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THE HARMONIC MAP COMPACTIFICATION OF TEICHMÜLLER SPACES FOR PUNCTURED RIEMANN SURFACES

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ABSTRACT. In the paper [*The Teichmüller theory of harmonic maps*, J. Differential Geom. 29 (1989), no. 2, 449–479], Wolf provided a global coordinate system of the Teichmüller space of a closed oriented surface S with the vector space of holomorphic quadratic differentials on a Riemann surface X homeomorphic to S. This coordinate system is via harmonic maps from the Riemann surface X to hyperbolic surfaces. Moreover, he gave a compactification of the Teichmüller space by adding a point at infinity to each endpoint of harmonic map rays starting from X in the space. Wolf also showed this compactification coincides with the Thurston compactification.

In this paper, we extend the harmonic map ray compactification to the case of punctured Riemann surfaces and show that it still coincides with the Thurston compactification.

1. INTRODUCTION

Let S be a finite type oriented surface of genus g with n punctures such that $\chi(S) = 2 - 2g - n < 0$. The *Teichmüller space* $\mathcal{T}_{g,n}$ of S is the space of isotopy classes of complete finite-area hyperbolic metrics on S.

Let σ and ρ be hyperbolic metrics on S. If S is closed, there exists a unique harmonic diffeomorphism from (S, σ) to (S, ρ) in the isotopy class of the identity map $\mathrm{id}\colon S \to S$. This follows from work of Eells and Sampson [ES64], Hartman [Har67], Schoen and Yau [SY78], and Sampson [Sam78]. Moreover, Lohkamp extended this unique existence result to a punctured surface [Loh91]. Let $h(\sigma, \rho)$ be the harmonic diffeomorphism $(S, \sigma) \to (S, \rho)$. Then, the $(2, 0)$-part of the pullback metric $h(\sigma, \rho)^*\rho$ is a finite L^1-norm holomorphic quadratic differential on (S, σ). The $(2, 0)$-part $(h(\sigma, \rho)^*\rho)^{2,0}$ is called the *Hopf differential* of $h(\sigma, \rho)$.

By fixing the metric σ of the domain and varying a metric ρ of the target, we obtain a well-defined map $\Phi\colon \mathcal{T}_{g,n} \to \mathrm{QD}(\sigma)$ given by $\Phi(\rho) = (h(\sigma, \rho)^*\rho)^{2,0}$, where $\mathrm{QD}(\sigma)$ denotes the vector space of holomorphic quadratic differentials on (S, σ) with finite L^1-norm. Wolf showed Φ is a homeomorphism if S is a closed surface [Wol89], and Lohkamp generalized it to a closed surface with finitely many punctures [Loh91].

Thus, we identify $\mathcal{T}_{g,n}$ with $\mathrm{QD}(\sigma)$ via the homeomorphism Φ. The vector space $\mathrm{QD}(\sigma)$ has a natural compactification obtained by adding a point at infinity to each ray starting from the origin. Therefore, we obtain a compactification of the

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Teichmüller space $\mathcal{T}_{g,n}$. The compactification is called the *harmonic map compactification* and denoted by $\overline{\mathcal{T}_{g,n}^{\mathrm{harm}}}$. The boundary of the harmonic map compactification is identified with the L^1-norm unit sphere $\mathrm{SQD}(\sigma)$ in $\mathrm{QD}(\sigma)$.

On the other hand, Thurston introduced a compactification of $\mathcal{T}_{g,n}$, which is called the *Thurston compactification* and denoted by $\overline{\mathcal{T}_{g,n}^{\mathrm{Th}}}$. Let $\mathcal{C} = \mathcal{C}(S)$ be the set of all isotopy classes of essential simple closed curves. The Thurston compactification is obtained by embedding the Teichmüller space into the projective space $(\mathbb{R}_{\geq 0}^{\mathcal{C}} - \{0\})/\mathbb{R}_{>0}$ as a length function and taking its closure in the projective space. Then, the boundary is the projective space $\mathcal{PMF}$ of measured foliations on S, and it is called the *Thurston boundary*.

In the case that S is closed (i.e. $n = 0$), Wolf showed that the homeomorphism Φ continuously extends to a homomorphism $\overline{\Phi}$ from $\overline{\mathcal{T}_{g,0}^{\mathrm{Th}}}$ to $\overline{\mathcal{T}_{g,0}^{\mathrm{harm}}}$. This implies that the restriction of $\overline{\Phi}$ to the Thurston boundary coincides with the canonical identification $\overline{F_v}$ of $\mathrm{SQD}(\sigma)$ and $\mathcal{PMF}$, where $\overline{F_v}$ is induced by the map $F_v\colon \mathrm{QD}(\sigma) \to \mathcal{MF}$ given by the vertical measured foliation $F_v(\Psi)$ for $\Psi \in \mathrm{QD}(\sigma)$. The purpose of this paper is to extend this identification to the case of punctured Riemann surfaces.

Main Theorem (Theorem 4.7, Proposition 4.6)**.** *The homeomorphism $\Phi\colon \mathcal{T}_{g,n} \to \mathrm{QD}(\sigma)$ continuously extends to a homeomorphism $\overline{\Phi}$ from $\overline{\mathcal{T}_{g,n}^{\mathrm{Th}}}$ to $\overline{\mathcal{T}_{g,n}^{harm}}$. Moreover, this extension on the boundary coincides with the inverse of the canonical identification $\overline{F_v}$.*

This theorem implies the coincidence of two compactifications constructed differently, so the harmonic map compactification is independent of the choice of the fixed metric σ. In addition, the homeomorphism $\overline{\Phi}$ provides a global parametrization of the Thurston compactification with the closed unit ball in $\mathrm{QD}(\sigma)$. The proof of the main theorem reproves that the Thurston compactification is homeomorphic to a closed ball of dimension $6g - 6 + 2n$. The proof is relatively easy compared to Thurston's original argument [FLP12], which uses difficult results in high-dimensional topology (the collar neighborhood theorem and generalized Schöenflies theorem).

Our strategy is based on the original proof for closed surfaces. However, the compactness of surfaces was essential in Wolf's proof. Moreover, in our setting, a holomorphic quadratic differential may have a simple pole at a puncture, so we need to substantially change his proof.

We outline our proof below, comparing it with Wolf's proof. Fixing a quadratic differential $\Phi_0 \in \mathrm{SQD}(\sigma)$, we set $\rho_t = \Phi^{-1}(t\Phi_0)$ for $t > 0$. The one-parameter family $\{\rho_t\}$ is called the *harmonic map ray* in the direction of Φ_0. We first show that the norm of the Beltrami differential of the harmonic diffeomorphism $h(t)\colon (S, \sigma) \to (S, \rho_t)$ converges monotonically to 1 as $t \to \infty$ (Proposition 3.7). For the proof of this convergence, [Wol89] uses the compactness of a closed surface. Therefore, using some results on a holomorphic energy function for punctured surfaces in [Loh91], we prove Proposition 3.7 without the compactness of a surface. From Proposition 3.7, we obtain the asymptotic length of each arc along a leaf of the horizontal or vertical measured foliation of Φ_0 (Proposition 3.8).

Let β be the identification of $\mathrm{QD}(\sigma)$ and $\mathcal{MF}$ given by $\beta\Phi = F_v(4\Phi)$. We next recall Wolf's fundamental lemma, which is an inequality of the intersection number function $i(\beta\Phi(\rho), \cdot)$ and the hyperbolic length function ℓ_ρ on $\mathcal{C}$ for a metric $\rho \in \mathcal{T}_{g,0}$.

Theorem ([Wol89, Lemma 4.1]). *Fix an isotopy class $[\gamma] \in \mathcal{C}$. Then, for every $\rho \in \mathcal{T}_{g,0}$, there exist positive constants $k_0 = k_0(\|\Phi(\rho)\|)$ and $\eta = \eta(\|\Phi(\rho)\|, [\gamma])$ such that*

$$i(\beta\Phi(\rho), [\gamma]) \leq \ell_\rho([\gamma]) \leq k_0 i(\beta\Phi(\rho), [\gamma]) + \eta,$$

where $k_0 \searrow 1$ and $\eta\|\Phi(\rho)\|^{-1/2} \to 0$ as $\|\Phi(\rho)\| \to \infty$.

The key to showing the inequality is constructing a "staircase" representative γ_Φ of an isotopy class $[\gamma]$ for each $\Phi \in \mathrm{SQD}(\sigma)$. The representative γ_Φ satisfies some conditions. One of the conditions is that γ_Φ consists of horizontal or vertical segments of Φ and the intersection number $i(\beta\Phi(\rho), \gamma_\Phi)$ is equal to $i(\beta\Phi(\rho), [\gamma])$. However, for punctured surfaces, we cannot construct γ_Φ in the same manner as [Wol89], since an integrable holomorphic quadratic differential on a punctured Riemann surface may have a simple pole at each puncture. Hence, we show an alternate theorem as follows.

Theorem (Theorem 4.1). *Fix an isotopy class $[\gamma] \in \mathcal{C}$ and $\varepsilon > 0$. Then, there exists a nonnegative number $c_0 = c_0([\gamma], \varepsilon) < \varepsilon$ satisfying the following: for every $\rho \in \mathcal{T}_{g,n}$, there exist positive constants $k_0 = k_0(\|\Phi(\rho)\|, \varepsilon)$ and $\eta = \eta(\|\Phi(\rho)\|, [\gamma], \varepsilon)$ such that*

$$i(\beta\Phi(\rho), [\gamma]) \leq \ell_\rho([\gamma]) \leq k_0 i(\beta\Phi(\rho), [\gamma]) + \eta,$$

where $k_0 \searrow 1$ and $\eta\|\Phi(\rho)\|^{-1/2} \to c_0$ as $\|\Phi(\rho)\| \to \infty$.

