$

Hence we have:

$$\begin{array}{l} 2R\left|\frac{\sum_{i=0}^{t+1}\ell_{\mathcal{F}}([f^N(b_i)])}{(3^N-2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w)}\right| + 2R\left|\frac{2\sum_{i=0}^{t+2}\ell_{\mathcal{F}}([f^N(a_i)])}{(3^n-2C)\delta\ell_{exp}(\gamma_w)}\right| \\ \leqslant 2R\left|\frac{C\lambda^N(1-\mathfrak{g}(\gamma_w))\ell_{exp}(\gamma_w)}{(3^N-2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w)}\right| + 2R\left|\frac{2C'\ell_{\mathcal{F}}(\gamma_w)n^k}{(3^N-2C)\delta\ell_{exp}(\gamma_w)}\right| \text{ by Equation (13)} \\ \leqslant 2R\left|\frac{C\lambda^N(1-\mathfrak{g}(\gamma_w))\ell_{exp}(\gamma_w)}{(3^N-2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w)}\right| + 2R\left|\frac{2C''n^k}{(3^N-2C)\delta}\right| \\ \leqslant \frac{2\epsilon}{6} \text{ by Equations (16) and (18).} \end{array}$$

Finally, using Equation (11) and the fact that for every $i \in \{1,\dots,t\}$ and every $j \in \{1,\dots,k_i\}$, the splitting unit $c_j^{(i)}$ is expanding, we have:

$$\begin{array}{l} \left|\frac{\sum_{i=1}^{t}\sum_{j=1}^{k_i}\left\langle\gamma,[f^N(c_j^{(i)})]\right\rangle}{\sum_{i=1}^{t}\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])} - \frac{\left\langle\gamma,\sum_{i=1}^{t}\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])\mu(c_j^{(i)})\right\rangle}{\sum_{i=1}^{t}\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])}\right| \\ = \left|\frac{\sum_{i=1}^{t}\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])\left(\frac{\left\langle\gamma,[f^N(c_j^{(i)})]\right\rangle}{\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])}-\left\langle\gamma,\mu(c_j^{(i)})\right\rangle\right)}{\sum_{i=1}^{t}\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])}\right| \\ \leqslant \frac{\frac{\epsilon}{6}\sum_{i=1}^{t}\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])}{\sum_{i=1}^{t}\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])} = \frac{\epsilon}{6}. \end{array}$$

Combining all inequalities, we have

$$\begin{aligned} \left|\frac{\left\langle\gamma,[f^N(\gamma_w)]\right\rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} - \frac{\left\langle\gamma,\sum_{i=1}^{t}\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])\mu(c_j^{(i)})\right\rangle}{\sum_{i=1}^{t}\sum_{j=1}^{k_i}\ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])}\right| & \\ \leqslant \frac{\epsilon}{4}+\frac{\epsilon}{6}+\frac{2\epsilon}{6}+\frac{\epsilon}{6} \leqslant \epsilon. \end{aligned}$$

This concludes the proof of Assertion (1) of Lemma 5.20 since for every $i \in \{1,\dots,t\}$ and every $j \in \{1,\dots,k_i\}$, we have $\mu(c_j^{(i)}) \in \Delta_+(\phi)$.

The proof of Assertion (2) is the same one as the proof of Assertion (1), replacing $\ell_{\mathcal{F}}$ and $\ell_{\mathcal{F}}^{\gamma}$ by ℓ_{exp} and ℓ_{exp}^{γ}, adding the following arguments. Let γ and $w \in F_{\mathtt{n}}$ be as in Assertion (2). Then γ is not contained in a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ by Lemma 5.18(3). If

$$\gamma_w = a_0 b_0 a_1 c_1^{(1)} c_2^{(1)} \dots c_{k_1}^{(1)} a_2 b_2 \dots a_t c_1^{(t)} c_2^{(t)} \dots c_{k_t}^{(t)} a_{t+1} b_{t+1} a_{t+2}$$

is the same decomposition of γ_w as in the proof of Assertion (1), then for every $m \in \mathbb{N}$ and every $i \in \{1,\dots,t+2\}$, the path γ is not contained in $[f^m(a_i)]$ by Lemma 3.10. Similarly, for every $m \in \mathbb{N}^*$ and every $i \in \{1,\dots,t+2\}$, we have $\ell_{exp}([f^m(a_i)]) = 0$. Hence we do not need Equation (16). By Lemma 5.18(5), we have

$$\ell_{exp}(\gamma) = \ell_{\mathcal{F}_1}(\gamma).$$

Moreover, by Lemma 5.18(5), for every current $[\mu] \in \Delta_+(\phi)$, we have $\Psi_0(\mu) = \|\mu\|_{\mathcal{F}_1}$. Replacing $\ell_{\mathcal{F}}$ and $\ell_{\mathcal{F}}^{\gamma}$ by ℓ_{exp} and ℓ_{exp}^{γ} in the equations in the proof of Assertion (1) concludes the proof. □

For Lemma 5.21, we need to compute the exponential length of incomplete segments in a circuit γ in G. Let $\ell_{exp}(\mathrm{Inc}(\gamma))$ be the sum of the exponential lengths of the incomplete segments of an optimal splitting of γ. Let $\ell_{exp}^{\gamma}(\mathrm{Inc}(\gamma))$ be the sum of the exponential lengths relative to γ of the incomplete segments of an optimal splitting of γ. Note that $\ell_{exp}^{\gamma}(\mathrm{Inc}(\gamma))$ do not depend on the choice of an optimal splitting. Note that

$$\ell_{exp}^{\gamma}(\mathrm{Inc}(\gamma)) = (1-\mathfrak{g}(\gamma))\ell_{exp}(\gamma) \leqslant \ell_{exp}(\gamma).$$

Lemma 5.21. *Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ and let $f\colon G \to G$ be as in Remark 5.15. Let $\delta \in (0,1)$, and let $R > 1$. There exists $n_0 \in \mathbb{N}^*$ such that for every $n \geqslant n_0$ and every nonperipheral element $w \in F_{\mathtt{n}}$ such that $\eta_{[w]} \notin K_{PG}(\phi)$, we either have*

$$\mathfrak{g}([f^n(\gamma_w)]) \geqslant \delta$$

or

$$\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)])) \leqslant \frac{10C}{R}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w))$$
$$\text{and } \ell_{exp}([f^n(\gamma_w)]) \leqslant \frac{10C}{(1-\delta)R}\ell_{exp}(\gamma_w).$$

Proof. Let $w \in F_{\mathtt{n}}$ be a nonperipheral element such that $\eta_{[w]} \notin K_{PG}(\phi)$. Suppose that $n \in \mathbb{N}^*$ is such that $\mathfrak{g}([f^n(\gamma_w)]) < \delta$. Assuming for now that we have proved that

$$\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)])) \leqslant \frac{10C}{R}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)),$$

we deduce that $\ell_{exp}([f^n(\gamma_w)]) \leqslant \frac{10C}{(1-\delta)R}\ell_{exp}(\gamma_w)$. Indeed, we have

$$\ell_{exp}^{[f^n(\gamma)]}(\mathrm{Inc}([f^n(\gamma)])) = (1-\mathfrak{g}([f^n(\gamma)]))\ell_{exp}([f^n(\gamma)]) \geqslant (1-\delta)\ell_{exp}([f^n(\gamma)]).$$

Thus we have

$$\begin{aligned} \ell_{exp}([f^n(\gamma_w)]) &\leqslant \tfrac{1}{1-\delta}\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)])) \leqslant \tfrac{10C}{(1-\delta)R}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)) \\ &\leqslant \tfrac{10C}{(1-\delta)R}\ell_{exp}(\gamma_w). \end{aligned}$$

Therefore, it suffices to prove that there exists $n_0 \in \mathbb{N}^*$ such that for every $n \geqslant n_0$, if $\mathfrak{g}([f^n(\gamma_w)]) < \delta$, then

$$\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)])) \leqslant \frac{10C}{R}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)).$$

Consider an optimal splitting $\gamma_w = \alpha_0'\beta_1'\alpha_1' \dots \alpha_m'\beta_m'$, where for every $i \in \{0, \dots, m\}$, the path α_i' is an incomplete factor of γ_w and for every $i \in \{0, \dots, m\}$, the path β_i' is a PG-relative complete factor of γ_w. We can modify the splitting of γ_w in a new splitting $\gamma_w = \alpha_0\beta_1\alpha_1 \dots \beta_k\alpha_k$ where:

(i) for every $i \in \{0, \dots, k\}$, the path α_i is a concatenation of incomplete factors and complete factors of zero exponential length relative to γ_w of the old splitting;

(ii) for every $i \in \{1, \dots, k\}$, the path β_i is a complete factor of positive exponential length relative to γ_w of the old splitting.

In the remainder of the proof, we still refer to the paths α_i as incomplete factors. By the last claim of Remark 5.15, we may suppose that $\mathfrak{g}(\gamma_w) < \delta$, that is:

$$\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)) = \sum_{i=0}^{k} \ell_{exp}^{\gamma_w}(\alpha_i) \geqslant (1-\delta)\ell_{exp}(\gamma_w). \tag{20}$$

Claim 1. For every $i \in \{0, \dots, k\}$ and every $m \in \mathbb{N}^*$, we have

$$\ell_{exp}^{[f^m(\gamma_w)]}(\mathrm{Inc}([f^m(\alpha_i)])) \leqslant 24C^2\ell_{exp}^{\gamma_w}(\alpha_i).$$

Similarly, for every $m \in \mathbb{N}^*$, we have

$$\ell_{exp}^{[f^m(\gamma_w)]}(\mathrm{Inc}([f^m(\gamma_w)])) \leqslant 24C^2\ell_{exp}(\gamma_w).$$

Proof. Since a reduced iterate of a complete factor is complete, every incomplete factor of $[f^m(\gamma_w)]$ is contained in a reduced iterate of some α_i. Thus, we have

$$\ell_{exp}^{[f^m(\gamma_w)]}(\mathrm{Inc}([f^m(\gamma_w)])) \leqslant \sum_{i=0}^{k} \ell_{exp}^{[f^m(\gamma_w)]}(\mathrm{Inc}([f^m(\alpha_i)])).$$

Hence it suffices to prove the result for the paths α_i with $i \in \{0, \dots, k\}$. By Property (ii) for every $i \in \{1, \dots, k\}$, the path β_i has positive exponential length relative to γ_w. Therefore, if there exists $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma_w)$ such that α_i intersects γ' nontrivially, then γ' is contained in $\beta_i\alpha_i\beta_{i+1}$. In particular, Lemma 5.13 applies and for every $i \in \{0, \dots, k\}$, if $\ell_{exp}^{\gamma_w}(\alpha_i) = 0$, then $\ell_{exp}(\alpha_i) = 0$.

Let $i \in \{0, \dots, k\}$. Suppose first that $\ell_{exp}^{\gamma_w}(\alpha_i) = 0$. By the above, we have $\ell_{exp}(\alpha_i) = 0$. By Lemma 5.12, there exists $N \in \mathbb{N}^*$ such that for every $m \geqslant N$, such that the total exponential length of incomplete factors in any optimal splitting of $[f^m(\alpha_i)]$ is equal to 0. Hence for every $m \geqslant N$, the path $[f^m(\alpha_i)]$ is PG-relative completely split. Up to taking a power of f, we may assume that $N = 1$. So this concludes the proof of the claim in the case when $\ell_{exp}^{\gamma_w}(\alpha_i) = 0$.

So we may assume that $\ell_{exp}^{\gamma_w}(\alpha_i) > 0$. By Lemma 5.12, for every $m \in \mathbb{N}^*$, the total exponential length of incomplete factors in $[f^m(\alpha_i)]$ is at most equal to $8C\ell_{exp}(\alpha_i)$. By Lemma 5.6, for every $i \in \{1, \dots, k\}$, we have

$$\ell_{exp}(\alpha_i) \leqslant \ell_{exp}^{\gamma_w}(\alpha_i) + 2C \leqslant 3C\ell_{exp}^{\gamma_w}(\alpha_i).$$

Hence by Lemma 5.6 again, we have

$$\ell_{exp}^{[f^m(\gamma_w)]}(\mathrm{Inc}([f^m(\alpha_i)])) \leqslant \ell_{exp}(\mathrm{Inc}([f^m(\alpha_i)])) \leqslant 24C^2\ell_{exp}^{\gamma_w}(\alpha_i).$$

This proves the claim. □

Let Λ_{γ_w} be the set consisting of all incomplete factors α_i of γ_w whose exponential length relative to γ_w is at least equal to $(3.10^8)R^6C^{12}+1$. Let Λ'_{γ_w} be the set consisting of all incomplete factors α_i of γ_w which are not in Λ_{γ_w}. Let $\ell_{exp}^{\gamma_w}(\Lambda_{\gamma_w})$ (resp. $\ell_{exp}^{\gamma_w}(\Lambda'_{\gamma_w})$) be the sum of the exponential lengths relative to γ_w of all incomplete factors of γ that belongs to Λ_{γ_w} (resp. Λ'_{γ_w}). We distinguish between two cases, according to the proportion of $\ell_{exp}^{\gamma_w}(\Lambda_{\gamma_w})$ in the exponential length relative to γ_w of incomplete factors in γ_w.

Case 1. Suppose that

$$\frac{\ell_{exp}^{\gamma_w}(\Lambda_{\gamma_w})}{\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w))}<\frac{1}{(24C^2R)^2}.$$

This implies that

$$\frac{\ell_{exp}^{\gamma_w}(\Lambda'_{\gamma_w})}{\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w))}\geqslant\frac{(24C^2R)^2-1}{(24C^2R)^2}. \tag{21}$$

Note that, by Lemma 5.6, every path in Λ'_{γ_w} has exponential length at most equal to $(3.10^8)C^{12}R^6+1+2C$. By Lemma 5.11, there exists $n_0\in\mathbb{N}^*$ such that, for every edge path β of exponential length at most equal to $(3.10^8)R^6C^{12}+1+2C$ and every $n\geqslant n_0$ either $[f^n(\beta)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ or $[f^{n_0}(\beta)]$ contains a complete factor of exponential length at least equal to $10C$. By Lemma 5.6, in the second case, the path $[f^{n_0}(\beta)]$ has a complete factor of positive exponential length relative to $[f^{n_0}(\beta)]$.

