

THE DYNAMICAL DEGREES OF RATIONAL SURFACE AUTOMORPHISMS

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ABSTRACT. The induced action on the Picard group of a rational surface automorphism with positive entropy can be identified with an element of the Coxeter group associated to $E_n, n \geq 10$ diagram. It follows that the set of dynamical degrees of rational surface automorphisms is a subset of the spectral radii of elements in the Coxeter group. This article concerns the realizability of an element of the Coxeter group as an automorphism on a rational surface with an irreducible reduced anticanonical curve. For any unrealizable element, we explicitly construct a realizable element with the same spectral radius. Hence, we show that the set of dynamical degrees and the set of spectral radii of the Coxeter group are, in fact, identical. This has been shown by Uehara [Ann. Inst. Fourier (Grenoble) 66 (2016), pp. 377–432] by explicitly constructing a rational surface automorphism. This construction depends on a decomposition of an element of the Coxeter group. Our proof is conceptual and provides a simple description of elements of the Coxeter group, which are realized by automorphisms on anticanonical rational surfaces.

1. INTRODUCTION

A compact complex surface S admitting an automorphism is rare. However, it is shown [2, 3, 17] that there are infinite families of rational surfaces admitting automorphisms. Several methods for constructing rational surfaces and studying the dynamics of automorphisms on them have been considered, and the dynamical properties of such automorphisms have been studied in several recent papers (for example, [4–6, 15]).

Suppose S is a rational surface obtained by blowing up a set P of n (possibly infinitely near) points in \mathbf{P}^2 . The Picard group $Pic(S)$ is generated by the class \mathbf{e}_0 of a strict transform of a generic line in \mathbf{P}^2 and the classes \mathbf{e}_i of total transforms of exceptional curves over points in P . There is a natural isomorphism ϕ between $\mathbb{Z}^{1,n}$ with a Minkowski inner product and a Picard group $Pic(S)$. By Nagata [18, 19], if $F : S \rightarrow S$ is an automorphism with infinite order, then there is an element ω of the Weyl group W_n generated by a set of reflections acting on $\mathbb{Z}^{1,n}$ such that the induced action $F_* : Pic(S) \rightarrow Pic(S)$ is equivalent to ω via the isomorphism ϕ , in

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other words, the following diagram commutes.

$$\begin{array}{ccc} \mathbb{Z}^{1,n} & \xrightarrow{\omega} & \mathbb{Z}^{1,n} \\ \phi \downarrow & & \downarrow \phi \\ \text{Pic}(S) & \xrightarrow{F_*} & \text{Pic}(S) \end{array}$$

In this case, we say F realizes $\omega \in W_n$. The generators of this Weyl group W_n are reflections through a set of vectors $\alpha_i \in \mathbb{Z}^{1,n}$, $i = 0, \dots, n-1$ such that

$$\alpha_0 = e_0 - e_1 - e_2 - e_3 \quad \text{and} \quad \alpha_i = e_i - e_{i+1}, \quad i = 1, \dots, n-1.$$

This Weyl group is isomorphic to the Coxeter group associated to the Coxeter diagram E_n . (See [12].)

If F is a rational surface automorphism, the dynamical degree $\delta(F)$ is given by

$$\delta(F) = \lim_{n \rightarrow \infty} (\text{algebraic degree of } F^n)^{1/n}.$$

Let Δ be the set of dynamical degrees of rational surface automorphisms and Λ be the set of spectral radii of Weyl group elements:

$$\begin{aligned} \Delta &= \{\delta(F) : F \text{ is a rational surface automorphism.}\} \quad \text{and} \\ \Lambda &= \{\lambda(\omega) : \omega \in W_n \text{ for some } n \geq 1\} \end{aligned}$$

where δ denotes the dynamical degree and λ denotes the spectral radius. According to Diller and Favre [8], the dynamical degree $\delta(F)$ of an automorphism F is given by the spectral radius of F_* . Thus, we have $\Delta \subset \Lambda$. In this article, we show $\Lambda = \Delta$.

Theorem A. *For any $\lambda \in \Lambda$, there is an element $\omega \in W_{n \geq 1}$ such that $\omega = F_*$ for a rational surface automorphism F with $\delta(F) = \lambda$. Thus we have*

$$\Delta = \Lambda.$$

This theorem was shown by Uehara [22] by constructing an automorphism with a given dynamical degree as a composition of quadratic birational maps. The main idea in [22] is identifying possible *orbit data* (See Section 7.3 for the definition.) for quadratic birational maps. The brief sketch of Uehara's construction is the following: Let κ be a reflection through α_0 , and κ_I be a reflection through $e_0 - e_i - e_j - e_k$ for $I \subset \{1, \dots, n\}$ with $|I| = 3$. Also, let Σ be a span of reflections through α_i , $i \geq 1$. With these notation, we see that for any $r \in \Sigma$, we have $r^{-1}\kappa r = \kappa_I$ for some $I \subset \{1, 2, \dots, n\}$. Thus for $\omega = r_0 \kappa r_1 \kappa r_2 \cdots \kappa r_m \in W_n$ with $r_i \in \Sigma$, $i = 1, \dots, m$ we have

$$\omega = r \kappa_{I_m} \cdots \kappa_{I_1},$$

where

$$r = r_0 r_1 \cdots r_m \in \Sigma, \quad \kappa_{I_j} = (r_j r_{j+1} \cdots r_m)^{-1} \kappa (r_j r_{j+1} \cdots r_m) \quad \text{for } j = 1, \dots, m$$

and $I_j = \{j_1, j_2, j_3\}$ is a set of three distinct integers in $\{1, 2, \dots, n\}$. By inspecting the above decomposition, for each j_i , one can obtain the smallest nonnegative integers t and s_k such that

$$\kappa_{I_{s-1}} \cdots \kappa_{I_1} \omega^t r \kappa_{I_m} \cdots \kappa_{I_{j+1}} e_{j_i} = e_{s_k}.$$

This allows to obtain *orbit data* (See Section 7.3 for the definition.) for a quadratic birational map f_j whose critical images correspond to $e_{j_1}, e_{j_2}, e_{j_3}$ for each $j = 1, \dots, m$. By assuming all f_j fix a given cusp cubic $C = \{\gamma(t)\}$, we have $e_{j_i} = \gamma(p_{j,i})$ and $f_j|_C \gamma(t) = \gamma(d_j t + c_j)$ for some $p_{j,j}, d_j, c_j \in \mathbb{C}$. Diller in [7] showed that the multiplier d_j and the parameters for three critical images uniquely determine (up to a linear conjugacy) a birational map properly fixing a cubic with arithmetic genus = 1. By repeatedly applying Diller's construction, Uehara obtained a system of equations of $p_{j,i}, d_j, c_j$ to get an explicit formula for the desired birational map $f = f_m \circ \dots \circ f_1$.

Uehara's construction required expressing $\omega \in W_n$ as a composition of generators of W_n . However, this is a difficult task. Furthermore, the construction depends on the orbit data, which is not uniquely determined by $\omega \in W_n$.

In this article, we generalize McMullen's construction in [17] by allowing infinitely near points in the base locus with the assumption of the existence of an anticanonical curve. The isomorphism $\phi : \mathbb{Z}^{1,n} \rightarrow \text{Pic}(S)$ together with the base points is called the *marked blowup* of S . McMullen [17] constructs a rational surface automorphism from a marked blowup with an anticanonical curve provided that all base points are distinct points in \mathbf{P}^2 . A brief sketch of McMullen's construction is as follows:

Suppose $\omega \in W_n$. If ϕ and $\phi \circ \omega$ give two markings for S , each marking gives a projection to \mathbf{P}^2 . Each marking would have a different base locus, and these two markings are associated with a birational map f , which does not necessarily lift to automorphism.

The most well-known example would be the case with $n = 3$, and ω is the Cremona involution κ , the reflection through the vector $e_0 - e_1 - e_2 - e_3$. Let (S, ϕ) be a marked blowups with $\phi(e_i) = \mathbf{e}_i$ where \mathbf{e}_0 is the class of the strict transform \tilde{H} of generic line H in \mathbf{P}^2 and \mathbf{e}_i is the class of the exceptional curve \mathcal{E}_i over a point $p_i \in \mathbf{P}^2$. Let $\pi : S \rightarrow \mathbf{P}^2$ be the corresponding blowup along three distinct points $\{p_1, p_2, p_3\} \in \mathbf{P}^2$. The marked blowup $(S, \phi \circ \kappa)$ with the projection $\pi' : S \rightarrow \mathbf{P}^2$ satisfies $\phi(\kappa e_i) = \mathbf{e}'_i$ and thus the class $\mathbf{e}'_i = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 + \mathbf{e}_i$, $i = 1, 2, 3$ is the class of exceptional curves over a point $q_i \in \mathbf{P}^2$.

$$\begin{array}{ccc} & S & \\ \pi \swarrow & & \searrow \pi' \\ \mathbf{P}^2 & \xrightarrow{f} & \mathbf{P}^2 \end{array}$$

Since $\omega e_0 = 2e_0 - e_1 - e_2 - e_3$, we have $\pi' \tilde{H} = Q$ where Q is a conic passing through p_1, p_2, p_3 while $\pi \tilde{H} = H$. Also, the exceptional curve \mathcal{E}_i over p_i satisfies $\pi(\mathcal{E}_i) = p_i$, and $\pi'(\mathcal{E}_i) = L'_{j,k}$ where $L'_{i,j}$ is the line joining q_i, q_j . The class of the strict transform of the line joining p_i, p_j , $L_{i,j}$ is given by $[L_{i,j}] = \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j = \mathbf{e}'_k$. Thus we see the birational map f is quadratic and satisfies

$$f : \pi(\mathcal{E}_i) = p_i \mapsto L'_{j,k}, \quad \text{and} \quad L_{i,j} \mapsto q_k \quad \{i, j, k\} = \{1, 2, 3\}.$$

In addition, if there is an automorphism g on \mathbf{P}^2 sending one base locus to another, by composition $f \circ g$ of an automorphism g and a birational map f , we can obtain an automorphism on S by lifting $f \circ g$.

$$\begin{array}{ccccc}
 & S & \xrightarrow{F} & S & \\
 \swarrow \pi' & & & & \searrow \pi' \\
 \mathbf{P}^2 & \xrightarrow{g} & \mathbf{P}^2 & \xrightarrow{f} & \mathbf{P}^2 \\
 & \swarrow \pi & & \swarrow \pi & \\
 & & & &
 \end{array}$$

To have a dynamical degree > 1 , as stated by McMullen [16], we need to blow up at least 10 points in \mathbf{P}^2 . Thus, if two sets of points P, P' with $|P| = |P'| \geq 10$ are in general position, an automorphism on \mathbf{P}^2 sending P to P' is special, or in fact, might not exist.

One can overcome this difficulty by imposing the condition of the existence of an anticanonical curve Y . If a point p in the base locus has no infinitely near points in the base locus, then p is in the strict transform of Y . Suppose $p^{(i)}, i = 1, \dots, k$ are in the base locus satisfying a point $p^{(i)}$ is an infinitely near point of $p^{(i-1)}$ of order 1 and no other points are infinitely near to $p^{(1)}$, then there is a successive blowups $\pi_i : S_i = \text{Bl}_{p^{(i)}} \rightarrow S_{i-1}$ of a single point $p^{(i)}$ with $S_0 = \mathbf{P}^2$. It follows that each strict transform of Y in S_i contains $p^{(i)}$. In either case, the set of base points in \mathbf{P}^2 belongs to the projection $X = \pi(Y)$. If X is a rational curve with arithmetic genus 1, it is easy to tell whether there is such an automorphism from P to P' .