This proposition is different from Wolf's lemma in that the limit of $\eta\|\Phi(\rho)\|^{-1/2}$ may not be 0. Setting the allowance for arbitrarily small $\varepsilon > 0$, we can construct such a "staircase" representative γ_Φ. However, unlike Wolf's lemma, we need to increase the horizontal measure of the representative γ_Φ from $i(\beta\Phi(\rho), [\gamma])$ by a little. To estimate the length of the additional horizontal segments, we use some results on a quadratic differential metric ([Min92, Wol89]). The theorem is a weak version of [Wol89, Lemma 4.1], but it can lead to the main theorem.

Last, we note a recent development on harmonic map rays. In [PW22], Pan and Wolf gave some asymptotic relations between harmonic map rays and two other types of rays in the Teichmüller space. One of them is a Teichmüller ray described from the viewpoint of complex analysis, and the other is a Thurston stretch ray described from the viewpoint of hyperbolic geometry.

2. Background

2.1. **Harmonic maps.** Let $\overline{S}$ be a closed oriented surface of genus g, and $S \coloneqq \overline{S} - P$, where $P \subset \overline{S}$ is a finite set. Let n be the number of the points of P. If $\chi(S) = 2 - 2g - n < 0$, the surface admits a (complete, finite area) hyperbolic metric. Throughout this paper, we fix S such that $\chi(S) < 0$ and S is not a thrice punctured surface. Let σ be a hyperbolic metric on S. The hyperbolic surface (S, σ) can be regarded as a (punctured) Riemann surface by taking the isothermal coordinate system for σ.

Let $\sigma|dz|^2$ and $\rho|dw|^2$ be hyperbolic metrics on S, where $z = x+iy$ and $w = u+iv$ denote the conformal structures of (S, σ) and (S, ρ), respectively. For a C^2 map $f\colon (S, \sigma|dz|^2) \to (S, \rho|dw|^2)$, we define the *energy density* of f by

$$e(f) \coloneqq \frac{\rho(f(z))}{\sigma(z)}(|f_z|^2 + |f_{\bar{z}}|^2),$$

the *total energy* of f by

$$\mathcal{E}(f) := \int_S \rho(f(z))(|f_z|^2 + |f_{\bar{z}}|^2)\frac{i}{2}dzd\bar{z}$$

and, the *holomorphic energy* $H(f)$ and the *anti-holomorphic energy* $L(f)$ by

$$H(f) := \frac{\rho(f(z))}{\sigma(z)}|f_z|^2,\ L(f) := \frac{\rho(f(z))}{\sigma(z)}|f_{\bar{z}}|^2.$$

Let $J(f)$ be the Jacobian of f, then $J(f) = H(f) - L(f)$. The Beltrami differential of f is defined by

$$\nu(f) := \frac{f_{\bar{z}}d\bar{z}}{f_z dz}.$$

The norm of the Beltrami differential of f is a well-defined function on (S, σ). Clearly, we have $|\nu(f)|^2 = L(f)/H(f)$.

A C^2 map $f\colon (S, \sigma) \to (S, \rho)$ is said to be *harmonic* if f is a critical point of the energy functional $\mathcal{E}$. As the other definition, f is a harmonic map if and only if f satisfies the Euler-Lagrange equation

$$f_{z\bar{z}} + \frac{\rho_w}{\rho} f_z f_{\bar{z}} = 0.$$

It is a well-known fact that if S is closed, there exists a unique harmonic diffeomorphism isotopic to the identity map from (S, σ) to (S, ρ). Eells and Sampson proved the existence of a harmonic map in each homotopy class [ES64]. Hartman proved its uniqueness [Har67]. Schoen-Yau and Sampson independently proved that the harmonic map is a diffeomorphism [SY78, Sam78].

In the case of a punctured surface, Lohkamp showed the following.

Theorem 2.1 ([Loh91])**.** *Let S be a surface of finite type, and σ, ρ be hyperbolic metrics on S. Then, there exists a unique harmonic diffeomorphism $h(\sigma, \rho)\colon (S, \sigma) \to (S, \rho)$ such that $h(\sigma, \rho)$ is homotopic to the identity and $\mathcal{E}(h(\sigma, \rho)) < +\infty$.*

2.2. **The harmonic map compactification of the Teichmüller space.** The *Teichmüller space* of the surface S is the quotient space of hyperbolic metrics on S by the pullback action of a diffeomorphism isotopic to the identity. Let $\mathcal{T}_{g,n}$ denote the Teichmüller space of S. We often write ρ simply for $[\rho] \in \mathcal{T}_{g,n}$.

A holomorphic quadratic differential Φ on (S, σ) is said to be *integrable*, if the $|\Phi|$-area of (S, σ) is finite, namely

$$\|\Phi\| = \int_S |\Phi| dxdy < +\infty.$$

Let $\mathrm{QD}(\sigma)$ denote the space of integrable holomorphic quadratic differentials on (S, σ). For a holomorphic quadratic differential Φ on (S, σ), Φ is integrable if and only if Φ has a pole of at most order one at each puncture.

Fixing the metric σ of the domain surface, we set $h(\rho) := h(\sigma, \rho)$ for ρ. Since $(h(\rho)^*\rho)^{2,0}$ is a holomorphic quadratic differential on (S, σ) with finite norm ([Loh91] Lemma 7), we can define the map

$$\Phi : \mathcal{T}_{g,n} \to \mathrm{QD}(\sigma)$$

as $\Phi([\rho]) = (h(\rho)^*\rho)^{2,0}$. The uniqueness of the harmonic diffeomorphism shows that the map Φ is well-defined. Here, we list very useful well-known formulae.

Proposition 2.2. *Let* $J(\rho) = J(h(\rho)), H(\rho) = H(h(\rho)), L(\rho) = L(h(\rho))$, *and* $\nu(\rho) = \nu(h(\rho))$. *Then the following hold:*

(I) $J(\rho) = H(\rho) - L(\rho)$
(II) $|\Phi(\rho)|/\sigma^2 = H(\rho)L(\rho)$
(III) $|\nu(\rho)|H(\rho)\sigma = |\Phi(\rho)|$
(IV) $\nu(\rho) = L(\rho)/H(\rho)$
(V) $\Delta_\sigma H(\rho) = 2H(\rho) - 2L(\rho) - 2$
(VI) $\Delta_\sigma L(\rho) = 2L(\rho) - 2H(\rho) - 2$ *on* $S - \{\Phi(\rho) = 0\}$

Here,

$$\Delta_\sigma := \frac{4}{\sigma}\frac{\partial^2}{\partial z \partial \bar{z}}$$

is the Laplace-Beltrami operator on (S, σ).

For the proof of (V) and (VI), see [Jos06, Lemma 3.10.1].

Theorem 2.3 is shown by Wolf for $n = 0$ and by Lohkamp for $n > 0$.

Theorem 2.3 ([Wol89], [Loh91]). Φ *is a homeomorphism.*

Using the Riemann-Roch theorem, we see that $\mathrm{QD}(\sigma)$ is a vector space of real dimension $6g - 6 + 2n$. Since a vector space can be compactified by adding a point at infinity to the endpoint of every ray from the origin, we can obtain the compactification of $\mathcal{T}_{g,n}$ through the homeomorphism Φ. The compactification of $\mathcal{T}_{g,n}$ is called the *harmonic map compactification* and denoted by $\overline{\mathcal{T}_{g,n}^{\mathrm{harm}}}$. In other words,

$$\overline{\mathcal{T}_{g,n}^{\mathrm{harm}}} = \mathrm{BQD}(\sigma) \cup \mathrm{SQD}(\sigma),$$

where $\mathrm{BQD}(\sigma) := \{\Phi \in \mathrm{QD}(\sigma) \mid \|\Phi\| < 1\}$ and $\mathrm{SQD}(\sigma) := \{\Phi \in \mathrm{QD}(\sigma) \mid \|\Phi\| = 1\}$.

2.3. **The Thurston compactification.** Here, we put a brief description of $\overline{\mathcal{T}_{g,n}^{\mathrm{Th}}}$ with reference to [FLP12]. A simple closed curve γ on S is *peripheral* if γ bounds a once-punctured disk. If a simple closed curve γ is neither null-homotopic nor peripheral, γ is said to be *essential*. Let $\mathcal{C} = \mathcal{C}(S)$ be the set of all isotopy classes of essential simple closed curves in S. Given functionals $f, g \in \mathbb{R}_{\geq 0}^{\mathcal{C}}$, f and g are said to be equivalent, if there exists a positive real number $\lambda > 0$ such that $f = \lambda g$. The quotient space of the functionals by the equivalence relation is denoted by $P(\mathbb{R}_{\geq 0}^{\mathcal{C}})$, and let $\pi : \mathbb{R}_{\geq 0}^{\mathcal{C}} - \{0\} \to P(\mathbb{R}_{\geq 0}^{\mathcal{C}})$ be the projection. Let $\mathcal{MF}^*$ be the set of nontrivial measured foliations on S which may have possibly one-pronged singularities at the punctures. Then, $\mathcal{MF}^*$ can be embedded into $\mathbb{R}_{\geq 0}^{\mathcal{C}} - \{0\}$ by the map I_* which is defined by

$$I_*(F) := (i(F, \cdot) \colon \mathcal{C} \to \mathbb{R}_{\geq 0}) \qquad (F \in \mathcal{MF}^*),$$

where $i(F, [\gamma])$ is the infimum of the transverse measure of representatives of $[\gamma] \in \mathcal{C}$. Thus we identify $\mathcal{MF}^*$ with its image in $\mathbb{R}_{\geq 0}^{\mathcal{C}}$. Thurston showed that $\pi \circ I_*(\mathcal{MF}^*)$ is homeomorphic to a sphere of dimension $6g - 7 + 2n$.

