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DYNAMICAL SYSTEMS OF CORRESPONDENCES ON THE PROJECTIVE LINE I: MODULI SPACES AND MULTIPLIER MAPS

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Abstract. We consider moduli spaces of dynamical systems of correspondences over the projective line as a generalization of moduli spaces of dynamical systems of endomorphisms on the projective line. We define the moduli space $\mathrm{Dyn}_{d,e}$ of degree (d,e) correspondences. We construct a family $\rho_c : \mathrm{Dyn}_{d,e} \dashrightarrow \mathrm{Dyn}_{1,d+e-1}$ of rational maps representation-theoretically. Here we note that $\mathrm{Dyn}_{1,d+e-1}$ is identical to the moduli space of the usual dynamical systems of degree $d+e-1$. We show that the moduli space $\mathrm{Dyn}_{d,e}$ is rational by using ρ_c. Moreover, the multiplier maps for the fixed points factor through ρ_c. Furthermore, we show the Woods Hole formulae for correspondences of different degrees are also related by ρ_c and obtain another representation-theoretically simplified form of the formula.

1. Introduction

Silverman [32] studied moduli spaces of dynamical systems over the projective line $\mathbb{P}^1$, which parameterizes endomorphisms up to the conjugations by the automorphisms on $\mathbb{P}^1$ by using geometric invariant theory (GIT for short). Self-correspondence is a generalization of endomorphism. Some important concepts on a dynamical system of endomorphism have natural generalization for a dynamical system of (self-)correspondence. An example is Woods Hole formula, which was originally stated for correspondence by Atiyah-Bott [1] and Illusie [13] and used for dynamical system of self-maps by Ueda [36]. Other examples are the canonical measure and the canonical height, which were originally stated for self-map and generalized to correspondence by Dinh-Kaufmann-Wu [4] and Ingram [14] respectively.

In this paper, we construct moduli spaces of dynamical systems of correspondences on the projective line as an analogue of Silverman's construction [32]. We firstly construct the moduli space $\mathrm{Corr}_{d,e}$ of correspondences of degree (d,e), which parameterizes the closed subschemes $C \subset \mathbb{P}^1_x \times \mathbb{P}^1_y$ defined by an equation $\sum_{i=0}^{d}\sum_{j=0}^{e} a_{ij}x^iy^j = 0$. To construct moduli spaces of dynamical systems up to coordinate changes, we consider the diagonal action of $\mathrm{Aut}(\mathbb{P}^1) \simeq \mathrm{PGL}_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$, which is equivalent to the conjugation action on the graph variety (for example, see [30]). We give characterization of stable points and semistable points. The stable and semistable loci were given in [32] for the case $d = 1$, that is, the case of moduli

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spaces of rational maps. We obtain a simple generalization of this result to our moduli spaces of correspondences as follows.

Theorem 1.1 (Theorem 6.12). *The point of* $\mathrm{Corr}_{d,e}$ *which represents a correspondence* C *is a stable point (resp. a semistable point) if and only if* C *has no point of multiplicity* $\geq \frac{d+e}{2}$ *(resp. of multiplicity* $> \frac{d+e}{2}$*) on the diagonal of* $\mathbb{P}^1 \times \mathbb{P}^1$.

Corollary 1.2. *The semistable locus* $\mathrm{Corr}_{d,e}^{ss}$ *coincides with the stable locus* $\mathrm{Corr}_{d,e}^{s}$ *if and only if* $d+e$ *is odd.*

Remark 1.3. GIT ensures the existence of a uniform geometrical quotient of the stable locus and a compactification of the quotient as a universal categorical quotient of the semistable locus.

The compactified moduli space of dynamical systems $\mathrm{Dyn}_{d,e} := \mathrm{Corr}_{d,e}^{ss} /\!\!/ \mathrm{PGL}_2$, of dimension $(d+1)(e+1)-4$, is constructed as the projective spectrum of a graded invariant ring.

By computing the composition of correspondence explicitly, we construct the iteration map $\Psi_n : \mathrm{Corr}_{d,e} \dashrightarrow \mathrm{Corr}_{d^n,e^n}, C \mapsto C \circ C \circ \cdots \circ C$ (Definition 5.21). We check that the iteration map on $\mathrm{Dyn}_{d,e}$ is well-defined by using Theorem 6.12.

Corollary 1.4 (Corollary 6.13). *The iteration map* $\Psi_n : \mathrm{Corr}_{d,e} \dashrightarrow \mathrm{Corr}_{d^n,e^n}$ *induces the rational map*

$$\Phi_n : \mathrm{Dyn}_{d,e} \dashrightarrow \mathrm{Dyn}_{d^n,e^n} .$$

Representation theory is an effective tool to study graded invariant ring (see [3],[28],[29]), which is applied for $\mathrm{Dyn}_{1,d}$ in [37]. In this paper, we also construct rational maps parametrized by points c of $\mathbb{A}^1 \setminus \{0\}$

$$\rho_c : \mathrm{Dyn}_{d,e} \dashrightarrow \mathrm{Dyn}_{1,d+e-1} \tag{1.1}$$

using the Clebsch-Gordan decomposition in representation theory. The rationality of the moduli space $\mathrm{Dyn}_{1,d+e-1}$ was shown by Levy [19]. The method used in the same paper also gives that these rational maps ρ_c are generically affine space bundle. Thus, we can deduce the rationality of $\mathrm{Dyn}_{d,e}$:

Proposition 1.5 (Proposition 6.17). $\mathrm{Dyn}_{d,e}$ *is rational for* $d, e \geq 1$ *with* $(d,e) \neq (1,1)$.

Moduli spaces as above can be applied to so-called inverse problems, which concern the existence and the classification of dynamical systems with prescribed invariants. A typical example of such invariant is multipliers of periodic orbits. For a dynamical system $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d, we denote the elementary symmetric polynomials of the fixed point multipliers by $\sigma_k(f)$, that is,

$$1 + \sum_{i=1}^{d+1} \sigma_k(f) t^k = \prod_{x : f(x) = x} (1 + f'(x)t) \tag{1.2}$$

for a formal variable t. The rational map

$$\lambda_{1,(1,d)} : \mathrm{Dyn}_{1,d} \dashrightarrow \mathbb{P}^{d+1},\ \lambda_{1,(1,d)}([f]) := [1 : \sigma_1(f) : \cdots : \sigma_{d+1}(f)]$$

is called *the fixed point multiplier map.* This is used to show the rationality of the moduli space $\mathrm{Dyn}_{1,2}$ (see [26], [32], [33], [34]), as well as to study inverse problems for multipliers (see [6], [7], [9], [10], [12],[25],[35]). A fundamental relation among multipliers, holomorphic Lefschetz formula (see for example [11]), is obtained as an application of the Woods Hole formula

$$\sum_{x:f(x)=x} \frac{1}{1-f'(x)} = 1, \text{ or equivalently, } \sum_{i=0}^{d+1}(-1)^i(d-i)\sigma_i(f) = 0. \tag{1.3}$$

This formulation is given in [33],[36] for a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$. We construct the fixed point multiplier map for correspondence

$$\lambda_{1,(d,e)} : \mathrm{Dyn}_{d,e} \dashrightarrow \mathbb{P}^{d+e},$$

interpreting the fixed points of the correspondence defined by $f(x, y) = 0$ as the points $\{z \in \mathbb{P}^1 | f(z, z) = 0\}$ and the derivative as the implicit function derivative $df := -\partial_x f/\partial_y f \in \mathbb{P}^1$. The Woods Hole formula for a correspondence is given in [1] and [13]. Our convention of fixed points and multipliers suits to express the Woods Hole formula in a form generalizing (1.3),

$$\sum_{z:f(z,z)=0} \frac{1}{1-df(z,z)} = d, \text{ or equivalently, } \sum_{i=0}^{d+e}(-1)^i(e-i)\sigma_i(f) = 0, \tag{1.4}$$

where $\sigma_i(f)$'s are the elementary symmetric forms of multipliers, we define on (7.4) in Subsection 7.1. For correspondences of different degrees, these Woods Hole formulae were not strongly related except that they can be deduced by parallel arguments.

We show that the map $\rho_c : \mathrm{Dyn}_{d,e} \dashrightarrow \mathrm{Dyn}_{1,d+e-1}$ mentioned above also gives unexpected equivalences between (1.3) and (1.4).

Proposition 1.6 (Proposition 7.5). *There exists a projective linear morphism $A_c \in \mathrm{Aut}(\mathbb{P}^{d+e}) = \mathrm{PGL}_{d+e}$ which makes the following diagram commutative:*

$$\begin{tikzcd} \mathrm{Dyn}_{d,e} \arrow[r, dashed, "\lambda_{1,(d,e)}"] \arrow[d, dashed, "\rho_c"] & \mathbb{P}^{d+e} \arrow[d, "A_c"] \\ \mathrm{Dyn}_{1,d+e-1} \arrow[r, dashed, "\lambda_{1,(1,d+e-1)}"] & \mathbb{P}^{d+e}. \end{tikzcd}$$

Moreover, by using an explicit coordination $\mathbb{P}^{d+e} \simeq \mathbb{P}(k[Z_0, Z_1]_{d+e})$ depending on the degree (d, e) (more precise construction is on Sections 3 and 7), we can write the images of the multiplier map explicitly.

Theorem 1.7 (Theorem 7.7). *For any $d, e \geq 1$, the image of $\lambda_{1,(d,e)}$ on $\mathbb{P}^{d+e} \simeq \mathbb{P}(k[Z_0, Z_1]_{d+e})$ is the hyperplane ([the coefficient of $Z_0^{d+e-1}Z_1] = 0$).*

Combining this argument with a known elementary proof of (1.3), we obtain another proof (Corollary 7.9) of the Woods Hole formula for correspondences (1.4).

As the construction of the moduli space of dynamical systems of correspondences, there are two problems unsolved in this paper. See corresponding remarks for more precise information.

Problem 1.8 (Remark 6.15). How do indeterminacy loci of iteration maps behave?

Problem 1.9 (Remark 7.2). Is the n-th multiplier map $\lambda_{n,(d,e)} := \lambda_{1,(d^n,e^n)} \circ \Psi_n$ on $\mathrm{Corr}_{d,e}$ well-defined? If (d, e) are coprime and n is odd, then $\lambda_{n,(d,e)}$ is well-defined.

Remark 1.10. The fixed point of $\Psi_n([f(x,y)])$ is a generalization of periodic points of period n of usual dynamical system. If the n-th multipliers of f (i.e. the fixed point multipliers of $\Psi_n([f]))$ are well-defined, then (1.4) gives the correspondence-analogues of multiplier formulae for periodic points.

By writing down the multiplier map $\lambda_{1,(d,e)}$ on $\mathrm{Corr}_{d,e}$ in a representationally simplified coordinate, we obtain another result about a universal polynomial function called resultant. For a pair of polynomials $f(x) = a_0 + a_1x + \ldots + a_dx^d$ and $g(x) = b_0 + b_1x + \ldots + b_ex^e$, the resultant $\mathrm{res}_x(f(x), g(x))$ is a polynomial function of $a_0, \ldots, a_d, b_0, \ldots, b_e$ which vanishes if and only if f and g have any common root. For more details on resultant, see Subsection 5.2. We prove Theorem 1.11.

Theorem 1.11 (Corollary 7.10)**.** *For an arbitrary field k and any polynomials $f, g \in k[x]$ such that $\deg f \geq 3$ and $\deg f \geq \deg g + 2$, we have*

$$\frac{\partial}{\partial t}\,\mathrm{res}_x(f(x), f'(x) + tg(x))\bigg|_{t=0} = 0.$$

Remark 1.12. In Remark 7.11, we give another, nondynamical theoretic method to prove Theorem 1.11 by regarding the above resultant as a perturbation of the discriminant $\Delta(f) = \mathrm{res}_x(f(x), f'(x))$.

This paper is organized as follows: In Section 2, we set up notation and terminology. In Section 3, we see a sketch of the proofs. In Section 4, we review representation theory of SL_2, including Clebsch-Gordan decomposition that we use later. In Section 5, we construct moduli spaces of correspondences and rewrite the composition and conjugation of correspondences as maps and actions on the moduli spaces respectively. In Section 6, we construct the moduli spaces of dynamical systems of correspondences, characterize the stable and semistable loci of the conjugation action and show the rationality of the moduli spaces. In Section 7, we construct multiplier maps and reformulate the Woods Hole Formula representation-theoretically.

2. Notation and terminology

Throughout this paper, we follow [20] for the terminology of algebraic geometry.