Let Γ_{γ_w} be the set consisting of all incomplete paths α_i of γ_w such that $\alpha_i\in\Lambda'_{\gamma_w}$ and $[f^{n_0}(\alpha_i)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Let Γ'_{γ_w} be the set consisting in all incomplete paths α_i of γ_w such that $\alpha_i\in\Lambda'_{\gamma_w}$ and $[f^{n_0}(\alpha_i)]$ has at least one complete factor of positive exponential length relative to $[f^{n_0}(\alpha_i)]$. Note that $\Lambda'_{\gamma_w}=\Gamma_{\gamma_w}\cup\Gamma'_{\gamma_w}$. Let $\ell_{exp}^{\gamma_w}(\Gamma_{\gamma_w})$ (resp. $\ell_{exp}^{\gamma_w}(\Gamma'_{\gamma_w})$) be the sum of the exponential lengths relative to γ_w of paths in Γ_{γ_w} (resp. Γ'_{γ_w}).

Subcase 1. Suppose that

$$\frac{\ell_{exp}^{\gamma_w}(\Gamma_{\gamma_w})}{\ell_{exp}^{\gamma_w}(\Lambda'_{\gamma_w})}\geqslant\frac{24C^2R}{24C^2R+1}.$$

Then

$$\ell_{exp}^{\gamma_w}(\Gamma_{\gamma_w})\geqslant\frac{24C^2R}{24C^2R+1}\ell_{exp}^{\gamma_w}(\Lambda'_{\gamma_w})\geqslant\frac{24C^2R-1}{24C^2R}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)).$$

Note that, for every $n\geqslant n_0$ and every path $\alpha_i\in\Gamma_{\gamma_w}$, we have $\ell_{exp}([f^n(\alpha_i)])=0$ by Lemma 3.18. By Claim 1, for every path α_i such that $\alpha_i\in\Lambda'_{\gamma_w}$ and $\alpha_i\notin\Gamma_{\gamma_w}$, and for every $n\in\mathbb{N}^*$, the total exponential length of incomplete factors in $[f^n(\alpha_i)]$ relative to $[f^n(\alpha_i)]$ is at most equal to $24C^2\ell_{exp}^{\gamma_w}(\alpha_i)$. Recall that, by Equation (20), we have $\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w))=\sum_{\alpha_i\in\Lambda_{\gamma_w}\cup\Lambda'_{\gamma_w}}\ell_{exp}^{\gamma_w}(\alpha_i)$. Thus, for every $n\geqslant n_0$, we have:

$$\begin{aligned}
\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)])) &\leqslant \sum_{\alpha_i\in\Lambda_{\gamma_w}\cup\Lambda'_{\gamma_w}}\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\alpha_i)]))\\
&\leqslant \sum_{\alpha_i\in\Lambda_{\gamma_w}\cup(\Lambda'_{\gamma_w}-\Gamma_{\gamma_w})}24C^2\ell_{exp}^{\gamma_w}(\alpha_i)\\
&\leqslant 24C^2\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w))-24C^2\tfrac{24C^2R-1}{24C^2R}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w))\\
&\leqslant \tfrac{1}{R}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)).
\end{aligned}$$

This concludes the proof of Lemma 5.21 when Subcase 1 occurs.

Subcase 2. Suppose that

$$\frac{\ell_{exp}(\Gamma_{\gamma_w})}{\ell_{exp}(\Lambda'_{\gamma_w})} < \frac{24C^2R}{24C^2R+1}.$$

Note that the assumption of Subcase 2 and Equation (21) imply that

$$\ell_{exp}^{\gamma_w}(\Gamma'_{\gamma_w}) \geqslant \frac{1}{24C^2R+1}\ell_{exp}^{\gamma_w}(\Lambda'_{\gamma_w}) \geqslant \frac{(24C^2R)^2-1}{(24C^2R)^2}\frac{1}{24C^2R+1}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)).$$

Since every path in Γ'_{γ_w} has exponential length at most equal to $(3.10^8)R^6C^{12}+1+2C$, by Lemma 5.7, up to taking a larger n_0, for every path $\alpha_i \in \Gamma'_{\gamma_w}$ such that $\ell_{exp}(\alpha_i) > 0$ and every $n \geqslant n_0$, the exponential length of a complete factor in $[f^n(\alpha_i)]$ is at least equal to $3^{n-n_0}\ell_{exp}(\alpha_i)$. Moreover, for every path $\alpha_i \in \Gamma'_{\gamma_w}$ such that $\ell_{exp}(\alpha_i) = 0$ and every $n \geqslant n_0$, the exponential length of a complete factor in $[f^n(\alpha_i)]$ is at least equal to 3^{n-n_0}. By Lemma 5.6, for every $n \geqslant n_0$ and every path $\alpha_i \in \Gamma'_{\gamma_w}$ such that $\ell_{exp}(\alpha_i) > 0$, the exponential length relative to $[f^n(\alpha_i)]$ of a complete factor in $[f^n(\alpha_i)]$ is at least equal to

$$3^{n-n_0}\ell_{exp}(\alpha_i) - 2C \geqslant (3^{n-n_0}-2C)\ell_{exp}(\alpha_i).$$

Thus, for every $n \geqslant n_0$ and every path $\alpha_i \in \Gamma'_{\gamma_w}$, the exponential length relative to $[f^n(\alpha_i)]$ of a complete factor in $[f^n(\alpha_i)]$ is at least equal to

$$(3^{n-n_0}-2C)\ell_{exp}(\alpha_i).$$

Therefore, for every $n \geqslant n_0$, the sum of the exponential lengths of complete factors in $[f^n(\gamma_w)]$ is at least equal to

$$(3^{n-n_0}-2C)\ell_{exp}^{\gamma_w}(\Gamma'_{\gamma_w}) \geqslant (3^{n-n_0}-2C)\frac{(24C^2R)^2-1}{(24C^2R)^2}\frac{1}{24C^2R+1}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)). \tag{22}$$

By Claim 1, for every $n \in \mathbb{N}^*$, we have $\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)])) \leqslant 24C^2\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w))$. Recall that the goodness function is a supremum over splittings of the considered path. Thus, by Equation (22) for every $n \geqslant n_0$, since the maps $t \mapsto \frac{t}{t+a}$ are nonincreasing for every $a > 0$, we have

$$\begin{aligned}
&\mathfrak{g}([f^n(\gamma_w)]) \\
&\quad\geqslant \frac{(3^{n-n_0}-2C)\frac{(24C^2R)^2-1}{(24C^2R)^2}\frac{1}{24C^2R+1}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w))}{(3^{n-n_0}-2C)\frac{(24C^2R)^2-1}{(24C^2R)^2}\frac{1}{24C^2R+1}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)) + \ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)]))} \\
&\quad\geqslant \frac{(3^{n-n_0}-2C)\frac{(24C^2R)^2-1}{(24C^2R)^2}\frac{1}{24C^2R+1}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w))}{(3^{n-n_0}-2C)\frac{(24C^2R)^2-1}{(24C^2R)^2}\frac{1}{24C^2R+1}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)) + 24C^2\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w))} \\
&\quad\geqslant \frac{(3^{n-n_0}-2C)\frac{(24C^2R)^2-1}{(24C^2R)^2}\frac{1}{24C^2R+1}}{(3^{n-n_0}-2C)\frac{(24C^2R)^2-1}{(24C^2R)^2}\frac{1}{24C^2R+1}+24C^2},
\end{aligned}$$

which goes to 1 as n goes to infinity. Hence there exists $n_1 \in \mathbb{N}$ which is independent of γ_w, such that, for every path γ_w as in Subcase 2 and every $n \geqslant n_1$, we have: $\mathfrak{g}([f^n(\gamma_w)]) \geqslant \delta$. This concludes the proof of Lemma 5.21 when Case 1 occurs.

Case 2. Suppose that, contrarily to Case 1, we have

$$\frac{\ell_{exp}^{\gamma_w}(\Lambda_{\gamma_w})}{\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w))} \geqslant \frac{1}{(24C^2R)^2}.$$

Let $\alpha \in \Lambda_{\gamma_w}$ and consider the decomposition of the reduced path α into maximal subsegments $\alpha^{(1)} \dots \alpha^{(k_\alpha)}$ of exponential length relative to γ_w equal to $2000R^3C^6$, except possibly the last one of exponential length relative to γ_w less than or equal to $2000R^3C^6$. Let

$$\Lambda_{\gamma_w}^{(1)} = \left\{\alpha^{(j)} \mid \alpha \in \Lambda_{\gamma_w}, j \in \{1, \dots, k_\alpha\}, \ell_{exp}^{\gamma_w}(\alpha^{(j)}) = 2000R^3C^6\right\},$$

$$\Lambda_{\gamma_w}^{(2)} = \left\{\alpha^{(j)} \mid \alpha \in \Lambda_{\gamma_w}, j \in \{1, \dots, k_\alpha\}, \ell_{exp}^{\gamma_w}(\alpha^{(j)}) < 2000R^3C^6\right\}.$$

Note that, since for every $\alpha \in \Lambda_{\gamma_w}$, we have $\ell_{exp}^{\gamma_w}(\alpha) \geqslant (3.10^8)R^6C^{12} + 1$, we see that

$$|\Lambda_{\gamma_w}^{(1)}| \geqslant 120000R^3C^6|\Lambda_{\gamma_w}^{(2)}|. \tag{23}$$

Note that every element in $\Lambda_{\gamma_w}^{(1)} \cup \Lambda_{\gamma_w}^{(2)}$ has exponential length at most equal to $2000R^3C^6 + 1 + 2C$ by Lemma 5.6. By Lemma 5.11, there exists $M \in \mathbb{N}^*$ depending only on f such that for every $n \geqslant M$ and every reduced edge path α of exponential length at most equal to $(3.10^8)R^6C^{12} + 1 + 2C$, either $[f^n(\alpha)]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ or the following hold

(a) there exists a complete factor of $[f^n(\alpha)]$ whose exponential length is at least equal to $10C$;
(b) the exponential length of an incomplete factor of $[f^n(\alpha)]$ is at most equal to $8C$.

This applies in particular to every element $\alpha \in \Lambda_{\gamma_w}^{(1)} \cup \Lambda_{\gamma_w}^{(2)}$ and to every element $\alpha \in \Lambda'_{\gamma_w}$. For every $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(1)}$ and every $n \geqslant M$, let $\alpha^{(j,n)}$ be the (possibly degenerate) subpath of $[f^n(\alpha^{(j)})]$ contained in $[f^n(\alpha)]$. Let $\Lambda_{\gamma_w}^{(3)}$ be the subset of $\Lambda_{\gamma_w}^{(1)}$ consisting of all $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(1)}$ such that $\ell_{exp}(\alpha^{(j,M)}) \leqslant 80C^2$, and let $\Lambda_{\gamma_w}^{(4)} = \Lambda_{\gamma_w}^{(1)} - \Lambda_{\gamma_w}^{(3)}$.

Suppose first that

$$|\Lambda_{\gamma_w}^{(4)}| > \frac{1}{30000R^3C^6}|\Lambda_{\gamma_w}^{(3)}|. \tag{24}$$

Therefore, as $|\Lambda_{\gamma_w}^{(1)}| = |\Lambda_{\gamma_w}^{(3)}| + |\Lambda_{\gamma_w}^{(4)}|$, by Equation (23), we have

$$|\Lambda_{\gamma_w}^{(2)}| \leqslant \frac{30001R^3C^6}{120000R^3C^6}|\Lambda_{\gamma_w}^{(4)}| = K_0|\Lambda_{\gamma_w}^{(4)}|,$$

where K_0 is a constant depending only on C and R. Note that $\Lambda_{\gamma_w} = \Lambda_{\gamma_w}^{(2)} \cup \Lambda_{\gamma_w}^{(3)} \cup \Lambda_{\gamma_w}^{(4)}$ and for every $j \in \{2, 3, 4\}$, every path in $\Lambda_{\gamma_w}^{(j)}$ has exponential length at most equal to $2000R^3C^6$. Thus, we see that

$$\ell_{exp}^{\gamma_w}(\Lambda_{\gamma_w}) \leqslant 2000R^3C^6(|\Lambda_{\gamma_w}^{(2)}| + |\Lambda_{\gamma_w}^{(3)}| + |\Lambda_{\gamma_w}^{(4)}|) \leqslant K'_0|\Lambda_{\gamma_w}^{(4)}|$$

for some constant K'_0 depending only on C and R.

Recall that if $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(4)}$, then $\ell_{exp}(\alpha^{(j,M)}) > 80C^2$. Suppose towards a contradiction that $[f^M(\alpha^{(j)})]$ is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Since $\alpha^{(j,M)}$ is a subpath of $[f^M(\alpha^{(j)})]$, we have $\ell_{exp}^{[f^M(\alpha^{(j)})]}(\alpha^{(j,M)}) = 0$. By Lemma 5.6, we see that $\ell_{exp}(\alpha^{(j,M)}) \leqslant \ell_{exp}^{[f^M(\alpha^{(j)})]}(\alpha^{(j,M)}) + 2C = 2C$, which leads to a contradiction. Hence $[f^M(\alpha^{(j)})]$ satisfies Assertions (a) and (b).