We say $\omega \in W_n, n \geq 10$ is *essential* if ω is not conjugate into a proper subgroup of W_n generated by α_i 's. It is shown in [13] that there is a nonessential element $\omega \in W_{14}$ such that ω is not realizable by a rational surface automorphism. This is shown by counting periodic points using the Lefschetz fixed point formula. In fact, this ω is conjugate to the Coxeter element $\omega_{10} := s_0 \cdots s_{10} \in W_{10}$, and it is known [2, 17] that ω_{10} is realized by a rational surface automorphism. In this article, we let W_n^{ess} denote the set of essential elements in W_n and focus on realizing essential elements in $\cup_{n \geq 10} W_n^{ess}$.

Let $\mathbb{C}^{1,n} = \mathbb{Z}^{1,n} \otimes \mathbb{C}$ with the complex Minkowski form. For each $v \in \mathbb{C}^{1,n}$ we defined a *marked blowup* $(S^v, Y^v, \phi^v) = \text{Bl}(X, \rho^v)$ (See Section 6 for the explicit construction.) by blowing up points on a cuspidal cubic. Let us denote Φ_n^+ as the set of positive roots and $\Phi_n^- = -\Phi_n^+$ as the set of negative roots. Also, let Φ_v^+ be a set of nodal roots in (S^v, Y^v, ϕ^v) and let Σ_v be the span of all reflections through nodal roots:

$$\begin{aligned}
 \Sigma_v &= \text{Span}\{s_\alpha : \alpha \in \Phi_v^+\}, \\
 \text{where } s_\alpha(x) &:= x + (x \cdot \alpha)\alpha \quad \text{for } x \in \mathbb{Z}^{1,n}.
 \end{aligned}$$

Theorem B. *Let $v \in \mathbb{C}^{1,n}$ and let ω be an essential element in $W_{n \geq 10}$ with the spectral radius λ . Suppose v is a leading eigenvector for ω . Then we have:*

- (1) ω is realized by an automorphism on (S^v, Y^v, ϕ^v) if and only if $\omega^{-1}(\alpha) \in \Phi_n^+$ for all $\alpha \in \Phi_v^+$.
- (2) If ω is not realized by an automorphism on (S^v, Y^v, ϕ^v) , there is $s \in \Sigma_v$ such that $s \circ \omega$ is realized by a rational surface automorphism on (S^v, Y^v, ϕ^v) and we have $\lambda(s \circ \omega) = \lambda(\omega)$.

Also, via an example (see Theorems 8.4 and 8.5), we show that there is an essential element $\omega \in W_{n \geq 10}$ such that ω cannot be realized by any automorphisms on anticanonical rational surfaces. Then, using Theorem B, we find a realizable ω' with the same spectral radius. The method in Theorem B can be done in finitely many steps due to Corollary 3.6.

Theorem C. *There is an essential element $\omega \in \cup_{n \geq 10} W_n$ such that ω cannot be realized by an automorphism on an anticanonical rational surface.*

This article is organized as follows: Section 2 discusses Coxeter groups and their essential elements. In Sections 3–5, we generalize McMullen’s definitions [17] of marked blowups and marked cubics by allowing infinitely near points as base points. Also, we show that the set of nodal roots is finite, provided a unique anticanonical curve exists. In Section 6, we prove Theorem A and Theorem B. A couple of basic properties of quadratic birational maps and their orbit data are discussed in Section 7. In this section, we also briefly explain Diller’s construction in [7] and a few more cases omitted in [7]. In Section 8, using a specific example, we prove Theorem C and show how to use Theorem B.

2. COXETER GROUP

Let $\mathbb{Z}^{1,n}$ be a lattice of signature $(1, n)$ with the basis (e_0, \dots, e_n) such that

$$e_0 \cdot e_0 = 1, \quad e_i \cdot e_i = -1, \quad i = 1, \dots, n, \quad \text{and} \quad e_i \cdot e_j = 0, \quad i \neq j.$$

Using this basis as coordinates, the Minkowski inner product is given by

$$x \cdot x = x_0^2 - x_1^2 - x_2^2 - \dots - x_n^2 \quad \text{for } x = \sum x_i e_i.$$

The canonical vector is given by $\kappa_n = (-3, 1, 1, \dots, 1) \in \mathbb{Z}^{1,n}$. The subgroup $L_n = \kappa_n^\perp \subset \mathbb{Z}^{1,n}$ orthogonal to κ_n is generated by a set of vectors

$$\alpha_0 = e_0 - e_1 - e_2 - e_3 \quad \text{and} \quad \alpha_i = e_i - e_{i+1}, \quad i = 1, \dots, n-1.$$

The generating vectors $\alpha_i, i = 0, \dots, n-1$ satisfy

$$\kappa_n \cdot \alpha_i = 0 \quad \text{and} \quad \alpha_i \cdot \alpha_i = -2, \quad \text{for } 0 \leq i \leq n-1.$$

For each $i = 0, \dots, n-1$, we can define a reflection s_i on $\mathbb{Z}^{1,n}$:

$$s_i(x) = x + (x \cdot \alpha_i) \alpha_i.$$

We see that $s_i(\kappa_n) = \kappa_n$, $s_i(\alpha_i) = -\alpha_i$, and $s_i(\alpha_j) = \alpha_j$ for $j \neq i$. The group W_n generated by those reflections s_0, \dots, s_{n-1} fixes κ_n and preserves the Minkowski product, and we have

$$W_n = \langle s_0, s_1, \dots, s_{n-1} \mid (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where

$$m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } \{i, j\} = \{0, 3\} \text{ or } |i - j| = 1, \quad i, j \neq 0, \\ 2 & \text{otherwise.} \end{cases}$$

This group $W_n \subset O(L_n)$ is called the Coxeter group. Let V_n be a inner product space over \mathbb{R} with the basis $\{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$. Then each $s_i \in W_n$ gives the unique reflection in $O(V_n)$ preserving L_n . The root system $\Phi_n = \cup_i W_n \alpha_i$ is the W_n -orbits in V_n of generating vectors α_i 's.

2.1. Root system and simple roots. Each vector in $\alpha \in \Phi_n$ is called a root, and each generating vector α_i is called a simple root. Each $\alpha \in \Phi_n$ is a linear combination $\sum_i c_i \alpha_i$ with either $c_i \geq 0$ for all i or $c_i \leq 0$ for all i . We say $\alpha \in \Phi_n$ is positive if $\alpha = \sum_i c_i \alpha_i$ with $c_i \geq 0$ for all $i = 0, \dots, n-1$ and α is negative if $c_i \leq 0$ for all i . Let Φ_n^+ denote the set of all positive roots and Φ_n^- for the set of negative roots.

Let us list a couple of useful facts about root systems. (See [12] for details.)

- The group homomorphism $W_n \rightarrow GL(\mathbb{Z}^{1,n})$ is injective.
- $\Phi_n = \Phi_n^+ \cup \Phi_n^-$.
- For all α_i , $s_i(\Phi_n^+ \setminus \{\alpha_i\}) = \Phi_n^+ \setminus \{\alpha_i\}$.

2.2. Length function. For each $\omega \in W_n$, ω is a word in $s_i, i = 0, \dots, n-1$ and the length $\ell(\omega)$ is given by the minimum possible length of a word in s_i for ω . Note that $\omega \in W_n$ can have several different words of length $\ell(\omega)$. One useful fact of the length of ω is its relation to the number of positive roots sent to negative roots [12], that is, for $\omega \in W_n$, $\ell(\omega) = |\Pi(\omega)|$ where $\Pi(\omega)$ is the set of positive roots sent by ω to negative roots.

$$\ell(\omega) = |\{\alpha \in \Phi_n^+ : \omega(\alpha) \in \Phi_n^-\}|$$

2.3. Essential elements. Let $S = \{s_0, s_1, \dots, s_{n-1}\}$ be a set of generators of W_n .

Definition 2.1. We say $\omega \in W_n$ is *essential* if there is no subgroup $W' \subset W_n$ generated by a proper subset $S' \subset S$ such that

$$\omega W' \omega^{-1} = \omega^{-1} W' \omega = W'.$$

Let us denote W_n^{ess} the set of essential elements of W_n .

Thus if $\omega \in W_n$ is nonessential, then ω can be identified as an element of a smaller Coxeter group generated by S' . If $S' \subset \{s_1, s_2, \dots, s_{n-1}\}$, then W' is finite and its Coxeter graph is given by A_m for some $m \leq n$ [12, Chapter 2]. If $s_0 \in S'$, we may assume that $S' \subset \{s_0, s_1, \dots, s_{n-2}\}$. Then we can treat ω as an element of W_{n-1} . If $\omega \in W_n$ is generated by s_1, \dots, s_n , then the spectral radius $\lambda(\omega)$ of $\omega = 1$. Thus, from the definition, we see that:

Lemma 2.2. *If $\omega \in W_n$ with the spectral radius $\lambda(\omega) > 1$, then there is $n' \leq n$ such that ω is conjugate to an essential element $\omega' \in W_{n'}^{ess}$ and $\lambda(\omega) = \lambda(\omega')$.*

The essential elements of $W_n, n \geq 10$ and their properties were considered by Krammer [14] and Paris [20].

Proposition 2.3 ([14]). *Suppose a Coxeter group $W = W_n, n \geq 10$. Then*

- every essential element is of infinite order,
- ω is essential if and only if $\omega^m, m \neq 0$ is essential, and
- if ω is essential then $\langle \omega \rangle = \{\omega^m, m \in \mathbb{Z}\}$ is a finite index subgroup of the centralizer of ω in W .

It is known [16, 20] that the Coxeter element $\omega = s_0 s_1 \cdots s_{n-1} \in W_n$ is essential, and the product of s_0, \dots, s_{n-1} in any other is conjugate to ω . In fact, the Coxeter element has the smallest spectral radius among all essential elements in W_n .

Theorem 2.4 ([16, Theorem 1.2]). *Let $\omega \in W_n^{ess}$. Then the spectral radius $\lambda(\omega) \geq \lambda(s_0 s_1 \cdots s_{n-1})$.*

In this article, we will focus on essential elements.

Lemma 2.5. *If $\omega \in W_n^{ess}$, then for each $i \neq 0$, there is $k \in \mathbb{Z}$ such that $\omega^k(e_i) \cdot e_0 \neq 0$.*

Proof. Suppose there is e_i such that $\omega^k(e_i) \cdot e_0 = 0$ for all k . Then $\omega^k(e_i) = \sum_{j \neq 0} m_j e_j$ for some $m_j \in \mathbb{Z}$. Since ω^k preserves the bilinear form, we have

$$\omega^k(e_i) \cdot \omega^k(e_i) = - \sum_{j \neq 0} m_j^2 = -1.$$

It follows that $m_j = 0$ for all but one $1 \leq j \leq n$. Thus there is a subset $I \subset \{1, 2, \dots, n\}$ such that

$$\{\omega^k(e_i), k \in \mathbb{Z}\} = \{e_j, j \in I\}.$$

We may assume that $I = \{k+1, \dots, n\}$ for some $k \leq n-1$. Then there must be $m \in \mathbb{Z}$ such that $\omega^m e_j = e_j$ for all $j \in I$. Thus it is clear that $\omega^m W_k \omega^{-m} = W_k$, for some $k \leq n-1$ and ω^m is not essential. Using the second statement of Proposition 2.3, we see that ω is not essential. \square

2.4. Periodic roots. Suppose $\omega \in W_n$. Let us denote λ_ω , the spectral radius of ω .

Proposition 2.6. *Let $\omega \in O(V_n)$. If $v \in V_n$ is an eigenvector corresponding to an eigenvalue $\lambda \neq \pm 1$, then $v \cdot v = 0$.*

Proof. Since $\omega v = \lambda v$, we have

$$v \cdot v = \omega v \cdot \omega v = \lambda^2 (v \cdot v).$$

Since $\lambda^2 \neq 1$, we have $v \cdot v = 0$. \square

Suppose $\lambda_\omega > 1$ and let S be a subspace of V_n generated by eigenvectors whose eigenvalues are Galois conjugate of λ_ω . Then, any periodic roots are in the orthogonal complement S^\perp .