For $\rho \in \mathcal{T}_{g,n}$, the length functional $\ell_*(\rho) \in \mathbb{R}_{\geq 0}^{\mathcal{C}}$ is given by

$$\ell_*(\rho)([\gamma]) := \ell_\rho([\gamma]) = \inf_{\gamma \in [\gamma]} \ell_\rho(\gamma) \qquad ([\gamma] \in \mathcal{C}).$$

It is known that ℓ_* is an embedding of $\mathcal{T}_{g,n}$ into $\mathbb{R}_{\geq 0}^{\mathcal{C}} - \{0\}$ and $\pi \circ \ell_*$ is still an embedding of $\mathcal{T}_{g,n}$ into $P(\mathbb{R}_{\geq 0}^{\mathcal{C}})$.

Let $\overline{\mathcal{T}_{g,n}^{\mathrm{Th}}}$ be the subset $\pi\circ\ell_*(\mathcal{T}_{g,n})\cup\pi\circ I_*(\mathcal{MF}^*)$ in $P(\mathbb{R}^{\mathcal{C}}_{\geq 0})$. In fact, it is homeomorphic to a closed ball of dimension $6g-6+2n$. The boundary $\pi\circ I_*(\mathcal{MF}^*)$ of the Thurston compactification is denoted by $\mathcal{PMF}$. By the construction, the mapping class group action on $\mathcal{T}_{g,n}$ extends continuously to the Thurston compactification.

3. Deformation Along Harmonic Map Rays

3.1. **Norm functions of Beltrami differentials.** We denote the inverse homeomorphism $\Phi^{-1}:\mathrm{QD}(\sigma)\to\mathcal{T}_{g,n}$ by ρ .

Definition 3.1. Let $\Phi_0\in\mathrm{SQD}(\sigma)$. The *harmonic map ray* in the direction of Φ_0 is the ray defined by $\{\rho_t:=\rho(t\Phi_0)\}_{t>0}$ in $\mathcal{T}_{g,n}$.

Let $h(t)$ denote the unique harmonic diffeomorphism $h(\rho_t)$ homotopic to the identity with $\mathcal{E}(h(\rho_t))<+\infty$. We denote the holomorphic energy by $H(t)$, anti-holomorphic energy by $L(t)$, and the Beltrami differentials of $h(t)$ by $\nu(t)$. Clearly $\Phi(\rho_t)=t\Phi_0$ by the definition. The main purpose of this subsection is to prove Proposition 3.2.

Proposition 3.2. *For any $\Phi_0\in\mathrm{SQD}(\sigma)$, let $\{\rho_t\}_{t>0}$ be the harmonic map ray in the direction of Φ_0. Then, for every nonzero point p of Φ_0, we have*

$$|\nu(t)(p)|^2\nearrow 1\qquad (\textit{as } t\to\infty).$$

Remark 3.3. Let M be the domain surface (S,σ). In the case of closed surfaces, Wolf [Wol89] proved this proposition in the following three steps.

Step 1: Show that $|\nu(t)(p)|^2$ converges to 1 almost everywhere on M.
Step 2: Show that $(|\nu(t)(p)|^2)'>0$ on $M-\{\Phi_0(p)=0\}$.
Step 3: Exclude the possibility that $|\nu(t)(p)|^2\to\delta\neq 1$ as $t\to\infty$ for a nonzero point p of Φ_0.

If the surface S has punctures, we can proceed Step 1 and Step 3 in essentially the same way as the paper [Wol89]. Therefore we give the proof of Step 2 here. Since the compactness of the surface is critical for Step 2 in [Wol89], We prove this inequality for punctured surfaces.

Proposition 3.4. *For every nonzero point $p\in M-\{\Phi_0(q)=0\}$, the derivative $(|\nu(t)(p)|^2)'$ with respect to t is positive.*

Proof. By the formula (II) in Proposition 2.2, we have

$$(H(t)L(t))'=H'(t)L(t)+L'(t)H(t)=\frac{2t|\Phi_0|^2}{\sigma^2}=\frac{2}{t}H(t)L(t).$$

Therefore, for every $p\in M-\{\Phi_0(p)=0\}$,

$$\frac{H'(t)}{H(t)}(p)+\frac{L'(t)}{L(t)}(p)=\frac{2}{t}. \tag{3.1}$$

Next, we introduce some results on the holomorphic energy functions $H(t)$. The following is shown by Lohkamp.

Lemma 3.5 ([Loh91] Lemma 14)**.** *For $t_1,t_2>0$,*

$$\min\left\{1,\frac{t_2}{t_1}\right\}\cdot H(t_1)\leq H(t_2)\leq\max\left\{1,\frac{t_2}{t_1}\right\}\cdot H(t_1)\quad \textit{on } M.$$

Using Lemma 3.5, we show the following.

Lemma 3.6. *For every $t > 0$ and $p \in M$, we have*

$$0 \leq \frac{H'(t)}{H(t)}(p) < \frac{1}{t},$$

where H' is a derivative with respect to t.

Proof. Since the holomorphic energy $H(t)$ is positive, by Lemma 3.5,

$$\min\left\{0, \log\frac{t_2}{t_1}\right\} \leq \log\frac{H(t_2)}{H(t_1)} \leq \max\left\{0, \log\frac{t_2}{t_1}\right\}.$$

Therefore, we obtain

$$0 \leq \frac{\log H(t_2) - \log H(t_1)}{t_2 - t_1} \leq \frac{\log t_2 - \log t_1}{t_2 - t_1}.$$

Setting $t_1 = t, t_2 = t + h$ and tending h to 0, we find

$$0 \leq \frac{H'(t)}{H(t)} \leq \frac{1}{t}. \tag{3.2}$$

Next, we show that the second inequality of (3.2) is strict. If there exists a point $p_0 \in M$ such that $H'(t)/H(t)(p_0) = 1/t$, since p_0 maximizes $H'(t)/H(t)$, we have

$$0 \geq \Delta_\sigma \frac{H'(t)}{H(t)}(p_0) = \Delta_\sigma(\log H(t))' = 2(H'(t) - L'(t)).$$

Therefore, we find $H'(t)(p_0) \leq L'(t)(p_0)$.

This implies that p_0 is not a zero of Φ_0. If p_0 is a zero, by $H(t) > 0$ and $H(t)L(t) = t^2|\Phi_0(p_0)|^2/\sigma^2 = 0$,

$$L(t)(p_0) = 0$$

holds for any $t > 0$. Hence we have $H'(t)(p_0) \leq 0$, however this is impossible since $H(t)(p_0) > 0$ and $H'(t)/H(t)(p_0) = 1/t$. Thus, we find that $\Phi_0(p_0) \neq 0$.

By $H(t) - L(t) > 0$, we have

$$\frac{L'(t)}{L(t)}(p_0) \geq \frac{H'(t)}{L(t)}(p_0) > \frac{H'(t)}{H(t)}(p_0).$$

However, this contradicts $H'(t)/H(t)(p_0) = 1/t$ and (3.1). □

By Lemma 3.6 and (3.1), we have

$$\frac{L'(t)}{L(t)} > \frac{H'(t)}{H(t)} \quad \text{on } M - \{\Phi_0(p) = 0\}.$$

Thus, we obtain

$$(|\nu(t)|^2)' = \left(\frac{L(t)}{H(t)}\right)' = \frac{L'(t)H(t) - L(t)H'(t)}{H(t)^2} > 0 \quad \text{on } M - \{\Phi_0(p) = 0\}.$$

This completes the proof. □

3.2. **Asymptotic length of horizontal and vertical arcs.** Given hyperbolic structures σ, ρ on S, we can isotope ρ so that the identity map $\mathrm{id}\colon (S, \sigma) \to (S, \rho)$ is harmonic. For the rest of this paper, we always take such a representative in its isotopy class.

A holomorphic quadratic differential Φ defines two measured foliations on M which are orthogonal to each other away from the zeros. By changing a conformal coordinate z on M to the natural coordinate $\zeta = \xi + i\eta$ of Φ, the representation of Φ in terms of ζ is identically 1 away from zero, that is

$$\Phi\, dz^2 = d\zeta^2.$$

Then lines parallel to ξ-axis (resp. η-axis) define a singular foliation on M such that its singular points are zeros of Φ_0, and also $|d\eta|$ (resp. $|d\xi|$) defines a transverse measure for the singular foliation. We call the measured foliation the *horizontal* (resp. *vertical*) *foliation* of Φ_0, and it is denoted by $F_h(\Phi)$ (resp. $F_v(\Phi)$). If Φ_0 has a pole of order one at a puncture on S, the foliation is one-pronged at the puncture. An arc along a leaf of horizontal (resp. vertical) measured foliation of Φ is called a *horizontal* (resp. *vertical*) *arc* of Φ_0.