We fix a field k of characteristic zero. Unless otherwise stated, we suppose that every scheme is a scheme over k.

For a ring R and a free R-module M of finite rank, we denote by $R[M]$ the symmetric tensor algebra $\mathrm{Sym}^\bullet_R M := \bigoplus_{n=0}^{\infty} M^{\otimes n}/\langle v \otimes w - w \otimes v\rangle$. We note that if we choose an R-basis $\{x_1, \ldots, x_r\}$ of M, $R[M]$ is identified with the polynomial ring $R[x_1, \ldots, x_r]$. When a group G and a representation $\rho : G \to \mathrm{Aut}_R(M)$ are also given, we write $R[M]^G$ for the invariant ring.

We denote by $\mathrm{Sym}_n M$ the permutation-invariant part of the n-th tensor power.

3. Sketch of proofs

Our main aims in this paper are to prove Theorem 6.12, Proposition 6.17, Proposition 7.5 and Theorem 7.7. Moreover, there are some secondary aims to prepare fundamental concepts for the moduli-theoretic treatment of dynamical systems of correspondences on $\mathbb{P}^1$.

Theorem 6.12 is the theorem which gives a description of the stable and semistable loci of the moduli space $\mathrm{Corr}_{d,e}$ of the correspondences of degree (d, e).

This is shown in Subsection 6.1 by applying the numerical criterion of GIT to the explicit description of the PGL_2-action on $\mathrm{Corr}_{d,e}$. The description of the action is well-known for experts of moduli theory as similar action is used with no appropriate mention (e.g. [30], [21], [22], [23], [24] and [31]). However, we confirm the description precisely, because it is a key of this paper. In Subsection 5.1, we construct $\mathrm{Corr}_{d,e}$ as the complete linear system

$$\mathrm{Corr}_{d,e} := |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d,e)| = \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d,e))).$$

The diagonal action of SL_2 on $\Gamma(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d,e))$ is isomorphic to the representation $V_d\otimes V_e$, where $V_n := \mathrm{Sym}_n(k^{\oplus 2})$. That is, we have an SL_2-equivariant isomorphism $\mathrm{Corr}_{d,e} \simeq \mathbb{P}(V_d\otimes V_e)$. To see there is not so much difference between the SL_2-action and the PGL_2-action to use GIT, we briefly review GIT in Subsection 6.1. In particular, we state the numerical criterion, which is used to describe the PGL_2-action.

Proposition 6.17 establishes the rationality of the moduli space $\mathrm{Dyn}_{d,e} := \mathrm{Corr}_{d,e}^{ss} /\!\!/ \mathrm{PGL}_2$. This proposition follows from Levy's theorem [19], which shows the rationality of $\mathrm{Dyn}_{1,d}$. The most important step is to construct a PGL_2-equivariant rational map

$$\rho : \mathrm{Corr}_{d,e} \dashrightarrow \mathrm{Corr}_{1,d+e-1}\,.$$

This ρ induces the rational map $\overline{\rho} : \mathrm{Dyn}_{d,e} \dashrightarrow \mathrm{Dyn}_{1,d+e-1}$, which inherits surjectivity and rationality of generic fibers from ρ. Combining this fact with Levy's theorem, we obtain the rationality of $\mathrm{Dyn}_{d,e}$.

The construction of the morphism ρ is representation-theoretic. The morphism ρ is derived from the Clebsch-Gordan decomposition, which is the isomorphism of SL_2-representation

$$V_d\otimes V_e \simeq V_{d+e}\oplus V_{d+e-2}\oplus\cdots\oplus V_{|d-e|}$$

introduced in Subsection 4.2. The morphism

$$\rho : \mathrm{Corr}_{d,e} \simeq \mathbb{P}(V_d\otimes V_e) \dashrightarrow \mathbb{P}(V_1\otimes V_{d+e-1}) \simeq \mathrm{Corr}_{1,d+e-1}$$

is the projectivisation of the morphism of representation

$$\begin{aligned} &V_d\otimes V_e \simeq V_{d+e}\oplus V_{d+e-2}\oplus\cdots\oplus V_{|d-e|}\\ &\xrightarrow{(\mathrm{id}_{V_{d+e}},\,c\,\mathrm{id}_{V_{d+e-2}},0,\ldots,0)} V_{d+e}\oplus V_{d+e-2} \simeq V_1\otimes V_{d+e-1}. \end{aligned} \tag{3.1}$$

Here we can take an arbitrary constant $c\in k^\times$.

Other two aims are about relations between Clebsch-Gordan decomposition and multiplier maps. Let

$$\Omega^i : V_d\otimes V_e \simeq V_{d+e}\oplus V_{d+e-2}\oplus\cdots\oplus V_{|d-e|} \to V_{d+e-2i}\ (i=0,1,\ldots,\min(d,e))$$

be the projection defined from the Clebsch-Gordan decomposition. In Subsection 7.2, we show that the fixed point multiplier map $\lambda_{1,(d,e)} : \mathrm{Corr}_{d,e} \dashrightarrow \mathbb{P}^{d+e}$ for degree (d,e) correspondences is given by

$$\lambda_{1,(d,e)}([f]) = [\mathrm{res}_z(\Omega^0 f(z), (\Omega^0 f)'(z)Z_0 + (\Omega^1 f)(z)Z_1)] \in \mathbb{P}(k[Z_0,Z_1]_{d+e}) \tag{3.2}$$

for $[f] \in \mathrm{Corr}_{d,e} = \mathbb{P}(V_d \otimes V_e)$, where $k[Z_0, Z_1]_{d+e}$ is the vector space of the homogeneous polynomials of degree $d+e$. Then we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Dyn}_{d,e} \arrow[r, dashed, "\lambda_{1,(d,e)}"] \arrow[d, dashed, "\overline{\rho}_c"] & \mathbb{P}(k[Z_0, Z_1]_{d+e}) \arrow[d, "A_c"] \\ \mathrm{Dyn}_{1,d+e-1} \arrow[r, dashed, "\lambda_{1,(1,d+e-1)}"] & \mathbb{P}(k[Z_0, Z_1]_{d+e}) \end{tikzcd}$$

where $c \in k^{\times}$ is the constant taken in (3.1) and A_c is the isomorphism induced from the variable transformation $Z_0 \mapsto Z_0, Z_1 \mapsto cZ_1$. This commutativity is the assertion of Proposition 7.5.

We note that the rational map $\lambda_{1,(d,e)}$ is originally defined as the function which gives the multipliers of the fixed points. In Subsection 7.1, we define $\lambda_{1,(d,e)}$ along this original meaning, with a little modification using resultant. At the beginning of Subsection 7.2, we transform its expression to the above form (3.2) by using the definition of the Clebsch-Gordan decomposition introduced in Subsection 4.2 and a property of resultant introduced in Subsection 5.2.

Theorem 7.7 is that the coefficient of $Z_0^{d+e-1}Z_1$ of $\lambda_{1,(d,e)}(f)$ vanishes in (3.2). This theorem is a variation of known Woods Hole Formula for the map case (1.3). As other variations, this theorem implies Corollaries 7.9 and 7.10.

Secondary aims of this paper are to define composition maps (Definition 5.13), iteration maps (Definition 5.21, Corollary 6.13) and n-th multiplier maps (Definition 7.1, Remark 7.2). The composition of a generic pair of correspondences (C, D) is the closure of the variety $C \circ D$ such that

$$C \circ D(K) = \{(x, y) \in (\mathbb{P}^1)^2 \mid \exists z \in \mathbb{P}^1(K) \text{ s.t. } (x, z) \in C(K),\ (z, y) \in D(K)\}$$

for any algebraically closed field K over the base field k. In Subsection 5.3, we see that the composition maps on the moduli space are given by

$$\begin{aligned} \circ : \mathrm{Corr}_{d,e} \times \mathrm{Corr}_{d',e'} &\dashrightarrow \mathrm{Corr}_{dd',ee'}, \\ ([f(x,y)], [g(x,y)]) &\mapsto [\mathrm{res}_z(f(x,z), g(z,y))]. \end{aligned}$$

Iteration maps and n-th multiplier maps are constructed from the composition maps, the quotient $\mathrm{Corr}_{d,e} \dashrightarrow \mathrm{Dyn}_{d,e}$ and the fixed point multiplier map $\lambda_{1,(d,e)}$, these are all rational maps. We check compatibility and well-definedness in each step of constructions.

Compatibility is mainly reduced to associativity of the composition maps (Proposition 5.19), this is shown by an abstract argument. To show well-definedness, we restrict indeterminacy loci of rational maps we use to construct, by writing down the rational maps in some resultants. Then, we show that the correspondence given by $x^d - y^e$ or $x^d y^e - 1$ avoids the indeterminacy loci. Unfortunately, these examples are not enough to define n-th multiplier maps, this is Problem 1.9.

4. Representation theory for SL_2 and PGL_2

In this section, we review some known materials from representation theory, the Cayley operator and the Clebsch-Gordan decomposition in Subsection 4.2. Contents in this section are found for example in [2].

For the special linear group $\mathrm{SL}_2(k)$, we denote the trivial representation on k by V_0, the natural representation on k^2 by V_1 and the symmetric tensor representation $\mathrm{Sym}_n(V_1)$ by V_n. From the construction, we have $\dim V_n = n + 1$.

Proposition 4.1. *If n is an even number, then there exists an action of* $\mathrm{PGL}_2(k)$ *on the vector space V_n such that the pullback action to* $\mathrm{SL}_2(k)$ *is V_n.*

Proof. We consider the representation $\Delta_n(-\frac{n}{2})$ of $\mathrm{GL}_2(k)$ on $(k^2)^{\otimes n}$ which is given by

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_n) := (\det g)^{-\frac{n}{2}} (gv_1 \otimes gv_2 \otimes \cdots \otimes gv_n) \text{ (for } g \in \mathrm{GL}_2(k)).$$

By $\Delta_n(-\frac{n}{2})$, any scalar matrix $\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right) \in \mathrm{GL}_2(k)$ acts trivially. Therefore, $\Delta_n(-\frac{n}{2})$ is a representation of $\mathrm{PGL}_2(k)$. Moreover, the action of $\mathrm{PGL}_2(k)$ by $\Delta_n(-\frac{n}{2})$ commutes with the action of $\mathfrak{S}_n$ by permuting components. This gives a subrepresentation of $\mathrm{PGL}_2(k)$ on $\mathrm{Sym}_n k^2$. The restriction of the subrepresentation on $\mathrm{SL}_2(k)$ is the n-th symmetric tensor of V_1, therefore this is a required representation. $\square$

4.1. **Weight.** Weight theory is a theory which measures representations of a group scheme G over k by looking at the action of the multiplication group $\mathbb{G}_m$ through morphisms $\mathbb{G}_m \to G$. For any integer n, we write $k(n)$ for the one-dimensional representation of $\mathbb{G}_m$ given by $t \cdot v = t^n v$ $(t \in \mathbb{G}_m(k), v \in k(n))$.

Definition 4.2. Let V be a finite dimensional rational representation of $\mathbb{G}_m$. If $V \simeq \bigoplus_i k(n_i)$, then we define the *weight* $w(V)$ of V by the Laurent polynomial $w(V) = \sum_i q^{n_i}$.

For a homomorphism $\lambda : \mathbb{G}_m \to G$ between group schemes, the λ-*weight* $w_\lambda(V)$ of a finite dimensional rational representation V of G is the weight of the $\mathbb{G}_m$-action induced by λ.

Remark 4.3. From a diagonalization of the action of an element which is not a root of unity, we can see the weight of a representation is well-defined if it exists. Moreover, by the diagonalization, we can see that if a representation V has the weight, then any subrepresentation of V has its weight.

We fix the group morphism

$$c : k^\times \ni t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in \mathrm{SL}_2(k).$$

Example 4.4.

(i) For the canonical basis $\{e_1, e_2\}$ of $V_1 \simeq k^{\oplus 2}$, we have $c(t) \cdot e_1 = t \cdot e_1$ and $c(t) \cdot e_2 = t^{-1} \cdot e_2$. Therefore, the representation of $k^\times$ on V_1 induced by c is isomorphic to $k(1) \oplus k(-1)$. So we have $w_c(V_1) = q + q^{-1}$.