Proposition 2.7. *Suppose $\omega \in W_n$ and there exists $v \in V_n$ such that $\omega v = \lambda_\omega v$ with $\lambda_\omega > 1$. Then for all periodic roots $\alpha \in \Phi_n$, we have*

$$\alpha \cdot v = 0.$$

Conversely, if $\alpha \cdot v = 0$ then α is periodic.

Proof. Suppose α is a periodic root with period $p \geq 1$. Since ω preserves the inner product, we have

$$\alpha \cdot v = \omega(\alpha) \cdot \omega(v) = \lambda_\omega \omega(\alpha) \cdot v = \lambda_\omega^2 \omega^2(\alpha) \cdot v = \cdots = \lambda_\omega^p \omega^p(\alpha) \cdot v.$$

But since $\omega^p \alpha = \alpha$ and $\lambda_\omega > 1$ we have

$$\alpha \cdot v = \lambda_\omega^p \alpha \cdot v \Rightarrow \alpha \cdot v = 0.$$

The converse follows from Theorem 2.7 in [17]. \square

3. MARKED BLOWUPS

Let $\pi : S \rightarrow \mathbf{P}^2$ be a blowup of \mathbf{P}^2 along a finite set $P = \{p_1, \dots, p_n\}$ of (possibly infinitely near) points of \mathbf{P}^2 . Then the birational morphism π factors into a sequence of blowups of a point

$$\pi : S = S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \dots \xrightarrow{\pi_3} S_2 \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = \mathbf{P}^2,$$

where $\pi_i : S_i \rightarrow S_{i-1}$ is a blowup of a point $p_i \in S_{i-1}$.

Let \mathcal{E}_i be the total transformation of the exceptional curve over a point $p_i \in S_{i-1}$ and $\mathbf{e}_i = [\mathcal{E}_i]$ be its class in $Pic(S)$.

$$\mathcal{E}_i = (\pi_i \circ \dots \circ \pi_n)^* p_i \quad \text{and} \quad \mathbf{e}_i = [\mathcal{E}_i].$$

Then $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ together with $\mathbf{e}_0 = [H]$, the class of the pullback of a generic line in \mathbf{P}^2 , form a *geometric basis* for $Pic(S)$ equipped with inherited intersection pairing.

$$Pic(S) = \langle \mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n \rangle,$$

$$\text{where } \mathbf{e}_0 \cdot \mathbf{e}_0 = 1, \quad \mathbf{e}_i \cdot \mathbf{e}_i = -1, \quad i \geq 1, \quad \text{and} \quad \mathbf{e}_i \cdot \mathbf{e}_j = 0, \quad i \neq j.$$

Thus we can identify $Pic(S)$ to a lattice $\mathbb{Z}^{1,n}$ with the Minkowski product. And we can consider the Weyl group W_n as a subgroup of $GL(Pic(S))$.

3.1. Marked blowup. A marked blowup was defined in [17]. A marked blowup in [17] only allows blowing up a set of distinct points in \mathbf{P}^2 . Let us define a slightly modified version to allow blowing up infinitely near points.

Definition 3.1. A marked blowup (S, ϕ) is a smooth projective surface S equipped with an isomorphism $\phi : \mathbb{Z}^{1,n} \rightarrow Pic(S)$ such that

- (1) the marking ϕ sends the Minkowski inner product to the intersection pairing,
- (2) there exists a set P of n (possibly infinitely near) points p_1, \dots, p_n such that p_j is infinitely near to p_i implies $j > i$, and each p_i has at most one infinitely near point of order 1,
- (3) there exists a sequence of blowups $\pi : S \rightarrow \mathbf{P}^2$ along a set P , and
- (4) the marking ϕ satisfies $\phi(\mathbf{e}_0) = [H]$ and $\phi(\mathbf{e}_i) = [\mathcal{E}_i]$ for $i = 1, \dots, n$, where H is the pullback of a generic line in \mathbf{P}^2 and $\mathcal{E}_i \subset S$ is the total transform of the exceptional curve over p_i .

Notice that the set P of base points can be written as

$$\{p_1, \dots, p_n\} = \{q_i^{(j)}, 1 \leq j \leq h_i, 1 \leq i \leq k\},$$

where $\sum_i h_i = n$, $q_i^{(j+1)}$ is infinitely near of order one to $q_i^{(j)}$, and $q_1^{(1)}, q_2^{(1)}, \dots, q_k^{(1)}$ are distinct points in \mathbb{P}^2 . We call h_i the *height* of a point $q_i^{(1)}$ in \mathbb{P}^2 . Thus each \mathcal{E}_i is either an irreducible (-1) curve if there is no infinitely near point to p_i or a tower of $h-1$ (-2) curves with (-1) curve at the top if p_i has an infinitely near point of order j for $j = 1, \dots, h-1$. The canonical class of S is $\kappa_S = -3\mathbf{e}_0 + \sum_{i \geq 1} \mathbf{e}_i = \phi(\kappa_n)$ where κ_n is the canonical vector in $\mathbb{Z}^{1,n}$. Thus the set of elements in $Pic(S)$ orthogonal to κ_S corresponds to the root system Φ_n in $\mathbb{Z}^{1,n}$.

3.2. Effective irreducible and reduced anticanonical divisor. Suppose S has an effective irreducible and reduced anticanonical divisor. Let C be a reduced irreducible curve such that $[C] = -\kappa_S$ and suppose

$$P = \{p_1, \dots, p_n\} = \{q_i^{(j)}, 1 \leq j \leq h_i, 1 \leq i \leq k\}.$$

Then $C_i := (\pi_i \circ \dots \circ \pi_n)C$ in S_{i-1} contains p_i for each $1 \leq i \leq n$. It follows that:

Lemma 3.2. *Suppose S has an effective, irreducible, and reduced anticanonical divisor and suppose (S, ϕ) is a marked blowup. Let C be a reduced irreducible curve such that $[C] = -\kappa_S$. If p_j is an infinitely near point to p_i of order 1, then $j > i$ and $p_j = C_j \cap \mathcal{F}(p_i)$ where $\mathcal{F}(p_i)$ is the strict transform of the exceptional curve over a point p_i in S_{j-1} .*

Also, we have:

Lemma 3.3. *Suppose S has an effective, irreducible, and reduced anticanonical divisor and suppose (S, ϕ) is a marked blowup. Let C be a reduced irreducible curve such that $[C] = -\kappa_S$. The base locus P is completely determined by a set of pairs (or base points with heights) $\{(q_i, h_i), 1 \leq i \leq k\} \subset \pi C \times \mathbb{N}$ where $q_i = q_i^{(1)}$ with height h_i .*

Proof. Let C be the anticanonical curve. By Lemma 3.2, we have that $q_i^{(j)}$ is the unique intersection of the strict transform of πC and the strict transform of the exceptional curve over $q_i^{(j-1)}$ for all $j = 2, \dots, h_i$ and $i = 1, \dots, k$. \square

Notice that all the base points with heights for a marked blowup belong to some strict transform of $\pi(C)$. Thus we have:

Lemma 3.4. *Suppose S has an effective, irreducible, and reduced anticanonical divisor and suppose (S, ϕ) is a marked blowup. Let C be a reduced irreducible curve such that $[C] = -\kappa_S$. If $X \subset S$ is a smooth rational curve of degree d and the intersection $\pi(X) \cap \pi(C)$ consists of $m_i \cdot q_i^{(1)}$ and $n_j \cdot x_j$ (counted with multiplicity) where $x_j \notin P$, then*

$$(3.1) \quad X \cdot X = d^2 - \sum_i \min\{m_i, h_i\} \geq d(d-3).$$

Proof. Since x_j 's are not in the base locus P , x_j would not change the self-intersection number. Also, the total number of the intersection of πX and πC is $3d$, we have $\sum_i m_i \leq 3d$. Since X is a smooth curve in S , the singularities (if there is any) of $\pi(X)$ are from the transversal intersections.

Now let us focus on $q_i^{(1)} \in P$. For simplicity, let us assume that $p_i = q_1^{(i)}$ for $i = 1, \dots, h_1$ and thus p_i is the intersection of exceptional curve \mathcal{F}_i over p_{i-1} and the strict transform of $\pi(C)$ in S_{i-1} . Since computation for any base points is identical, we may assume that $m_j = 0$ for all $j \geq 2$. Suppose $m_1 = 1$. In S_1 , the strict transform of πX intersects \mathcal{F}_1 but is disjoint from $(\pi_2 \circ \dots \circ \pi_n)C$. Thus the strict transform of X in S_i doesn't contain p_{i+1} for all $i \geq 2$. It follows that $X \cdot X = d^2 - 1$. If $m_1 = 2$, because only singular points $\pi X \in \mathbf{P}^2$ are transversal self-intersections, there are only two possibilities: (1) the strict transform of πX in S_1 is disjoint from $(\pi_2 \circ \dots \circ \pi_n)C$ or (2) the strict transform of πX in S_1 intersects $(\pi_2 \circ \dots \circ \pi_n)C$ at p_2 . In either case, the strict transform of πX in S_2 is disjoint from $(\pi_3 \circ \dots \circ \pi_n)C$. Continuing this procedure, we see that for $1 \leq m_1 \leq h_1$,

we have $X \cdot X = d^2 - m_1$. If $m_1 = h_1 + s$ for some $s > 0$, then $X \cdot X = d^2 - h_1$ since there are no more blowups over an infinitely near point of p_{h_1} . Thus we have $X \cdot X = d^2 - \min\{m_1, h_1\}$. Repeating this for $j > 1$ if necessary, we have $X \cdot X = d^2 - \sum_i \min\{m_i, h_i\}$. Combining the fact that $\sum_i m_i \leq 3d$, we have the inequality in (3.1). \square

3.3. Nodal roots. We say $\alpha \in \Phi_n^+$ is a *nodal root* if $\alpha \in \Phi_n^+$ and $\phi(\alpha)$ is effective and irreducible. Let

$$\Phi(\phi)^+ := \{\alpha \in \Phi_n^+; \phi(\alpha) \text{ is effective}\}$$

be the set of nodal roots. There are two possibilities. If $\alpha \in \Phi(\phi)^+$, then either

- $\alpha = e_i - e_j$ with $i < j$ and p_j is infinitely near to p_i of the first order, or
- $\phi(\alpha) = [D]$, where D projects to a smooth rational curve in S . In this case, we say α is a geometric nodal root.