Let $\Phi_0 \in \mathrm{SQD}(\sigma)$ and $\{\rho_t\}$ be the harmonic map ray in the direction of Φ_0. In this subsection, we consider the asymptotic ρ_t-length of horizontal and vertical arcs of Φ_0. Here, we recall Wolf's setting in [Wol89]. For the ray $\{\rho_t\}$, we define conformal coordinates $z = x + iy$ on M, such that $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ give an orthonormal frame field on M and they are respectively maximum and minimum stretching directions of the differential map $dh(t)$. Then, they are tangent to the horizontal and vertical foliations of Φ_0, respectively. The conformal coordinate z is defined away from all zeros of Φ_0. By the definitions of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, they are orthogonal to each other in the ρ_t-metric. Thus, we have

$$t\Phi_0\, dz^2 = \frac{1}{4}\left(\left\| \frac{\partial}{\partial x} \right\|_{\rho_t}^2 - \left\| \frac{\partial}{\partial y} \right\|_{\rho_t}^2 \right) dz^2.$$

Therefore, the $|\Phi(\rho_t)|$ metric length of a tangent vector $\frac{\partial}{\partial x}$ is

$$\left\| \frac{\partial}{\partial x} \right\|_{\Phi(\rho_t)}^2 = \frac{1}{4}\left(\left\| \frac{\partial}{\partial x} \right\|_{\rho_t}^2 - \left\| \frac{\partial}{\partial y} \right\|_{\rho_t}^2 \right). \tag{3.3}$$

Moreover, by computation, we find

$$|\nu(t)| = \frac{1 - \|\frac{\partial}{\partial y}\|_{\rho_t} / \|\frac{\partial}{\partial x}\|_{\rho_t}}{1 + \|\frac{\partial}{\partial y}\|_{\rho_t} / \|\frac{\partial}{\partial x}\|_{\rho_t}}. \tag{3.4}$$

Hence, by Proposition 3.2, the following holds.

Proposition 3.7. *Let $\{\rho_t\}_{t>0}$ be the harmonic map ray in the direction Φ_0. Then*

(1) *for every p with $\Phi_0(p) \neq 0$, $\|\frac{\partial}{\partial y}\|_{\rho_t} / \|\frac{\partial}{\partial x}\|_{\rho_t} \searrow 0$ as $t \to \infty$, and*
(2) *for every p with $\Phi_0(p) \neq 0$, $\|\frac{\partial}{\partial x}\|_{|4\Phi(\rho_t)|} / \|\frac{\partial}{\partial x}\|_{\rho_t} \nearrow 1$ as $t \to \infty$.*

Proposition 3.8. *Let $\{\rho_t\}_{t>0}$ be a harmonic map ray in the direction Φ_0. For an arc γ on M, let $\ell_{\rho_t}(\gamma)$ denote the ρ_t-length of γ.*

(1) *Let γ be a compact horizontal arc of Φ_0 containing no zeros of Φ_0. Then, there exist constants $c_0, c_1 > 0$ depending only on γ such that for every $t > 1$*

$$0 < c_0 < \ell_{\rho_t}(\gamma) t^{-1/2} < c_1 < \infty.$$

(2) *Let γ be a compact vertical arc of Φ_0 (that may contain zeros of Φ_0). Then*

$$\ell_{\rho_t}(\gamma)t^{-1/2} \to 0 \quad \text{as } t \to \infty.$$

Proof. This proposition can be shown similarly to [Wol89], using the fact that the holomorphic energy function is bounded [Loh91, Corollary 3 and Lemma 9] instead of the compactness for a closed surface. □

Lemma 3.9 for closed surfaces is proved by Minsky in [Min92, Lemma 3.3]. Since Minsky's proof works for Lemma 3.9, we omit the proof.

Lemma 3.9. *Let $\Phi \in \mathrm{QD}(\sigma)$. Suppose that $p \in M$ is at least distance d away from poles and zeros of Φ, and*

$$\log|\nu(\Phi)|^{-1} < b \quad \text{on } B_{|\Phi|}(p, d),$$

where $B_{|\Phi|}(p, d)$ denotes the $|\Phi|$-radius d disk centered at p, and $\nu(\Phi) = \nu(\rho(\Phi))$. Then

$$\log|\nu(\Phi)(p)|^{-1} < \frac{b}{\cosh d}.$$

By using Lemma 3.9 and some estimates in [Wol91], we can bound the asymptotic ρ_t-length of horizontal arcs of Φ_0 from above by the $|\Phi_0|$-length.

Proposition 3.10. *Let $\Phi_0 \in \mathrm{SQD}(\sigma)$, and let $\{\rho_t\}_{t>0}$ be the harmonic map ray in the direction of Φ_0. If γ is a compact horizontal arc of Φ_0 containing no zeros, then there exist $C, D > 0$ depending only on γ such that*

$$\ell_{\rho_t}(\gamma) < t^{1/2}\ell_{|4\Phi_0|}(\gamma)(1 + Ce^{-D\sqrt{t}}) \text{ for all } t > 0.$$

Proof. Let $\Sigma(\Phi_0)$ denote the set consisting of punctures and zeros of Φ_0. We set

$$d(t) = \inf\{d_{t|\Phi_0|}(p, q) \mid p \in \gamma, q \in \Sigma(\Phi_0)\}.$$

Then we find that $d(t) = t^{1/2}d(1)$ by the definition of quadratic differential metrics. We define a constant C_0 as

$$C_0 = \sup\{\log|\nu(1)(q)|^{-1} \mid q \in N_{|\Phi_0|}(\gamma, d(1)/2)\},$$

where $N_{|\Phi_0|}(\gamma, d(1)/2)$ denote the $(d(1)/2)$-neighborhood of γ in the $|\Phi_0|$ metric. Since $|\nu(t)|$ increases monotonically in t, we see that for every $q \in N_{|\Phi_0|}(\gamma, d(1)/2)$ and every $t > 1$,

$$\log|\nu(t)(q)|^{-1} < \log|\nu(1)(q)|^{-1} \leq C_0.$$

Therefore, by Lemma 3.9,

$$\log|\nu(t)|^{-1} < \frac{C_0}{\cosh(d(t)/2)} = \frac{C_0}{\cosh\big(t^{1/2}d(1)/2\big)} < C_0 e^{-\sqrt{t}d(1)/2}$$

holds on γ. Then, we have

$$\begin{aligned} |\nu(t)|^{-1/2} - 1 &< \frac{1}{|\nu(1)|^{1/2}}(1 - |\nu(t)|^2) \\ &\leq \frac{1}{\min_\gamma |\nu(1)|^{1/2}} \log|\nu(t)|^{-2} \\ &\leq \frac{2}{\min_\gamma |\nu(1)|^{1/2}} C_0 e^{-\sqrt{t}d(1)/2}. \end{aligned}$$

Setting

$$C = \frac{2}{\min_\gamma |\nu(1)|^{1/2}} C_0 \quad \text{and} \quad D = d(1)/2,$$

we have

$$\begin{aligned}
\ell_{\rho_t}(\gamma) &= \int_\gamma \left\| \frac{\partial}{\partial x} \right\|_{\rho_t} ds \\
&= \int_\gamma \{H(t)^{1/2} + L(t)^{1/2}\}\, ds_\sigma \\
&= \int_\gamma H(t)^{1/2}(1 + |\nu(t)|)\, ds_\sigma \\
&= \int_\gamma \frac{t^{1/2}|\Phi_0|^{1/2}}{|\nu(t)|^{1/2}}(1 + |\nu(t)|)\frac{ds_\sigma}{\sigma^{1/2}} \\
&= t^{1/2} \int_\gamma (1 + (|\nu(t)|^{-1/2} - 1))(2 - (1 - |\nu(t)|))\, ds_{|\Phi_0|} \\
&< 2t^{1/2} \int_\gamma (1 + (|\nu(t)|^{-1/2} - 1))\, ds_{|\Phi_0|} \\
&< t^{1/2} \ell_{|4\Phi_0|}(\gamma)(1 + Ce^{-D\sqrt{t}}),
\end{aligned}$$

as $2\ell_{|\Phi_0|}(\gamma) = \ell_{|4\Phi_0|}(\gamma)$. We obtain the desired inequality. □

Corollary 3.11. *Under the assumptions of Proposition* 3.10*,*

$$\lim_{t\to\infty} t^{-1/2}\ell_{\rho_t}(\gamma) \le \ell_{|4\Phi_0|}(\gamma)$$

holds.

Lemma 3.12 will be used in the proof of Proposition 4.3.

Lemma 3.12. *Let $q \in \overline{S} - S$, a puncture of S. Then for every $\Phi \in \mathrm{QD}(\sigma)$,*

$$\ell_{|\Phi|}(\partial B_{|\Phi|}(q, R)) \le L_1 R,$$

where L_1 is a constant which depends only on the topological type of S.

Remark 3.13. Note for closed Riemann surfaces that Lemma 3.12 is a special case of Lemma 4.1 in [Min92]. For every point $p \in \overline{S}$, let $\mathrm{ord}_p\,\Phi$ denote the order of Φ at p. We know that a singularity p has a cone angle of $(n+2)\pi$, or concentrated curvature $-n\pi$, where $n = \mathrm{ord}_p\,\Phi$. Note that, if p is a pole of Φ, then $\mathrm{ord}_p\,\Phi = -1$.

Proof. For a number r with $0 \le r \le R$, we set $\gamma_r = \partial B_{|\Phi|}(q, r)$. Then, we have

$$\frac{d}{dr}\ell_{|\Phi|}(\gamma_r) = \kappa(\gamma_r),$$

where $\kappa(\gamma_r)$ is a total curvature of γ_r in the $|\Phi|$-metric. By the Gauss-Bonnet theorem, we have

$$\begin{aligned}
\kappa(\gamma_r) &= 2\pi\chi(B_{|\Phi|}(q,r)) - \sum_{p \in B_{|\Phi|}(q,R)} (-(\mathrm{ord}_p\,\Phi)\pi) \\
&\le 2\pi + (4g - 4 + n)\pi.
\end{aligned}$$

Therefore, $\ell_{|\Phi|}(\gamma_R) = \int_0^R \kappa(\gamma_r)\, dr \le (4g - 2 + n)\pi R$ follows. □

4. The Identification of the Compactifications

4.1. **The fundamental lemma.** As described at the beginning of the previous section, a holomorphic quadratic differential Φ on M defines the vertical measured foliation on M. In fact, by the Hubbard-Masur theorem, the map $\Phi \mapsto F_v(\Phi)$ is a homeomorphism from $\mathrm{QD}(\sigma)$ to $\mathcal{MF}$ ([HM79], also see [Gar87, p.206] for punctured surfaces), where $\mathcal{MF}$ denotes the set of measured foliations on S (here $\mathcal{MF}$ contains the empty measured foliation). We define a homeomorphism $\beta\colon \mathrm{QD}(\sigma) \to \mathcal{MF}$ by $\beta\Phi = F_v(4\Phi)$. Then, by the definition of the transverse measure of $\beta\Phi$, we find that, for every $[\gamma] \in \mathcal{C}$

$$i(\beta\Phi, [\gamma]) = \|\Phi\|^{1/2} i(\beta(\Phi/\|\Phi\|), [\gamma]).$$

The main purpose of this subsection is to prove the following proposition analogous to [Wol89, Lemma 4.6].