(ii) For any representation V which has the weight $w_c(V) = \sum_{i=1}^{\dim V} q^{n_i}$, the generating function (called Hilbert Series) of the symmetric products is given by

$$\sum_{n=0}^{\infty} w_c(\mathrm{Sym}_n V) \cdot t^n = \prod_{i=1}^{\dim V} \frac{1}{1 - q^{n_i} \cdot t}.$$

In particular, we have

$$w_c(V_n) = \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} = q^n + q^{n-2} + \ldots + q^{-(n-2)} + q^{-n}.$$

For any representation V with the λ-weight $w_\lambda(V) = f(q)$, the weight of the dual representation V^* is given by $w_\lambda(V^*) = f(q^{-1})$. In particular, by Example 4.4(ii), the representation V_n and its dual has the same weight. In fact, they are isomorphic in characteristic zero.

Proposition 4.5. *The dual representation of V_n is isomorphic to V_n.*

Proof. Since k is of characteristic 0, it is enough to show the proposition for $n = 1$ (see Remark 4.6).

Let $[e_1, e_2]$ be the basis of V_1 and $[f_1, f_2]$ the dual basis of V_1^*. The dual representation is defined by $A \cdot f := f \circ A^{-1}$ ($f \in V_1^*,\ A \in \mathrm{SL}_2(k)$). Therefore the transpose $(A^{-1})^T$ is the representation matrix of the action of A by the dual representation. We have $(A^{-1})^T = IAI^{-1}$ for $I = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Therefore, an isomorphism $V_1 \to V_1^*$ is given by $e_1 \mapsto -f_2,\ e_2 \mapsto f_1$. $\square$

Remark 4.6. The canonical morphism $\mathrm{Sym}_n(V^*) \to (\mathrm{Sym}_n V)^*$ induced from the inclusion morphism

$$\mathrm{Sym}_n V = (V^{\otimes n})^{\mathfrak{S}_n} \to V^{\otimes n}$$

is an isomorphism because our field k has characteristic 0 and the binomial coefficients are invertible. In positive characteristic p, the canonical morphism $\mathrm{Sym}_n(V^*) \to (\mathrm{Sym}_n V)^*$ is not isomorphism for $n \geq p$. In fact, the representations $V_n = \mathrm{Sym}_n(V_1)$ and $V_n^* = \mathrm{Sym}_n(V_1)^*$ are not isomorphic [2].

4.2. **Clebsch-Gordan decomposition.** For a finite dimensional vector space V, the space of n-ic forms, that is, the vector space of the all degree n homogeneous polynomials in $k[V]$, is naturally isomorphic to $(\mathrm{Sym}_n(V^*))^*$ in arbitrary characteristic. Therefore, by Proposition 4.5, the representation V_n is identified with the space of n-ic binomial forms in characteristic zero. The variables are the standard basis of $V_1 = k^2$ indeed, and we write the basis as $\{x_0, x_1\}, \{y_0, y_1\}$ or $\{z_0, z_1\}$ in this subsection.

Proposition 4.7. *The representation V_n of* SL_2 *is irreducible.*

Proof. We note that $c(t) \cdot x_0^{n-i} x_1^i = t^{n-2i} x_0^{n-i} x_1^i$ under the identification between V_n and the space of n-ic binomial forms.

Let W be an arbitrary nonzero SL_2-stable subspace of V_n. By Remark 4.3, we have

$$W = \bigoplus_{i \in I_W} k x_0^{n-i} x_1^i$$

for some nonempty $I_W \subset \{0, 1, \ldots, n\}$. We take a monomial $x_0^{n-i} x_1^i \in W$. Since W is SL_2-stable, we have $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right) \cdot x_0^{n-i} x_1^i = (x_0 + x_1)^{n-i}(x_0 + 2x_1)^i \in W$. Therefore we have $I_W \supset \{0, \ldots, n\}$, that is, $V = W$. $\square$

If we have two binomial forms $f(x_0, x_1)$ and $g(y_0, y_1)$ of degree d and e respectively, the Cayley operator $\Omega_{xy} := \partial_{x_0}\partial_{y_1} - \partial_{y_0}\partial_{x_1}$ gives a new binomial form

$$\left(\Omega_{xy}^m f(x_0, x_1) \cdot g(y_0, y_1)\right)\big|_{(x_0,x_1)=(y_0,y_1)=(z_0,z_1)}$$

of variables (z_0, z_1) and degree $d+e-2m$, for $0 \leq m \leq \min(d, e) = \frac{1}{2}(d+e-|d-e|)$. This linear map is SL_2-equivariant, that is, we have a morphism of representation

$$\Omega^m : V_d \otimes V_e \to V_{d+e-2m}$$

given by

$$\Omega^m(f(x_0, x_1) \otimes g(y_0, y_1)) := \left(\Omega_{xy}^m f(x_0, x_1) \cdot g(y_0, y_1)\right)\big|_{(x_0,x_1)=(y_0,y_1)=(z_0,z_1)}. \tag{4.1}$$

Proposition 4.8 (Clebsch-Gordan decomposition, [2, Theorem 3.2.4]). *The morphism*

$$\bigoplus_{i=0}^{|d-e|} \Omega^i : V_d \otimes V_e \to V_{d+e} \oplus V_{d+e-2} \oplus \cdots \oplus V_{|d-e|+2} \oplus V_{|d-e|}$$

is an isomorphism.

Proof. It is enough to show that the morphism $\Omega^m : V_d \otimes V_e \to V_{d+e-2m}$ is surjective for $0 \leq m \leq \min(d, e)$. By Proposition 4.7, it is sufficient to show that the morphism $\Omega^m : V_d \otimes V_e \to V_{d+e-2m}$ is nonzero, which follows from the explicit calculation

$$\Omega^m(x_0^d \otimes y_0^{e-m} y_1^m) = \frac{d!m!}{(d-m)!} z_0^{d+e-2m}.$$

□

Remark 4.9 (Schur's lemma). Let V and W be two finite dimensional irreducible representations of SL_2. Then there exists nonzero homomorphism from V to W if and only if they are isomorphic. In particular, we have

$$\dim_k \mathrm{Hom}_{\mathrm{SL}_2(k)} \left(\bigoplus_i V_{a_i}, \bigoplus_j V_{b_j} \right) = \sum_{i,j} \delta_{a_i, b_j}.$$

5. Correspondence

Let X and Y be schemes. A closed subscheme of $X \times Y$ is called an algebraic correspondence between X and Y. We can regard a morphism $X \to Y$ as an algebraic correspondence given by the graph of the morphism. Therefore algebraic correspondence is a generalization of morphism.

5.1. The moduli space of correspondences over $\mathbb{P}^1$.

Over $\mathbb{P}^1 \times \mathbb{P}^1$, we denote the line bundle $p_1^* \mathcal{O}(d) \otimes p_2^* \mathcal{O}(e)$ by $\mathcal{O}(d, e)$ and the set of its global sections by $V_{d,e}$. We fix homogeneous coordinates of each component, choosing canonical bases $x_0, x_1 \in V_{1,0}$ and $y_0, y_1 \in V_{0,1}$. Using these coordinates we have

$$V_{d,e} = \left\{ \sum_{0 \leq i \leq d,\ 0 \leq j \leq e} a_{i,j} x_0^{d-i} x_1^i y_0^{e-j} y_1^j \;\middle|\; a_{i,j} \in k \right\}.$$

Later we will see that we can identify $V_{d,e}$ with $V_d \otimes V_e$ (Corollary 5.23).

Definition 5.1. A nonzero element f of $V_{d,e}$ is called *a bihomogeneous polynomial of bidegree* (d, e). A $(d+1) \times (e+1)$-matrix $A = (a_{ij})_{0 \leq i \leq d, 0 \leq j \leq e}$ is called *the coefficient matrix* of f if

$$f = \sum_{0 \leq i \leq d,\ 0 \leq j \leq e} a_{ij} x_0^{d-i} x_1^i y_0^{e-j} y_1^j. \tag{5.1}$$

We often abbreviate a bihomogeneous polynomial (5.1) as

$$f = f(x, y) = \sum_{i,j} a_{ij} x^i y^j$$

if the degree of correspondence (d, e) is apparent.

Remark 5.2. Despite fixing a notation of canonical bases of $V_{1,0}$ and $V_{0,1}$, we sometimes regard a bihomogeneous polynomial as just a polynomial of variables x_0, x_1, y_0, y_1. Moreover, we sometimes substitute a pair of variables x_0, x_1 or y_0, y_1 by another pair of variables z_0, z_1.

Definition 5.3. A closed subscheme C of $\mathbb{P}^1 \times \mathbb{P}^1$ is said to be *a divisorial correspondence*, or simply *a correspondence*, if $\mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}/\mathcal{I}$ and $\mathcal{I}$ is a locally free sheaf of rank one. A divisorial correspondence C given by an ideal sheaf $\mathcal{I}$ is *of degree* (d, e) if $\mathcal{I}$ is isomorphic to $\mathcal{O}(-d, -e)$ as an $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$-module.

Remark 5.4. We abbreviated the term "effective", the condition which we required for divisorial correspondence as a divisor on $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, we sometimes abbreviate the term "divisorial" and simply call correspondence.

The fine moduli space of divisorial correspondences of degree (d, e), denoted by $\mathrm{Corr}_{d,e}$, is given by the complete linear system

$$\mathrm{Corr}_{d,e} \simeq (\mathbb{A}(V_{d,e}) \setminus \{0\})/\mathbb{G}_m \simeq \mathbb{P}(V_{d,e}).$$

We sometimes abuse a symbol for a bihomogeneous polynomial to the correspondence given by the polynomial and the point of $\mathrm{Corr}_{d,e}$ indicating the correspondence.

5.2. **Resultant.**

Definition 5.5. Let R be a commutative ring R and x a variable. For $R[x]_d := \{f \in R[x] \mid \deg_x f \le d\}$, the resultant $\mathrm{res}_{x,(d,e)} : R[x]_d \times R[x]_e \to R$ is defined as the determinant of the Sylvester matrix

$$\mathrm{res}_{x,(d,e)}\left(\sum_{i=0}^{d} f_i x^i, \sum_{j=0}^{e} g_i x^i\right) := \begin{vmatrix} f_0 & f_1 & & \cdots & f_d & & & \\ & f_0 & f_1 & & \cdots & f_d & & \\ & & \ddots & \ddots & & \ddots & \ddots & \\ & & & f_0 & f_1 & & \cdots & f_d \\ g_0 & g_1 & & \cdots & g_e & & & \\ & g_0 & g_1 & & \cdots & g_e & & \\ & & \ddots & \ddots & & \ddots & \ddots & \\ & & & g_0 & g_1 & & \cdots & g_e \end{vmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} a\\a\\a\\a \end{matrix}}\right\} e \\ \left.\vphantom{\begin{matrix} a\\a\\a\\a \end{matrix}}\right\} d. \end{array}$$

For homogeneous polynomials of two variables x_0, x_1, $F(x_0, x_1) = x_0^d f(\frac{x_1}{x_0})$ and $G(x_0, x_1) = x_0^e g(\frac{x_1}{x_0})$ of degree d and e respectively, we define the *homogeneous resultant*

$$\mathrm{res}_{[x_0,x_1]}(F(x_0, x_1), G(x_0, x_1)) := \mathrm{res}_{x,(d,e)}(f(x), g(x)).$$

Example 5.6. *The discriminant* of a polynomial $f(x)$ (with respect to its variable x) is the resultant of the polynomial $f(x)$ and its derivative $f'(x)$. For example, the discriminant of a cubic polynomial $f(x) = x^3 + ax + b$ is

$$\mathrm{res}_{x,(3,2)}(f(x), f'(x)) = \begin{vmatrix} b & a & & 1 & \\ & b & a & & 1 \\ a & & 3 & & \\ & a & & 3 & \\ & & a & & 3 \end{vmatrix} = 4a^3 + 27b^2.$$

By Proposition 5.7(i), a polynomial $g(x)$ has a multiple divisor if and only if its discriminant is zero.

We need the following fundamental properties of the resultant.

Proposition 5.7 ([33, Proposition 2.13]). *Let R be an integral domain and let $\bar{K}$ be an algebraic closure of the fractional field* $\mathrm{Frac}(R)$.