Note that $\alpha^{(j,M)}$ is a subpath of $[f^M(\alpha^{(j)})]$. Since $\ell_{exp}(\alpha^{(j,M)}) > 80C^2$, since every incomplete factor of $[f^M(\alpha^{(j)})]$ has exponential length at most equal to $8C$ by (b) and since an incomplete factor of $[f^M(\alpha^{(j)})]$ is followed by a complete factor of $[f^M(\alpha^{(j)})]$, we see that $\alpha^{(j,M)}$ contains a subpath of a complete factor of $[f^M(\alpha^{(j)})]$. Since $\ell_{exp}(\alpha^{(j,M)}) > 80C^2$ and since every incomplete subpath of $[f^M(\alpha^{(j)})]$ has exponential length at most equal to $8C$, the path $\alpha^{(j,M)}$ must contain a subpath $\alpha^{(j,M)'}$ such that the total exponential length of complete factors of $\alpha^{(j,M)'}$ is at least equal to $10C$.

Let $\alpha_0^{(j,M)}$ be the minimal concatenation of splittings of a fixed optimal splittings of $[f^m(\alpha^{(j)})]$ which contains $\alpha^{(j,M)'}$. Let $\tau_1^{(j,M)}$ and $\tau_2^{(j,M)}$ be paths such that $[f^M(\alpha^{(j)})] = \tau_1^{(j,M)}\alpha_0^{(j,M)}\tau_2^{(j,M)}$.

By Lemma 5.8 applied twice (once with $\gamma = \alpha_0^{(j,M)}\tau_2^{(j,M)}[f^M(\alpha^{(j+1)}\dots\alpha_k^{(k_{\alpha_k})})]$ and $\gamma_1 = \alpha_0^{(j,M)}$ and once with $\gamma^{-1} = [f^M(\alpha_1^{(1)}\dots\alpha^{(j-1)})]\tau_1^{(j,M)}\alpha_0^{(j,M)}$ and $\gamma_1^{-1} = \alpha_0^{(j,M)}$), we see that $\alpha^{(j,M)}$ contains a complete factor of $[f^M(\gamma_w)]$ of exponential length at least equal to $10C - 4C = 6C$. By Lemma 5.6, the path $\alpha^{(j,M)}$ contains a complete factor of $[f^M(\gamma_w)]$ of exponential length relative to $[f^M(\gamma_w)]$ at least equal to C. By Lemma 5.7 (with γ a complete factor contained in $\alpha^{(j,M)}$), for every $n \geqslant M$ and every $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(4)}$, the path $\alpha^{(j,n)}$ contains a complete subpath of $[f^n(\gamma_w)]$ of exponential length at least equal to $3^{n-M}C$. By Lemma 5.6, for every $n \geqslant M$ and every $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(4)}$, the path $\alpha^{(j,n)}$ contains a complete subpath of $[f^n(\gamma_w)]$ of exponential length relative to $[f^n(\gamma_w)]$ at least equal to $3^{n-M}C - 2C$. Hence for every $n \geqslant M$, the sum of the exponential length relative to $[f^n(\gamma_w)]$ of complete factors contained in $[f^n(\gamma_w)]$ is at least equal to $(3^{n-M}C - 2C)|\Lambda_{\gamma_w}^{(4)}|$.

By Claim 1, for every $n \geqslant M$, we have

$$\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)])) \leqslant 24C^2\ell_{exp}^{\gamma_w}(\gamma_w) \leqslant 24C^2\frac{1}{1-\delta}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)),$$

where the last inequality holds by Equation (20). Using the above equations and the assumptions of Case 2, we see that

$$\begin{array}{rcl} \ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)])) & \leqslant & 24C^2\frac{1}{1-\delta}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)) \\ & \leqslant & 24C^2\frac{1}{1-\delta}(24C^2R)^2\ell_{exp}^{\gamma_w}(\Lambda_{\gamma_w}) \\ & \leqslant & 24C^2\frac{1}{1-\delta}(24C^2R)^2K_0'|\Lambda_{\gamma_w}^{(4)}| = K_1|\Lambda_{\gamma_w}^{(4)}|, \end{array}$$

where K_1 is a constant depending only on C, R and δ. Thus, since the goodness function is a supremum over all splittings of the considered path, for every $n \geqslant M$, we have:

$$\begin{array}{rcl} \mathfrak{g}([f^n(\gamma_w)]) & \geqslant & \frac{(3^{n-M}C-2C)|\Lambda_{\gamma_w}^{(4)}|}{(3^{n-M}C-2C)|\Lambda_{\gamma_w}^{(4)}|+\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\alpha)]))} \\ & \geqslant & \frac{(3^{n-M}C-2C)|\Lambda_{\gamma_w}^{(4)}|}{(3^{n-M}C-2C)|\Lambda_{\gamma_w}^{(4)}|+K_1|\Lambda_{\gamma_w}^{(4)}|} \\ & = & \frac{3^{n-M}C-2C}{3^{n-M}C-2C+K_1}, \end{array}$$

which converges to 1 as n goes to infinity. Hence there exists $M' \in \mathbb{N}^*$ depending only on f such that for every $n \geqslant M$, we have $\mathfrak{g}([f^n(\gamma_w)]) \geqslant \delta$. This proves Lemma 5.21 in this case.

Suppose now that contrarily to Equation (24), we have

$$|\Lambda_{\gamma_w}^{(4)}| \leqslant \frac{1}{30000R^3C^6}|\Lambda_{\gamma_w}^{(3)}|. \tag{25}$$

Then

$$|\Lambda_{\gamma_w}^{(1)}| = |\Lambda_{\gamma_w}^{(3)}| + |\Lambda_{\gamma_w}^{(4)}| \leqslant \left(1 + \frac{1}{30000R^3C^6}\right)|\Lambda_{\gamma_w}^{(3)}|.$$

Claim 2. Let $n \geqslant M$, let $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(2)} \cup \Lambda_{\gamma_w}^{(4)}$. The total exponential length of incomplete factors of $[f^n(\gamma_w)]$ contained in $\alpha^{(j,n)}$ is at most equal to $12C\ell_{exp}(\alpha^{(j)})$.

Proof. Let σ be an incomplete factor of $[f^n(\gamma_w)]$ which is contained in $\alpha^{(j,M)}$. Then one of the following holds:

(i) the path σ is an incomplete factor of $[f^n(\alpha^{(j)})]$;
(ii) the path σ contains a subpath which is complete in $[f^n(\alpha^{(j)})]$.

Note that the total exponential length of incomplete factors of $[f^n(\gamma_w)]$ which satisfy (i) is bounded by the total exponential length of incomplete factors of $[f^n(\alpha^{(j)})]$. Thus, by Lemma 5.12, the total exponential length of incomplete factors of $[f^n(\gamma_w)]$ which satisfy (i) is bounded by $8C\ell_{exp}(\alpha^{(j)})$.

Suppose that σ satisfies (ii). Let $\alpha^{(j,n)} = a_1ca_2$ be a decomposition of $\alpha^{(j,n)}$ where for every $i \in \{1,2\}$, the total exponential length of complete factors of $[f^n(\alpha^{(j)})]$ contained in a_i is equal to $2C$. By Lemma 5.8 applied to

$$\gamma = [f^n(\alpha^{(j)})][f^n(\alpha^{(j+1)} \dots \alpha_k^{(k_{\alpha_k})})] \text{ and } \gamma_1 = [f^n(\alpha^{(j)})]$$

and to

$$\gamma^{-1} = [f^n(\alpha_1^{(1)} \dots \alpha^{(j-1)})][f^n(\alpha^{(j)})] \text{ and } \gamma_1^{-1} = [f^n(\alpha^{(j)})],$$

the path σ is contained in either a_1 or a_2. For every $t \in \{1,2\}$, let $a_t = b_1^{(t)}b_1^{(t)'} \dots b_s^{(t)}b_{s_t}^{(t)'}$ be a decomposition of a_t where, for every $i \in \{1,\dots,s_t\}$, the path $b_i^{(t)}$ is an incomplete factor of $[f^n(\alpha^{(j)})]$ and for every $i \in \{1,\dots,s_t\}$, the path $b_i^{(t)'}$ is a complete factor of $[f^n(\alpha^{(j)})]$ contained in a_t.

Suppose that there exists $i \in \{1,\dots,s_1\}$ such that $b_i^{(1)'}$ is a complete factor of $[f^n(\gamma_w)]$. We claim that for every $j \geqslant i+1$, the path $b_j^{(1)'}$ is a complete factor of $[f^n(\gamma_w)]$. Indeed, let $n' \geqslant n$ and let $j \geqslant i+1$. Then there is no identification between an initial segment of $[f^{n'}(b_j^{(1)'})]$ and an initial segment of $[f^n(\gamma_w)]$ not intersecting $\alpha^{(j,n')}$ as otherwise there would exist identifications with $[f^{n'}(b_i^{(1)'})]$, contradicting the fact that $b_i^{(1)'}$ is complete. Similarly, there is no identification between a terminal segment of $[f^{n'}(b_j^{(1)'})]$ and a terminal segment of $[f^n(\gamma_w)]$ not intersecting $\alpha^{(j,n')}$ as otherwise there would exist identifications with $[f^{n'}(c)]$. The claim follows. Similarly, if there exists $i \in \{1,\dots,s_2\}$ such that $b_i^{(2)'}$ is a complete factor of $[f^n(\gamma_w)]$, then for every $j < i$, the path $b_j^{(2)'}$ is a complete factor of $[f^n(\gamma_w)]$.

Hence we may assume that for every $t \in \{1,2\}$ and every $s \in \{1,\dots,s_t\}$, the path $b_s^{(t)'}$ is incomplete in $[f^n(\gamma_w)]$. Therefore, for every $t \in \{1,2\}$, the whole path a_t is incomplete in $[f^n(\gamma_w)]$. Thus, in order to prove the claim, it suffices to bound the exponential lengths of a_1 and a_2. Let $t \in \{1,2\}$. By Lemma 3.17, we have

$$\ell_{exp}(a_t) \leqslant \sum_{i=1}^{s_t} \ell_{exp}(b_i^{(t)}) + \ell_{exp}(b_i^{(t)'}).$$

For every $i \in \{1\dots,s_t\}$, the path $b_i^{(t)}$ satisfies (i) and we already have a bound on the total exponential length of such paths. Moreover, since the total exponential length of complete factors of $\alpha^{(j,n)}$ contained in a_t is at most equal to $2C$, we have

$$\sum_{i=1}^{s_t} \ell_{exp}(b_i^{(t)'}) \leqslant 2C.$$

Thus, the total exponential length of incomplete factors of $[f^n(\gamma_w)]$ contained in $\alpha^{(j,M)}$ is at most equal to

$$8C\ell_{exp}(\alpha^{(j)}) + \sum_{t=1}^{2}\sum_{i=1}^{s_t} \ell_{exp}(b_i^{(t)'}) \leqslant 8C\ell_{exp}(\alpha^{(j)}) + 4C \leqslant 12C\ell_{exp}(\alpha^{(j)}),$$

where the last inequality follows from the fact that every element of $\Lambda^{(2)}_{\gamma_w} \cup \Lambda^{(4)}_{\gamma_w}$ has positive exponential length. $\square$

By Claim 2 and Lemma 5.6, for every $n \geqslant M$ and every $\alpha^{(j)} \in \Lambda^{(2)}_{\gamma_w} \cup \Lambda^{(4)}_{\gamma_w}$, the total exponential length relative to $[f^n(\gamma_w)]$ of incomplete factors in the subpath of $[f^n(\gamma_w)]$ contained in $[f^n(\alpha^{(j)})]$ is at most equal to $12C\ell^{\gamma_w}_{exp}(\alpha^{(j)}) + 2C \leqslant 14C\ell^{\gamma_w}_{exp}(\alpha^{(j)})$. Hence by definition, for every $n \geqslant M$ and every path $\alpha^{(j)} \in \Lambda^{(2)}_{\gamma_w} \cup \Lambda^{(4)}_{\gamma_w}$, we have

$$\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)]) \cap \alpha^{(j,n)}) \leqslant 14C\ell_{exp}(\alpha^{(j)}).$$

We claim that, for every $n \geqslant M$, every element in $\Lambda_{[f^n(\gamma_w)]}$ is contained in an iterate of an element in Λ_{γ_w}. Indeed, note that, by the choice of M (in the above application of Lemma 5.11), for every element $\alpha \in \Lambda'_{\gamma_w}$, the exponential length of an incomplete factor in $[f^n(\alpha)]$ is at most equal to $8C$. Hence an incomplete factor of $[f^n(\alpha)]$ whose exponential length is at least equal to $(3.10^8)R^6C^{12}+1$ cannot be contained in an iterate of an element of Λ'_{γ_w}. The claim follows.