Theorem 4.1. *Fix $[\gamma] \in \mathcal{C}$ and $\varepsilon > 0$. Then, for every $\rho \in \mathcal{T}_{g,n}$, there exist nonnegative constants $c_0 = c_0([\gamma], \varepsilon)$, $k_0 = k_0(\varepsilon, \|\Phi(\rho)\|)$ and $\eta = \eta([\gamma], \varepsilon, \|\Phi(\rho)\|)$ such that*

$$i(\beta\Phi(\rho), [\gamma]) \le \ell_\rho([\gamma]) \le k_0 i(\beta\Phi(\rho), [\gamma]) + \eta,$$

and

$$k_0 \searrow 1, \quad \eta\|\Phi(\rho)\|^{-1/2} \to c_0 < \varepsilon \quad \text{as } \|\Phi(\rho)\| \to \infty.$$

Remark 4.2. In the statement of Lemma 4.6 in [Wol89], it is written that the constant k_0 depends on $[\gamma]$. However, from his proof, one can observe it is actually independent.

Proof. The lower bound $i(\beta\Phi(\rho), [\gamma]) \le \ell_\rho([\gamma])$ is shown in the same manner as [Wol89], so we omit the proof here.

In order to show the upper bound, we first show Proposition 4.3.

Proposition 4.3. *For each $\Phi \in \mathrm{SQD}(\sigma)$, we can construct a representative $\gamma_\Phi \in [\gamma]$ so that there exist domains $R_0 = R_0([\gamma], \varepsilon)$ and $R_1 = R_1(\varepsilon)$ containing all punctures of M, constants $\delta = \delta(\varepsilon) > 0, K = K([\gamma]) > 0$, and positive integers $L = L([\gamma]), m = m([\gamma], \varepsilon)$ satisfying the following conditions:*

(1) *The curve γ_Φ consists of horizontal and vertical arcs of Φ and does not intersect with the neighborhood $R_0 = R_0([\gamma], \varepsilon)$ of the punctures of M.*
(2) *The horizontal arcs of γ_Φ are divided into the main part γ^h_Φ and the additional part $\widetilde{\gamma}^h_\Phi$ such that*
   - *the main part γ^h_Φ is disjoint from*
   $$\left(\bigcup_p B_\sigma(p, \delta)\right) \cup R_1,$$
   where the union is over all zeros p of Φ.
   - *the number of the connected segments constituting the additional part $\widetilde{\gamma}^h_\Phi$ is at most $m = m([\gamma], \varepsilon)$, and they are disjoint from some neighborhood of zeros of Φ.*
(3) *$i(\beta\Phi, \gamma^h_\Phi) = i(\beta\Phi, [\gamma])$ and $i(\beta\Phi, \widetilde{\gamma}^h_\Phi) < \varepsilon$.*
(4) *The total $|\Phi|$-length of the vertical arcs of γ_Φ is uniformly bounded by $K = K([\gamma])$, i.e. $i(\beta(-\Phi), \gamma^v_\Phi) < K$, where γ^v_Φ is the union of the vertical arcs. Moreover, the number of the connected segments of the vertical arcs which contain a zero of Φ is at most $L = L([\gamma])$.*

Proof. We fix $\Phi_0 \in \mathrm{SQD}(\sigma)$. By the compactness of $\mathrm{SQD}(\sigma)$, it is enough to show that the claim holds on a neighborhood of Φ_0.

For a quadratic differential $\Phi \in \mathrm{SQD}(\sigma)$, we let

$$\begin{aligned} Z(\Phi) &= \{\text{zeros of } \Phi \text{ in } M\}, \\ P_{\mathrm{reg}}(\Phi) &= \{\text{punctures which are regular points or zeros of } \Phi\}, \\ P_{\mathrm{pole}}(\Phi) &= \{\text{punctures which are poles of } \Phi\}, \\ \Sigma(\Phi) &= Z(\Phi) \cup P_{\mathrm{reg}}(\Phi) \cup P_{\mathrm{pole}}(\Phi), \text{the singular set of } \Phi. \end{aligned}$$

Furthermore, we pick a positive constant $\delta' = \delta'(\Phi_0)$ such that, for all distinct $p_i, p_j \in \Sigma(\Phi_0)$, the $|\Phi_0|$-distance between $B_{|\Phi_0|}(p_i, 2\delta')$ and $B_{|\Phi_0|}(p_j, 2\delta')$ is at least $2\delta'$, and also

$$\delta'(\Phi_0) < \Big(\min_{\Phi \in \mathrm{SQD}(\sigma)} \mathrm{inj}\,|\Phi|\Big)/(8g-5+2n),$$

where we set $\mathrm{inj}\,|\Phi| \coloneqq \inf_{[\gamma]\in\mathcal{C}} \ell_{|\Phi|}([\gamma])/2$. We may assume $\varepsilon < \delta'$, since $\varepsilon > 0$ is arbitrary. Then, we pick a sufficiently small neighborhood $\mathcal{N}$ of Φ_0 in $\mathrm{SQD}(\sigma)$ so that every $\Phi \in \mathcal{N}$ satisfies the following conditions:

(1) The connected components of

$$N(\Sigma(\Phi_0), \delta') \coloneqq \bigcup_{p \in \Sigma(\Phi_0)} B_{|\Phi_0|}(p, \delta')$$

bijectively correspond to the connected components of

$$N(\Sigma(\Phi), \delta') \coloneqq \bigcup_{p \in \Sigma(\Phi)} B_{|\Phi|}(p, \delta')$$

by the correspondence between $\Sigma(\Phi)$ and $\Sigma(\Phi_0)$.

(2) If we fill the punctures, every connected component C of $N(\Sigma(\Phi), \delta')$ is topologically a disk, and the $|\Phi|$-distance between C and the other connected components is at least δ'.

(3) The total length of critical vertical leaves of Φ contained in $N(\Sigma(\Phi), \delta')$ is uniformly bounded from above by a constant $K_1 = K_1(\Phi_0)$.

(4) The zeros of Φ splitting off (see Figure 1) from $q \in P_{\mathrm{reg}}(\Phi_0)$ is contained in $B_{|\Phi|}(q, r/2)$, where r is a sufficiently small constant which depends only on $[\gamma]$, ε and Φ_0 and is defined later in (4.3).

Under the preparation, we describe the construction of the representative γ_Φ for each $\Phi \in \mathcal{N}$. First, we begin with the $|\Phi|$-geodesic representative Γ_Φ of $[\gamma]$. Note that Γ_Φ may touch some punctures and cannot be realized on M in a strict sense (see Figure 2). (If all of punctures of S are simple poles of Φ, there exists a strict $|\Phi|$-geodesic representative of $[\gamma]$ in M, see [Sya96].) Each segment of the geodesic representative Γ_Φ outside of $N(\Sigma(\Phi), \delta')$ is a Euclidean straight line segment, so we replace such a straight segment of Γ_Φ with a Φ-*staircase curve* which is a union of horizontal and vertical arcs of Φ. Let Γ'_Φ be the resulting curve. Then

$$i(\beta\Phi, \Gamma'_\Phi) = i(\beta\Phi, [\gamma]) \text{ and } i(\beta(-\Phi), \Gamma'_\Phi) < \max_{\Phi \in \mathrm{SQD}(\sigma)} \ell_{|4\Phi|}([\gamma]).$$

Next, let C be a connected component of $N(\Sigma(\Phi), \delta')$. Then, the number of punctures contained in C is at most one.

Claim 4.4. The number of connected components of $\Gamma'_\Phi \cap C$ is, at most, $a = a([\gamma], \delta')$.

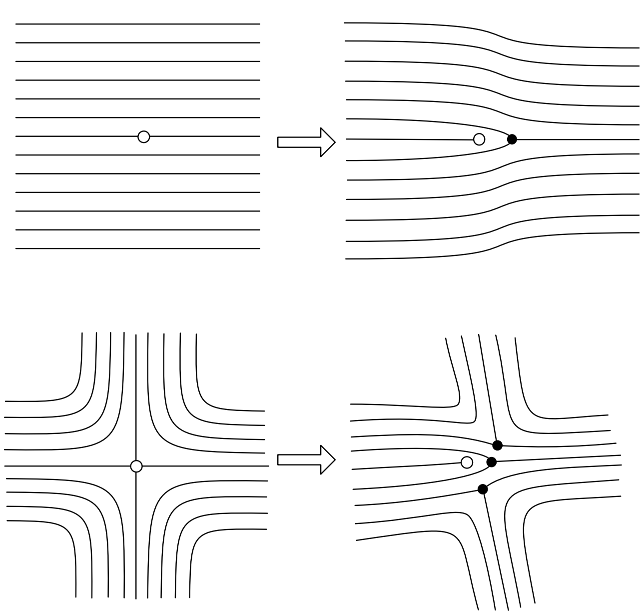

FIGURE 1. The zeros split off from a puncture.