(i) *The homogeneous resultant of two homogeneous polynomials on R is* 0 *if and only if the polynomials have a common factor as homogeneous polynomials over $\bar{K}$.*

(ii) *For $f, g \in R[x]$, $d = \deg_x f$ and $e = \deg_x g$, we have*

$$R \cap (fR[x] + gR[x]) = \mathrm{res}_{x,(d,e)}(f(x), g(x))R.$$

(iii) *For homogeneous polynomials $F(x,y)$ and $G(x,y)$ such that*

$$F(x,y) = f_0 \prod_{i=1}^{d}(x - \alpha_i y), \; G(x,y) = g_0 \prod_{j=1}^{e}(x - \beta_i y) \; (\alpha_i, \beta_j, \in \bar{K}, f_0, g_0 \in R),$$

we have

$$\mathrm{res}_{[x,y]}(F(x,y), G(x,y)) = f_0^e g_0^d \prod_{i=1}^{d}\prod_{j=1}^{e}(\alpha_i - \beta_j) = f_0^e \prod_{i=1}^{d} G(\alpha_i, 1).$$

(iv) *The homogeneous resultant is a unique family of maps which satisfies*

$$\mathrm{res}_{[x,y]}(ax + by, cx + dy) = bc - ad \; (a, b, c, d \in \bar{K}),$$
$$\mathrm{res}_{[x,y]}(F_1F_2, G) = \mathrm{res}_{[x,y]}(F_1, G)\,\mathrm{res}_{[x,y]}(F_2, G) \text{ and}$$
$$\mathrm{res}_{[x,y]}(G, F) = (-1)^{\deg F \deg G}\,\mathrm{res}_{[x,y]}(F, G)$$

for any homogeneous polynomial F, F_1, F_2 and G.

Proof. The assertions (i), (ii) and (iii) are the assertions (a), (c) and (b) of [33, Proposition 2.13] respectively. The assertion (iv) follows from (iii). ☐

Corollary 5.8. *Let F, G and H be homogeneous polynomials of variables x, y.*

(i) *If* $\deg F + \deg H = \deg G$, *then we have*

$$\mathrm{res}_{[x,y]}(F, G + FH) = \mathrm{res}_{[x,y]}(F, G).$$

(ii) [8, Chapter 12.1] *For any matrix* $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(R)$, *we have*

$$\mathrm{res}_{[x,y]}(F(g \cdot (x,y)), G(g \cdot (x,y))) = (\det g)^{\deg F \cdot \deg G}\,\mathrm{res}_{[x,y]}(F(x,y), G(x,y)),$$

where $g \cdot (x,y) := (ax + by, cx + dy)$.

Proof. (i) This is evident from Proposition 5.7(iii).

(ii) This is evident from Proposition 5.7(iv). ☐

Remark 5.9. In [8, Chapter 12.1], Corollary 5.8(ii) is shown by using the universality of the resultant and covariance.

In the theory of arithmetic dynamics, homogeneous resultants are used for various purposes:

- to give a Lipschitz constant with respect to the chordal metric [33, Theorem 2.4];
- to determine well-definedness of a rational self-map over $\mathbb{P}^1$ [33, Theorem 2.5];

- to compute the image of multiplier map (this is perhaps well-known to experts, but the author could not find suitable reference);
- to compute the composition of correspondences ([15], [18]).

We use homogeneous resultants for the latter two purposes.

5.3. Composition.

Proposition 5.10. *Let* $C = \operatorname{Spec} k[x,y]/(f(x,y))$ *and* $D = \operatorname{Spec} k[x,y]/(g(x,y))$ *be closed subschemes of* $\mathbb{A}^2$ *and let* i_C *and* i_D *be their inclusion morphisms respectively. Let* $p : \widetilde{C \circ D} \to \mathbb{A}^2$ *be the morphism defined as the composition of the morphisms in upper row of the following diagram*

$$\begin{tikzcd} \widetilde{C \circ D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathbb{A}^3 \arrow[r, "p_{13}"] \arrow[d, "\mathrm{id}\times\Delta\times\mathrm{id}"] & \mathbb{A}^2 \\ C \times D \arrow[r, "i_C \times i_D"] & \mathbb{A}^4 & \end{tikzcd} \tag{5.2}$$

where $\Delta : \mathbb{A}^1 \to \mathbb{A}^2$ *is the diagonal morphism. Then the scheme-theoretic image of* p*, denoted by* $C \circ D \subset \mathbb{A}^2$*, is written as*

$$C \circ D \simeq \operatorname{Spec} k[x,y]/(\operatorname{res}_{z,(d,e)}(f(x,z), g(z,y))) \to \mathbb{A}^2.$$

Remark 5.11. For any algebraically closed field K over k, the pullback diagram (5.2) yields

$$\widetilde{C \circ D}(K) = \{(x,z,y) \in \mathbb{A}^3 \mid (x,z) \in C(K),\ (z,y) \in D(K)\}.$$

Therefore, the support of $C \circ D$ is the Zariski closure of the points

$$\{(x,y) \in \mathbb{A}^2 \mid \text{ there exists } z \in \mathbb{A}^1(K) \text{ such that } (x,z) \in C(K) \text{ and } (z,y) \in D(K)\}.$$

Proof. The following diagrams

$$\begin{tikzcd} C \arrow[r, "i_C"] \arrow[d, "\mathrm{id}\times(p_2\circ i_C)"'] & \mathbb{A}^2 \arrow[d, "\mathrm{id}\times\Delta"] \\ C \times \mathbb{A}^1 \arrow[r, "i_C\times\mathrm{id}"] & \mathbb{A}^3 \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "i_D"] \arrow[d, "(p_1\circ i_D)\times\mathrm{id}"'] & \mathbb{A}^2 \arrow[d, "\Delta\times\mathrm{id}"] \\ \mathbb{A}^1 \times D \arrow[r, "\mathrm{id}\times i_D"] & \mathbb{A}^3 \end{tikzcd}$$

are pullback diagrams. On the other hand, we have a pullback diagram

$$\begin{tikzcd} C \times D \arrow[r, "i_C\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times i_D"] & C \times \mathbb{A}^2 \arrow[d, "\mathrm{id}\times i_C"] \\ \mathbb{A}^2 \times D \arrow[r, "\mathrm{id}\times i_D"] & \mathbb{A}^2 \times \mathbb{A}^2. \end{tikzcd} \tag{5.3}$$

By base-changing (5.3) by $\mathbb{A}^3 \xrightarrow{\mathrm{id}\times\Delta\times\mathrm{id}} \mathbb{A}^4$, we obtain the pullback diagram

$$\begin{tikzcd} \widetilde{C \circ D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & C \times \mathbb{A}^1 \arrow[d, "i_C\times\mathrm{id}"] \\ \mathbb{A}^1 \times D \arrow[r, "\mathrm{id}\times i_D"] & \mathbb{A}^3. \end{tikzcd}$$

This means $\widetilde{C \circ D}$ is equal to $\operatorname{Spec} k[x,y,z]/(f(x,z), g(z,y))$. The assertion follows from Proposition 5.7 (ii). ☐

Proposition 5.12. *Let $C, D \subset \mathbb{P}^1 \times \mathbb{P}^1$ be divisorial correspondences, $F = (f_{ij})$, $G = (g_{kl})$ be the coefficient matrices of C and D respectively. Let us consider the following diagram*

$$\begin{tikzcd} \widetilde{C \circ D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "p_{13}"] \arrow[d, "\mathrm{id}\times\Delta\times\mathrm{id}"] & \mathbb{P}^1 \times \mathbb{P}^1 \\ C \times D \arrow[r, "\subset\times\subset"] & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \end{tikzcd} \tag{5.4}$$

Then the composition of the morphisms in upper row $\widetilde{C \circ D} \to \mathbb{P}^1 \times \mathbb{P}^1$ factors through the divisorial correspondence given by the coefficient matrix $H = (h_{mn})$ such that

$$\sum_{m,n} h_{mn} x_0^{dd'-m} x_1^m y_0^{ee'-n} y_1^n = \mathrm{res}_{[z_0,z_1]} \left(\sum_{i,j} f_{ij} x_0^{d-i} x_1^i z_0^{e-j} z_1^j, \sum_{k,l} g_{kl} z_0^{d'-k} z_1^k y_0^{e'-l} y_1^l \right)$$

if $H \neq 0$.

Proof. For a standard open covering of $\mathbb{P}^1$, $\{U_0 = \mathbb{P}^1 \setminus \{0\}, U_1 = \mathbb{P}^1 \setminus \{\infty\}\}$, we denote $U_{\alpha_1\alpha_2\ldots\alpha_n}$ for the open subscheme $U_{\alpha_1} \times \cdots \times U_{\alpha_n}$ of $(\mathbb{P}^1)^n$. The closed subscheme $\mathrm{Im}(\mathrm{id} \times \Delta \times \mathrm{id})$ of $(\mathbb{P}^1)^4$ is covered by the open subschemes $U_{\alpha\beta\beta\gamma}$ of $(\mathbb{P}^1)^4$. Proposition 5.10 gives the construction of the upper row of (5.4) over each $U_{\alpha\beta\beta\gamma} \simeq \mathbb{A}^4$, where the codomain is restricted to $U_{\alpha\gamma} \simeq \mathbb{A}^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$. By combining these constructions, we obtain the assertion. $\square$

Definition 5.13. *The composition map* $\circ : \mathrm{Corr}_{d,e} \times \mathrm{Corr}_{d',e'} \dashrightarrow \mathrm{Corr}_{dd',ee'}$ is the rational map which is induced from the map

$$V_{d,e} \times V_{d',e'} \ni (f(x,y), g(x,y)) \mapsto (f \circ g)(x,y) := \mathrm{res}_z(f(x,z), g(z,y)) \in V_{dd',ee'}.$$

We write $C \circ D$ for $\circ(C, D)$.

Lemma 5.14. *Let* $+ : \mathrm{Corr}_{d,e} \times \mathrm{Corr}_{d',e'} \to \mathrm{Corr}_{d+d',e+e'}$, $(C, D) \mapsto C + D$ *be the map of the addition of divisors. Then we have*

$$(C + D) \circ E = (C \circ E) + (D \circ E) \text{ and } C \circ (D + E) = (C \circ D) + (C \circ E).$$

Proof. The addition of divisors is given by the multiplication of homogeneous polynomials. Therefore, the assertion follows from Proposition 5.7(iv). $\square$

Remark 5.15. Compositions of smooth integral correspondences are not always reduced and irreducible. For example:

(i) Nonreduced case is given in [18, (2.5)],

$$\mathrm{res}(f(x, y^k), y^k - z) = f(x, y)^k.$$

(ii) Reducible case is given by symmetric correspondences, the bihomogeneous polynomials such that $f(x, y) = f(y, x)$. In this case, we have the diagonal correspondence $x - y (= x_0y_1 - y_0x_1)$ as a nonreduced irreducible component, that is,

$$(x - y)^d \mid \mathrm{res}(f(x, z), f(z, y)).$$

By Proposition 5.12 and Proposition 5.7(i), the composition map is defined except on the locus where $f(x,z)$ and $g(z,y)$ have a common divisor $h(z)$. Indeed, this locus is the image of the morphism

$$\begin{array}{lllll} \Phi\colon \mathrm{Corr}_{d,e-1} \times & \mathrm{Corr}_{d'-1,e'} \times & \mathbb{P}^1 & \to & \mathrm{Corr}_{d,e} \quad \times \quad \mathrm{Corr}_{d',e'}, \\ ([f(x,y)], & [g(x,y)], & [\alpha:\beta]\ ) & \mapsto & ([f(x,y)(\beta y_0 - \alpha y_1)], [g(x,y)(\beta x_0 - \alpha x_1)]), \\ (\quad C, & D, & P\ ) & \mapsto & (\quad C \cup (P \times \mathbb{P}^1), \quad D \cup (\mathbb{P}^1 \times P)\quad). \end{array}$$

Proposition 5.16. *The indeterminacy locus of the composition map*

$$\circ : \mathrm{Corr}_{d,e} \times \mathrm{Corr}_{d',e'} \dashrightarrow \mathrm{Corr}_{dd',ee'}$$

is $\mathrm{Im}(\Phi)$.

Proof. Let $(\tilde{C}, \tilde{D}) := \Phi(C, D, P)$ be a point of $\mathrm{Im}(\Phi)$. We show the indeterminacy of the point under the composition map by taking lines

$$\mathbb{A}^1 \to \mathrm{Corr}_{d,e} \times \mathrm{Corr}_{d',e'} (= \mathbb{P}(V_{d,e}) \times \mathbb{P}(V_{d',e'}))$$

through $(\tilde{C}, \tilde{D})$ and comparing their image by the composition map.