Proposition 3.5. *Suppose S has an effective, irreducible, and reduced anticanonical divisor and suppose (S, ϕ) is a marked blowup. Let C be a reduced irreducible curve such that $[C] = -\kappa_S$. If α is a geometric nodal root with $\phi(\alpha) = [D]$, then either $\deg \pi D = 1$ or $\deg \pi D = 2$.*

Proof. Let $d = \deg \pi D$. By Lemma 3.4, we see that $D \cdot D \geq d(d-3)$. Since α is a nodal root, $D \cdot D = -2$. Thus only possibilities are $d = 1$ or $d = 2$. \square

Corollary 3.6. *Suppose S has an effective, irreducible, and reduced anticanonical divisor and suppose (S, ϕ) is a marked blowup. Then there are, at most, finitely many nodal roots.*

Proof. Since there are only finitely many blowups, the set

$$\mathcal{A} = \{e_i - e_j, i < j\} \cup \{de_0 - \sum m_i e_i : d = 1, 2, \sum m_i \leq 3d, m_i \geq 0\}$$

is finite. By Proposition 3.5, we see all nodal roots are in \mathcal{A} . \square

By Dolgachev and Ortland [9, Proposition 4], we see that a geometric nodal root α is given by

$$\alpha = e_0 - (e_{i_1} + e_{i_2} + e_{i_3}) \quad \text{or} \quad \alpha = 2e_0 - (e_{i_1} + \cdots + e_{i_6}) \quad i_j \neq i_k \text{ if } j \neq k.$$

Lemma 3.7. *If $\alpha \neq \beta$ are two nodal roots, then either $\alpha \cdot \beta = 0$ or $\alpha \cdot \beta = 1$.*

Proof. The statement is obvious if $\alpha = e_i - e_j$.

Now suppose $\phi(\alpha) = D_\alpha, \phi(\beta) = D_\beta$. If both D_α and D_β are lines, $\alpha \cdot \beta = 1$ or 0 because two points determine a line.

Suppose $\alpha = e_0 - (e_{i_1} + e_{i_2} + e_{i_3})$ and $\beta = 2e_0 - (e_{j_1} + \cdots + e_{j_6})$. If $\{i_1, i_2, i_3\} \cap \{j_1, \dots, j_6\} = \emptyset$. Since a cubic curve is determined by 9 points, we have that $C = D_\alpha + D_\beta$ is reducible. Thus we have $\alpha \cdot \beta = 1$ or 0.

Lastly suppose $\alpha = 2e_0 - (e_{i_1} + \cdots + e_{i_6})$ and $\beta = 2e_0 - (e_{j_1} + \cdots + e_{j_6})$. If $|\{i_1, \dots, i_6\} \cap \{j_1, \dots, j_6\}| \leq 3$ then we may assume $\{i_1, \dots, i_6\} = \{1, \dots, 6\}$ and $\{j_1, \dots, j_6\} = \{7, \dots, 9, j_1, j_2, j_3\}$ where $j_1, j_2, j_3 \in \{1, \dots, n\}$. Let us consider two degree 4 curves such that $Q_\alpha = D_\alpha + Q'_\alpha$ and $Q_\beta = D_\beta + Q'_\beta$ where Q'_α is a conic passing through p_4, \dots, p_7 and p_{10} and Q'_β is a conic passing through p_6, \dots, p_{10} where p_i 's are the base points corresponding to e_i . By [21], there exists a cubic passing through p_1, \dots, p_{10} if and only if six of $\{p_1, \dots, p_{10}\}$ are in $Q_\alpha \cap Q_\beta$. However $|Q_\alpha \cap Q_\beta| \leq 5$. It follows that $\alpha \cdot \beta = 0$ because $\alpha \neq \beta$. \square

We say two nodal roots α, β are *linked* if either $\alpha \cdot \beta = 1$ or there exists a finite set of nodal roots $\{\alpha_i, i = 1, \dots, k\}$ for some $k \in \mathbb{N}$ such that

$$\alpha \cdot \alpha_1 = \alpha_i, \alpha_{i+1} = \alpha_k \cdot \beta = 1.$$

Let us denote $\mathcal{N}^+(\alpha)$ the set of all nodal roots linked to α and let $\ell^+(\alpha) = |\mathcal{N}^+(\alpha)|$, the number of linked nodal roots linked to α including α . Since any root α satisfies $\alpha \cdot \kappa_n = 0$, it forms a Hirzebruch-Jung string. In fact it is known [1, Section III.2] that the intersection form on $\mathcal{N}^+(\alpha)$ is given by Dynkin diagram A_n, D_n, E_6, E_7 or E_8 with $n \geq 1$.

Also let $\Sigma(\alpha)$ be the span of reflections in $\mathcal{N}^+(\alpha)$:

$$\Sigma(\alpha) = \text{Span}\{s_\beta : \beta \in \mathcal{N}^+(\alpha)\}.$$

Lemma 3.8. *Suppose α is a nodal root such that $\alpha \cdot \beta = 0$ for all nodal roots $\beta \neq \alpha$. Let $s = s_\alpha$ be the reflection through α . Then*

$$s(\alpha) = -\alpha, \quad s(\beta) = \beta \quad \text{for all } \beta \in \Phi^+(\phi) \setminus \{\alpha\}.$$

Proof. This is trivial since $\alpha \cdot \beta = 0$. □

Lemma 3.9. *Let $\omega \in W_n$. Suppose α is a nodal root with $\ell^+(\alpha) = 2$ satisfying $\omega(\alpha) \in \Phi_n^-$. Then there is $s \in \Sigma(\alpha)$ such that*

$$\omega \circ s(\alpha) \in \Phi_n^+ \quad \text{and}$$

either $\omega \circ s(\beta), \omega(\beta) \in \Phi_n^+$ or $\omega \circ s(\beta), \omega(\beta) \in \Phi_n^-$ for all $\beta \in \Phi^+(\phi) \setminus \{\alpha\}$.

Proof. Let $\mathcal{N}^+(\alpha) = \{\alpha_1, \alpha_2\}$. We may assume $\alpha = \alpha_1$. Let s_i be the reflection through α_i . Note that $s_1(\alpha_2) = \alpha_1 + \alpha_2$. Thus $\alpha_1 + \alpha_2 \in \Phi_n$. Since all roots not in $\mathcal{N}^+(\alpha)$ are orthogonal to α_1 , we only need to check for α_1, α_2 .

- If $\omega(\alpha_2) \in \Phi_n^+$ and $\omega(\alpha_1 + \alpha_2) \in \Phi_n^-$, then let $s = s_1 \circ s_2$. We have

$$s(\alpha_1) = \alpha_2, \quad s(\alpha_2) = -(\alpha_1 + \alpha_2) \quad \text{and}$$

$$\omega \circ s(\alpha_1) = \omega(\alpha_2) \in \Phi_n^+, \quad \omega \circ s(\alpha_2) = -\omega(\alpha_1 + \alpha_2) \in \Phi_n^+.$$

- If $\omega(\alpha_2) \in \Phi_n^+$ and $\omega(\alpha_1 + \alpha_2) \in \Phi_n^+$, then let $s = s_1$. We have

$$s(\alpha_1) = -\alpha_1, \quad s(\alpha_2) = \alpha_1 + \alpha_2 \quad \text{and}$$

$$\omega \circ s(\alpha_1) = -\omega(\alpha_1) \in \Phi_n^+, \quad \omega \circ s(\alpha_2) = \omega(\alpha_1 + \alpha_2) \in \Phi_n^+.$$

- If $\omega(\alpha_2) \in \Phi_n^-$, then let $s = s_1$. We have

$$s(\alpha_1) = -\alpha_1, \quad s(\alpha_2) = (\alpha_1 + \alpha_2) \quad \text{and}$$

$$\omega \circ s(\alpha_1) = -\omega(\alpha_1) \in \Phi_n^+, \quad \omega \circ s(\alpha_2) = \omega(\alpha_1 + \alpha_2) = \omega(\alpha_1) + \omega(\alpha_2) \in \Phi_n^-.$$

We covered all possibilities. □

Notice that when $\ell^+(s)$ increases from 1 to 2, s has to be changed by multiplying s_2 on the right if two conditions ($\omega(\alpha_2) \in \Phi_n^+$ and $\omega(\alpha_1 + \alpha_2) \in \Phi_n^-$) hold.

Lemma 3.10. *Let $\omega \in W_n$. Suppose α is a nodal root with $\ell^+(\alpha) = 3$ satisfying $\omega(\alpha) \in \Phi_n^-$. Then there is $s \in \Sigma(\alpha)$ such that*

$$\omega \circ s(\alpha) \in \Phi_n^+ \quad \text{and}$$

either $\omega \circ s(\beta), \omega(\beta) \in \Phi_n^+$ or $\omega \circ s(\beta), \omega(\beta) \in \Phi_n^-$ for all $\beta \in \Phi^+(\phi) \setminus \{\alpha\}$.

Proof. Let $\mathcal{N}^+(\alpha) = \{\alpha_j, j = 1, 2, 3\}$. Notice that $s_1(\alpha_2) = \alpha_1 + \alpha_2$ and $s_2(s_1(\alpha_3)) = \alpha_1 + \alpha_2 + \alpha_3$. Thus, using other indexes, we see that sums of distinct α_i 's are in Φ_n . We have two cases:

Case 1 ($\alpha = \alpha_1$). There are three possibilities:

- If $\omega(\alpha_2) \in \Phi_n^+$, $\omega(\alpha_3) \in \Phi_n^+$, $\omega(\alpha_1 + \alpha_2) \in \Phi_n^-$, and $\omega(\alpha_1 + \alpha_2 + \alpha_3) \in \Phi_n^-$, then let $s = s_1 \circ s_2 \circ s_3$. We have

$$\begin{aligned} s(\alpha_1) &= \alpha_2, s(\alpha_2) = \alpha_3, s(\alpha_3) = -(\alpha_1 + \alpha_2 + \alpha_3) \quad \text{and} \\ \omega \circ s(\alpha_1) &= \omega(\alpha_2) \in \Phi_n^+, \omega \circ s(\alpha_2) = \omega(\alpha_3) \\ &\in \Phi_n^+, \omega \circ s(\alpha_3) = -\omega(\alpha_1 + \alpha_2 + \alpha_3) \\ &\in \Phi_n^+. \end{aligned}$$

Since it is not hard to check, we will give only the formula for s for the remaining cases. Also, for all cases, the possibilities are listed such that each condition assumes that the conditions of the above possibilities have failed.

- If $\omega(\alpha_2) \in \Phi_n^+$, $\omega(\alpha_3) \in \Phi_n^-$, and $\omega(\alpha_1 + \alpha_2) \in \Phi_n^-$, then $s = s_1 \circ s_2$.
- All other cases: We have $s = s_1$.

Case 2 ($\alpha = \alpha_2$). There are five possibilities:

- If $\omega(\alpha_1), \omega(\alpha_3) \in \Phi_n^+$ and $\omega(\alpha_1 + \alpha_2 + \alpha_3) \in \Phi_n^-$, then

$$s = s_2 \circ s_1 \circ s_3 \circ s_2.$$

- If $\omega(\alpha_1), \omega(\alpha_3) \in \Phi_n^+$, $\omega(\alpha_1 + \alpha_2), \omega(\alpha_2 + \alpha_3) \in \Phi_n^-$, and $\omega(\alpha_1 + \alpha_2 + \alpha_3) \in \Phi_n^+$, then

$$s = s_2 \circ s_3 \circ s_1.$$

- If $\omega(\alpha_3) \in \Phi_n^+$ and $\omega(\alpha_2 + \alpha_3) \in \Phi_n^-$, but not in the above two cases, then

$$s = s_2 \circ s_3.$$

- If $\omega(\alpha_1) \in \Phi_n^+$ and $\omega(\alpha_1 + \alpha_2) \in \Phi_n^-$, but not in the above three cases, then

$$s = s_2 \circ s_1.$$

- All other cases: We have $s = s_2$.