Therefore, using Equation (25) for the third inequality, the value of $\ell_{exp}^{[f^M(\gamma_w)]}(\Lambda_{[f^M(\gamma_w)]})$ is at most equal to

$$\begin{aligned}
&\sum_{\alpha^{(j)}\in\Lambda^{(3)}_{\gamma_w}} \ell_{exp}(\alpha^{(j,M)}) + \sum_{\alpha^{(j)}\in\Lambda^{(4)}_{\gamma_w}} \ell_{exp}^{[f^M(\gamma_w)]}(\mathrm{Inc}([f^M(\gamma_w)]) \cap \alpha^{(j,M)})\\
&+ \sum_{\alpha^{(j)}\in\Lambda^{(2)}_{\gamma_w}} \ell_{exp}^{[f^M(\gamma_w)]}(\mathrm{Inc}([f^M(\gamma_w)]) \cap \alpha^{(j,M)})\\
&\leqslant 80C^2|\Lambda^{(3)}_{\gamma_w}| + 14C\sum_{\beta\in\Lambda^{(4)}_{\gamma_w}}\ell_{exp}(\beta) + 14C\sum_{\alpha\in\Lambda^{(2)}_{\gamma_w}}\ell_{exp}(\alpha)\\
&\leqslant 80C^2|\Lambda^{(3)}_{\gamma_w}| + 14C(2000R^3C^6)|\Lambda^{(4)}_{\gamma_w}| + 14C\sum_{\alpha\in\Lambda^{(2)}_{\gamma_w}}\ell_{exp}(\alpha)\\
&\leqslant 80C^2|\Lambda^{(3)}_{\gamma_w}| + C|\Lambda^{(3)}_{\gamma_w}| + 14C\sum_{\alpha\in\Lambda^{(2)}_{\gamma_w}}\ell_{exp}(\alpha)\\
&\leqslant 81C^2|\Lambda^{(3)}_{\gamma_w}| + 14C\sum_{\alpha\in\Lambda^{(2)}_{\gamma_w}}\ell_{exp}(\alpha).
\end{aligned}$$

Since by Equation (23)

$$\left(1+\frac{1}{30000R^3C^6}\right)|\Lambda^{(3)}_{\gamma_w}| \geqslant |\Lambda^{(1)}_{\gamma_w}| \geqslant 120000R^3C^6|\Lambda^{(2)}_{\gamma_w}|,$$

we have $|\Lambda^{(3)}_{\gamma_w}| \geqslant 60000R^3C^6|\Lambda^{(2)}_{\gamma_w}|$. Hence we have

$$\begin{aligned}\ell_{exp}^{[f^M(\gamma_w)]}(\Lambda_{[f^M(\gamma_w)]}) &\leqslant 81C^2|\Lambda^{(3)}_{\gamma_w}| + 14C\textstyle\sum_{\alpha\in\Lambda^{(2)}_{\gamma_w}}\ell_{exp}(\alpha)\\ &\leqslant 81C^2|\Lambda^{(3)}_{\gamma_w}| + (14C)(2000R^3C^6)|\Lambda^{(2)}_{\gamma_w}|\\ &\leqslant 81C^2|\Lambda^{(3)}_{\gamma_w}| + 2C|\Lambda^{(3)}_{\gamma_w}| = 83C^2|\Lambda^{(3)}_{\gamma_w}|.\end{aligned}$$

Suppose first that

$$\frac{\ell_{exp}^{[f^M(\gamma_w)]}(\Lambda_{[f^M(\gamma_w)]})}{\ell_{exp}^{[f^M(\gamma_w)]}(\mathrm{Inc}([f^M(\gamma_w)]))} < \frac{1}{(24C^2R)^2}.$$

Then we can apply Case 1 to conclude the proof of Lemma 5.21. Indeed, Case 1 gives a larger $M' \geqslant M$ such that for every $n \geqslant M'$, either

$$\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)])) \leqslant \frac{1}{R}\ell_{exp}^{[f^M(\gamma_w)]}(\mathrm{Inc}([f^M(\gamma_w)]))$$

(this is the conclusion of Subcase 1) or else $\mathfrak{g}([f^n(\gamma_w)]) \geqslant \delta$ (this is the conclusion of Subcase 2). Recall that, by Lemma 5.12 and Lemma 5.6, we have

$$\ell_{exp}^{[f^M(\gamma_w)]}(\mathrm{Inc}([f^M(\gamma_w)])) \leqslant 10C\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)).$$

Hence, if the first conclusion occurs, we have

$$\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)])) \leqslant \frac{1}{R}\ell_{exp}^{[f^M(\gamma_w)]}(\mathrm{Inc}([f^M(\gamma_w)])) \leqslant \frac{10C}{R}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)),$$

which gives the desired result.

Otherwise, we have

$$(24C^2R)^2\ell_{exp}^{[f^M(\gamma_w)]}(\Lambda_{[f^M(\gamma_w)]}) \geqslant \ell_{exp}^{[f^M(\gamma_w)]}(\mathrm{Inc}([f^M(\gamma_w)])).$$

Let $n \geqslant M$. By Lemma 5.12 and Lemma 5.6, we have

$$\begin{aligned}\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)])) &\leqslant \ell_{exp}(\mathrm{Inc}([f^n(\gamma_w)])) \leqslant 8C\ell_{exp}(\mathrm{Inc}([f^M(\gamma_w)]))\\ &\leqslant 10C\ell_{exp}^{[f^M(\gamma_w)]}(\mathrm{Inc}([f^M(\gamma_w)])).\end{aligned}$$

Recall that the exponential length of every path $\alpha \in \Lambda^{(3)}_{\gamma_w}$ is equal to $2000R^3C^6$. Hence we have

$$\begin{aligned}\frac{\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)]))}{\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_\mathrm{w}))} &= \frac{\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)]))}{\ell_{exp}^{[f^M(\gamma_w)]}(\mathrm{Inc}([f^M(\gamma_w)]))}\frac{\ell_{exp}^{[f^M(\gamma_w)]}(\mathrm{Inc}([f^M(\gamma_w)]))}{\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_\mathrm{w}))}\\ &\leqslant \frac{10C(24C^2R)^2\ell_{exp}^{[f^M(\gamma_w)]}(\Lambda_{[f^M(\gamma_w)]})}{\ell_{exp}^{\gamma_w}(\Lambda_{\gamma_w})}\\ &\leqslant \frac{10C(24C^2R)^2(83C^2|\Lambda^{(3)}_{\gamma_w}|)}{2000R^3C^6|\Lambda^{(3)}_{\gamma_w}|}\\ &\leqslant \frac{10C}{R}.\end{aligned}$$

This concludes the proof of Lemma 5.21. □

In Proposition 5.22, we need to work with CT maps that represent both an (almost) atoroidal outer automorphism and its inverse. We therefore introduce the following conventions.

Let $f'\colon G' \to G'$ be a CT map representing ϕ^{-M}, which exists by Theorem 2.10. We denote by K' the constant similar to the constant K given above Lemma 5.6 and by $C_{f'}$ the bounded cancellation constant given by Lemma 4.9. We set $C' = \max\{K', C_{f'}\}$ as in Equation (6). We denote by $G_{p'}$ the invariant subgraph of G' such that $\mathcal{F}(G_{p'}) = \mathcal{F}$, by $\ell_{\mathcal{F}'}$ the corresponding $\mathcal{F}$-length and by $\ell_{exp'}$ the corresponding exponential length. Let $\mathfrak{g}'$ be the corresponding goodness function. If $w \in F_{\mathtt{n}}$, we denote by γ'_w the corresponding circuit in G'.

We also need a result which shows that the exponential length is invariant by $F_{\mathtt{n}}$-equivariant quasi-isometry. In order to prove this, we need some additional definitions. Let G be a connected (pointed) graph whose fundamental group is isomorphic to $F_{\mathtt{n}}$ and let $\widetilde{G}$ be the universal cover of G. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}})$ be an exponentially growing outer automorphism.

Let $\widehat{G}$ be the graph obtained from $\widetilde{G}$ as follows. We add one vertex v_{gA} for every left class gA, with $g \in F_{\mathtt{n}}$ and A is a subgroup of $F_{\mathtt{n}}$ such that $[A] \in \mathcal{A}(\phi)$ and we add one edge between v_{gA} and a vertex v of $\widetilde{G}$ if and only if the vertex v is contained in the tree $T_{gAg^{-1}}$. The graph $\widehat{G}$ is known as the *electrification of* $\widetilde{G}$ (see for instance [Bow]).

For a path γ in G, we denote by $\widetilde{\gamma}$ a lift of γ in $\widetilde{G}$. Let $\widehat{\gamma}$ be the path in $\widehat{G}$ constructed as follows. Let $\widetilde{\gamma} = a_1b_1\dots a_kb_k$ be the decomposition of $\widetilde{\gamma}$ such that, for every $i \in \{1,\dots,k\}$, the path b_i is contained in some tree $T_{g_iA_ig_i^{-1}}$ with $g_i \in F_{\mathtt{n}}$, A_i a subgroup of $F_{\mathtt{n}}$ such that $[A_i] \in \mathcal{A}(\phi)$ and b_i is maximal for the property of being contained in such a tree $T_{g_iA_ig_i^{-1}}$. Then $\widehat{\gamma}$ is a path $\widehat{\gamma} = a_1c_1\dots a_kc_k$ where, for every $i \in \{1,\dots,k\}$, the path c_i is the two-edge path whose endpoints are the endpoints of b_i and the middle vertex of c_i is $v_{g_iA_i}$.

Note that the path $\widehat{\gamma}$ is not uniquely determined. Indeed, it is possible that there exists $i \in \{1,\dots,k\}$ such that b_i is contained in two distinct trees T_A and T_B with $[A],[B] \in \mathcal{A}(\phi)$. However, if $\widehat{\gamma}$ and $\widehat{\gamma}'$ are two such paths associated with $\widetilde{\gamma}$, then $\ell(\widehat{\gamma}) = \ell(\widehat{\gamma}')$.

Note that if $\gamma = ab$ for some reduced edge paths a, b, then

$$\ell(\widehat{\gamma}) \leqslant 2\ell(\widehat{a}) + 2\ell(\widehat{b}).$$

Indeed, a maximal subpath of γ contained in some T_A with $[A] \in \mathcal{A}(\phi)$ is either contained in a, in b or is a concatenation of paths of a and b contained in T_A. Moreover, if e is an edge of G contained in some T_A with $[A] \in \mathcal{A}(\phi)$, then $\ell(\widehat{e}) = 2$. Thus, the inequality holds.

Proposition 5.22. *Let $\mathtt{n} \geqslant 3$, let $\phi \in \mathrm{Out}(F_{\mathtt{n}})$ and let $f\colon G \to G$ be a CT map representing a power of ϕ.*

(1) *There exists a constant $B_0 \geqslant 1$ such that, for every element $w \in F_{\mathtt{n}}$ with $\ell_{exp}(\gamma_w) > 0$, we have:*

$$\frac{1}{B_0}\ell_{exp}(\gamma_w) \leqslant \ell(\widehat{\gamma_w}) \leqslant B_0\ell_{exp}(\gamma_w).$$

(2) *Let $f'\colon G' \to G'$ be a CT map representing a power of ϕ^{-1}. There exists a constant $B > 0$ such that, for every element $w \in F_{\mathtt{n}}$, we have:*

$$\frac{1}{B}\ell_{exp'}(\gamma'_w) \leqslant \ell_{exp}(\gamma_w) \leqslant B\ell_{exp'}(\gamma'_w).$$

Proof. (1) Recall the definition of the graph G^* from just above Lemma 3.12. We can turn the graph G^* into a metric graph by assigning, to every edge $e \in \vec{E}G^*$, the length equal to the length of the path $p_{G^*}(e)$ in G. Since the graph G^* is finite, there exists a constant B' such that the diameter of every maximal subtree of G^* is at most B'. Let $B_0 = 2B' + 2$.

Let $w \in F_{\mathtt{n}}$. Let $\gamma_w = a_1 b_1 \dots a_k b_k$ be the decomposition of γ_w with a_1 and b_k possibly empty such that, for every $i \in \{1, \dots, k\}$, the path b_i is a maximal concatenation of paths in G'_{PG} and in $\mathcal{N}_{PG}$ and, for every $i \in \{1, \dots, k\}$ and every edge e of a_i, we have $\ell_{exp}^{\gamma_w}(e) = 1$. Note that by the definition of the exponential length we have

$$\ell_{exp}(\gamma_w) = \sum_{i=1}^{k} \ell(a_i).$$

Let A be a subgroup of $F_{\mathtt{n}}$ such that $[A] \in \mathcal{A}(\phi)$. Let $i \in \{1, \dots, k\}$ and let α be a subpath of a_i whose lift is contained in T_A. By Proposition 3.14, the subpath α is contained in a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Since a_i does not contain any concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$, the path α is a proper subpath of an EG INP. By the definition of C (see Equation (6)), we see that $\ell(\alpha) \leqslant C$. Thus, we have: $\ell(a_i) \leqslant C\ell(\widehat{a}_i)$ and

$$\ell_{exp}(\gamma_w) \leqslant C \sum_{i=1}^{k} \ell(\widehat{a}_i).$$

Claim. Let A be a subgroup of $F_{\mathtt{n}}$ such that $[A] \in \mathcal{A}(\phi)$. Let β be a subpath of γ_w such that a lift of β is contained in T_A. There does not exist $i \in \{1, \dots, k\}$ such that both $\beta \cap b_i$ and $\beta \cap b_{i+1}$ are not reduced to a point.

Proof. Suppose towards a contradiction that such an element $i \in \{1, \dots, k\}$ exists. Then a_{i+1} is contained in β. By the above, the path a_{i+1} is contained in an EG INP σ. Since both b_i and b_{i+1} are concatenations of paths in G'_{PG} and $\mathcal{N}_{PG}$, the path a_{i+1} must contain the initial or the terminal segment of σ. Since β is contained in a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ by Proposition 3.14, the EG INP σ must be contained in β and $\beta \cap a_{i+1} \subseteq \sigma$. This contradicts the maximality of the paths b_i and b_{i+1}. ☐

Hence β is either contained in $b_i a_{i+1}$ or in $a_{i+1} b_{i+1}$. Let $i \in \{1, \dots, k\}$ and let β be a maximal subpath of γ_w containing edges of a_i and such that a lift of β is contained in some T_A with A a subgroup of $F_{\mathtt{n}}$ such that $[A] \in \mathcal{A}(\phi)$. By the claim, the path a_i has a decomposition $a_i = c_i^+ d_i c_i^-$ such that c_i^+ and c_i^- are possibly trivial, lifts of c_i^+ and c_i^- are contained in trees T_{A_+} and T_{A_-} with A_+ and A_- subgroups of $F_{\mathtt{n}}$ such that $[A_+], [A_-] \in \mathcal{A}(\phi)$ and one of the following holds:

(a) $\beta \subseteq d_i$;
(b) $\beta \cap a_i \neq \beta$ and $\beta \cap a_i \in \{c_i^+, c_i^-\}$.