By Lemma 5.14, we can reduce the problem to the case that the pair (C, D) is not in the indeterminacy locus. By Proposition 5.8(ii), we can assume that $P = [1:0]$ without loss of generality. Let $h_1(x)$ and $h_2(x)$ be sections of $\mathcal{O}_{\mathbb{P}^1}(d)$ and $\mathcal{O}_{\mathbb{P}^1}(e')$ respectively and let l be a line on $\mathrm{Corr}_{d,e} \times \mathrm{Corr}_{d',e'}$ through the point $(\tilde{C}, \tilde{D})$ such that

$$\begin{aligned} l = l_{h_1,h_2} &: \mathbb{A}^1 \to \mathrm{Corr}_{d,e} \times \mathrm{Corr}_{d',e'}, \\ &a \mapsto ([f(x,y)y_1 + ah_1(x)y_0^e], [g(x,y)x_1 + ah_2(y)x_0^{d'}]). \end{aligned}$$

We write

$$f(x,y)y_0 = \sum_{i=1}^{e} f_i(x) y_0^i y_1^{e-i} \text{ and } g(y,z)y_0 = \sum_{j=1}^{d'} g_i(z) y_0^j y_1^{d'-j}.$$

Then we have

$$\begin{aligned} &\mathrm{res}_{[y_0,y_1]}(f(x,y)y_1 + ah_1(x)y_0^e, g(y,z)y_1 + ah_2(z)y_0^{d'}) \\ &= \begin{vmatrix} ah_1(x) & f_1(x) & & \cdots & f_e(x) & & & \\ & ah_1(x) & f_1(x) & & \cdots & f_e(x) & & \\ & & \ddots & \ddots & & \ddots & \ddots & \\ & & & ah_1(x) & f_1(x) & & \cdots & f_e(x) \\ ah_2(z) & g_1(z) & & \cdots & g_{d'}(z) & & & \\ & ah_2(z) & g_1(z) & & \cdots & g_{d'}(z) & & \\ & & \ddots & \ddots & & \ddots & \ddots & \\ & & & ah_2(z) & g_1(z) & & \cdots & g_{d'}(z) \end{vmatrix} \\ &= a(h_1(x)g_1(z) - h_2(z)f_1(x))\,\mathrm{res}_{[y_0,y_1]}(f(x,y), g(y,z)) \\ &\quad + a^2 \cdot (\text{polynomial}). \end{aligned}$$

Therefore we have

$$\circ(l(0)) = [(h_1(x)g_1(z) - h_2(z)f_1(x))\,\mathrm{res}_{[y_0,y_1]}(f(x,y), g(y,z))].$$

Since the pair $(C, D) = ([f(x,y)], [g(y,z)])$ has no common factor, we have $f_1(x) \neq 0$ or $g_1(z) \neq 0$. Thus the point $\circ(l(0))$ depends on the choice of h_1 and h_2. Therefore, at the point $(\tilde{C}, \tilde{D})$, the map $\circ$ has indeterminacy. $\square$

Definition 5.17. A *horizontal* (resp. *vertical*) component of a correspondence C is an irreducible component of degree $(i, 0)$ (resp. $(0, i)$) for some $i \geq 1$.

Lemma 5.18. *If a composition $C_1 \circ C_2$ of correspondences has any horizontal (resp. vertical) component, then C_1 or C_2 has a horizontal (resp. vertical) component.*

Proof. We note that the first projection $p_1 : C \to \mathbb{P}^1$ has a point $P \in \mathbb{P}^1$ with non-finite inverse image if and only if C has a vertical component. Thus, by Remark 5.11, we obtain the assertion. The case of horizontal component is similar. $\square$

Proposition 5.19. *The composition map is associative, that is, the following diagram is commutative:*

$$\begin{tikzcd} \mathrm{Corr}_{d,e} \times \mathrm{Corr}_{d',e'} \times \mathrm{Corr}_{d'',e''} \arrow[r, dashed, "\circ \times \mathrm{id}"] \arrow[d, dashed, "\mathrm{id} \times \circ"] & \mathrm{Corr}_{dd',ee'} \times \mathrm{Corr}_{d'',e''} \arrow[d, dashed, "\circ"] \\ \mathrm{Corr}_{d,e} \times \mathrm{Corr}_{d'd'',e'e''} \arrow[r, dashed, "\circ"] & \mathrm{Corr}_{dd'd'',ee'e''} \end{tikzcd} \tag{5.5}$$

Proof. By Proposition 5.16 and Lemma 5.18, the compositions of the two diagonal paths in (5.5) are rational map. By Proposition 5.12, the images through the two paths of a general point of $\mathrm{Corr}_{d,e} \times \mathrm{Corr}_{d',e'} \times \mathrm{Corr}_{d'',e''}$, which indicates the tuple of correspondences (C, C', C''), are both given by the upper row of the following diagram:

$$\begin{tikzcd} C \widetilde{\circ C' \circ} C'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & (\mathbb{P}^1)^4 \arrow[r, "p_{14}"] \arrow[d, "\mathrm{id} \times \Delta \times \Delta \times \mathrm{id}"] & (\mathbb{P}^1)^2 \\ C \times C' \times C'' \arrow[r, "\subset \times \subset \times \subset"] & (\mathbb{P}^1)^6 & \end{tikzcd}$$

Therefore (5.5) is commutative. $\square$

Remark 5.20. In terms of the resultant, this property is known as "associativity law of resultants" (in [18]). If we admit this fact, we can show Proposition 5.19 by checking well-definedness of the rational maps

$$\mathrm{Corr}_{d,e} \times \mathrm{Corr}_{d',e'} \times \mathrm{Corr}_{d'',e''} \dashrightarrow \mathrm{Corr}_{dd'd'',ee'e''}$$

in (5.5) at a point in the domain. An example of such a point is $(x^d - y^e, x^{d'} - y^{e'}, x^{d''} - y^{e''})$.

Definition 5.21. *The iteration map $\Psi_n : \mathrm{Corr}_{d,e} \dashrightarrow \mathrm{Corr}_{d^n,e^n}$ is the map which sends a bihomogeneous polynomial $f(x, y)$ to the bihomogeneous polynomial $(f \circ f \circ \cdots \circ f)(x, y)$.*

From the direct computation of resultants by using Sylvester matrix, or using the equivariance of resultants in Corollary 5.8(ii), we obtain Proposition 5.22.

Proposition 5.22. *Let $g \in \mathrm{PGL}_2 \simeq \mathrm{Aut}(\mathbb{P}^1)$ be a morphism given by $g([x_0 : x_1]) = [ax_0 + bx_1 : cx_0 + dx_1]$ and $f(x, y)$ be a bihomogeneous polynomial. Then we have*

$$(g \circ f \circ g^{-1})(x, y) = f(ax_0 + bx_1, cx_0 + dx_1, ay_0 + by_1, cy_0 + dy_1).$$

Corollary 5.23. *For the conjugation action*

$$\mathrm{Corr}_{d,e} \times \mathrm{PGL}_2 \to \mathrm{Corr}_{d,e} : (C, g) \mapsto g \circ C \circ g^{-1}, \tag{5.6}$$

the induced action of SL_2 on $\mathrm{Corr}_{d,e} \simeq \mathbb{P}(V_{d,e})$ is given by a representation on $V_{d,e}$ and the representation is isomorphic to $V_d \otimes V_e$.

Proof. By Proposition 5.22, the action of SL_2 on $V_{d,e}$ is isomorphic to the one on the tensor space of the space of d-ic forms and the space of e-ic forms, $\mathrm{Sym}_d(V_1^*)^* \otimes \mathrm{Sym}_e(V_1^*)^*$. By Proposition 4.5, it is isomorphic to $V_d \otimes V_e$. □

6. Fundamental properties of the moduli space of correspondence

In this section, we give simple generalizations of the results in [32] and [19], a characterization of the stable/semistable locus of the group action and the rationality of the moduli spaces.

6.1. **Stability of group action.** First, we briefly review the geometric invariant theory [27].

Definition 6.1 ([27, Definition 1.6])**.** Let G be a reductive group scheme and X a scheme with G-action $\sigma : G \times X \to X$. For an invertible sheaf $\mathcal{L}$ over X, an isomorphism $\phi : \sigma^*\mathcal{L} \simeq p_2^*\mathcal{L}$ is said to be *G-linearization* if ϕ satisfies the cocycle condition

$$p_{23}^*\phi \circ (\mathrm{id}_G \times \sigma)^*\phi = (\mu \times \mathrm{id}_X)^*\phi \text{ (on } G \times G \times X).$$

Remark 6.2. If $\mathcal{L}$ is very ample and G is affine, then G-linearization is described as the $G(\mathcal{O}(X))$-action on $\mathcal{L}(X)$ compatible with σ.

Remark 6.3. For a G-linearization ϕ of an invertible sheaf $\mathcal{L}$ over a normal scheme X, $\phi^n : \sigma^*\mathcal{L}^n \simeq p_2^*\mathcal{L}^n$ is a G-linearization of $\mathcal{L}^n$.

Remark 6.4 ([27, Proposition 1.4]). If there exists no surjective homomorphism $G \to \mathbb{G}_m$ of group schemes and $X \times_k \bar{k}$ is normal, G-linearization ϕ of an invertible sheaf $\mathcal{L}$ is unique if exists.

For a given action and a given invertible sheaf, G-linearization may not be unique, for instance, if the action is trivial, any regular homomorphism $G \to \mathrm{Aut}(\mathcal{L})$ gives a G-linearization.

Definition 6.5 ([27, Definition 1.7])**.** Let G be a reductive group, X an algebraic variety with G-action and P a geometric point of X.

(i) P is said to be *pre-stable* if the stabilizer group of P is finite and there exists a G-stable affine open neighborhood of P.

Moreover, we suppose that $\mathcal{L}$ is an ample invertible sheaf over X with G-linearization.

(ii) P is said to be *$\mathcal{L}$-semistable* if for some positive integer $n > 0$, there exists $f \in H^0(X, \mathcal{L}^n)^G$ such that $f(P) \neq 0$ and X_f is affine.

(iii) P is said to be *(proper) $\mathcal{L}$-stable* if P is $\mathcal{L}$-semistable and pre-stable.

The set of pre-stable (resp. $\mathcal{L}$-semistable, $\mathcal{L}$-stable) geometric points is the set of geometric points of an open subscheme of X called *pre-stable (resp. $\mathcal{L}$-semistable, $\mathcal{L}$-stable) locus.* We denote the loci by $X^s(\mathrm{Pre})$ (resp. $X^{ss}(\mathcal{L})$, $X^s(\mathcal{L})$).

Remark 6.6. For a G-variety X which is isomorphic to a projective space $\mathbb{P}(V)$, we sometimes write X^s and X^{ss} for the stable locus and semistable locus of any $\mathcal{O}(n)$ with G-linearization.

Remark 6.7 ([27, Converse 1.12]). If the categorical (resp. the geometric) quotient of X by G exists, then $X = X^{ss}(\mathcal{L})$ (resp. $X = X^s(\mathcal{L})$) for some ample invertible sheaf $\mathcal{L}$ over X with G-linearization.

Theorem 6.8 ([27, Theorem 1.1])**.** *Let G be a reductive group, $X = \operatorname{Spec} R$ an affine algebraic variety with G-action. Then the categorical quotient $X /\!\!/ G$ is constructed as* $\operatorname{Spec} R^G$*.*

Theorem 6.9 ([27, p. 40])**.** *Let G be a reductive group, X a proper algebraic variety with G-action, $\mathcal{L}$ a very ample invertible sheaf with G-linearization. Then the categorical quotient $X^{ss}(\mathcal{L}) /\!\!/ G$ is constructed as* $\operatorname{Proj} \bigoplus_{i=0}^{\infty} H^0(X, \mathcal{L}^i)^G$*.*

Definition 6.10. Let G be a reductive group, X a proper algebraic variety with G-action and $\mathcal{L}$ a very ample invertible sheaf over X with G-linearization.

(i) If $G = \mathbb{G}_m$ and $x = x_0$ is a fixed closed point of the $\mathbb{G}_m$-action, then the *weight* $\mu^{\mathcal{L}}(x_0)$ of x_0 is $-n$ if the $k(x_0)^\times$-representation $\mathcal{L}|_{x_0} \to \mathcal{L}|_{x_0}$ (Remark 6.2) is isomorphic to $(k(x_0))(n)$.

(ii) If $G = \mathbb{G}_m$ and x is a closed point, then we take the extension $l_x : \mathbb{A}^1 \to X$ of the $\mathbb{G}_m$-orbit of x by the valuative criterion. The *weight* $\mu^{\mathcal{L}}(x)$ of x is the weight of the fixed point $l_x(0)$ of the $\mathbb{G}_m$-action.

(iii) For any homomorphism of group schemes $\lambda : \mathbb{G}_m \to G$ and a closed point x, the *λ-weight* $\mu^{\mathcal{L}}(x, \lambda)$ of x is the weight of x by the $\mathbb{G}_m$-action on X induced by λ.