□

Lemma 3.11. *Let $\omega \in W_n$. Suppose α is a nodal root with $\ell^+(\alpha) = 4$ such that the intersection form of $\mathcal{N}^+(\alpha)$ is given by D_4 . Suppose $\omega(\alpha) \in \Phi_n^-$. Then there is $s \in \Sigma(\alpha)$ such that*

$$\omega \circ s(\alpha) \in \Phi_n^+ \quad \text{and}$$

either $\omega \circ s(\beta), \omega(\beta) \in \Phi_n^+$ or $\omega \circ s(\beta), \omega(\beta) \in \Phi_n^-$ for all $\beta \in \Phi^+(\phi) \setminus \{\alpha\}$.

Proof. Let $\mathcal{N}^+(\alpha) = \{\alpha_i, i = 1, 2, 3, 4\}$ and $\alpha_3 \cdot \alpha_i = 1$ for all $i \neq 3$. Due to the symmetry, we have two cases:

Case 1 ($\alpha = \alpha_1$). There are six possibilities:

- If $\omega(\alpha_i) \in \Phi_n^+$ for $i \neq 1$ and $\omega(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \in \Phi_n^-$, then

$$s = s_1 \circ s_3 \circ s_2 \circ s_4 \circ s_1 \circ s_3.$$

- If $\omega(\alpha_i) \in \Phi_n^+$ for $i \neq 1$ and $\omega(\alpha_1 + \alpha_2 + \alpha_3), \omega(\alpha_1 + \alpha_3 + \alpha_4) \in \Phi_n^-$, then

$$s = s_1 \circ s_3 \circ s_2 \circ s_4.$$

- If $\omega(\alpha_i) \in \Phi_n^+$ for $i = 3, 4$ and $\omega(\alpha_1 + \alpha_3 + \alpha_4) \in \Phi_n^-$, then

$$s = s_1 \circ s_3 \circ s_4.$$

- If $\omega(\alpha_i) \in \Phi_n^+$ for $i = 2, 4$ and $\omega(\alpha_1 + \alpha_2 + \alpha_4) \in \Phi_n^-$, then

$$s = s_1 \circ s_4 \circ s_2.$$

- If $\omega(\alpha_3) \in \Phi_n^+$ and $\omega(\alpha_1 + \alpha_3) \in \Phi_n^-$, then

$$s = s_1 \circ s_3.$$

- All other cases: We have $s = s_1$.

Case 2 ($\alpha = \alpha_3$). There are twelve possibilities:

- If $\omega(\alpha_i) \in \Phi_n^+$ for $i \neq 3$ and $\omega(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4) \in \Phi_n^-$, then

$$s = s_3 \circ s_2 \circ s_1 \circ s_4 \circ s_3.$$

- If $\omega(\alpha_i) \in \Phi_n^+$ and $\omega(\alpha_i + \alpha_3) \in \Phi_n^-$ for $i \neq 3$, then

$$s = s_1 \circ s_3 \circ s_2 \circ s_1 \circ s_4.$$

- If $\omega(\alpha_i) \in \Phi_n^+$ for $i = 1, 2$ and $\omega(\alpha_1 + \alpha_2 + \alpha_3) \in \Phi_n^-$, then

$$s = s_3 \circ s_2 \circ s_1 \circ s_3.$$

- If $\omega(\alpha_i) \in \Phi_n^+$ for $i = 1, 4$ and $\omega(\alpha_1 + \alpha_4 + \alpha_3) \in \Phi_n^-$, then

$$s = s_3 \circ s_1 \circ s_4 \circ s_3.$$

- If $\omega(\alpha_i) \in \Phi_n^+$ for $i = 2, 4$ and $\omega(\alpha_2 + \alpha_4 + \alpha_3) \in \Phi_n^-$, then

$$s = s_3 \circ s_2 \circ s_4 \circ s_3.$$

- If $\omega(\alpha_i) \in \Phi_n^+$ for $i = 1, 2$ and $\omega(\alpha_1 + \alpha_3), \omega(\alpha_2 + \alpha_3) \in \Phi_n^-$, then

$$s = s_3 \circ s_2 \circ s_1.$$

- If $\omega(\alpha_i) \in \Phi_n^+$ for $i = 1, 4$ and $\omega(\alpha_1 + \alpha_3), \omega(\alpha_4 + \alpha_3) \in \Phi_n^-$, then

$$s = s_3 \circ s_4 \circ s_1.$$

- If $\omega(\alpha_i) \in \Phi_n^+$ for $i = 2, 4$ and $\omega(\alpha_2 + \alpha_3), \omega(\alpha_4 + \alpha_3) \in \Phi_n^-$, then

$$s = s_3 \circ s_2 \circ s_4.$$

- If $\omega(\alpha_1) \in \Phi_n^+$ and $\omega(\alpha_1 + \alpha_3) \in \Phi_n^-$, then

$$s = s_3 \circ s_1.$$

- If $\omega(\alpha_2) \in \Phi_n^+$ and $\omega(\alpha_2 + \alpha_3) \in \Phi_n^-$, then

$$s = s_3 \circ s_2.$$

- If $\omega(\alpha_4) \in \Phi_n^+$ and $\omega(\alpha_4 + \alpha_3) \in \Phi_n^-$, then

$$s = s_3 \circ s_4.$$

- All other cases: We have $s = s_1$.

□

Lemma 3.12. *Let $\omega \in W_n$. Suppose α is a nodal root with $\ell^+(\alpha) = k \geq 4$ satisfying $\omega(\alpha) \in \Phi_n^-$. Then there is $s \in \Sigma(\alpha)$ such that*

$$\omega \circ s(\alpha) \in \Phi_n^+ \quad \text{and}$$

either $\omega \circ s(\beta), \omega(\beta) \in \Phi_n^+$ or $\omega \circ s(\beta), \omega(\beta) \in \Phi_n^-$ for all $\beta \in \Phi^+(\phi) \setminus \{\alpha\}$.

Proof. Let $\mathcal{N}^+(\alpha) = \{\alpha_j, j = 1, \dots, k\}$. In this case, the intersection form of $\mathcal{N}^+(\alpha)$ is given by the Dynkin diagram extending one arm of A_3 in Lemma 3.10 or D_4 in Lemma 3.11. We use induction on $\ell^+(\alpha)$. Suppose the statement is true for $\ell^+(\alpha) = k - 1$. Assume $\ell^+(\alpha) = k$. Since $\alpha_i \cdot \alpha_{i \pm 2} = 0$, we use the same s in $\ell^+(\alpha) = k$ by changing indexes if necessary. If there are three arms in the Dynkin diagram, then one of the arms has to have a length of 2. Because of symmetry, it is sufficient to consider the case $\alpha = \alpha_1$ or $\alpha = \alpha_2$. Let s' be the composition of reflections in case $\mathcal{N}^+(\alpha) = \{\alpha_j, j = 1, \dots, k - 1\}$.

Case 1 ($\alpha = \alpha_1$).

- If s_{k-1} is in s' , $\omega(\alpha_k) \in \Phi_n^+$, and $\omega(\sum_{i=1}^k \alpha_i) \in \Phi_n^-$, then

$$s = s' \circ s_k.$$

- Otherwise $s = s'$.

Case 2 ($\alpha = \alpha_2$).

- If s_{k-1} is in s' , $\omega(\alpha_j) \in \Phi_n^+$ for all $j \neq 2$, and $\omega(\sum_{i=1}^k \alpha_i) \in \Phi_n^-$, then

$$s = s' \circ s_k \circ s_{k-1}.$$

- If s_{k-1} is in s' , $\omega(\alpha_k) \in \Phi_n^+$, and $\omega(\sum_{i=2}^j \alpha_i) \in \Phi_n^-$ but not in the above, then

$$s = s' \circ s_k.$$

- Otherwise $s = s'$.

□

For a maked blowup (S, ϕ) , let Σ_ϕ be the span of all reflections through nodal roots:

$$\Sigma_\phi = \text{Span}\{s_\alpha : \alpha \in \Phi^+(\phi)\}.$$

Proposition 3.13. *Suppose S has an effective, irreducible, and reduced anticanonical divisor and suppose (S, ϕ) is a marked blowup. Let C be a reduced irreducible curve such that $[C] = -\kappa_S$ and $\pi(C)$ is a cubic curve with one singularity. Let $\omega \in W_n$. Then there is $s \in \Sigma_\phi$ such that $\omega \circ s(\alpha) \in \Phi_n^+$ for all nodal roots α .*

Proof. By Lemmas 3.8–3.12, there is $s \in \Sigma_\phi$ such that $\omega \circ s$ changes the positivity of exactly one nodal root. Since there are only finitely many nodal roots, by changing one at a time, we can map all nodal roots to positive roots. □

3.4. Isomorphisms. Let (S, ϕ) and (S', ϕ') be two marked blowups. If there is a biholomorphism $F : S \rightarrow S'$ such that the induced map $F_* : \text{Pic}(S) \rightarrow \text{Pic}(S')$ satisfies $F_* \circ \phi = \phi'$, then we say (S, ϕ) and (S', ϕ') are *isomorphic*, $(S, \phi) \cong (S', \phi')$. It is known [9, 19] that if (S, ϕ) and $(S, \phi \circ \omega)$ are marked blowups, then $\omega \in W_n$.

Let

$$W(S, \phi) = \{\omega \in W_n : (S, \phi \circ \omega) \text{ is a marked blowup}\}.$$

McMullen [17] showed that $W(S, \phi) = W_n$ provided that there is no nodal root.

Theorem 3.14 ([17, Theorem 5.4]). *If (S, ϕ) has no nodal roots, then $W(S, \phi) = W_n$.*

Proof. If (S, ϕ) has no nodal roots, then the set P is a set of distinct points; in other words, there are no nongeometric nodal roots. So we can directly apply McMullen's result. \square

In case there are nodal roots, Harbourne [11] gives an equivalent condition for $\omega \in W_n$ to be an element in $W(S, \phi)$.

Theorem 3.15 ([11, Proposition 2.4]). *Let $\omega \in W_n$. Then $\omega \in W(S, \phi)$ if and only if $\omega^{-1}\Phi(\phi)^+ \subset \Phi_n^+$.*

Proof. If there is a sequence of blowups S :

$$\pi : S = S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_3} S_2 \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = \mathbf{P}^2$$

where $\pi_i : S_i \rightarrow S_{i-1}$ is a blowup of a point $p_i \in S_{i-1}$, then the collection of $\{\mathbf{e}_0, \dots, \mathbf{e}_n\} \subset \text{Pic}(S)$ is called an *exceptional configuration* in [11]. Thus, if a collection $\mathcal{E} = \{\mathbf{e}_0, \dots, \mathbf{e}_n\}$ is an exceptional configuration, there is a marked blowup (S, ϕ) such that $\mathbf{e}_i = \phi(e_i)$ for $i = 0, \dots, n$. Also, each marked blowup has its exceptional configuration. The assertion is stated and proved using exceptional configurations in [11]. \square

Suppose S has an effective, irreducible, and reduced anticanonical divisor, and suppose (S, ϕ) is a marked blowup. Let $\omega \in W_n$ such that there exist a vector $v \in \mathbb{Z}^{1,n}$ and a real number $\lambda > 1$ such that $\omega v = \lambda v$. Then it is known [11] that $W(S, \phi)$ is infinite. Theorem 3.16 is written using the notations in this article.