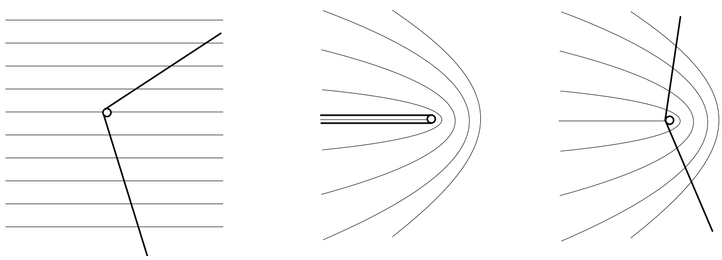

FIGURE 2. Left and middle: we illustrate examples of the geodesic representative in a slightly broad sense around a puncture. Note that there is no such a representative in the right of the figures.

Proof.

Case I. We suppose that C contains a puncture q. Then, C contains, at most, $4g-4+n$ zeros of Φ. Therefore, C is contained in the disk

$$D := B_{|\Phi|}(q, (8g-7+2n)\delta'). \tag{4.1}$$

The disk D does not contain the other puncture and D is embedded into M, since $(8g-7+2n)\delta' < \operatorname{inj}|\Phi|$. Let p be a zero of Φ contained in C. Let Γ''_Φ denote a connected component of $\Gamma_\Phi \cap D$. Notice that $B_{|\Phi|}(p,\delta')$ may not be a convex set, since it can contain a pole of Φ at the puncture q. However, considering a double branched covering of D, we find that the number of the connected components of $\Gamma''_\Phi \cap B_{|\Phi|}(p,\delta')$ is at most two (see Figure 3). Thus, the number of connected components of $\Gamma''_\Phi \cap C$ is at most $2(4g-3+n)$, where $4g-3+n$ is the maximum of the number of the disks constituting C. Let γ' denote a subarc of Γ_Φ which leaves

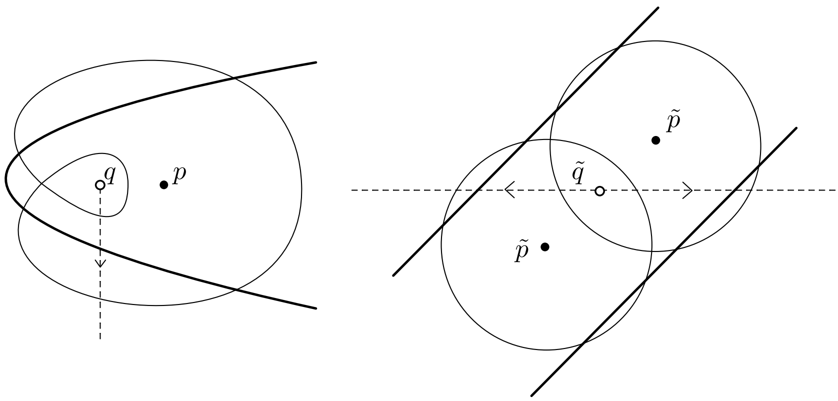

FIGURE 3. Left: it is difficult to directly understand the δ'-neighborhood of a zero near a pole of Φ. Right: the lift of the double cover branched at the puncture q.

D and comes back to D. The endpoints of γ' are on ∂D. Then, since D is convex and Γ_Φ is geodesic, γ' is not isotopic to ∂D rel the endpoints. Thus, we see that the $|\Phi|$-length of γ' is at least δ'. By the above discussion, we find that, every time Γ_Φ intersects D, the number of the connected components of $C \cap \Gamma_\Phi$ increases by at most $8g-6+2n$ and also the number of the connected components of $\Gamma_\Phi \cap D$ is at most

$$\left\lceil \max_{\Phi \in \mathrm{SQD}(\sigma)} \{\ell_{|4\Phi|}([\gamma])\}/\delta'(\Phi_0) \right\rceil. \tag{4.2}$$

Case II. Suppose next that C contains no punctures. Then, C contains, at most $4g-4+n$, zeros of Φ. Therefore, the $|\Phi|$-diameter of C is, at most, $(8g-8+2n)\delta'$. Hence, C is contained in a ball $B = B_{|\Phi|}(p_0, (4g-4+n)\delta')$ for a point $p_0 \in M$. Then the ball contains at most one puncture, since $(8g-8+2n)\delta' < (8g-5+2n)\delta' < \operatorname{inj}|\Phi|$.

Then we have two cases. First, suppose that B contains a puncture q. Then

$$D = B_{|\Phi|}(q, (8g-7+2n)\delta')$$

contains B, and D contains no other punctures. Therefore D is convex, and thus we can apply the argument of Case I to D. However, $B_{|\Phi|}(p, \delta')$ is now convex for a zero p of Φ contained in C, so we do not need to take a double branched covering. Thus we obtain the desired upper bounds. Second, suppose that B contains no punctures. If the neighborhood $N_{|\Phi|}(B, \delta')$ of B contains a puncture q, B is contained in the disk

$$D = B_{|\Phi|}(q, (8g-7+2n)\delta').$$

Therefore we can show the desired claim as in the first case that C contains a puncture q. If the neighborhood $N_{|\Phi|}(B, \delta')$ of B does not contain a puncture, each connected component of $\Gamma_\Phi \setminus B$ has length at least δ'. Then, setting $D = B$, we again apply the argument of Case I to D.

□

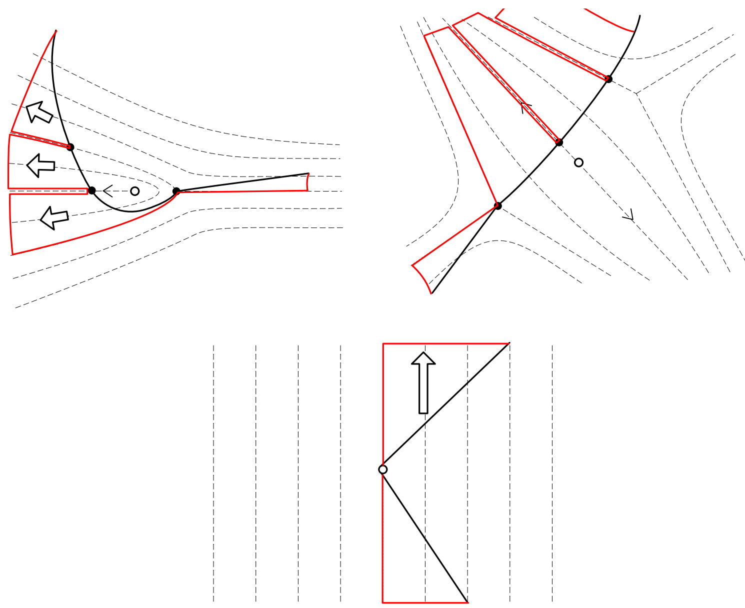

FIGURE 4. The dotted lines denote vertical leaves of Φ. The black points denote intersections of critical vertical arcs and $\overline{\gamma}$. The upper right figure illustrates the lift of the foliation in the upper left figure.

Let $\overline{\gamma}$ denote a connected component of $\Gamma'_\Phi \cap C$. If a vertical arc of Φ contains a singular point of Φ as its endpoint, then it is called a *critical vertical arc*. Then, either (i) $\overline{\gamma}$ has at most one intersection with each critical vertical arc contained in C, or (ii) $\overline{\gamma}$ is a union of critical vertical arcs. We leave the curves in Case (ii), and deform the curves in Case (i). The number of the critical vertical arcs contained in C is, at most, $3(4g-4+n)+1$, so the number of the intersections of $\overline{\gamma}$ and the critical vertical arcs is clearly finite. Then, we divide $\overline{\gamma}$ at these intersection points into finitely many segments. For each such segment of $\overline{\gamma}$, we replace it with a curve consisting of horizontal arcs and vertical arcs of Φ. Then, we drag the horizontal measure out of C by adding vertical arcs (Figure 4). Then we set

$$R_1 = \bigcap_{\Phi \in \mathcal{N}} \left(\bigcup_{q \in \overline{S}-S} B_{|\Phi|}(q, \delta') \right).$$

Let $\widehat{\gamma}$ denote the resulting curve from $\overline{\gamma}$.

Next, we add the following operation, which is denoted by $(*)$, for each connected component α of $\widehat{\gamma} \cap B_{|\Phi|}(q, r)$:

(1) isotope α to $\partial B_{|\Phi|}(q, r)$ rel the endpoints, and
(2) then further isotope α on $\partial B_{|\Phi|}(q, r)$ to a union of vertical and horizontal arcs tangent to $\partial B_{|\Phi|}(q, r)$ (see Figure 5).

By this operation $(*)$, γ_Φ does not enter the r-neighborhood of the puncture q. Since, by Lemma 3.12, the $|\Phi|$-length of $\partial B_{|\Phi|}(q, r)$ is at most $L_1 r$, the operation

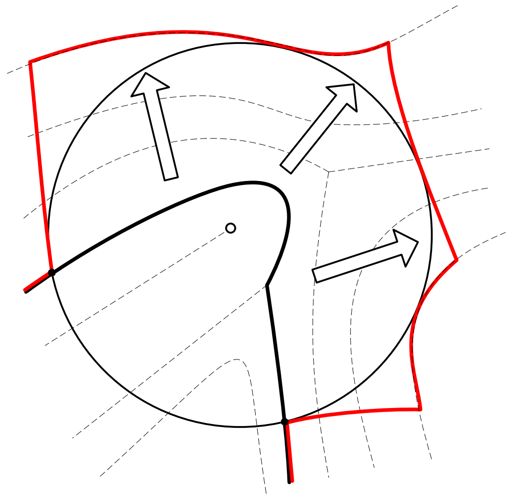

FIGURE 5. This is an example of a vertical arc in $B_{|\Phi|}(q,r)$. By the operation $(*)$, we let the curve not enter $B_{|\Phi|}(q,r)$.