Theorem 6.11 ([27, Theorem 2.1])**.** *Let G be a reductive group, X a proper algebraic variety with G-action and $\mathcal{L}$ a very ample invertible sheaf over X with G-linearization. Then, a closed point x of X is in $X^{ss}(\mathcal{L})$ (resp. $X^s(\mathcal{L})$) if and only if $\mu^{\mathcal{L}}(x, \lambda) \geq 0$ (resp. $\mu^{\mathcal{L}}(x, \lambda) > 0$) for all nontrivial group homomorphisms $\lambda : \mathbb{G}_m \to G$.*

By Corollary 5.23, the moduli space of correspondences $\mathrm{Corr}_{d,e}$ is isomorphic to $\mathbb{P}(V_d \otimes V_e)$ equivariantly with respect to SL_2-actions. By Remark 6.2, an SL_2-linearization is given by the natural representation over $H^0(\mathbb{P}(V_d \otimes V_e), \mathcal{O}(1)) \simeq (V_d \otimes V_e)^*$, the dual representation of $V_d \otimes V_e$. By Proposition 4.5, $(V_d \otimes V_e)^*$ is isomorphic to $V_d \otimes V_e$. By Theorem 6.9, the uniform categorical quotient $\mathrm{Corr}_{d,e}^{ss} /\!\!/ \mathrm{SL}_2$ is constructed as $\operatorname{Proj} k[V_d \otimes V_e]^{\mathrm{SL}_2}$.

The reductive group PGL_2 also has a PGL_2-linearization on $\mathcal{O}(\frac{2}{\gcd(2,d+e)})$ by Proposition 4.8 and Corollary 5.23. In fact, we have $k[V]^{\mathrm{SL}_2} = k[V]^{\mathrm{PGL}_2}$ for any finite dimensional representation V of PGL_2, therefore

$$\mathrm{Corr}_{d,e}^{ss} /\!\!/ \mathrm{PGL}_2 \simeq \operatorname{Proj} \bigoplus_{i=0}^{\infty} k[V_d \otimes V_e]_{2i}^{\mathrm{PGL}_2} \simeq \operatorname{Proj} \bigoplus_{i=0}^{\infty} k[V_d \otimes V_e]^{\mathrm{SL}_2}.$$

Theorem 6.12 generalizes a result in [32].

Theorem 6.12. *A divisorial correspondence C given by a bihomogeneous polynomial $\sum_{i=0}^{d} \sum_{j=0}^{e} a_{ij} x^i y^j$ is* not *a stable point (resp.* not *a semistable point) of $\mathrm{Corr}_{d,e}$ if and only if there exists an SL_2-conjugate of the coefficient matrix (b_{ij}) such that $b_{ij} = 0$ for all $i + j < \frac{d+e}{2}$ (resp. $i + j \leq \frac{d+e}{2}$).*

Proof. By [32], any maximal subtorus of $\mathrm{SL}_2(k)$ is conjugate to $c : \mathbb{G}_m \to \mathrm{SL}_2(k)$ such that $c(t) := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. For a bihomogeneous polynomial f of degree (d, e), we have

$$c(t) \cdot f = f(tx_0, t^{-1}x_1, ty_0, t^{-1}y_1) = \sum_{i=0}^{d} \sum_{j=0}^{e} t^{(d+e)-2(i+j)} a_{ij} x^i y^j.$$

Therefore by Theorem 6.11, we obtain the claim. $\square$

6.2. **Unstable locus of iteration map.** Proposition 5.19 implies that $(g \circ C \circ g^{-1})^n = g \circ C^n \circ g^{-1}$, so the iteration map $\Psi_n : \mathrm{Corr}_{d,e} \dashrightarrow \mathrm{Corr}_{d^n,e^n}$ (Definition 5.21) is PGL_2-equivariant. Therefore, we can define iteration map $\mathrm{Dyn}_{d,e} \dashrightarrow \mathrm{Dyn}_{d^n,e^n}$ if the composition $\mathrm{Corr}_{d,e} \dashrightarrow \mathrm{Corr}_{d^n,e^n} \dashrightarrow \mathrm{Dyn}_{d^n,e^n}$ is well-defined, that is, the image of $\Psi_n : \mathrm{Corr}_{d,e} \dashrightarrow \mathrm{Corr}_{d^n,e^n}$ is not contained in the indeterminacy locus of $\mathrm{Corr}_{d^n,e^n} \dashrightarrow \mathrm{Dyn}_{d^n,e^n}$. Using Theorem 6.12, we check this by seeing the simplest case as in Remark 5.20.

Corollary 6.13. *The iteration map $\Psi_n : \mathrm{Corr}_{d,e} \dashrightarrow \mathrm{Corr}_{d^n,e^n}$ induces the rational map*

$$\Phi_n : \mathrm{Dyn}_{d,e} \dashrightarrow \mathrm{Dyn}_{d^n,e^n}.$$

Proof. Put $l := \gcd(d, e)$, $d' := d/l$ and $e' := e/l$. Then, by direct computation, we have

$$\Psi_n([x^d - y^e]) = [(x^{(d')^n l} - y^{(e')^n l})^{l^{n-1}}].$$

The largest multiplicity of this iterated correspondence at the point on the diagonal is $\min(d^n, e^n)$ of $(0, 0)$ and (∞, ∞). By Theorem 6.12, the point $\Psi_n([x^d - y^e])$ on the Corr_{d^n,e^n} is in the semistable locus. Therefore, the composition $\mathrm{Corr}_{d,e} \dashrightarrow \mathrm{Corr}_{d^n,e^n} \dashrightarrow \mathrm{Dyn}_{d^n,e^n}$ is well-defined, and PGL_2-invariant by Proposition 5.19. By Theorem 6.9, we obtain the rational map $\Phi : \mathrm{Dyn}_{d,e} \dashrightarrow \mathrm{Dyn}_{d^n,e^n}$. □

Remark 6.14. The composition map does not induce a rational map

$$\mathrm{Dyn}_{d,e} \times \mathrm{Dyn}_{d',e'} \dashrightarrow \mathrm{Dyn}_{dd',ee'},$$

because for a general pair of correspondences (C, C') and a general $g \in \mathrm{PGL}_2$, we have $([C], [C']) = ([g \circ C \circ g^{-1}], [C'])$ as a point of $\mathrm{Dyn}_{d,e} \times \mathrm{Dyn}_{d',e'}$, but $C \circ C'$ is not PGL_2-conjugate to $g \circ C \circ g^{-1} \circ C'$.

Remark 6.15. To describe the indeterminacy locus of each $\Phi_n : \mathrm{Dyn}_{d,e} \dashrightarrow \mathrm{Dyn}_{d^n,e^n}$ is a problem. A conjecture is that each Φ_n has indeterminacy locus which $\Phi_m (m < n)$ does not have. The case of quadratic map $(d, e) = (1, 2)$ is shown in [5] and the case of maps $d = 1$ in arbitrary degree e is shown in [17].

6.3. **Rationality.** Let $V = V' \oplus V''$ be a representation of a reductive group G. Then we have the inclusion morphism $k[V'] \to k[V]$. This morphism is G-equivariant by definition, therefore leads to the morphism $k[V']^G \to k[V]^G$ and the rational map $\mathbb{P}(V^*)^{ss} /\!\!/ G \dashrightarrow \mathbb{P}((V')^*)^{ss} /\!\!/ G$. If the action of the group G on $(V')^*$ is free for general point, then the fiber of a general point of $\mathbb{P}((V')^*)^{ss} /\!\!/ G$ via $\mathbb{P}(V^*)^{ss} /\!\!/ G \dashrightarrow \mathbb{P}((V')^*) /\!\!/ G$ is naturally isomorphic to $(V/V')^*$. Therefore we have Proposition 6.16.

Proposition 6.16. *For a representation of a reductive group G and a representation V, $\mathbb{P}(V^*) /\!\!/ G$ is rational if there exists a subrepresentation $V' \subset V$ such that the action of G on V' is generically free and $\mathbb{P}((V')^*) /\!\!/ G$ is rational.*

We recall that the field k we fixed is infinite. Let $1 \leq d, e$ be positive integers. By the Clebsch-Gordan decomposition (Proposition 4.8) and the Schur's Lemma

(Remark 4.9), we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{SL}_2}(V_{d+e-1} \otimes V_1, V_d \otimes V_e) \\
&\simeq \mathrm{Hom}_{\mathrm{SL}_2}(V_{d+e} \oplus V_{d+e-2}, V_{d+e} \oplus V_{d+e-2} \oplus \cdots \oplus V_{|d-e|}) \\
&\simeq \mathrm{Hom}_{\mathrm{SL}_2}(V_{d+e}, V_{d+e}) \oplus \mathrm{Hom}_{\mathrm{SL}_2}(V_{d+e-2}, V_{d+e-2}) \\
&\simeq k \oplus k.
\end{aligned}$$

For a vector $c = (c_0, c_1) \in k \oplus k$ $(c_0, c_1 \neq 0)$, we have an injective homomorphism

$$\rho_c : V_{d+e-1} \otimes V_1 \to V_d \otimes V_e$$

of representations. Then it induces a surjective rational map

$$\rho_c^* : \mathrm{Corr}_{d,e} \dashrightarrow \mathrm{Corr}_{1,d+e-1}\,.$$

Here, homomorphism of representation is indeed equivariant, so we obtain

$$\bar{\rho_c^*} : \mathrm{Dyn}_{d,e} \dashrightarrow \mathrm{Dyn}_{1,d+e-1}\,.$$

Proposition 6.17. $\mathrm{Dyn}_{d,e}$ *is rational for* $d, e \geq 1$ *and* $(d, e) \neq (1, 1)$.

Proof. For the case $d = 1$, this is shown by Levy [19]. In the same paper, it is also shown $\mathrm{SL}_2(k)$ acts generically free on the representation $V_D \otimes V_1$ for $D \geq 3$. Therefore the general case follows from Proposition 6.16. □

Remark 6.18. In [16], [21], [22], [23], [24] and [31], the rationality of $\mathrm{Corr}_{d,e} /\!\!/ \mathrm{SL}_2 \times \mathrm{SL}_2$ is shown for some (d, e)'s. The rationality of SL_2 leads to the rationality of $\mathrm{Corr}_{d,e}$ for these cases.

Remark 6.19. In [19], the rationality of $\mathrm{Dyn}_{1,d} \simeq \mathbb{P}(V_1 \otimes V_d) /\!\!/ \mathrm{PGL}_2$ is shown by reducing to the rationality of $\mathrm{Fix} \simeq \mathbb{P}(V_{d+e}) /\!\!/ \mathrm{PGL}_2$ using the morphism

$$\Omega^0 : \mathbb{P}(V_{d+e-1} \otimes V_1) \simeq \mathbb{P}(V_{d+e} \oplus V_{d+e-2}) \dashrightarrow \mathbb{P}(V_{d+e}).$$

The isomorphism $\mathrm{Fix} \simeq \mathbb{P}(V_{d+e}) /\!\!/ \mathrm{PGL}_2$ is explained in Section 7.

7. Multiplier map

7.1. **Construction.** Let f be a bihomogeneous polynomial

$$f(x_0, x_1, y_0, y_1) = \sum_{0 \leq i \leq d,\ 0 \leq j \leq e} a_{ij} x_0^{d-i} x_1^i y_0^{e-j} y_1^j.$$

We need to define multipliers of f. We begin with a local argument. We fix an affine coordinate

$$\left(\bar{x} = \frac{x_1}{x_0}, \bar{y} = \frac{y_1}{y_0} \right) \tag{7.1}$$

of the open affine subscheme $\mathbb{A}^1_x \times \mathbb{A}^1_y = U^+_{\mathbb{P}^1_x}(x_0) \times U^+_{\mathbb{P}^1_y}(y_0)$ of $\mathbb{P}^1_x \times \mathbb{P}^1_y$.

The restriction of the correspondence f over $\mathbb{A}^1_x \times \mathbb{A}^1_y$ is given by

$$\bar{f}(\bar{x}, \bar{y}) := \sum_{0 \leq i \leq d,\ 0 \leq j \leq e} a_{ij} \bar{x}^i \bar{y}^j.$$

From the implicit function theorem, the derivative, which we denote by $\frac{dy}{dx}$ or $\left(\frac{dy}{dx}\right)_f(a,b)$, of the curve $V(\bar{f}(\bar{x},\bar{y}))$ around a point $(a,b) \in V(\bar{f}(\bar{x},\bar{y}))$ is given by

$$\left(\frac{dy}{dx}\right)_f (a,b) = -\left.\frac{\partial_{\bar{x}}\bar{f}(\bar{x},\bar{y})}{\partial_{\bar{y}}\bar{f}(\bar{x},\bar{y})}\right|_{(\bar{x},\bar{y})=(a,b)} \tag{7.2}$$

if $\partial_y \bar{f}(a,b) \neq 0$.