Theorem 3.16 ([11, Theorem 3.1]). *Suppose (S, ϕ) is a marked blowup and that $\text{rank Pic}(S) \geq 10$. Let d be the rank of the submodule of $\text{Pic}S$ spanned by the nodal roots. Then $W(S, \phi)$ is finite if and only if $d = -1 + \text{rank Pic}S$ and S has only finitely many nodal roots.*

3.5. Automorphisms. Suppose a rational surface S admits two marked blowups, (S, ϕ) and (S, ϕ') . Also suppose $(S, \phi) \cong (S, \phi')$. Then $\phi' = \phi \circ \omega$ for some $\omega \in W_n$ and there exists an automorphism on S . Let

$$\text{Aut}(S, \phi) = \{\omega \in W(S, \phi) : (S, \phi) \cong (S, \phi \circ \omega)\}.$$

4. MARKED CUSPIDAL CUBIC

Suppose the anticanonical class of $\pi : S \rightarrow \mathbf{P}^2$ is effective, irreducible, and reduced. Then there is an irreducible curve C such that $[C] = -\kappa_S$ and the every center of blowup, $p_i \in S_{i-1}$, is in the projection of C , $\pi_i \circ \cdots \circ \pi_n(C) \subset S_{i-1}$. Thus, a marked blowup (S, ϕ) is determined by a set of points $\{p_1, \dots, p_n\}$, the markings on C together with their heights. For an irreducible and reduced cubic curve $X \subset \mathbf{P}^2$, let us denote X^* , the set of smooth points of X .

4.1. Marked cubic curves. Similar to generalized marked blowups, we make a slight modification in the definition of marked cubic curves in [17] to allow iterated blowups.

Definition 4.1. A marked cubic curve (X, ρ) is an abstract curve X equipped with a homomorphism $\rho : \mathbb{Z}^{1,n} \rightarrow \text{Pic}(X)$ such that

- (1) $\rho(e_0)$ provides an embedding $X \hookrightarrow \mathbf{P}^2$, making X into a cubic curve, and
- (2) there are a positive integer $3 \leq k \leq n$ and a set of pairs

$$P = \{(p_i, h_i) : p_i \neq p_j \text{ for } 1 \leq i \neq j \leq k\} \subset X^* \times \mathbb{N}_{\geq 1}$$

such that

- the sum $\sum_{i=1}^k h_i = n$,
- there are k pairwise disjoint subsets I_i of $\{1, \dots, n\}$ such that $|I_i| = h_i$, and
- $\rho(e_j) = [p_i]$ for $j \in I_i$.

We call a positive integer $h(p_i) := h_i$ assigned to p_i above a *height* of p_i . As defined in [17], we say two marked cubics (X, ρ) and (X', ρ') are isomorphic, $(X, \rho) \cong (X', \rho')$ if there exists a biholomorphic map $f : X \rightarrow X'$ such that $\rho' = f_* \circ \rho$. Notice that a marked cubic in [17] only allows $h_i = 1$ for all $i = 1, \dots, n$. Since the heights of base points could be bigger than 1, we have the following:

Lemma 4.2. *If two marked cubics (X, ρ) and (X', ρ') are isomorphic, then there is 1 – 1 correspondence between the sets of base points with their heights $P = \{(p_i, h_i), 1 \leq i \leq k\}$ and $P' = \{(p'_i, h_i), 1 \leq i \leq k\}$ satisfying $p_i \neq p_j$ and $p'_i \neq p'_j$ for all $i \neq j$.*

Proof. If p_i has a height h_i , then there is a subset $I_i \subset \{1, \dots, n\}$ such that $\rho(e_j) = [p_i]$ for all $j \in I_i$. It follows that $\rho'(e_j) = \rho'(e_s)$ for all $i, s \in I_i$. \square

We also define

$$W(X, \rho) = \{\omega \in W_n : (X, \rho \circ \omega) \text{ is a marked cubic}\}$$

and

$$\text{Aut}(X, \rho) = \{\omega \in W(X, \rho) : (X, \rho) \cong (X, \rho \circ \omega)\}.$$

4.2. Anticanonical surfaces. The main object of this article is a rational surface with an effective, irreducible, and reduced anticanonical divisor. Let us denote (S, Y, ϕ) a marked blowup with a unique anticanonical curve Y . If $(S, \phi) \cong (S', \phi')$, then there is a biholomorphism $F : S \rightarrow S'$ such that F_* maps $\phi(e_i)$ to $\phi'(e_i)$.

Definition 4.3. We say (S, Y, ϕ) and (S', Y', ϕ') are isomorphic, $(S, Y, \phi) \cong (S', Y', \phi')$ if there is a biholomorphic map $F : S \rightarrow S'$ such that $F_* \circ \phi = \phi'$ and $F(Y) = Y'$.

We define

$$\text{Aut}(S, Y, \phi) = \{\omega \in W_n : (S, Y, \phi) \cong (S, Y, \phi \circ \omega)\}.$$

If $\omega \in \text{Aut}(S, Y, \phi)$, then there is an automorphism $F : S \rightarrow S$ preserving Y such that $\phi \circ \omega = F_* \circ \phi$ on $\mathbb{Z}^{1,n}$. In this case, we say ω is *realized* by $F \in \text{Aut}(S, Y)$.

4.3. Cuspidal cubic. Suppose that X is a cuspidal cubic with a set X^* of smooth points on X . Then $\text{Pic}_0(X) = \mathbb{C}$, an additive group. If $f \in \text{Aut}(X)$ then $f(t) = at + b$. We call a the *determinant* of f as in [17]. Note that if ω is a meromorphic two form on X , then $f^*\omega = a\omega$.

Suppose (X, ρ) is a marked cuspidal cubic. We have $\deg(\rho(u)) = -u \cdot \kappa_n$ for $u \in \mathbb{Z}^{1,n}$ and let ρ_0 be the restriction

$$\rho_0 : \text{Ker}(\deg \circ \rho) \rightarrow \text{Pic}_0(X).$$

4.3.1. *Blowup of a marked cuspidal cubic.* Let (X, ρ) be a marked cubic with markings $P = \{(p_i, h_i)\}$ and the embedding $X \subset \mathbf{P}^2$ determined by $\rho(e_0)$ be a cuspidal cubic. Let (S, ϕ) be the marked blowup with base points

$$P = \cup_i \{p_i^{(j)}, j = 1, \dots, h_i\},$$

where

$$p_i^{(1)} = p_i \in X \quad \text{and} \quad p_i^{(j+1)} = \mathcal{F}(p_i^{(j)}) \cap \tilde{X}, \quad j = 2, \dots, h_i - 1$$

where $\mathcal{F}(p_i^{(j)})$ is the exceptional curve over $p_i^{(j)}$ and \tilde{X} is the strict transform of X . Let Y be the strict transform of X in S . Then $[Y] = -\kappa_S$. Thus we have

$$(S, Y, \phi) = \text{Bl}(X, \rho)$$

and we call this the blowup of (X, ρ) (or a marked pair) as in [17].

On the other hand, suppose (S, Y, ϕ) is a marked blowup with the unique anticanonical curve Y such that $X = \pi(Y)$ is a cuspidal cubic. Suppose the base locus is given by

$$P = \{p_1, \dots, p_n\} = \{q_i^{(j)}, 1 \leq j \leq h_i, 1 \leq i \leq k\}.$$

Then with the restriction map $r : \text{Pic}(S) \rightarrow \text{Pic}(Y)$, we have a marking $\rho : \mathbb{Z}^{1,n} \rightarrow \text{Pic}(X)$ such that $\rho(e_m) = [\pi(p_m)]$ where $\pi(p_m) = q_i$ if $p_m = q_i^{(j)}$ for some $j = 1, \dots, h_i$. Thus (S, Y, ϕ) determines a marked cubic (X, ρ) .

4.3.2. *Nodal roots.* Suppose $(S, Y, \phi) = \text{Bl}(X, \rho)$. Suppose α is a nodal root, then we have either $\alpha = e_i - e_j$ with $i < j$ or α is a geometric nodal root. For the first case, we have $\deg \circ \rho(\alpha) = -\alpha \cdot \kappa_n = 0$ and for the second case we also have $\deg \circ \rho(\alpha) = 0$ by [17, Theorem 6.6]. Thus we have:

Proposition 4.4. *If α is a nodal root, then $\rho_0(\alpha) = 0$.*

5. AUTOMORPHISMS

Let $(S, Y, \phi) = \text{Bl}(X, \rho)$ be a marked blowup with the unique anticanonical curve Y such that $X = \pi Y$ is a cuspidal cubic. Let $\text{Aut}(S, Y)$ be the group of automorphism of S fixing Y . We have:

Theorem 5.1 ([17, Theorem 6.1]). *Suppose $(S, Y, \phi) = \text{Bl}(X, \rho)$. Then we have*

$$\text{Aut}(S, Y, \phi) = \text{Aut}(X, \rho) \cap W(S, \phi).$$

Proof. Since Y is the unique anticanonical curve and $X = \pi(Y)$ is irreducible, the proof is essentially the same as in [17]. Suppose $\omega \in \text{Aut}(X, \rho) \cap W(S, \phi)$. Then ϕ and $\phi' = \phi \circ \omega$ are two markings on S . Let π' be the projection corresponding to ϕ' . We can choose the embedding of $\pi'(Y) \subset \mathbf{P}^2$ so that $(S, Y, \phi') = \text{Bl}(X, \rho' = \rho \circ \omega)$. Since $\omega \in \text{Aut}(X, \rho)$, there is an automorphism g on X such that $g(p_i) = p'_j$ with $h(p_i) = h(p'_j)$. Thus g lifts to an automorphism F on S such that $\pi \circ g = F \circ \pi'$ and $F(Y) = Y$. Thus $\omega \in \text{Aut}(S, Y, \phi)$. If $\omega \in \text{Aut}(S, Y, \phi)$, then by definition $\omega \in W(S, \phi)$ and there is an automorphism F preserving Y . It follows that there is a birational map f on \mathbf{P}^2 such that the restriction $f|_X$ is an automorphism on X . \square

For any automorphism F on S preserving Y , we have a constant $\delta(F) = \det DF_p$ for all fixed points $p \in S \setminus Y$. We call this constant $\delta(F)$ the determinant of F . As a special case of Theorem 6.5 in [17], we have:

Theorem 5.2. *For any $F \in \text{Aut}(S, Y)$, we have the determinant of $f|X = \delta(F)$, that is*

$$f|X(t) = \delta(F)t + b, \quad t \in \mathbb{C}.$$

6. REALIZING WEYL SPECTRUM

Let $\mathbb{C}^{1,n} = \mathbb{Z}^{1,n} \otimes \mathbb{C}$ with the complex Minkowski form. For each $v \in \mathbb{C}^{1,n}$, we let

$$W_n^v = \{\omega \in W_n : [v] \in \mathbb{C}^{1,n}/\mathbb{C}\kappa_n \text{ is an eigenvector for } \omega\}.$$

6.1. Blowup of a marked cuspidal cubic. The following construction is essentially the same as the one in [17, Section 7]. One difference is that we allow the iterated blowups over a point in \mathbf{P}^2 . Let $X = \{p(t) = [1 : t : t^3], t \in \mathbb{C} \cup \{\infty\}\}$ be the cuspidal cubic with the smooth locus X^* . For each $v \in \mathbb{C}^{1,n}$, define

$$3t_0 = v \cdot e_0, \quad t_i = v \cdot e_i, \quad \text{and} \quad p_i = p(t_i - t_0) \quad \text{for } i > 0.$$

Let $\{q_i : 1 \leq i \leq k\}$ be the set of distinct points in $\{p_i : 1 \leq i \leq n\}$, and $h_i := h(q_i) = \#\{j : p_j = q_i\}$. Then the set of pairs $\{(q_i, h_i), 1 \leq i \leq k\}$ determines a marking $\rho^v : \mathbb{Z}^{1,n} \rightarrow \text{Pic}(X)$, and

$$\rho_0^v(u) = (u \cdot v)[p(1) - p(0)].$$

Since $\text{Pic}_0(X) \cong \mathbb{C}$ and the height of each base point is given by the occurrence of coordinates of v , from Lemma 4.2 we have

$$(X, \rho^v) \cong (X, \rho^u) \quad \Leftrightarrow \quad u = av + \mu\kappa_n, \quad \text{for some } a, \mu \in \mathbb{C}.$$

Thus, we have:

Proposition 6.1. *Let $v \in \mathbb{C}^{1,n}$. Then we have*

$$\text{Aut}(X, \rho^v) = W_n^v.$$

Let $(S^v, Y^v, \phi^v) = \text{Bl}(X, \rho^v)$ where Y^v is the strict transform of X .