Note that for every $i \in \{1, \dots, k\}$, we have $\ell(\widehat{a_i}) \leqslant \ell(\widehat{d_i}) + 4$. Then

$$\ell(\widehat{\gamma_w}) \geqslant \sum_{i=1}^{k} \ell(\widehat{d_i}) \geqslant \sum_{i=1}^{k} (\ell(\widehat{a_i}) - 4) = \sum_{i=1}^{k} \ell(\widehat{a}_i) - 4k.$$

Moreover, if β is a path which satisfies the hypothesis of the claim, then there exists at most one $i \in \{1, \dots, k\}$ such that $\beta \cap b_i$ is not reduced to a point. Therefore, we see that $\ell(\widehat{\gamma_w}) \geqslant k$. Thus, we have

$$\ell_{exp}(\gamma_w) \leqslant C \sum_{i=1}^{k} \ell(\widehat{a}_i) \leqslant C(\ell(\widehat{\gamma}_w) + 4k) \leqslant 5C\ell(\widehat{\gamma}_w).$$

This proves the first inequality of Assertion (1).

We now prove the second inequality. For every $i \in \{1, \dots, k\}$, there exists a unique path $b_i^* \subseteq G^*$ such that $p^*(b_i^*) = b_i$. Let $i \in \{1, \dots, k\}$. Since G^* is a finite graph, there exist (possibly trivial) reduced paths β_i^*, δ_i^* and $\delta_i^{*'}$ such that:

(i) the path β_i^* is a circuit;
(ii) the paths δ_i^* and $\delta_i^{*'}$ are contained in maximal trees of G^*;
(iii) we have $b_i^* = \delta_i^* \beta_i^* \delta_i^{*'}$.

By Lemma 3.12(1), the paths $p^*(\delta_i^*)$, $p^*(\beta_i^*)$ and $p^*(\delta_i^{*'})$ are reduced edge paths of G. By definition of B', we have $\ell(\delta_i^*), \ell(\delta_i^{*'}) \leqslant B'$. Since $p^*(\beta_i^*)$ is a circuit which is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$, by Proposition 3.14, there exists a subgroup H_i of $F_{\mathtt{n}}$ such that $[H_i] \in \mathcal{A}(\phi)$ and the conjugacy classes of elements of $F_{\mathtt{n}}$ represented by $p^*(\beta_i^*)$ are contained in $[H_i]$. Hence the length of $\widehat{p^*(\beta_i^*)}$ is bounded by 2 and the length of the path $\widehat{b}_i$ is bounded by $2 + 2B' = B_0$. Therefore, since $\ell_{exp}(\gamma_w) > 0$, we have

$$\begin{aligned}\ell(\widehat{\gamma}_w) \leqslant \sum_{i=1}^{k} 2\ell(\widehat{a}_i) + 2\ell(\widehat{b}_i) \leqslant \sum_{i=1}^{k} (4\ell(a_i) + 2B_0) \leqslant (2B_0 + 4) \sum_{i=1}^{k} \ell(a_i) \\ = (2B_0 + 4)\ell_{exp}(\gamma_w).\end{aligned}$$

This proves Assertion (1).
(2) Let f' be as in Assertion (2) and let $w \in F_{\mathtt{n}}$. Suppose first that $\ell_{exp}(\gamma_w) = 0$. Then γ_w is a concatenation of paths in G'_{PG} and in $\mathcal{N}_{PG}$. By Proposition 2.5(4) and Lemma 2.9, there does not exist an edge in a zero stratum which is adjacent to a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Since zero strata are contractible by Proposition 2.5(3), it follows that γ_w is a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. By Proposition 3.14, there exists a subgroup A of $F_{\mathtt{n}}$ such that $[A] \in \mathcal{A}(\phi)$ and $w \in A$. Since $\mathcal{A}(\phi) = \mathcal{A}(\phi^{-1})$ by Equation (1), by Proposition 3.14, we have $\ell_{exp'}(\gamma'_w) = 0$. So we may suppose that $\ell_{exp}(\gamma_w) > 0$ and that $\ell_{exp'}(\gamma'_w) > 0$. By Assertion (1), in order to prove Assertion (2), it suffices to prove that $\widehat{G}$ and $\widehat{G}'$ are $F_{\mathtt{n}}$-equivariantly quasi-isometric. Since $\mathcal{A}(\phi)$ is a malnormal subgroup system, this follows from [Bow, Theorem 7.11] and [Hru, proof of Theorem 5.1]. □

Proposition 5.23. *Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ and let $f\colon G \to G$ be as in Remark 5.15. Let $f'\colon G' \to G'$ be as in the above convention. Let $\delta \in (0,1)$ and let W be a neighborhood of $K_{PG}(\phi)$ in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$. There exists $n_0 \in \mathbb{N}^*$ such that for every $n \geqslant n_0$ and every nonperipheral element $w \in F_{\mathtt{n}}$ such that $\eta_{[w]} \notin W$, one of the following holds:*

$$\mathfrak{g}([f^n(\gamma_w)]) \geqslant \delta$$

or

$$\mathfrak{g}'([f'^n(\gamma'_w)]) \geqslant \delta.$$

Proof. Let $w \in F_{\mathtt{n}}$ be a nonperipheral element such that $\eta_{[w]} \notin W$. Let $R = \frac{10C}{(1-\delta)^2} 8C'B^2$. We use the alternative given by Lemma 5.21 with the constants δ and R. If the first alternative of Lemma 5.21 occurs, then we are done. Suppose that $\mathfrak{g}([f^n(\gamma_w)]) < \delta$. There exists $n_0 \in \mathbb{N}^*$ depending only on f such that for every $n \geqslant n_0$, we have

$$\ell_{exp}^{[f^n(\gamma_w)]}(\mathrm{Inc}([f^n(\gamma_w)])) \leqslant \frac{10C}{R} \ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)).$$

By Lemma 5.14, since $\mathfrak{g}([f^n(\gamma_w)]) < \delta$, we have $\mathfrak{g}(\gamma_w) < \delta$. Thus, we see that

$$\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)) \geqslant (1-\delta)\ell_{exp}(\gamma_w).$$

Let γ'' be the reduced circuit in G such that $[f^{n_0}(\gamma'')] = \gamma_w$. Since $\mathfrak{g}(\gamma_w) < \delta$ and $[\eta_{[w]}] \notin K_{PG}(\phi)$, by Lemma 5.21, we see that

$$\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)) \leqslant \frac{10C}{R} \ell_{exp}^{\gamma''}(\mathrm{Inc}(\gamma'')).$$

We have

$$\begin{array}{rcl} \ell_{exp'}([f'^{n_0}(\gamma'_w)]) & \geqslant & \frac{1}{B}\ell_{exp}(\gamma'') \geqslant \frac{1}{B}\ell_{exp}^{\gamma''}(\mathrm{Inc}(\gamma'')) \\ & \geqslant & \frac{1}{B}\frac{R}{10C}\ell_{exp}^{\gamma_w}(\mathrm{Inc}(\gamma_w)) \geqslant \frac{1}{B}\frac{(1-\delta)R}{10C}\ell_{exp}(\gamma_w) \\ & \geqslant & \frac{1}{B^2}\frac{(1-\delta)R}{10C}\ell_{exp'}(\gamma'_w) = 8C'\frac{1}{1-\delta}\ell_{exp'}(\gamma'_w). \end{array}$$

But by Lemma 5.12, we have:

$$\ell_{exp'}^{[f'^{n_0}(\gamma'_w)]}(\mathrm{Inc}(f'^{n_0}(\gamma'_w)) \leqslant \ell_{exp'}(\mathrm{Inc}(f'^{n_0}(\gamma'_w)) \leqslant 8C'\ell_{exp'}(\gamma'_w).$$

Therefore, we see that

$$\mathfrak{g}'([f'^{n_0}(\gamma'_w)]) = 1 - \frac{\ell_{exp'}^{[f'^{n_0}(\gamma'_w)]}(\mathrm{Inc}([f'^{n_0}(\gamma'_w)]))}{\ell_{exp'}([f'^{n_0}(\gamma'_w)])} \geqslant 1-(1-\delta) = \delta > 0.$$

By Lemma 5.16, we see that there exists $n_1 \geqslant n_0$ depending only on f' such that for every $n \geqslant n_1$,

$$\mathfrak{g}'([f'^n(\gamma'_w)]) \geqslant \delta.$$

This concludes the proof. □

Proposition 5.24. *Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ and let $f\colon G \to G$ be as in Remark 5.15. Let U_+ be a neighborhood of $\Delta_+(\phi)$, let U_- be a neighborhood of $\Delta_-(\phi)$, let V be a neighborhood of $K_{PG}(\phi)$. There exists $N \in \mathbb{N}^*$ such that for every $n \geqslant 1$ and every $\mathcal{F} \wedge \mathcal{A}(\phi)$-nonperipheral $w \in F_{\mathtt{n}}$ such that $\eta_{[w]} \notin V$, one of the following holds*

$$\phi^{Nn}(\eta_{[w]}) \in U_+ \qquad \textit{or} \qquad \phi^{-Nn}(\eta_{[w]}) \in U_-.$$

Proof. Let $\delta \in (0,1)$ and let $w \in F_{\mathtt{n}}$ be a nonperipheral element with $\eta_{[w]} \notin V$. By Proposition 5.23, there exists $n_0 \in \mathbb{N}^*$ such that for every $n \geqslant n_0$, we have $\mathfrak{g}([f^n(\gamma_w)]) \geqslant \delta$ or $\mathfrak{g}'([f'^n(\gamma'_w)]) \geqslant \delta$. By Lemma 5.20(1), there exists $n_1 \geqslant n_0$ such that for every $n \geqslant n_1$, we have

$$\phi^{Nn}(\eta_{[w]}) \in U_+ \qquad \text{or} \qquad \phi^{-Nn}(\eta_{[w]}) \in U_-.$$

This concludes the proof. □

Proposition 5.24 gives a result of North-South dynamics outside of a neighborhood of $K_{PG}(\phi)$. As $K_{PG}(\phi)$ is empty for a relative expanding outer automorphism by Lemma 3.28(1), we can now prove Theorem 5.1.

Proof of Theorem 5.1. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an expanding outer automorphism relative to $\mathcal{F}$. By Lemma 3.28, we have $K_{PG}(\phi) = \varnothing$. Let U_+ be a neighborhood of $\Delta_+(\phi)$ and let U_- be a neighborhood of $\Delta_-(\phi)$. By Proposition 5.24, there exists $N \in \mathbb{N}^*$ such that for every $n \geqslant 1$ and every nonperipheral element $w \in F_{\mathtt{n}}$, we have

$$\phi^{Nn}(\eta_{[w]}) \in U_+ \qquad \text{or} \qquad \phi^{-Nn}(\eta_{[w]}) \in U_-.$$

Recall that, by Proposition 2.15, the rational currents are dense in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$. Hence we can apply [LU2, Proposition 3.3] to see that ϕ^{2N} has generalized North-South dynamics. Then, using [LU2, Proposition 3.4], we conclude that ϕ has generalized North-South dynamics. □

6. North-South dynamics for almost atoroidal relative outer automorphism

Let $\mathtt{n} \geqslant 3$ and let $\mathcal{F}$ be a free factor system of $F_{\mathtt{n}}$. Let $\phi \in \mathrm{Out}(F_{\mathtt{n}}, \mathcal{F})$ be an almost atoroidal outer automorphism (see Definition 4.3). Let $\mathcal{F} \leqslant \mathcal{F}_1 \leqslant \mathcal{F}_2 = \{[F_{\mathtt{n}}]\}$ be a sequence of free factor system given in this definition. We use the convention of Remark 5.19. We will show a result of North-South type dynamics for ϕ (see Theorem 6.4). Note that if $\mathcal{A}(\phi) \neq \{[F_{\mathtt{n}}]\}$ the simplices $\Delta_\pm(\phi)$ are still defined. Note that, by Lemma 3.28(3) and Lemma 5.18(4), for every current $\mu \in \mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$, we have $\|\mu\|_{\mathcal{F}_1} > 0$. Let $K_{PG}(\phi)$ be the set of polynomially growing currents of ϕ. Note that, combining Lemma 4.8 and Lemma 5.18(5), we have $K_{PG}(\phi) \cap \Delta_\pm(\phi) = \varnothing$. Let

$$\widehat{\Delta}_\pm(\phi) = \{[t\mu + (1-t)\nu] \,|t \in [0,1], [\mu] \in \Delta_\pm(\phi), [\nu] \in K_{PG}(\phi), \|\mu\|_{\mathcal{F}_1} = \|\nu\|_{\mathcal{F}_1} = 1\}$$

be the *convexes of attraction and repulsion of* ϕ.

In order to promote a global North-South type dynamics, we need to construct contracting neighborhoods of $\widehat{\Delta}_\pm(\phi)$. To this end, following Clay and Uyanik [CU], we introduce a notion of *goodness for currents of* $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$.

Let $f \colon G \to G$ be as in Remark 5.15. By Lemma 3.22, there exists $N \in \mathbb{N}^*$ such that, for every edge e of $\overline{G - G'_{PG}}$, we have $\ell_{exp}([f^N(e)]) \geqslant 4C + 1$. Let $C_N = C_{f^N}$ be a constant associated with f^N given by Lemma 4.9. Let $L > 0$ be such that for every path γ of G of length at least L, we have $\ell([f^N(\gamma)]) \geqslant C_N + 1$. The constant L exists since f^N lifts to a quasi-isometry on the universal cover of G. Let $\mathcal{P}_{cs}$ be the finite set of paths of the form $\gamma = \gamma_1 e \gamma_2$, where, for every $i \in \{1, 2\}$, the path γ_i has length equal to L, the path e is an edge in $\overline{G - G'_{PG}}$ and $\gamma_1 e \gamma_2$ is a splitting of γ. In Lemma 6.1(2), we prove in particular that $\mathcal{P}_{cs}$ is not empty. We will denote by $\widehat{\gamma}$ the edge e.