We can use the bihomogeneous polynomial $f(x,y) = f(x_0,x_1,y_0,y_1)$ to compute the value (7.2). From the coordination (7.1), we have the following equations of rational functions of variables x_0, x_1, y_0, y_1:

$$\begin{aligned} x_0^d y_0^e \partial_{\bar{x}} \bar{f}(\bar{x},\bar{y}) &= \partial_{x_1} f(x,y) = d \cdot f(x,y) - \partial_{x_0} f(x,y) \text{ and} \\ x_0^d y_0^e \partial_{\bar{y}} \bar{f}(\bar{x},\bar{y}) &= \partial_{y_1} f(x,y) = d \cdot f(x,y) - \partial_{y_0} f(x,y). \end{aligned} \tag{7.3}$$

Therefore, at a point $(a,b) \in V^+(f) \setminus V^+(\partial_y f)$, we have

$$\frac{dy}{dx} = -\frac{\partial_{x_1} f(a,b)}{\partial_{y_1} f(a,b)} = -\frac{-\partial_{x_0} f(a,b)}{-\partial_{y_0} f(a,b)}$$

because $f(x,y) = 0$ for $(x,y) \in V^+(f)$. Therefore, for any pair of bihomogeneous polynomials g_0, g_1, we have

$$\frac{dy}{dx} = \frac{g_0(a,b)\partial_{x_0} f(a,b) + g_1(a,b)\partial_{x_1} f(a,b)}{g_0(a,b)\partial_{y_0} f(a,b) + g_1(a,b)\partial_{y_1} f(a,b)}.$$

We do not specify the derivation operator $g_0\partial_0 + g_1\partial_1$ and write the value given by them by

$$\frac{dy}{dx} = -\frac{\partial_x f(a,b)}{\partial_y f(a,b)}.$$

As we mentioned in Section 1, *the fixed point* of (the correspondence defined by) f is $\Delta_{\mathbb{P}^1} \times_{\mathbb{P}^1 \times \mathbb{P}^1} V_+(f) = \{z \in \mathbb{P}^1 \mid f(z,z) = 0\}$. To describe the tuple of the multipliers for the fixed points of f, we construct the symmetric form of the fixed point multipliers $\sigma_i(f)$ by

$$1 + \sum_{i=1}^{d+e} (-1)^i \sigma_i(f) t^i = \prod_{z : f(z,z)=0} \left(1 + \frac{(\partial_x f)(z,z)}{(\partial_y f)(z,z)} t\right). \tag{7.4}$$

The map $\mathrm{Corr}_{d,e} \dashrightarrow \mathbb{A}^{d+e}$ given by $f \mapsto (\sigma_i(f))_{i=1,\dots d+e}$ has indeterminacy locus which consists of the correspondences that have any y-critical fixed point. To incorporate correspondences with y-critical fixed points, we prefer to consider the following homogenized form of the multiplier map:

$$\mathrm{Corr}_{d,e} \ni f \mapsto \left[\prod_{z : f(z,z)=0} (\partial_y f(z,z) + \partial_x f(z,z) t)\right] \in \mathbb{P}(k[t]_{d+e}).$$

By Proposition 5.7(iii), we have

$$\prod_{z : f(z,z)=0} (\partial_y f(z,z) + \partial_x f(z,z) t) = \mathrm{res}_{[z_0,z_1]}(f(z,z), \partial_y f(z,z) + \partial_x f(z,z) t). \tag{7.5}$$

In Subsection 7.2, we need to substitute the variable t. Therefore we define the multiplier maps as the following.

Definition 7.1. *The fixed point multiplier map* is the rational map

$$\lambda_{1,(d,e)} : \mathrm{Corr}_{d,e} \ni f \mapsto [\mathrm{res}_z(f(z,z), \partial_x f(z,z)dx + \partial_y f(z,z)dy)] \in \mathbb{P}(D_{d+e}),$$

where we regard dx and dy as just variables and D_n is the space of n-ic forms in them. *The n-th multiplier map* is the rational map $\lambda_{n,(d,e)} := \lambda_{1,(d^n,e^n)} \circ \Psi_n$ if exists.

By the equivariance of resultant (Corollary 5.8(ii)) and the description of the conjugation action (Proposition 5.22), the multiplier map $\lambda_{n,(d,e)}$ is invariant under the conjugation action of PGL_2 on $\mathrm{Corr}_{d,e}$. Therefore, by Theorem 6.9, we obtain a rational map $\mathrm{Dyn}_{d,e} \dashrightarrow \mathbb{P}(D_{d^n+e^n})$. We also denote this map by $\lambda_{n,(d,e)}$.

Remark 7.2. Whether $\Psi_n \circ \lambda_{1,(d^n,e^n)}$ and $\Phi_n \circ \lambda_{1,(d^n,e^n)}$ are well-defined is a problem. From the expression of LHS in (7.5), the fixed point multiplier map is defined for correspondences with no singular fixed point. A simple way to show the well-definedness is to give a correspondence C of degree (d,e) such that C^n have no singular fixed point. If d and e are coprime and n is odd, $\Psi_n([x^d y^e - 1]) = [x^{d^n} y^{e^n} - 1]$ are examples. But the author does not have enough examples to give the well-definedness for all (d,e,n) yet.

Example 7.3. Consider the bihomogeneous polynomial of degree $(2,2)$,

$$f(x,y) = x^2y^2 - 2x^2y - x^2 + 2y = x_1^2y_1^2 - 2x_0^2y_0y_1 - x_1^2y_0^2 + 2x_0^2y_0y_1.$$

The fixed points of f are

$$\{z = [z_0, z_1] \in \mathbb{P}^1 \mid f(z,z) = 0\} = \{[z_0, z_1] \in \mathbb{P}^1 \mid z_1/z_0 = -1, 0, 1, 2\}.$$

The derivative $\frac{dy}{dx}$ around a fixed point z is given by

$$-\frac{\partial_{x_1} f(z,z)}{\partial_{y_1} f(z,z)} = -\frac{2z^3 - 4z^2 - 2z}{2z^3 - 2z^2 + 2} = -\frac{z^3 - 2z^2 - z}{z^3 - z^2 + 1},$$

thus the multipliers are $\left\{-2, 0, 2, \frac{2}{5}\right\}$. The symmetric form of the multiplier is given by

$$(t+2)\,t\,(t-2)\left(t - \frac{2}{5}\right) = t^4 - \frac{2}{5}t^3 - 4t^2 + \frac{8}{5}t = \frac{1}{5}(5t^4 - 2t^3 - 20t^2 + 8t).$$

Indeed, by Definition 7.1, we obtain the point in $\mathbb{P}(D_4)$ corresponding to the symmetric form:

$$\begin{aligned}\lambda_{1,(d,e)}(f) &= [\mathrm{res}_z(z^4 - 2z^3 - z^2 + 2z, (2z^3 - 4z^2 - 2z)dx + (2z^3 - 2z^2 + 2)dy)] \\ &= [-128(dx)^3(dy) + 320(dx)^2(dy)^2 + 32(dx)(dy)^3 - 80(dy)^4] \\ &= [8(dx)^3(dy) - 20(dx)^2(dy)^2 - 2(dx)(dy)^3 + 5(dy)^4].\end{aligned}$$

The sequence of coefficients $(0, 8, -20, -2, 5)$ satisfies the equation

$$2 \cdot 0 + 1 \cdot 8 + 0 \cdot (-20) - 1 \cdot (-2) - 2 \cdot 5 = 0,$$

this is the Woods Hole formula for this correspondence.

7.2. **Resultant form of the Woods Hole formula.** For any pair of variables (α_0, α_1), we define a derivative operator d_α by

$$d_\alpha := -\alpha_0 \partial_{\alpha_0} + \alpha_1 \partial_{\alpha_1}.$$

We also recall that the Cayley operator Ω_{xy} is defined by $\Omega_{xy} := \partial_{x_0}\partial_{y_1} - \partial_{y_0}\partial_{x_1}$ and the Clebsch-Gordan decomposition is defined by

$$\Omega^m f(z_0, z_1) := \left(\Omega_{xy}^m f(x_0, x_1, y_0, y_1)\right)\big|_{(x_0,x_1)=(y_0,y_1)=(z_0,z_1)}.$$

In particular, $\Omega^0 f(z) = f(z, z)$ gives the fixed points of f. Moreover, we can give a representation-theoretic decomposition of the multiplier map as follows.

Lemma 7.4. *Let*

$$f = f(x, y) = \sum_{i,j} a_{ij} x_0^{d-i} x_1^i y_0^{e-j} y_1^j$$

be a bihomogeneous polynomial of degree (d, e)*. Then we have*

$$\begin{pmatrix} (d_x f)(z, z) \\ (d_y f)(z, z) \end{pmatrix} = \begin{pmatrix} \frac{d}{d+e} & \frac{2}{d+e} \\ \frac{e}{d+e} & -\frac{2}{d+e} \end{pmatrix} \begin{pmatrix} d_z(\Omega^0 f)(z) \\ z_0 z_1 (\Omega^1 f)(z) \end{pmatrix}.$$

Proof. From the definition of Cayley operator (4.1) and the operators d_x, d_y and d_z, we have

$$\begin{aligned} (d_x f)(z, z) &= \sum_{i,j} (d - 2i) a_{ij} z^{i+j}, \\ (d_y f)(z, z) &= \sum_{i,j} (e - 2j) a_{ij} z^{i+j}, \\ d_z(\Omega^0 f)(z) &= \sum_{i,j} (d + e - 2i - 2j) a_{ij} z^{i+j} \text{ and} \\ z_0 z_1 (\Omega^1 f)(z) &= \sum_{i,j} (-ei + dj) a_{ij} z^{i+j}, \end{aligned}$$

where z^{i+j} is the abbreviation of $z_0^{d+e-i-j} z_1^{i+j}$. This leads to

$$\begin{pmatrix} d_z(\Omega^0 f)(z) \\ z_0 z_1 (\Omega^1 f)(z) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{e}{2} & \frac{-d}{2} \end{pmatrix} \begin{pmatrix} (d_x f)(z, z) \\ (d_y f)(z, z) \end{pmatrix}$$

and this is equivalent to the assertion. □

For any pair of positive integers d', e', we take the basis $(dz^0_{(d',e')}, dz^1_{d'+e'})$ of $D_1 = k \cdot dx \oplus k \cdot dy$ by

$$dz^0_{(d',e')} := \frac{d'}{d'+e'} dx + \frac{e'}{d'+e'} dy \text{ and } dz^1_{d'+e'} := \frac{2}{d'+e'} dx - \frac{2}{d'+e'} dy. \tag{7.6}$$

This coordinate change helps to give Proposition 7.5.