6.2. $\text{Aut}(S^v, Y^v, \phi^v)$. For a given $\omega \in W_n^{\text{ess}}$, let $\lambda(\omega)$, the spectral radius. It is known [2, 10] that the spectral radius of the Coxeter element of W_n is the largest real root of

$$\chi_n(t) = t^n(t^3 - t - 1) + (t^3 + t^2 - 1).$$

Thus if $\omega \in W_n^{\text{ess}}$ for $n \geq 10$, then $\lambda(\omega) > 1$. By Nagata [18, 19], it is well known that if $\lambda(\omega) > 1$, then $\lambda(\omega)$ is a Salem number, and eigenvalues of ω are either Galois conjugates of $\lambda(\omega)$ or a root of unity. A *Salem number* is an algebraic integer > 1 whose minimal polynomial is reciprocal of degree at least 4 and has exactly two roots outside the unit circle. In fact, we have,

Theorem 6.2 ([18, 19]). *Suppose $\omega \in W_n$ and let λ_ω denote the spectral radius of ω . If $\lambda_\omega > 1$, then the characteristic polynomial $\chi_\omega(t)$ is given by a product of a Salem polynomial $Q(t)$ and a product $R(t)$ of cyclotomic factors*

$$\chi_\omega(t) = Q(t)R(t).$$

Since $\lambda(\omega) > 1$ is a Salem number, we have:

Lemma 6.3. *Suppose $\omega \in W_n^{ess}$ for some $n \geq 10$. Let v be the eigenvector corresponding to the spectral radius $\lambda(\omega)$. Then, the marked cuspidal cubic (X, ρ^v) has at least three base points.*

Proof. Recall that the set of base points of (X, ρ^v) is given by the set of distinct points in $p(v \cdot e_i - \frac{1}{3}v \cdot e_0)$. Thus, the number of distinct $v \cdot e_i$ is the number of base points. If there is a unique base point, then $v = \{a, b, b, \dots, b\}$ for some $a, b \in \mathbb{C}$. Thus there are $n - 1$ linearly independent nodal roots $e_i - e_{i+1}, i = 1, \dots, n - 1$. Similarly, if there are two distinct base points, there are $n - 2$ linearly independent nodal roots. $\omega \in W_n^{ess}$ with $n \geq 10, \lambda(\omega) > 1$. Since every zero of a Salem polynomial is simple, we see that the subspace spanned by eigenvectors corresponding to $\lambda(\omega)$ and $1/\lambda(\omega)$ has dimension 2 in V . Thus, there cannot be more than $n - 3$ linearly independent nodal roots. \square

Including the trivial example, we have a rational surface automorphism with $\delta(f) = 1$. Hence, let us focus on the existence of a realizable element ω with the spectral radius $\lambda > 1$, that is, $\omega \in W_n^{ess}, n \geq 10$. Because of Theorem 5.1 and Proposition 6.1, we have $\text{Aut}(S^v, Y^v, \phi^v) = W_n^v \cap W(S^v, \phi^v)$. To determine $W(S, \phi)$, let us consider two cases.

Case 1 ($0 \notin v \cdot \Phi_n$). Let $v = [v_0, v_1, \dots, v_n] \in \mathbb{C}^{1,n}$. If $0 \notin v \cdot \Phi_n$, then $v_i \neq v_j$ for all $1 \leq i \neq j \leq n$. It follows that if α is a nodal root in (S^v, Y^v, ϕ^v) , then α is a geometric nodal root. Thus the same argument as in Theorem 6.6 in [17] gives us the following:

Theorem 6.4 ([17, Theorem 6.6]). *If α is a geometric nodal root, then $\rho_0^v(\alpha) = 0$.*

Proof. Suppose α is a nodal root in (S^v, Y^v, ϕ^v) , then α is a geometric nodal root. Thus there is a smooth rational curve D such that $[\pi(\alpha)] = [D]$. Since X is a cuspidal cubic, the strict transform Y^v is a singular rational curve. Thus $D \not\subset Y$ and $\rho^v(\alpha) = [D \cap Y]$ is degree zero, i.e., $\rho_0^v(\alpha) = 0$. \square

Theorem 6.5. *If $0 \notin v \cdot \Phi_n$, then $W(S^v, \phi^v) = W_n$.*

Proof. Since $0 \notin v \cdot \Phi_n$, we have $0 \neq \rho_0^v(u) = (u \cdot v)[p(1) - p(0)]$ for all $u \in L_n$. Suppose α is a nodal root in (S^v, Y^v, ϕ^v) , then α is a geometric nodal root. Thus, by Theorem 6.4, we see that there are no nodal roots. Since there are no nodal roots, by Theorem 3.15, we see that $W_n = W(S^v, \phi^v)$. \square

Case 2 ($0 \in v \cdot \Phi_n$). Again let $v = [v_0, v_1, \dots, v_n] \in \mathbb{C}^{1,n}$. If there are no nodal roots, then again, by Theorem 3.15, we see that $W_n = W(S^v, \phi^v)$. If α is a nodal root, there are two cases. If $\alpha = e_i - e_j$ then we have $\phi^v(e_j)\rho^v(e_j) = \phi^v(e_i) = \rho^v(e_i)$ and thus $v_i = v_j$. It follows that $\alpha \cdot v = 0$. If α is a geometric nodal root, then by Theorem 6.4, we have $\alpha \cdot v = 0$. Thus we have:

Lemma 6.6. *If α is a nodal root, then $\alpha \cdot v = 0$.*

Since the curve Y^v is the unique anticanonical curve, the set of nodal roots is finite. Thus, by Harbourne [11], it is easy to check if ω is realizable.

Theorem 6.7. *Suppose $0 \in v \cdot \Phi_n$ and $\omega \in W_n^v$. Then $\omega \in \text{Aut}(S^v, Y^v, \phi^v)$ if and only if there is no nodal root $\alpha \in \Phi(\phi)^+$ such that $\omega^{-1}\alpha \in \Phi^-$.*

Proof. This is an immediate consequence of Theorem 3.15 and Theorem 5.1. \square

Theorem 6.8. *If $0 \in v \cdot \Phi_n$ and $W_n^v \neq \emptyset$, there is $\omega \in W_n^v \cap W(S^v, \phi^v)$.*

Proof. Suppose $\omega \in W_n^v$. If α is a nodal root, then by Lemma 6.6 and Proposition 2.7, we see that α is a periodic root. Since X is irreducible and the spectral radius $\lambda_\omega > 1$, from Proposition 3.13, we know that there is $s \in \Sigma$ such that $(\omega^{-1} \circ s)\alpha$ is a positive root for every nodal root α where Σ is a span of all reflections through nodal roots for ω . In other words, $(\omega^{-1} \circ s)(\Phi(\phi)^+) \subset \Phi_n^+$. Thus by Theorem 3.15 and Theorem 6.7, we see that $s^{-1} \circ \omega \in W(S^v, \phi^v)$. Now since any nodal root α satisfies $\alpha \cdot v = 0$, thus $s_\alpha v = v$ for all reflection s_α through a nodal root α and

$$s^{-1} \circ \omega(v) = s^{-1}(\lambda v) = \lambda v.$$

Thus $s^{-1} \circ \omega \in W_n^v \cap W(S^v, \phi^v)$. \square

Let the Weyl spectrum

$$\Lambda = \cup_n \{\lambda_\omega : \omega \in W_n\},$$

where λ_ω is the spectral radius of ω . By Lemma 2.2, we see that $\Lambda = \cup_n \{\lambda_\omega : \omega \in W_n^{ess}\}$. Thus we have:

Proof of Theorem A. Suppose $\lambda \in \Lambda$. If $\lambda > 1$, then there is $\omega \in W_n^{ess}$ for some $n \geq 10$ such that $\lambda_\omega = \lambda$. Let v be the eigenvector of ω corresponding to λ . By Theorem 6.5 and Theorem 6.8 we have $\omega' \in \text{Aut}(S^v, Y^v, \phi^v)$ and thus there is an automorphism on S^v such that $\phi^v \circ F_* = \omega' \circ \phi^v$ and $\delta(F) = \lambda$. For $\lambda = 1$, since the Cremona involution $s_0 s_1 s_2 \in W_3$ lifts to an automorphism F and $\lambda_{s_0 s_1 s_2} = 1$, we have the desired result. \square

Proof of Theorem B. Suppose an essential element $\omega \in \cup_{n \geq 10} W_n$ with the spectral radius $\lambda = \lambda(\omega)$ and $\omega v = \lambda v$. Then $\omega \in W_n^v$. If no nodal roots are mapped to negative roots by ω^{-1} , by Theorems 6.5–6.7 together with Theorem 5.1, we see that ω is realized by an automorphism on (S^v, Y^v, ϕ^v) . On the other hand, Suppose there is a nodal root α such that $\omega^{-1}\alpha \in \Phi_n^-$ and let C_α be a projection of α on S^v . If ω is realized by an automorphism f on (S^v, Y^v, ϕ^v) , then the class $[f^{-1}C_\alpha] = f_*^{-1}\alpha$ is not effective which is impossible. Thus, we have the first assertion.

In case ω is not realizable, then by Lemmas 3.8–3.12, we can find $s \in \Sigma_v$ such that $s \circ \omega \in W_n^v$ and $s \circ \omega$ maps no nodal roots to negative roots. Notice that this can be done in finitely many steps because there are finitely many nodal roots in (S^v, Y^v, ϕ^v) . Thus, by Theorems 6.5–6.8, we see that $s \circ \omega$ is realizable. \square

7. QUADRATIC BIRATIONAL MAPS

Let $(S, Y, \phi) = \text{Bl}(X, \rho)$ be a marked blowup with the unique anticanonical curve Y such that $\pi Y = X$. Then by Diller [7] we have the following:

Theorem 7.1 ([7]). *Suppose $F : S \rightarrow S$ is an automorphism stabilizing Y and $\delta(F) > 1$, then X is either a cuspidal cubic, a conic with its tangent, or three lines joining at a single point.*

Proof. Suppose the automorphism F covers a birational map f on \mathbb{P}^2 . In [7], Diller showed that if X is not one of three kinds in the statement of the theorem, then the determinant, $D(f|_X)$ of f , has modulus 1. However by Theorem 5.2, we have $D(f|_X) = \delta(F) > 1$. \square

7.1. Cremona involution. Suppose i, j, k are three distinct integers between 1 and n . The Cremona involution $\kappa_{i,j,k} \in W_n$ is given by the reflection through the vector $\alpha_{i,j,k} = e_0 - e_i - e_j - e_k$:

$$\kappa_{i,j,k}(x) = x + (x \cdot \alpha_{i,j,k}) \alpha_{i,j,k}.$$

Recall that the reflection s_i is given by

$$s_i(x) = x + (x \cdot \alpha_i) \alpha_i, \quad \alpha_i = e_i - e_{i+1}.$$

Let us denote $\Sigma_n \subset W_n$ a subspace spanned by reflections s_i for $i = 1, \dots, n-1$.