$(*)$ increases the horizontal measure by at most $L_1 r$. Thus, if we take r so that

$$m_1 a L_1 r < \varepsilon, \tag{4.3}$$

then the increase of horizontal measure due to the operation $(*)$ is less than ε, where m_1 is the upper bounds on the total number of the operation $(*)$ all over each $\widehat{\gamma}$ and a is the constant as in Claim 4.4. Moreover, m_1 depends only on the topology of M. Thus, we set a domain R_0 by

$$R_0 = \bigcap_{\Phi \in \mathcal{N}} \left(\bigcup_{q \in \overline{S} - S} B_{|\Phi|}(q,r) \right)$$

and define γ_Φ as the resulting representative of $[\gamma]$. The constant δ and the domain R_1 in Proposition 4.3 is determined by δ'. □

Next, we describe the constant k_0. For $\rho \in \mathcal{T}_{g,n}$, we set

$$M' = M - R_1, \ M_\delta(\rho) = M' - \bigcup_{p \in Z(\Phi)} B_\sigma(p,\delta).$$

Furthermore, we define a function k_2 by

$$k_2(\rho) = \max_{p \in M_{\delta(\rho)}} \frac{\|(\frac{\partial}{\partial x})_p\|_\rho}{\|(\frac{\partial}{\partial x})_p\|_{|4\Phi(\rho)|}}.$$

Claim. The function k_2 is upper semicontinuous in $\mathcal{T}_{g,n}$.

Proof. This proof is similar to that in [Wol89, p.464]. □

Hence, defining a function $\kappa_A : \mathrm{SQD}(\sigma) \to \mathbb{R}_{\geq 0}$ for $A > 0$ by

$$\kappa_A(\Phi_0) := k_2(\rho(A\Phi_0)),$$

we find that κ_A is the upper semicontinuous function on $\mathrm{SQD}(\sigma)$. Since a upper semicontinuous function on a compact set has a maximum, we set

$$k_0(A,\varepsilon)=\max_{\Phi_0\in\mathrm{SQD}(\sigma)}\kappa_A(\Phi_0).$$

Then, by Proposition 3.7, for every $p\in M_\delta(\Phi_0)$,

$$\frac{\|(\frac{\partial}{\partial x})_p\|_{\rho(A\Phi_0)}}{\|(\frac{\partial}{\partial x})_p\|_{|4A\Phi_0|}}\searrow 1 \quad\text{as } A\to\infty.$$

Therefore $\kappa_A(\Phi_0)\searrow 1$ as $A\to\infty$. Since $\{\kappa_A\}$ converges uniformly on $\mathrm{SQD}(\sigma)$ by Dini's theorem, we find that $k_0\searrow 1$ as $A\to\infty$. Thus we have

$$\begin{aligned}\ell_\rho(\gamma^h_{\Phi_0(\rho)})&=\sum_{\gamma\subset\gamma^h_{\Phi_0}}\int_\gamma\left\|\frac{\partial}{\partial x}\right\|_\rho ds\\&\le\sup_{\gamma^h_{\Phi_0}}\frac{\|(\frac{\partial}{\partial x})_p\|_\rho}{\|(\frac{\partial}{\partial x})_p\|_{|4\Phi(\rho)|}}\sum\int_\gamma\left\|\frac{\partial}{\partial x}\right\|_{|4\Phi(\rho)|}ds\\&\le k_2(\rho)i(\beta\Phi(\rho),\gamma^h_{\Phi_0})\\&\le k_0(A,\varepsilon)i(\beta\Phi(\rho),\gamma^h_{\Phi_0}),\end{aligned}$$

where $\Phi_0(\rho)\coloneqq\Phi(\rho)/\|\Phi(\rho)\|$.

Finally, we describe the way of setting the constant $\eta=\eta([\gamma],\varepsilon,A)$. By Proposition 3.8, for any $\Phi_0\in\mathrm{SQD}(\sigma)$

$$\ell_{\rho(A\Phi_0)}(\gamma^v_{\Phi_0})A^{-1/2}\to 0 \quad\text{as } A\to\infty.$$

Since γ_{Φ_0} continuously changes with respect to Φ_0 and $\gamma^v_{\Phi_0}$ contains at most finite zeros of Φ_0, the total σ-length of $\gamma^v_{\Phi_0}$ is bounded from above. Therefore, we conclude that there exists a constant $\eta_1([\gamma],A)$ such that

$$\sum_{\gamma\subset\gamma^v_{\Phi_0(\rho)}}\ell_\rho(\gamma)<\eta_1 \text{ and } \eta_1A^{-1/2}\to 0 \quad\text{as } A\to\infty.$$

By Proposition 3.10 and the construction of the representative $\gamma_\Phi\in[\gamma]$ for $\Phi\in\mathrm{SQD}(\sigma)$, we can take constants C and D (see Remark 4.5) such that, for every $\Phi\in\mathrm{SQD}(\sigma)$,

$$\ell_{\rho(t\Phi)}(\widetilde{\gamma}^h_\Phi)<t^{1/2}i(\beta\Phi,\widetilde{\gamma}^h_\Phi)(1+Ce^{-D\sqrt{t}}). \tag{4.4}$$

Therefore, if we set

$$c_0=c_0([\gamma],\varepsilon)=\sup_{\Phi\in\mathrm{SQD}(\sigma)}i(\beta\Phi,\widetilde{\gamma}^h_\Phi)$$

and

$$\eta_2=\eta_2([\gamma],\varepsilon,A)=A^{1/2}c_0(1+Ce^{-D\sqrt{t}}),$$

then, for every $\Phi\in\mathrm{SQD}(\sigma)$,

$$\ell_{\rho(t\Phi)}(\widetilde{\gamma}^h_\Phi)<\eta_2 \text{ and } \eta_2A^{-1/2}\to c_0<\varepsilon$$

as $A \to \infty$. Hence, setting $\eta := \eta_1 + \eta_2$, we have $\eta A^{-1/2} \to c_0$ as $A \to \infty$. Thus, for every $\rho \in \rho(A \cdot \mathrm{SQD}(\sigma))$,

$$\begin{aligned}\ell_\rho([\gamma]) &\le \ell_\rho(\gamma_{\Phi_0(\rho)}) \\ &= \ell_\rho(\gamma^h_{\Phi_0(\rho)}) + \ell_\rho(\widetilde{\gamma}^h_{\Phi_0(\rho)}) + \ell_\rho(\gamma^v_{\Phi_0(\rho)}) \\ &< k_0 i(\beta\Phi(\rho), [\gamma]) + \eta,\end{aligned}$$

and the proof is complete. □

Remark 4.5. Here, we describe the setting of η_2, namely the constants C and D in (4.4). In the proof of Proposition 3.10, D is determined by the $|\Phi|$-distance from a horizontal arc to zeros of Φ. Now, for each $\Phi \in \mathrm{SQD}(\sigma)$, $\widetilde{\gamma}^h_\Phi$ is at least $r/2$ away from zeros of Φ in the $|\Phi|$ metric, so the constant D can be taken uniformly on $\mathrm{SQD}(\sigma)$. Moreover, we take the constant C as follows. We set $r' > 0$ so that, for every $\Phi \in \mathrm{SQD}(\sigma)$ and every $p \in Z(\Phi)$,

$$B_\sigma(p, r') \subset B_{|\Phi|}(p, r/2).$$

By the same manner as the upper semicontinuity of k_2, we conclude that the map

$$\Phi \mapsto \min_{p \in M_{r'}(\rho(\Phi))} |\nu(\Phi)|$$

is lower semicontinuous on $\mathrm{QD}(\sigma)$. Therefore, there exists a lower bound of $\min_{p \in M_{r'}(\rho(\Phi))} |\nu(\Phi)(p)|$ for $\Phi \in \mathrm{SQD}(\sigma)$.

Thus we find that $|\nu(\Phi)| \ge C_1 > 0$ and $\log |\nu(\Phi)|^{-1} \le C_2 < +\infty$ on $\widetilde{\gamma}^h_\Phi$ ($\subset M_{r'}(\rho(\Phi))$) for every $\Phi \in \mathrm{SQD}(\sigma)$. Therefore we obtain the desired constant C, since it is given by C_1 and C_2 as in the proof of Proposition 3.10.

4.2. **The extended homeomorphism.** The following lemma is an extension of [Wol89, Lemma 4.7] for punctured surfaces.

Proposition 4.6. *Let $\{\rho_i\} \subset \mathcal{T}_{g,n}$ be a sequence diverging to ∞ (i.e. it leaves every compact set in $\mathcal{T}_{g,n}$). Then $\{\pi \circ \ell(\rho_i)\}$ converges in $\mathcal{PMF}$ if and only if $\{\pi \circ \beta\Phi(\rho_i)\}$ converges in $\mathcal{PMF}$. Moreover, their limits in $\mathcal{PMF}$ coincide when they converge.*

Proof. We can take finitely many essential simple closed curves $[\gamma_1], \ldots, [\gamma_k] \in \mathcal{C}$ and a positive number $\delta > 0$ so that

$$\sum_{j=1}^{k} i(\beta\Phi_0, [\gamma_j]) > \delta > 0$$

holds for every $\Phi_0 \in \mathrm{SQD}(\sigma)$.