Let $[\mu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$. Recall the definition of Ψ_0 just above Definition 3.26. By Lemma 3.28(1), (2), we have $\phi(K_{PG}(\phi)) = K_{PG}(\phi)$. Hence, for every current $[\mu] \notin K_{PG}(\phi)$, we have $\Psi_0(\phi(\mu)) > 0$. Thus, for every current $[\mu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) - K_{PG}(\phi)$, we can define the *completely split goodness* $\overline{\mathfrak{g}}(\mu)$ *of* μ by

$$\overline{\mathfrak{g}}(\mu) = \frac{1}{\Psi_0(\phi^N(\mu))} \sum_{\gamma \in \mathcal{P}_{cs}} \langle \gamma, \mu \rangle.$$

Observe that the function $\overline{\mathfrak{g}}$ is continuous and that it defines a well-defined continuous function $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) - K_{PG}(\phi) \to \mathbb{R}$.

Lemma 6.1. *Let $f\colon G \to G$ be as in Remark* 5.15*.*

(1) *Let $w \in F_{\mathtt{n}}$ be such that $\ell_{exp}(\gamma_w) > 0$. We have $\mathfrak{g}([f^N(\gamma_w)]) \geqslant \overline{\mathfrak{g}}(\eta_{[w]})$.*
(2) *For every $[\mu] \in \Delta_+(\phi)$, we have $\overline{\mathfrak{g}}([\mu]) > 0$.*

Proof. (1) The proof of this assertion is similar to the one of [CU, Lemma 4.9 (2)]. Let $\gamma \in \mathcal{P}_{cs}$ be such that $\langle \gamma, \eta_{[w]} \rangle > 0$. Then $\gamma \subseteq \gamma_w$. For every occurrence of γ in γ_w, by the choice of L, C_N and by Lemma 4.9, the path $[f^N(\gamma_w)]$ contains $[f^N(\widehat{\gamma})]$, which has exponential length at least equal to $4C_N + 1$. Therefore, Lemma 5.8 implies that the path $[f^N(\gamma_w)]$ contains a subpath of $[f^N(\widehat{\gamma})]$ of exponential length at least 1 which is a complete factor of $[f^N(\gamma_w)]$ relative to G_{PG}. Hence we have:

$$\ell_{exp}([f^N(\gamma_w)])\mathfrak{g}([f^N(\gamma_w)]) \geqslant \sum_{\gamma \in \mathcal{P}_{cs}} \langle \gamma, \eta_{[w]} \rangle.$$

By Lemma 3.27, we have

$$\Psi_0(\phi^N(\eta_{[w]})) = \ell_{exp}([f^N(\gamma_w)]) = \Psi_0(\eta_{[\phi^N(w)]}) = \ell_{exp}(\gamma_{\phi^N([w])}).$$

Therefore, we have

$$\mathfrak{g}([f^N(\gamma_w)]) \geqslant \overline{\mathfrak{g}}(\eta_{[w]}).$$

(2) Let $[\mu] \in \Delta_+(\phi)$. Since $[\mu]$ is a convex combination of extremal points of $\Delta_+(\phi)$ and since for every element $\gamma \in \mathcal{P}_{cs}$, the application $\langle \gamma, . \rangle$ is linear, it suffices to prove the result for every extremal point of $\Delta_+(\phi)$. So we may suppose that $[\mu]$ is an extremal point of $\Delta_+(\phi)$.

Let G_i be the minimal subgraph of G such that $\mathcal{F}(G_i) = \mathcal{F}_1$. Since $[\mu]$ is extremal and since $\phi|_{\mathcal{F}_1}$ is expanding relative to $\mathcal{F}$, by Proposition 4.4, there exists an expanding splitting unit σ in G_i whose initial direction is fixed by f and such that, for every path $\gamma \in \mathcal{P}(\mathcal{F}_1 \wedge \mathcal{A}(\phi))$, we have

$$\langle \gamma, \mu \rangle = \mu(C(\gamma)) = \lim_{n \to \infty} \frac{\langle \gamma, [f^n(\sigma)] \rangle}{\ell_{\mathcal{F}_1}([f^n(\sigma)])}.$$

By Lemma 5.18(5), since the path $[f^n(\sigma)]$ is contained in G_i and, for every path $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$, the above limit is finite, we have

$$\lim_{n \to \infty} \frac{\langle \gamma, [f^n(\sigma)] \rangle}{\ell_{\mathcal{F}_1}([f^n(\sigma)])} = \lim_{n \to \infty} \frac{\langle \gamma, [f^n(\sigma)] \rangle}{\ell_{exp}([f^n(\sigma)])}.$$

Hence it suffices to prove that there exists $\gamma \in \mathcal{P}_{cs}$ such that

$$\lim_{n \to \infty} \frac{\langle \gamma, [f^n(\sigma)] \rangle}{\ell_{exp}([f^n(\sigma)])} > 0.$$

Let e be an edge of $\overline{G - G'_{PG}}$. Note that, since σ is a splitting unit, for every $m \in \mathbb{N}^*$, the path $[f^m(\sigma)]$ is completely split. Hence an occurrence of e in $\lim_{m \to \infty}[f^m(\sigma)]$ is contained in a splitting unit of $\lim_{m \to \infty}[f^m(\sigma)]$ which is either an INP or is equal to e. By Lemma 3.8 if an INP γ' contains e, there exists $\gamma'_0 \in \mathcal{N}_{PG}$ such that $e \subseteq \gamma'_0 \subseteq \gamma'$. For every $m \in \mathbb{N}^*$, we denote by $N(m, e)$ the number of occurrences of e or e^{-1} in $[f^m(\sigma)]$ which are splitting units of $[f^m(\sigma)]$ and by $EGINP(e)$ the set of all EG INPs containing e. Note that, since the set $\mathcal{N}_{PG}$ is finite by Lemma 3.5, so is the limit

$$\lim_{n \to \infty} \sum_{\gamma \in EGINP(e)} \frac{\langle \gamma, [f^n(\sigma)] \rangle}{\ell_{exp}([f^n(\sigma)])}.$$

Since for every $m \in \mathbb{N}^*$, we have

$$\langle e, [f^m(\sigma)] \rangle = N(m,e) + \sum_{\gamma \in EGINP(e)} \langle \gamma, [f^n(\sigma)] \rangle,$$

we see that the limit

$$\lim_{m \to \infty} \frac{N(m,e)}{\ell_{exp}([f^m(\sigma)])}$$

exists. We claim that there exists an edge e of $\overline{G - G'_{PG}}$ such that

$$\lim_{m \to \infty} \frac{N(m,e)}{\ell_{exp}([f^m(\sigma)])} > 0.$$

Indeed, note that, by Lemma 3.24, for every $m \in \mathbb{N}^*$, since $[f^m(\sigma)]$ is PG-relative completely split, we have

$$\ell_{exp}([f^m(\sigma)]) = \sum_{e \in \vec{E}\left(\overline{G - G'_{PG}}\right)} N(m,e).$$

Hence

$$\sum_{e \in \vec{E}\left(\overline{G - G'_{PG}}\right)} \lim_{m \to \infty} \frac{N(m,e)}{\ell_{exp}([f^m(\sigma)])} = 1,$$

which implies the claim.

Let e_0 be an edge of $\overline{G - G'_{PG}}$ which satisfies the claim. Since, for every $m \in \mathbb{N}^*$, the path $[f^m(\sigma)]$ is completely split, if an occurrence of e_0 or e_0^{-1} in $[f^m(\sigma)]$ is a splitting unit and if γ is a path in $[f^m(\sigma)]$ of the form $\gamma = \gamma_1 e_0 \gamma_2$ or $\gamma = \gamma_1 e_0^{-1} \gamma_2$, then such a decomposition of γ is a splitting of γ. Thus, if $\ell(\gamma_1) = \ell(\gamma_2) = L$, then the path γ is in $\mathcal{P}_{cs}$ and it contains the occurrence of e_0. Hence since $\mu = \mu(\sigma)$, we have

$$\lim_{m \to \infty} \frac{N(m,e_0)}{\ell_{exp}([f^m(\sigma)])} = \sum_{\gamma \in \mathcal{P}_{cs}, e_0 \subseteq \gamma} \langle \gamma, \mu \rangle > 0.$$

Therefore, there exists $\gamma \in \mathcal{P}_{cs}$ such that $\langle \gamma, \mu \rangle > 0$ and $\overline{\mathfrak{g}}([\mu]) > 0$. □

Lemma 6.2. *Let $f\colon G \to G$ be as in Remark 5.15. Let $U_\pm$ be open neighborhoods of $\Delta_\pm(\phi)$. There exist open neighborhoods $U'_\pm \subseteq U_\pm$ of $\Delta_\pm(\phi)$ such that $\phi^{\pm 1}(U'_\pm) \subseteq U'_\pm$.*

Proof. The proof is similar to the one of [CU, Lemma 4.13]. We prove the result for $\Delta_+(\phi)$, the proof for $\Delta_-(\phi)$ being symmetric.

By Lemma 6.1(2), for every $[\mu] \in \Delta_+(\phi)$, we have $\overline{\mathfrak{g}}([\mu]) > 0$. By compactness of $\Delta_+(\phi)$ and continuity of $\overline{\mathfrak{g}}$, there exists $\delta_0 > 0$ such that, for every $\mu \in \Delta_+(\phi)$, we have $\overline{\mathfrak{g}}(\mu) \geqslant \delta_0$. Let $\delta \in (0, \delta_0)$. Let U_+ be a neighborhood of $\Delta_+(\phi)$. Since the function $\overline{\mathfrak{g}}$ is continuous, there exists an open neighborhood $U_+^0 \subseteq U_+$ of $\Delta_+(\phi)$ such that, for every $[\mu] \in U_+^0$, we have $\overline{\mathfrak{g}}([\mu]) > \delta$. Up to taking a smaller U_+^0, we may suppose that $K_{PG}(\phi) \cap U_+^0 = \varnothing$ (this is possible since $K_{PG}(\phi)$ is compact and $\Delta_+(\phi) \cap K_{PG}(\phi) = \varnothing$). In particular, by Lemma 3.27, for every nonperipheral element $w \in F_\mathfrak{n}$ such that $\eta_{[w]} \in U_0^+$, we have $\ell_{exp}(\gamma_w) > 0$.

Let $w \in F_\mathfrak{n}$ be a nonperipheral element such that $\eta_{[w]} \in U_0^+$. By Lemma 6.1(1), we have

$$\mathfrak{g}([f^N(\gamma_w)]) \geqslant \overline{\mathfrak{g}}(\eta_{[w]}) > \delta.$$

By Lemma 5.20(1), there exists $M \geqslant N$ such that, for every $w \in F_{\mathtt{n}}$ such that $\eta_{[w]} \in U_+^0$, we have $\phi^M([\eta_{[w]}]) \in U_+^0$. Let

$$U'_+ = \bigcap_{i=0}^{M-1} \phi^i(U_+^0).$$

Since $\phi(\Delta_+(\phi)) = \Delta_+(\phi)$ by Proposition 4.12, the set U'_+ is an open neighborhood of $\Delta_+(\phi)$ which is stable by ϕ by density of rational currents (see Proposition 2.15) and continuity of ϕ. This concludes the proof. □

Lemma 6.3. *Let $f\colon G \to G$ be as in Remark* 5.15. *Suppose that the outer automorphism ϕ is almost atoroidal relative to $\mathcal{F}$. Let $\mathcal{F} \leqslant \mathcal{F}_1 \leqslant \mathcal{F}_2 = \{F_{\mathtt{n}}\}$ be as in the beginning of this section. Let $i \in \{1, \dots, k-1\}$ be such that $\mathcal{F}(G_i) = \mathcal{F}_1$. Let $\widehat{V}_\pm$ be open neighborhoods of $\widehat{\Delta}_\pm(\phi)$. There exist open neighborhoods $\widehat{V}'_\pm$ of $\widehat{\Delta}_\pm(\phi)$ contained in $\widehat{V}_\pm$ such that $\phi^{\pm}(\widehat{V}'_\pm) \subseteq \widehat{V}'_\pm$.*

Proof. The proof follows [CU, Lemma 4.14]. We prove the result for $\widehat{\Delta}_+(\phi)$, the proof for $\widehat{\Delta}_-(\phi)$ being symmetric.