Proposition 7.5. *For each* $c \in (k^\times)^2$*, let* A_c *be the automorphism of* $\mathbb{P}(D_{d+e})$ *induced by the linear automorphism*

$$\begin{pmatrix} dz^0_{(d,e)} \\ dz^1_{d+e} \end{pmatrix} \mapsto \begin{pmatrix} c_0 dz^0_{(1,d+e-1)} \\ c_1 dz^1_{d+e} \end{pmatrix}$$

on D_1. Then, the following diagram is commutative:

$$\begin{tikzcd} \mathrm{Dyn}_{d,e} \arrow[r, dashed, "\lambda_{1,(d,e)}"] \arrow[d, dashed, "\bar{\rho}_c"] & \mathbb{P}(D_{d+e}) \arrow[d, "A_c"] \\ \mathrm{Dyn}_{1,d+e-1} \arrow[r, dashed, "\lambda_{1,(1,d+e-1)}"] & \mathbb{P}(D_{d+e}). \end{tikzcd}$$

Proof. By Lemma 7.4, we have

$$d_xf(z,z)dx + d_yf(z,z)dy = d_z(\Omega^0 f)(z)dz^0_{(d',e')} + z_0z_1\Omega^1 f(z)dz^1_{d'+e'}.$$

For a vector $c = (c_0, c_1) \in (k^\times)^2$, the morphism $\rho_c : V_{d,e} \to V_{d+e-1,1}$ of representation is defined as $\Omega^i\rho_c(f) = c_i \cdot \Omega^i f$ $(i = 0, 1)$ for any $f \in V_{d,e}$. Therefore, for any bihomogeneous polynomial $f \in V_{d,e}$, we have

$$\lambda_{1,(d,e)}(f) = \mathrm{res}_z(\Omega^0 f(z), d_z(\Omega^0 f)(z)dz^0_{(d,e)} + z_0z_1\Omega^1 f(z)dz^1_{d+e}) \text{ and}$$
$$\lambda_{1,(1,d+e-1)} \circ \rho_c(f) = \mathrm{res}_z(\Omega^0 f(z), c_0d_z(\Omega^0 f)(z)dz^0_{(d+e-1,1)} + c_1z_0z_1\Omega^1 f(z)dz^1_{d+e}).$$

This shows the assertion. $\square$

Remark 7.6. For a bihomogeneous polynomial $f = \sum_{i,j} a_{ij}x^iy^j$, we have

$$\lambda_{1,(d,e)}(f) = [\mathrm{res}_z((\Omega^0 f)(z), d_z(\Omega^0 f)(z)dz^0_{(d,e)} + z_0z_1(\Omega^1 f)(z)dz^1_{d+e})] \in \mathbb{P}(D_{d+e}).$$

From the definition of resultant, we can see that each coefficient is divisible by $a_{00}a_{de}$. As a point of $\mathbb{P}(D_{d+e})$, we have

$$\lambda_{1,(d,e)}(f) = [\mathrm{res}_z((\Omega^0 f)(z), d_z(\Omega^0 f)(z)dz^0_{(d,e)} + z_0z_1(\Omega^1 f)(z)dz^1_{d+e})/a_{00}a_{de}]. \tag{7.7}$$

As a $(d+e)$-ic form of variables $dz^0_{(d,e)}$ and dz^1_{d+e}, the coefficient of $(dz^1_{d+e})^{d+e}$ on (7.7) is

$$\mathrm{res}_z(\Omega^0 f(z), z_0z_1\Omega^1 f(z))/a_{00}a_{de} = \mathrm{res}_z(\Omega^0 f(z), \Omega^1 f(z)),$$

this is SL_2-invariant by Proposition 5.8(ii). From the invariance of the multiplier map, the other coefficients of (7.7) are SL_2-invariant on $\mathrm{Corr}_{d,e}$ of degree $2(d+e-1)$.

For any basis $\{ds, dt\}$ of D_1, the coefficient function $[(ds)^i(dt)^{n-i}] \in D_n^*$ is the dual base of $(ds)^i(dt)^{n-i}$ with respect to the basis $\{(ds)^i(dt)^{n-i}\}_{0\leq i\leq n}$ of D_n. We also note that $D_n^* \simeq H^0(\mathbb{P}(D_n), \mathcal{O}(1))$.

Theorem 7.7. *The image of $\lambda_{1,(d,e)} : \mathrm{Dyn}_{d,e} \dashrightarrow \mathbb{P}(D_{d+e})$ is the hyperplane defined by*

$$[(dz^0)^{d+e-1}(dz^1)^1] = 0, \tag{7.8}$$

where $[(dz^0)^{d+e-1}dz^1]$ is the coefficient function of $(dz^0_{(d,e)})^{d+e-1}dz^1_{d+e}$.

We see an example before the proof.

Example 7.8. Let

$$f(x,y) := 2x^2y^2 + x^2y + 4xy^2 - 2x - 3y - 2$$

be the bihomogeneous polynomial of degree $(2, 2)$. Then we have

$$\Omega^0 f(z) = 2z^4 + 5z^3 - 5z - 2,$$
$$d_z(\Omega^0 f)(z) = 8z^4 + 10z^3 + 10z + 8 \text{ and}$$
$$z\Omega^1 f(z) = z(-6z^2 + 2).$$

Therefore, we have

$$\begin{aligned}\lambda_{1,(2,2)}(f) &= \mathrm{res}_z(2z^4+5z^3-5z-2,(8z^4+10z^2+10z+8)dz^0+z(-6z^2+2)dz^1)\\ &= 2^4(5832(dz^0)^4-7344(dz^0)^2(dz^1)^2+1600(dz^0)(dz^1)^3-88(dz^1)^4)\end{aligned}$$

and the coefficient of $(dz^0)^3(dz^1)^1$ is 0.

Proof of Theorem 7.7. By Proposition 7.5, it is enough to show the assertion for $e=1$, the case of the moduli space of maps. We show the index theorem (1.3) is equivalent to the hyperplane $[(dz^0)^d(dz^1)^1]=0$ by the coordinate change. The coordinate change is induced by the coordinate change

$$\begin{pmatrix} dz^0 \\ dz^1\end{pmatrix} = \begin{pmatrix} \frac{d}{d+1} & \frac{2}{d+1} \\ \frac{1}{d+1} & -\frac{2}{d+1}\end{pmatrix}\begin{pmatrix} dx \\ dy\end{pmatrix}$$

on D_1, which induces a coordinate change

$$\begin{pmatrix} [dz^0] \\ [dz^1]\end{pmatrix} = \begin{pmatrix} \frac{d}{d+1} & \frac{2}{d+1} \\ \frac{1}{d+1} & -\frac{2}{d+1}\end{pmatrix}^{-1}\begin{pmatrix} [dx] \\ [dy]\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{-d}{2}\end{pmatrix}\begin{pmatrix} [dx] \\ [dy]\end{pmatrix}$$

on D_1^*.

We note that D_n is naturally isomorphic to $(\mathrm{Sym}_n(D_1^*))^*$, so D_n^* is naturally isomorphic to $\mathrm{Sym}_n(D_1^*)$. The vector space of n-ic form of D_1^* is isomorphic to the vector space $(\mathrm{Sym}_n(D_1^{**}))^* \simeq (\mathrm{Sym}_n D_1)^*$ and the canonical morphism $\beta_n : \mathrm{Sym}_n(D_1^*) \to (\mathrm{Sym}_n D_1)^*$ is given by $\beta_n([(ds)^i(dt)^{n-i}]) = \binom{n}{i}[ds]^i[dt]^{n-i}$ for any basis $\{ds, dt\}$ of D_1 (Remark 4.6). Therefore, we have

$$\begin{aligned}\beta_{d+1}([(dz^0)^d(dz^1)]) &= \binom{d+1}{1}[dz^0]^d[dz^1]\\ &= (d+1)([dx]+[dy])^d(\frac{1}{2}[dx]-\frac{d}{2}[dy])\\ &= \frac{d+1}{2}\left(\sum_{i=0}^{d+1}\left(d\cdot\binom{d}{i-1}-\binom{d}{i}\right)[dx]^i[dy]^{d+1-i}\right)\\ &= \frac{d+1}{2}\left(\sum_{i=0}^{d+1}(i-1)\binom{d+1}{i}[dx]^i[dy]^{d+1-i}\right)\\ &= \beta_{d+1}\left(\frac{d+1}{2}\sum_{i=0}^{d+1}(i-1)[(dx)^i(dy)^{d+1-i}]\right),\end{aligned}$$

where we used

$$d\cdot\binom{d}{i-1}-\binom{d}{i} = \binom{d+1}{i}\left(d\frac{i}{d+1}-\frac{d+1-i}{d+1}\right) = (i-1)\binom{d+1}{i}$$

and $\binom{d}{i}=0$ for $i<0, d<i$.

Therefore, the hyperplane defined by $[(dz^0)^d(dz^1)]=0$ is the one defined by $\sum_{i=0}^{d+1}(i-1)[(dx)^i(dy)^{d+1-i}]=0$. By (7.4) and Definition 7.1, this is also the same as one which the index theorem (1.3) defines.

The surjectivity of the multiplier map onto the hyperplane is shown in [10]. $\square$

The only linear relation between the elementary symmetric polynomials of fixed point multipliers is the one obtained by the coordinate change from (7.8). The Woods Hole formula for correspondences (1.4) should be such a relation. It is easy to check this fact by a similar computation to the above proof. Moreover, if starting this argument from the Woods Hole Formula for rational maps (1.3), this gives an alternative proof of (1.4).

Corollary 7.9. *Let $d, e \geq 2$ be positive integers and let C be a correspondence defined by the bihomogeneous polynomial $f(x, y) = 0$ of degree (d, e). Then, we have the Woods Hole formula*

$$\sum_{i=0}^{d+e} (-1)^i (e-i) \sigma_i(f) = 0. \tag{7.9}$$

Proof. Let us fix the vector $c = (1, 1)$ for the morphism ρ_c. Then, by the proof of Proposition 7.5, we have

$$\begin{pmatrix} dz^0 \\ dz^1 \end{pmatrix} = \begin{pmatrix} \frac{d}{d+e} & \frac{2}{d+e} \\ \frac{e}{d+e} & -\frac{2}{d+e} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} \text{ and } \begin{pmatrix} [dz^0] \\ [dz^1] \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{e}{2} & \frac{-d}{2} \end{pmatrix} \begin{pmatrix} [dx] \\ [dy] \end{pmatrix}.$$

Therefore, the equation

$$d \cdot \binom{d+e-1}{i-1} - e \cdot \binom{d+e-1}{i} = \binom{d+e}{i} \left(d \frac{i}{d+e} - e \frac{d+e-i}{d+e} \right) = (i-e) \binom{d+e}{i}$$

of binomial coefficients leads to

$$\beta_{d+e}([(dz^0)^{d+e-1}(dz^1)]) = \beta_{d+e} \left(\frac{d+e}{2} \sum_{i=0}^{d+e} (i-e)[(dx)^i (dy)^{d+e-i}] \right).$$

Then, Theorem 7.7 leads to

$$\sum_{i=0}^{d+e} (e-i)[(dx)^i (dy)^{d+e-i}](\lambda_{1,(d,e)}(f)) = 0. \tag{7.10}$$

By Definition 7.1 and (7.4), we have

$$\lambda_{1,(d,e)}(f) = \left[\sum_{i=0}^{d+e} (-1)^i \sigma_i(f) (dx)^i (dy)^{d+e-i} \right] \in \mathbb{P}(D_{d+e}). \tag{7.11}$$

By applying (7.10) to (7.11), we obtain

$$\sum_{i=0}^{d+e} (-1)^i (i-e) \sigma_i(f) = 0.$$

□

The representational simplification gives the following result from the Woods Hole formula.

Corollary 7.10. *For an arbitrary field K and any polynomials $f, g \in K[x]$ of degree $\deg f \geq 3$ and $\deg f \geq \deg g + 2$,*

$$\frac{\partial}{\partial t} \operatorname{res}_x(f(x), f'(x) + tg(x)) \Big|_{t=0} = 0.$$

Proof. We first consider the case where $K = k$ is a field of characteristic zero. We put $d := \deg f$. By Proposition 4.8, there exists a bihomogeneous polynomial $F(x, y)$ of degree $(d-1, 1)$ such that $\Omega^0 F(z) = f(z)$ and $\Omega^1 F(z) = g(z)$. By Remark 7.6 and Theorem 7.7, we have the k-coefficient of t of the polynomial

$\mathrm{res}_z(f(z), zf'(z) + tzg(z)) \in k[t]$ is 0. Since resultant is $\mathbb{Z}$-polynomial of the coefficients of f and g, the coefficient of t is 0 in arbitrary characteristics. This leads to the assertion. □

Remark 7.11. Corollary 7.10 also can be proved by the following idea. We can expand the resultant as

$$\mathrm{res}_x(f(x), f'(x) + tg(x)) = \Delta(f) + t \cdot F_1(f, g) + t^2 \cdot (\text{polynomial}), \tag{7.12}$$

where $\Delta(f) = \mathrm{res}_x(f(x), f'(x))$ is the discriminant of f. From another expression of resultant

$$\mathrm{res}_x\left(a_d \prod_{i=1}^{d}(x - \alpha_i), b_e \prod_{j=1}^{e}(x - \beta_j)\right) = a_d^e b_e^d \prod_{i=1}^{d}\prod_{j=1}^{e}(\alpha_i - \beta_j),$$

we have $\Delta(f) = 0$ if and only if f have a multiple root. If we perturb the coefficients of f' by t, each solution of f' moves $O(t)$ $(t \to 0)$. Then, if f have a multiple root α, we have two factors of the form $(\alpha - (\alpha + O(t)))$ $(t \to 0)$ on the resultant of (7.12) factored by the above equation. Thus, we have $F_1 = 0$ in (7.12) if $\Delta(f) = 0$. By looking at degrees of each variables on $\Delta(f)$ and $F_1(f, g)$, we obtain $F_1 = 0$.

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