Suppose that $\pi \circ \ell(\rho_n)$ converges. Then, there exists a sequence $\{\lambda_n\} \subset \mathbb{R}_{>0}$ such that $\lambda_n \ell(\rho_n)$ converges in $\mathbb{R}^{\mathcal{C}}_{\ge 0}$. Therefore, there exists $B > 0$ such that, for each $n \in \mathbb{N}$,

$$\begin{aligned} B &> \sum_{j=1}^{k} \lambda_n \ell_{\rho_n}([\gamma_j]) \\ &> \lambda_n \sum_{j=1}^{k} i(\beta\Phi(\rho_n), [\gamma_j]) \quad \text{(By Theorem 4.1)} \\ &= \lambda_n \|\Phi(\rho_n)\|^{1/2} \sum_{j=1}^{k} i(\beta(\Phi(\rho_n)/\|\Phi(\rho_n)\|), [\gamma_j]) \\ &> (\lambda_n \|\Phi(\rho_n)\|^{1/2})\delta. \end{aligned} \tag{4.5}$$

Hence, we see that

$$\lambda_n \cdot \eta([\gamma], \varepsilon, \|\Phi(\rho_n)\|) < (B/\delta) \cdot \eta \|\Phi(\rho_n)\|^{-1/2}.$$

By Theorem 4.1,

$$\lim_{n\to\infty} \eta \|\Phi(\rho_n)\|^{-1/2} = c_0([\gamma], \varepsilon) < \varepsilon.$$

Moreover, for any $\varepsilon > 0$ and $[\gamma] \in \mathcal{C}$,

$$\begin{aligned} \lambda_n i(\beta\Phi(\rho_n), [\gamma]) &= \lambda_n \|\Phi(\rho_n)\|^{1/2} i(\beta(\Phi(\rho_n)/\|\Phi(\rho_n)\|), [\gamma]) \\ &< (B/\delta) \max_{\Phi_0 \in \mathrm{SQD}(\sigma)} i(\beta\Phi_0, [\gamma]). \end{aligned}$$

Thus by Theorem 4.1, for any $\varepsilon > 0$ and $[\gamma] \in \mathcal{C}$,

$$|\lambda_n \ell_{\rho_n}([\gamma]) - \lambda_n i(\beta\Phi(\rho_n), [\gamma])| < (k_0 - 1)\lambda_n i(\beta\Phi(\rho_n), [\gamma]) + \lambda_n \eta$$

and

$$(k_0 - 1)\lambda_n i(\beta\Phi(\rho_n), [\gamma]) \to 0$$

as $n \to \infty$. Therefore, for any $\varepsilon > 0$ and $[\gamma] \in \mathcal{C}$,

$$\lim_{n\to\infty} (\lambda_n \ell_{\rho_n}([\gamma]) - \lambda_n i(\beta\Phi(\rho_n), [\gamma])) < \varepsilon.$$

Since we can take arbitrarily small $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \lambda_n \ell(\rho_n) = \lim_{n\to\infty} \lambda_n \beta\Phi(\rho_n) \ \text{ in } \mathbb{R}^{\mathcal{C}}_{\geq 0}.$$

Thus we conclude that

$$\lim_{n\to\infty} \pi \circ \beta\Phi(\rho_n) = \lim_{n\to\infty} \pi \circ \ell(\rho_n).$$

We can show the converse in the same manner by starting with (4.5). □

Define a map $\psi : \overline{\mathcal{T}^{\mathrm{Th}}_{g,n}} \to \overline{\mathcal{T}^{\mathrm{harm}}_{g,n}}$ by

$$\psi(x) = \begin{cases} \left(\lim\limits_{n\to\infty} \dfrac{\Phi(x_n)}{\|\Phi(x_n)\|}, 1 \right) & (x \in \partial_{\mathrm{Th}} \mathcal{T}_{g,n},\ \{x_n\} \subset \mathcal{T}_{g,n} \text{ with } x_n \to x) \\ \left(\dfrac{\Phi(x)}{\|\Phi(x)\|}, \dfrac{4\|\Phi(x)\|}{1 + 4\|\Phi(x)\|} \right) & (x \in \mathcal{T}_{g,n}), \end{cases}$$

where we use the polar coordinates in $\overline{\mathrm{BQD}(\sigma)}$ (i.e. for $(r, \theta) \in \overline{\mathrm{BQD}(\sigma)}$, $\theta \in \mathrm{SQD}(\sigma)$ and $r \in [0, 1]$).

Our main theorem, Theorem 4.7, is shown from Proposition 4.6. The proof is similar to that of [Wol89, Theorem 4.1]. However we write the proof here for the sake of completeness.

Theorem 4.7. *The map ψ is a homeomorphism.*

Proof. We first show that ψ is well-defined on $\mathcal{PMF}$. For sequences $\{x_n\}$, $\{x'_n\}$ with $x_n, x'_n \to x \in \mathcal{PMF}$, there exist

$$\lim_{n\to\infty} \pi \circ \beta\Phi(x_n) \text{ and } \lim_{n\to\infty} \pi \circ \beta\Phi(x'_n),$$

and they coincide by Proposition 4.6. Therefore there exist sequences $\{\lambda_n\}$, $\{\lambda'_n\}$ such that

$$\lim_{n\to\infty} \lambda_n \beta\Phi(x_n) = \lim_{n\to\infty} \lambda'_n \beta\Phi(x'_n).$$

Since β is a homeomorphism,

$$\lim_{n\to\infty} \lambda_n^2 \Phi(x_n) = \lim_{n\to\infty} {\lambda'_n}^2 \Phi(x'_n).$$

Hence, we have

$$\lim_{n\to\infty} \frac{\Phi(x_n)}{\|\Phi(x_n)\|} = \lim_{n\to\infty} \frac{\lambda_n^2 \Phi(x_n)}{{\lambda_n}^2 \|\Phi(x_n)\|} = \lim_{n\to\infty} \frac{{\lambda'_n}^2 \Phi(x'_n)}{{\lambda'_n}^2 \|\Phi(x'_n)\|} = \lim_{n\to\infty} \frac{\Phi(x'_n)}{\|\Phi(x'_n)\|}.$$

Thus, ψ is well-defined.

We secondly show that ψ is continuous. In particular, we need to show ψ is continuous at a point $x \in \mathcal{PMF}$. Let $\{x_n\}$ be a sequence in $\mathcal{T}_{g,n}$ with $x_n \to x$. Then, since x_n diverges to ∞,

$$\frac{4\|\Phi(x_n)\|}{1 + 4\|\Phi(x_n)\|} \to 1 \quad \text{as} \quad n \to \infty.$$

Thus, we find that the second component of ψ is continuous. The first component is continuous by the definition of ψ.

We thirdly show that ψ is injective. The injectivity on $\mathcal{T}_{g,n}$ follows from the injectivity of Φ. Therefore, we need to show ψ is injective on $\mathcal{PMF}$. Suppose that $\psi(x) = \psi(x')$ for $x, x' \in \mathcal{PMF}$. Let x_n and x'_n be sequences in $\mathcal{T}_{g,n}$ such that $x_n \to x \in \mathcal{PMF}$ and $x'_n \to x' \in \mathcal{PMF}$ as $n \to \infty$. Since $\psi(x) = \psi(x')$, we have

$$\lim_{n\to\infty} \frac{\Phi(x_n)}{\|\Phi(x_n)\|} = \lim_{n\to\infty} \frac{\Phi(x'_n)}{\|\Phi(x'_n)\|}.$$

Therefore, we have

$$\lim_{n\to\infty} \pi \circ \beta\Phi(x_n) = \lim_{n\to\infty} \pi \circ \beta\Phi(x'_n).$$

By Proposition 4.6, we have

$$\lim_{n\to\infty} \pi \circ \ell(x_n) = \lim_{n\to\infty} \pi \circ \ell(x'_n).$$

This implies $x = x'$. Thus we find that ψ is injective on $\mathcal{PMF}$.

We next show that ψ is surjective. The restriction $\psi|_{\mathcal{T}_{g,n}}$ is surjective, since Φ is a homeomorphism, so we need to show the surjectivity for $\mathrm{SQD}(\sigma)$. Taking each $\theta \in \mathrm{SQD}(\sigma)$, we set $x_n = \Phi^{-1}(n\theta)$. Then $\pi \circ \beta\Phi(x_n) = \pi \circ \beta(n\theta) = [F_v(\theta)] \in \mathcal{PMF}$ for every n. Therefore, we have

$$\lim_{n\to\infty} \pi \circ \ell(x_n) = [F_v(\theta)]$$

by Proposition 4.6. Thus,

$$\psi([F_v(\theta)]) = \left(\lim_{n\to\infty} \frac{\Phi(x_n)}{\|\Phi(x_n)\|}, 1 \right) = (\theta, 1).$$

Thus, ψ is surjective.

We finally show that ψ^{-1} is continuous. Let $(\theta_n, r_n) \to (\theta, 1)$ as $n \to \infty$ for each $\theta \in \mathrm{SQD}(\sigma)$, where $\theta_n \in \mathrm{SQD}(\sigma)$ and $r_n \in (0,1)$. Then

$$\psi^{-1}(\theta_n, r_n) = \Phi^{-1}\left(\frac{r_n\theta_n}{4(1-r_n)}\right) \text{ and } \psi^{-1}(\theta, 1) = [F_v(\theta)].$$

Therefore, setting $x_n = \psi^{-1}(\theta_n, r_n)$, we have

$$\pi \circ \beta\Phi(x_n) = [F_v(\theta_n)].$$

Thus,

$$\psi^{-1}(\theta, 1) = [F_v(\theta)] = \lim_{n\to\infty}[F_v(\theta_n)] = \lim_{n\to\infty} \pi \circ \beta\Phi(x_n) = \lim_{n\to\infty} \psi^{-1}(\theta_n, r_n),$$

and this completes the proof. $\square$

Remark 4.8. This is another proof that the Thurston compactification $\overline{\mathcal{T}_{g,n}^{\mathrm{Th}}}$ is a closed ball of dimension $6g - 6 + 2n$.

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