Given $[\mu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) - K_{PG}(\phi)$, a finite set of reduced edge paths $\mathcal{P}$ in G and some $\epsilon > 0$ determine an open neighborhood $N([\mu], \mathcal{P}, \epsilon)$ of $[\mu]$ in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) - K_{PG}(\phi)$ as follows:

$$N([\mu], \mathcal{P}, \epsilon) = \left\{ [\nu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) - K_{PG}(\phi) \,\middle|\, \forall \gamma \in \mathcal{P}, \left| \frac{\langle \gamma, \nu \rangle}{\Psi_0(\nu)} - \frac{\langle \gamma, \mu \rangle}{\Psi_0(\mu)} \right| < \epsilon \right\}.$$

Since $K_{PG}(\phi)$ is compact, if ϵ is small enough, this defines an open neighborhood of $[\mu]$ in $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$. For a subset $X \subseteq \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) - K_{PG}(\phi)$, let

$$N(X, \mathcal{P}, \epsilon) = \bigcup_{[\mu] \in X} N([\mu], \mathcal{P}, \epsilon).$$

For $L > 0$, let $\mathcal{P}_+(L)$ be the set of reduced edge paths in G_i of length at most equal to L which are not contained in any concatenation of paths in $G_{PG,\mathcal{F}_1}$ and $\mathcal{N}_{PG,\mathcal{F}_1}$. By Lemma 5.18(3), the set $\mathcal{P}_+(L)$ is also the set of reduced edge paths in G_i of length at most equal to L which are not contained in any concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$. Let $[\mu] \in \Delta_+(\phi)$ and let $t \in [0, 1]$. Let

$$K_{PG}([\mu], t) = \{[(1-t)\nu + t\mu] \mid [\nu] \in K_{PG}(\phi), \|\nu\|_{\mathcal{F}_1} = \|\mu\|_{\mathcal{F}_1} = 1\}.$$

Remark that

$$\widehat{\Delta}_+(\phi) = \bigcup_{[\mu] \in \Delta_+(\phi), t \in [0,1]} K_{PG}([\mu], t).$$

Let $\epsilon > 0$. Let $V_{poly}(\epsilon) = [\Psi_0^{-1}((-\epsilon, \epsilon))]$. It is clear, by the continuity of Ψ_0 and Definition 3.26 of $K_{PG}(\phi)$, that $\bigcap_{\epsilon > 0} V_{poly}(\epsilon) = K_{PG}(\phi)$. Let $t \in (0, 1]$ and let $[\mu] \in \Delta_+(\phi)$ be such that $\|\mu\|_{\mathcal{F}_1} = 1$. By Lemma 5.18(5), we have $\Psi_0(\mu) = 1$. Let

$$V_{poly}([\mu], t, \epsilon) = \left\{ [\nu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \,\middle|\, \begin{array}{c} \|\nu\|_{\mathcal{F}_1} = \|\mu\|_{\mathcal{F}_1} = 1, \\ t(1+\epsilon) > \Psi_0(\nu) > t(1-\epsilon) \end{array} \right\}.$$

Note that, since $\Psi_0(\mu) = 1$, we have $[\nu] \in V_{poly}([\mu], t, \epsilon)$ if for $[\nu]$ such that $\|\nu\|_{\mathcal{F}_1} = 1$, we have

$$t\Psi_0(\mu)(1+\epsilon) > \Psi_0(\nu) > t\Psi_0(\mu)(1-\epsilon).$$

Let

$$V_\infty([\mu],t) = \bigcap_{L\to\infty,\epsilon\to 0} N(K_{PG}([\mu],t),\mathcal{P}_+(L),\epsilon) \cap V_{poly}([\mu],t,\epsilon).$$

Claim 1. For every $[\mu] \in \Delta_+(\phi)$ and every $t \in (0,1]$, we have $V_\infty([\mu],t) = K_{PG}([\mu],t)$.

Proof. The inclusion $K_{PG}([\mu],t) \subseteq V_\infty([\mu],t])$ being immediate since Ψ_0 is linear and vanishes on $K_{PG}(\phi)$, we prove the converse inclusion. Let $\nu \in V_\infty([\mu],t)$. By Definition 4.5 of $\Delta_+(\phi)$, for every $[\mu'] \in \Delta_+(\phi)$ and for every reduced edge path γ not contained in G_i, we have $\langle \gamma,\mu'\rangle = 0$. Hence, by Lemma 5.18(4), the current $[\mu]$ is entirely determined by the cylinder sets determined by reduced edge paths contained in G_i which are not contained in concatenation of paths in $G_{PG,\mathcal{F}_1}$ and $\mathcal{N}_{PG,\mathcal{F}_1}$. By Lemma 5.18(3), the current $[\mu]$ is entirely determined by the cylinder sets determined by reduced edge paths contained in G_i which are not contained in concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$.

Let γ be a reduced edge path which is contained in G_i and which is not contained in a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$. By Lemma 3.28, for every projective current $[\nu'] \in K_{PG}(\phi)$, the support of ν' is contained in $\partial^2\mathcal{A}(\phi)$. By Proposition 3.14, if $g \in F_{\mathtt{n}}$ is such that there exists a subgroup A of $F_{\mathtt{n}}$ such that $[A] \in \mathcal{A}(\phi)$ and $g \in A$, then γ_g is a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$. In particular, if γ' is a path of G such that $\{g^{+\infty},g^{-\infty}\} \in C(\gamma')$, then γ' is contained in a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. In particular, since γ is not contained in a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$, for every projective current $[\nu'] \in K_{PG}(\phi)$, we have $\langle \gamma,\nu'\rangle = 0$.

Suppose that $\|\nu\|_{\mathcal{F}_1} = \|\mu\|_{\mathcal{F}_1} = 1$. By Lemma 5.18(5), we also have $\Psi_0(\mu) = 1$. There exists $\lambda > 0$ such that for every path γ which is contained in G_i and which is not contained in a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$, we have $\langle\gamma,\nu\rangle = \langle\gamma,\lambda t\mu\rangle$. We claim that $\nu-\lambda t\mu \in \mathrm{Curr}(F_{\mathtt{n}},\mathcal{F}\wedge\mathcal{A}(\phi))$ and that $[\nu-\lambda t\mu] \in K_{PG}(\phi)$. Indeed, for the first part, it suffices to show that for every path $\gamma \in \mathcal{P}(\mathcal{F}_1\wedge\mathcal{A}(\phi))$, we have $(\nu-\lambda t\mu)(C(\gamma)) \geqslant 0$. This follows from the fact that, for every path $\gamma \in \mathcal{P}(\mathcal{F}_1\wedge\mathcal{A}(\phi))$ such that $\gamma \subseteq G_i$, the path γ is not contained in a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$. Hence we have $\langle\gamma,\nu\rangle = \langle\gamma,\lambda t\mu\rangle$. Moreover, if $\gamma \in \mathcal{P}(\mathcal{F}_1\wedge\mathcal{A}(\phi))$, then we have $\mu(C(\gamma)) = 0$. This shows that $\nu-\lambda t\mu \in \mathrm{Curr}(F_{\mathtt{n}},\mathcal{F}\wedge\mathcal{A}(\phi))$.

We now prove that $[\nu-\lambda t\mu] \in K_{PG}(\phi)$. Otherwise, by Lemma 3.28, the support of $\nu-\lambda t\mu$ is not contained in $\partial^2\mathcal{A}(\phi)$. By Proposition 3.14, there exists a path γ which is not contained in a concatenation of paths in G_{PG} and in $\mathcal{N}_{PG}$ such that

$$\langle\gamma,\nu-\lambda t\mu\rangle > 0.$$

Consider a decomposition of $\gamma = a_1b_1\ldots a_kb_k$ where, for every $j \in \{1,\ldots,k\}$, the path a_j is contained in $\overline{G-G_i}$ and, for every $j \in \{1,\ldots,k\}$, the path b_j is contained in G_i with a_1 and b_k possibly empty. By Lemma 5.18(1), (2) and Remark 5.19, up to taking a larger path γ, we may suppose that b_1 is nontrivial. By Lemma 5.18(2) and Remark 5.19, for every $j \in \{1,\ldots,k\}$, the path a_j is contained in G_{PG}. Since γ is not contained in a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$, there exists $j \in \{1,\ldots,k\}$ such that b_j is not contained in a concatenation of paths in G_{PG} and $\mathcal{N}_{PG}$. But then $\langle b_j,\nu\rangle = \langle b_j,\lambda t\mu\rangle$, that is $\langle b_j,\nu-\lambda t\mu\rangle = 0$. By additivity of $\nu-\lambda t\mu$, we have

$$\langle\gamma,\nu-\lambda t\mu\rangle \leqslant \langle b_j,\nu-\lambda t\mu\rangle = 0.$$

This contradicts the choice of γ. Hence $[\nu - \lambda t\mu] \in K_{PG}(\phi)$. Therefore, we have $\Psi_0(\nu - \lambda t\mu) = 0$. Since $[\nu] \in V_\infty([\mu], t)$ and since $\|\nu\|_{\mathcal{F}_1} = \|\mu\|_{\mathcal{F}_1} = 1$, we see that

$$\Psi_0(\nu) = t\Psi_0(\mu).$$

By linearity of Ψ_0 and the fact that $\Psi_0(\mu) = 1$, we have

$$t = t\Psi_0(\mu) = \Psi_0(\nu) = \lambda t\Psi_0(\mu) = \lambda t.$$

Since $t > 0$ and $\Psi_0(\mu) = 1$, we have $\lambda = 1$. Suppose first that $t \neq 1$. Let $\nu' = \frac{1}{1-t}(\nu - t\mu)$, so that $[\nu'] \in K_{PG}(\phi)$ and $\|\nu'\|_{\mathcal{F}} = 1$. Then $[\nu] = [(1-t)\nu' + t\mu] \in K_{PG}([\mu], t)$. Thus, we have $V_\infty([\mu], t) = K_{PG}([\mu], t)$.

Suppose now that $t = 1$. Then $\Psi_0(\nu) = 1 = \|\nu\|_{\mathcal{F}}$. We claim that if $\gamma \in \mathcal{P}(\mathcal{F}_1 \wedge \mathcal{A}(\phi))$ is such that $\nu(C(\gamma)) > 0$, then $\gamma \subseteq G_i$. Indeed, otherwise there would exist an edge e contained in $\overline{G - G_i}$ such that $\nu(C(e)) > 0$. By the description of $\overline{G - G_i}$ given in Lemma 5.18(1), (2) and additivity of the current ν, we can choose the edge $e \in \overline{G - G_i}$ in such a way that $e \in G_{PG}$. This would imply that $\|\nu\|_{\mathcal{F}_1} > \Psi_0(\nu) = 1$, a contradiction. The claim follows. But, since for every path $\gamma \in \mathcal{P}(\mathcal{F}_1 \wedge \mathcal{A}(\phi))$ such that $\gamma \subseteq G_i$, we have $\nu(C(\gamma)) = \mu(C(\gamma))$, we see that $\nu = \mu$ and that $\nu \in K_{PG}([\mu], 1)$. This concludes the proof of the claim. ☐

Since $\widehat{\Delta}_+(\phi)$ is compact, there exist $L > 0$ and $\epsilon > 0$ such that, for every $[\mu] \in \Delta_+(\phi)$ and every $t \in (0, 1]$, we have

$$V([\mu], t, L, \epsilon) = N(K_{PG}([\mu], t), \mathcal{P}_+(L), \epsilon) \cap V_{poly}([\mu], t, \epsilon) \subseteq \widehat{V}_+.$$

When $t = 0$, there exists $\epsilon > 0$ such that $V_{poly}(\epsilon) \subseteq \widehat{V}_+$. Let $s \in (0, 1)$, and let V be an open neighborhood of $K_{PG}(\phi)$ such that, for every $[\nu] \in V$ with $\|\nu\|_{\mathcal{F}_1} = 1$, we have:

$$\Psi_0(\nu) < s. \tag{26}$$

For every $[\mu] \in \left(N(\widehat{\Delta}_+(\phi), \widehat{\mathcal{P}}_+(L), \epsilon) - V\right) \cap \widehat{\Delta}_+(\phi)$, there exist $[\mu_{poly}] \in K_{PG}(\phi)$, $[\mu_{exp}] \in \Delta_+(\phi)$ and $t \in (0, 1]$ such that

$$[\mu] = [t\mu_{exp} + (1-t)\mu_{poly}].$$

By Lemma 6.1(2), for every $[\mu] \in \Delta_+(\phi)$, we have $\overline{\mathfrak{g}}([\mu]) > 0$. By compactness of $\Delta_+(\phi)$ and continuity of $\overline{\mathfrak{g}}$, there exists $\delta_1 > 0$ such that, for every $\mu \in \Delta_+(\phi)$, we have $\overline{\mathfrak{g}}(\mu) \geqslant \delta_1$. Since $\overline{N(\widehat{\Delta}_+(\phi), \widehat{\mathcal{P}}_+(L), \epsilon) - V} \cap \widehat{\Delta}_+(\phi)$ is compact, and since the function $\overline{\mathfrak{g}}$ is continuous, there exists $\delta_0' > 0$ such that the set $U = \overline{\mathfrak{g}}^{-1}((\delta_0', +\infty))$ is an open neighborhood of $(N(\widehat{\Delta}_+(\phi), \widehat{\mathcal{P}}_+(L), \epsilon) - V) \cap \widehat{\Delta}_+(\phi)$ intersecting V. Note that $U \cap K_{PG}(\phi) = \varnothing$. We set

$$\widehat{V}_+' = \left(\bigcup_{[\mu] \in \Delta_+(\phi), t \in (0,1]} V([\mu], t, L, \epsilon) \cup V_{poly}(\epsilon) \right) \cap (U \cup V).$$

Let δ_0 and M_0 be the constants given by Lemma 5.20(2) for the above choices of $\epsilon > 0$ and $L > 0$. Up to replacing δ_0 with a smaller constant and M_0 with a larger one, we may suppose that δ_0 and M_0 also satisfy the conclusion of Lemma 5.20(1) for U as well (where the open neighborhood W of $K_{PG}(\phi)$ needed in Lemma 5.20(1) is such that $W \subseteq V - U$).

Claim 2. There exists $N \in \mathbb{N}^*$ such that $\phi^N(\widehat{V}_+') \subseteq \widehat{V}_+'$.

Proof. Let $w \in F_{\mathtt{n}}$ be a nonperipheral element such that $\eta_{[w]} \in \widehat{V}'_+$. Suppose first that $\eta_{[w]} \in U \cap \widehat{V}'_+$. Since $\eta_{[w]} \notin K_{PG}(\phi)$, by Lemma 3.27, we have $\ell_{exp}(\gamma_w) > 0$. By Lemma 6.1(1), we have:

$$\mathfrak{g}([f^N(\gamma_w)]) \geqslant \overline{\mathfrak{g}}(\eta_{[w]}) > \delta'_0.$$

By Lemma 5.20(1), there exists $M \geqslant M_0 + N$ such that, for every $w \in F_{\mathtt{n}}$ such that $\eta_{[w]} \in U \cap \widehat{V}'_+$ and every $n \geqslant 1$, we have $\phi^{Mn}([\eta_{[w]}]) \in U \cap \widehat{V}'_+ \subseteq \widehat{V}'_+$.

Suppose now that $\eta_{[w]} \in V \cap \widehat{V}'_+$. By Lemma 3.28(3) and Lemma 5.18(4) for every projective current $[\mu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$, we have $\|\mu\|_{\mathcal{F}_1} > 0$. For a projective current $[\mu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$, let

$$\Psi_{\mathcal{F}_1}([\mu]) = \frac{\Psi_0(\mu)}{\|\mu\|_{\mathcal{F}_1}}.$$

Then, by definition of V and by Lemma 3.27, we have

$$\Psi_{\mathcal{F}_1}([\eta_{[w]}]) = \frac{\ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}_1}(\gamma_w)} < s.$$

If $[\eta_{[w]}] \in K_{PG}(\phi)$, then since $\phi(K_{PG}(\phi)) = K_{PG}(\phi)$, we are done. Therefore, we may suppose that $[\eta_{[w]}] \notin K_{PG}(\phi)$ and, by Lemma 3.27, for every $n \in \mathbb{N}^*$, we have $\ell_{exp}([f^n(\gamma_w)]) \geqslant 1$. Let $R > 1$ be such that $\frac{1}{1+\frac{R(1-\delta_0)}{10C}(1-s)} \leqslant \epsilon$. By Lemma 5.21, one of the following assertions holds:

(1) $\mathfrak{g}([f^M(\gamma_w)]) \geqslant \delta_0$,
(2) $\ell_{exp}([f^M(\gamma_w)]) \leqslant \frac{10C}{(1-\delta_0)R}\ell_{exp}(\gamma_w)$.

First assume that Assertion (1) holds. Let $[\mu_{\phi^M([w])}] \in \Delta_+(\phi)$ be the projective current associated with $\phi^M([w])$ given by Lemma 5.20(2). Let

$$t = \Psi_{\mathcal{F}_1}([\eta_{\phi^M([w])}]).$$

We claim that $[\eta_{\phi^M([w])}] \in V([\mu_{\phi^M([w])}], t, L, \epsilon)$. Indeed, we clearly have

$$[\eta_{\phi^M([w])}] \in V_{poly}([\mu_{\phi^M([w])}], t, \epsilon).$$

By Lemma 5.20(2), for every reduced edge path $\gamma \in \mathcal{P}_+(L)$, we have

$$\left| \frac{\langle \gamma, \eta_{\phi^M([w])} \rangle}{\Psi_0(\eta_{\phi^M([w])})} - \frac{\langle \gamma, \mu_{\phi^M([w])} \rangle}{\Psi_0(\mu_{\phi^M([w])})} \right| < \epsilon.$$

Therefore we have $[\eta_{\phi^M([w])}] \in N(K_{PG}([\mu_{\phi^M([w])}], t), \mathcal{P}_+(L), \epsilon)$. The claim follows by definition of $V([\mu_{\phi^M([w])}], t, L, \epsilon)$. By definition of $\widehat{V}'_+$, we see that $\phi^M([\eta_{[w]}]) = [\eta_{\phi^M([w])}] \in \widehat{V}'_+$.

Suppose now that Assertion (2) holds. We claim that $[\eta_{\phi^M([w])}] \in V_{poly}(\epsilon)$. By Lemma 5.18(1), (2) and Remark 5.19, the graph $\overline{G - G_i}$ consists in edges in G_{PG}. By Lemma 5.18(6), we have

$$\ell_{\mathcal{F}_1}([f^M(\gamma_w)]) - \ell_{exp}([f^M(\gamma_w)]) = \ell_{\mathcal{F}_1}(\gamma_w) - \ell_{exp}(\gamma_w).$$

Hence we have

$$\begin{aligned}\Psi_{\mathcal{F}_1}([\eta_{\phi^M(\gamma_w)}]) &= \frac{\ell_{exp}([f^M(\gamma_w)])}{\ell_{\mathcal{F}_1}([f^M(\gamma_w)])} \\ &= \frac{\ell_{exp}([f^M(\gamma_w)])}{\ell_{exp}([f^M(\gamma_w)]) + \ell_{\mathcal{F}_1}([f^M(\gamma_w)]) - \ell_{exp}([f^M(\gamma_w)])} \\ &= \frac{\ell_{exp}([f^M(\gamma_w)])}{\ell_{exp}([f^M(\gamma_w)]) + \ell_{\mathcal{F}_1}(\gamma_w) - \ell_{exp}(\gamma_w)} \\ &= \frac{1}{1+\frac{\ell_{\mathcal{F}_1}(\gamma_w)-\ell_{exp}(\gamma_w)}{\ell_{exp}([f^M(\gamma_w)])}} \leqslant \frac{1}{1+\frac{R(1-\delta_0)}{10C}\frac{\ell_{\mathcal{F}_1}(\gamma_w)-\ell_{exp}(\gamma_w)}{\ell_{exp}(\gamma_w)}} \\ &\leqslant \frac{1}{1+\frac{R(1-\delta_0)}{10C}\frac{\ell_{\mathcal{F}_1}(\gamma_w)-\ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}_1}(\gamma_w)}} \leqslant \frac{1}{1+\frac{R(1-\delta_0)}{10C}(1-s)} \leqslant \epsilon.\end{aligned}$$

Note that $\Psi_{\mathcal{F}_1}^{-1}((0,\epsilon)) \subseteq V_{poly}(\epsilon)$. Thus, we have

$$\phi^M([\eta_{[w]}]) = [\eta_{\phi^M([w])}] \in V_{poly}(\epsilon) \subseteq \widehat{V}'_+.$$

Therefore, by density of the rational currents (see Proposition 2.15) and continuity of ϕ, we have $\phi^M(\widehat{V}'_+) \subseteq \widehat{V}'_+$. This proves Claim 2. □

Let

$$\widehat{V}''_+ = \bigcap_{i=0}^{M-1} \phi^i(\widehat{V}'_+).$$

Since $\phi(\widehat{\Delta}_+(\phi)) = \widehat{\Delta}_+(\phi)$, the set $\widehat{V}''_+$ is an open neighborhood of $\widehat{\Delta}_+(\phi)$ which is stable by ϕ by construction. This concludes the proof. □

Theorem 6.4. *Let $\mathtt{n} \geqslant 3$. Let $\mathcal{F} \leqslant \mathcal{F}_1 \leqslant \{F_\mathtt{n}\}$ be a sequence of free factor systems such that the extension $\mathcal{F}_1 \leqslant \{F_\mathtt{n}\}$ is sporadic. Let $\phi \in \mathrm{Out}(F_\mathtt{n}, \mathcal{F})$ be such that ϕ preserves $\mathcal{F} \leqslant \mathcal{F}_1 \leqslant \{F_\mathtt{n}\}$ and $\phi|_{\mathcal{F}_1}$ is an expanding automorphism relative to $\mathcal{F}$.*

Let $\widehat{\Delta}_\pm(\phi)$ be the convexes of attraction and repulsion of ϕ and $\Delta_\pm(\phi)$ be the simplices of attraction and repulsion of ϕ. Let $U_\pm$ be open neighborhoods of $\Delta_\pm(\phi)$ in $\mathbb{P}\mathrm{Curr}(F_\mathtt{n}, \mathcal{F} \wedge \mathcal{A}(\phi))$ and $\widehat{V}_\pm$ be open neighborhoods of $\widehat{\Delta}_\pm(\phi)$ in $\mathbb{P}\mathrm{Curr}(F_\mathtt{n}, \mathcal{F} \wedge \mathcal{A}(\phi))$. There exists $M \in \mathbb{N}^*$ such that for every $n \geqslant M$, we have

$$\phi^{\pm n}(\mathbb{P}\mathrm{Curr}(F_\mathtt{n}, \mathcal{F} \wedge \mathcal{A}(\phi)) - \widehat{V}_\mp) \subseteq U_\pm.$$

Proof. The proof is similar to [CU, Theorem 4.15]. We replace ϕ by a power so that ϕ satisfies Remark 5.15. By Lemmas 6.2 and 6.3, we may suppose that $\phi(U_+) \subseteq U_+$ and that $\phi(\widehat{V}_+) \subseteq \widehat{V}_+$. Let M be the exponent given by Proposition 5.24 by using $U_+ = U_+$ and $U_- = V = \widehat{V}_-$. For every current $[\mu] \in \mathbb{P}\mathrm{Curr}(F_\mathtt{n}, \mathcal{F} \wedge \mathcal{A}(\phi)) - \phi^M(\widehat{V}_\mp)$, we have $\phi^M([\mu]) \in U_+$ since $\phi^{-M}([\mu]) \notin \widehat{V}_-$. Therefore, for every $[\mu] \in \mathbb{P}\mathrm{Curr}(F_\mathtt{n}, \mathcal{F} \wedge \mathcal{A}(\phi)) - \widehat{V}_-$, we have $\phi^{2M}([\mu]) \in U_+$ and for every $n \geqslant M$, we have $\phi^{2n}([\mu]) \in U_+$ since $\phi(U_+) \subseteq U_+$. Therefore for every $n \geqslant M$, we see that

$$\phi^{2n}(\mathbb{P}\mathrm{Curr}(F_\mathtt{n}, \mathcal{F} \wedge \mathcal{A}(\phi)) - \widehat{V}_-) \subseteq U_+.$$

A symmetric argument for ϕ^{-1} shows that ϕ^2 acts with generalized North-South dynamics. By [LU2, Proposition 3.4], we see that ϕ acts with generalized North-South dynamics. This concludes the proof. □

Corollary 6.5. *For every open neighborhood $\widehat{V}_- \subseteq \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ of $\widehat{\Delta}_-$, there exist $M \in \mathbb{N}^*$ and a constant L_0 such that, for every current $[\mu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) - \widehat{V}_-$, and every $m \geqslant M$, we have*

$$\|\phi^m(\mu)\|_{\mathcal{F}} \geqslant 3^{m-M} L_0 \|\mu\|_{\mathcal{F}}.$$

Proof. Let $f\colon G \to G$ be as in Remark 5.15. By Lemma 6.1(2), there exist a constant $\delta > 0$ and an open neighborhood U of $\Delta_+(\phi)$ such that, for every projective current $[\mu] \in U$, we have $\overline{\mathfrak{g}}([\mu]) \geqslant \delta$. We first prove Corollary 6.5 for currents $[\mu] \in U$. By Proposition 2.15, it suffices to prove the result for rational currents. By Lemma 6.1(1), since $U \cap K_{PG}(\phi) = \varnothing$, for every element $w \in F_{\mathtt{n}}$ such that $[\eta_{[w]}] \in U$, we have $\mathfrak{g}([f^N(\gamma_w)]) \geqslant \delta$. By Lemma 5.16(1) and Lemma 5.3, there exists a constant $K_1 > 0$ depending on δ such that for every $m \geqslant N$ and for every element $w \in F_{\mathtt{n}}$ such that $[\eta_w] \in U$, we have

$$\ell_{exp}([f^m(\gamma_w)]) \geqslant TEL(m-N, [f^N(\gamma_w)]) \geqslant 3^{m-N} K_1 \ell_{exp}([f^N(\gamma_w)]).$$

Since $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) - \widehat{V}_-$ is compact and since $K_{PG}(\phi) \subseteq \widehat{V}_-$, by Lemma 3.27 and Lemma 3.28(3), there exists a constant $K_2 > 0$ such that for every $m \geqslant N$ and for every element $w \in F_{\mathtt{n}}$ such that $[\eta_{[w]}] \in U$, we have $\frac{\ell_{exp}([f^N(\gamma_w)])}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} \geqslant K_2$. Thus, we have

$$\begin{aligned}\ell_{\mathcal{F}}([f^m(\gamma_w)]) &\geqslant \ell_{exp}([f^m(\gamma_w)]) \\ &\geqslant 3^{m-N} K_1 \ell_{exp}([f^N(\gamma_w)]) \geqslant 3^{n-M} K_1 K_2 \ell_{\mathcal{F}}([f^N(\gamma_w)]).\end{aligned}$$

We set $K_3 = K_1 K_2$. By compactness of $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ and Lemma 3.28(3), there exists $L > 0$ such that for every current $[\mu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$, we have $\frac{\|\phi^N(\mu)\|_{\mathcal{F}}}{\|\mu\|_{\mathcal{F}}} \geqslant L$. Hence for every $m \geqslant N$ and for every element $w \in F_{\mathtt{n}}$ such that $[\eta_{[w]}] \in U$, we have

$$\ell_{\mathcal{F}}([f^m(\gamma_w)]) \geqslant 3^{m-N} K_3 L \ell_{\mathcal{F}}(\gamma_w).$$

Hence the proof follows when $[\mu] \in U$.

We now prove the general case. By Theorem 6.4, there exists $M_1 \in \mathbb{N}^*$ such that, for all $m \geqslant M_1$ and $[\mu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) - \widehat{V}_-$, we have $\phi^m([\mu]) \in U$. Let $M = M_1 + N$. By the above, Lemma 3.27, the density of rational currents (see Proposition 2.15) and continuity of ϕ, for every current $[\mu] \notin \widehat{V}_-$, for every $n \geqslant M$, we have

$$\|\phi^n(\mu)\|_{\mathcal{F}} \geqslant 3^{n-M} K_3 L \|\phi^{M_1}(\mu)\|_{\mathcal{F}}.$$

By compactness of $\mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ and Lemma 3.28(3), there exists $L' > 0$ such that for every current $[\mu] \in \mathbb{P}\mathrm{Curr}(F_{\mathtt{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$, we have $\frac{\|\phi^{M_1}(\mu)\|_{\mathcal{F}}}{\|\mu\|_{\mathcal{F}}} \geqslant L'$. Hence for every $n \geqslant M$, we have

$$\|\phi^n(\mu)\|_{\mathcal{F}} \geqslant 3^{n-M} K_3 L L' \|\mu\|_{\mathcal{F}}.$$

This concludes the proof. □

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