7.2. Quadratic birational maps. Suppose i, j, k are three distinct integers in $\{1, \dots, n\}$ and $s \in \Sigma_n$. Let

$$\omega = \kappa_{i,j,k} \circ s \in W_n^{ess}, \quad n \geq 10.$$

Suppose there exists a marked blowup $(S, Y, \phi) = Bl(X, \rho)$ such that ω is realized by $F \in Aut(S, Y)$, i.e., $\omega = F_*$ through the isomorphism ϕ . Then

- there exists a set of (possibly infinitely) points $p_i \in X'$ in the smooth locus of X such that

$$\pi \mathcal{E}_i = p_i, \quad \text{where } \phi(e_i) = [\mathcal{E}_i],$$

- there exists a quadratic birational map $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ such that

$$\pi \circ F = \pi \circ f, \quad \text{and}$$

- the determinant of $f|_X = \delta(F) = \lambda$, where λ is the spectral radius of $\omega > 1$, i.e.,

$$f|_X(t) = \lambda t + b, \quad t \in \mathbb{C}.$$

Since ω is a composition of exactly one Cremona involution and a reflection in Σ_n , both f and f^{-1} have exactly three distinct (but possibly infinitely near) points of indeterminacy.

Lemma 7.2. *The sets of indeterminacy points for f and f^{-1} are*

$$\text{Ind}(f) = \{p_{i'}, p_{j'}, p_{k'}\} \quad \text{and} \quad \text{Ind}(f^{-1}) = \{p_i, p_j, p_k\},$$

where $s(e_{i'}) = e_i, s(e_{j'}) = e_j, s(e_{k'}) = e_k$.

Proof. Since F is an automorphism if $p \in \mathbb{P}^2$ is a point of indeterminacy for f , p must be a base point for the marked blowup $(S, Y, \phi) = Bl(X, \rho)$. Suppose $p = p_\ell$ for some $\ell = 1, \dots, n$. Since $F_* e_\ell = \omega e_\ell$ and $F_* e_\ell \cdot e_0 \neq 0$ if $s(e_\ell) \in \{e_i, e_j, e_k\}$. Thus we have $\text{Ind}(f) = \{p_{i'}, p_{j'}, p_{k'}\}$ with $s(e_{i'}) = e_i, s(e_{j'}) = e_j, s(e_{k'}) = e_k$. With the similar argument, we have $\text{Ind}(f^{-1}) = \{p_i, p_j, p_k\}$. \square

7.3. Orbit data. Since $\omega \in W_n^{ess}$ for each $\ell = 1, \dots, n$, there is a smallest positive integer n_ℓ such that $\omega^{n_\ell} e_\ell \cdot e_0 > 0$, in other words

$$\omega^{n_\ell-1} e_\ell \in \{e_{i'}, e_{j'}, e_{k'}\}.$$

Lemma 7.3. *There is a permutation σ of $\{i, j, k\}$ such that*

$$\omega^{n_s} e_s = e_{\sigma(s)'}, \quad s \in \{i, j, k\}.$$

Proof. It is sufficient to show that $\omega^{n_i} e_i \neq \omega^{n_j} e_j$ for $i \neq j$. Suppose $\omega^{n_i} e_i = \omega^{n_j} e_j$ and $n_i \leq n_j$. First we have

$$-1 = \omega^{n_i} e_i \cdot \omega^{n_j} e_j = e_i \cdot \omega^{n_j-n_i} e_j \quad \Rightarrow \quad \omega^{n_j-n_i} e_j = e_i \quad \Rightarrow \quad n_j - n_i \geq 0.$$

Then, we have

$$\omega^{n_j-n_i-1} e_j = \omega^{-1} e_i = s^{-1}(e_0 - e_j - e_k) \quad \Rightarrow \quad \omega^{n_j-n_i-1} e_j \cdot e_0 \neq 0.$$

Since n_j is the smallest positive integer such that $\omega^{n_\ell} e_\ell \cdot e_0 > 0$ and since $e_j \cdot e_0 = 0$, we have

$$n_j - n_i - 1 \leq 0 \quad \text{and} \quad n_j - n_i - 1 \neq 0.$$

It follows that $n_j - n_i \leq 0$. □

Lemma 7.4. *The sum $n_i + n_j + n_k = n$. Furthermore, the sets $O_i = \{\omega^s e_i, 0 \leq s \leq n_i - 1\}$, $O_j = \{\omega^s e_j, 0 \leq s \leq n_j - 1\}$ and $O_k = \{\omega^s e_k, 0 \leq s \leq n_k - 1\}$ are pairwise disjoint.*

Proof. It is sufficient to show that O_i, O_j, O_k are pairwise disjoint. Suppose there are $0 \leq s \leq n_i - 1$ and $0 \leq t \leq n_j - 1$ such that $\omega^s e_i = \omega^t e_j$. We may assume $s < t$. It follows that $\omega^{t-s} e_j = e_i$. Since n_i is the smallest positive integer such that $\omega^{n_i} e_i \cdot e_0 > 0$ and since $t - s < n_j$, we have $n_j = t - s + n_i$ and $\omega^{n_j-1} e_j = \omega^{n_i-1} e_i$. This is a contradiction because of Lemma 7.3. □

We can rewrite Lemma 7.4 in terms of the birational map f associated with ω .

Lemma 7.5. *There are three positive integers n_i, n_j, n_k and a permutation σ on $\{i, j, k\}$ such that for each $\ell \in \{i, j, k\}$*

$$f^{n_\ell-1}(p_\ell) = p_{\sigma(\ell)'}, \quad f^s(p_\ell) \notin \text{Ind}(f) \quad \text{for all } 0 \leq s \leq n_\ell - 2.$$

We call these numerical data n_i, n_j, n_k and a permutation σ on $\{i, j, k\}$ defined in Lemma 7.5 *the orbit data* of f . Also since

$$\omega : e_0 - e_{i'} - e_{j'} \mapsto (e_0 - e_i - e_j - e_k) - (e_0 - e_j - e_k) - (e_0 - e_i - e_k) = e_k,$$

we see that

$$f : \begin{cases} E_i \rightarrow p_i \rightarrow f(p_i) \rightarrow \cdots \rightarrow f^{n_i-1}(p_i) = p_{\sigma(i)'} \\ E_j \rightarrow p_j \rightarrow f(p_j) \rightarrow \cdots \rightarrow f^{n_j-1}(p_j) = p_{\sigma(j)'} \\ E_k \rightarrow p_k \rightarrow f(p_k) \rightarrow \cdots \rightarrow f^{n_k-1}(p_k) = p_{\sigma(k)'}, \end{cases}$$

where E_i is the line joining $e_{j'}$ and $e_{k'}$, E_j is the line joining $e_{i'}$ and $e_{k'}$, and E_k is the line joining $e_{i'}$ and $e_{j'}$.

7.4. Fixing a cubic. For any curve $C \in \mathbf{P}^2$, we define

$$f(C) = \overline{f(C \setminus \text{Ind}(f))}.$$

The degree of $f(C)$ is determined by $F_*(\tilde{C}) \cdot e_0$ where \tilde{C} is the strict transform of C in S . Thus we have

$$\deg f(C) = 2\deg(C) - \sum_{p \in \text{Ind}(f)} \nu_p(C),$$

where $\nu_p(C)$ is multiplicity of C at p . Using this, Diller [7] gives sufficient conditions on the orbit data for f to preserve the projection X of the anticanonical curve Y .

Theorem 7.6 ([7, Theorem 4.1]). *Suppose X is three lines L_1, L_2, L_3 joining at a single point, and X is preserved by f . Then, each line contains exactly one point of indeterminacy for f and exactly one point of indeterminacy for f^{-1} . Furthermore, we have one of the following:*

- (1) σ is an identity permutation,
- (2) $\sigma = (ij)$ is a transposition exchanging i and j and both n_i and n_j are odd, or
- (3) σ is a cyclic permutation, then $n_i \equiv n_j \equiv n_k \equiv 1 \pmod{3}$.

Proof. To make the notation simple, let us assume that $\{i, j, k\} = \{1, 2, 3\}$. Since each irreducible component of X is a line, $\deg f^\pm(L_i) = 1$ for all $i = 1, 2, 3$. It follows that each line contains exactly one point of indeterminacy for f^\pm . It follows that all three points of indeterminacy are distinct. Suppose $p_{i'} \in L_i$ for $i = 1, 2, 3$. Thus, for each $i = 1, 2, 3$, only L_i intersects E_i at a nonindeterminate point, say q_i . Since $f|_X$ is an automorphism, we have $f|_X(q_i) = p_i$. We have three possibilities:

- *Case 1.* $f(L_i) = L_i, i = 1, 2, 3$. Since $q_i, p_{i'} \in L_i$, the permutation $\sigma = Id$.
- *Case 2.* $f : L_1 \leftrightarrow L_2$. Since $q_1 \in L_1$, we have $p_1 \in L_2$. Note that $p_{2'} \in L_2$. Thus if $\sigma = (12)$, then n_1 has to be odd and so does n_2 ; and if $\sigma = Id$, then both n_1 and n_2 are even.
- *Case 3.* f permutes L_i cyclicly. Suppose $f : L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow L_1$. Similar to case 2, $p_1 \in L_2$. Since $p_{i'} \in L_i, i = 1, 2, 3$, we have
 - (1) if $\sigma(1) = 1$, then $n_1 \equiv 0 \pmod{3}$;
 - (2) if $\sigma(1) = 2$, then $n_1 \equiv 1 \pmod{3}$; and
 - (3) if $\sigma(1) = 3$, then $n_1 \equiv 2 \pmod{3}$.

Thus if σ is a cyclic permutation, then $n_i \equiv n_j \equiv n_k \equiv 1 \text{ or } 2 \pmod{3}$; if $\sigma = Id$, then $n_i \equiv n_j \equiv n_k \equiv 0 \pmod{3}$; and if $\sigma = (12)$, then $n_3 \equiv 0 \pmod{3}$ and $n_1 \equiv n_2 \equiv 1 \text{ or } 2 \pmod{3}$.

The assertion follows from the above three cases. The detailed proof is also available in [7, 13]. \square

Also, we have:

Theorem 7.7. *If X is a union of a conic Q and its tangent L , and X is preserved by f , then exactly one of the following occurs:*

- (1) there exists a unique $\ell \in \{i, j, k\}$ such that $p_\ell, p_{\ell'} \in L$ and $\sigma(\ell) = \ell$, or
- (2) three positive integers n_i, n_j, n_k are all odd, and every point of indeterminacy belongs to $Q \setminus L$

$$\text{Ind}(f) \cup \text{Ind}(f^{-1}) \subset Q \setminus L.$$

Proof. Again, let us assume $\{i, j, k\} = \{1, 2, 3\}$. We have either $f(L) = Q$ or $f(L) = L$. If $f(L) = L$, then L contains exactly one point of indeterminacy, say $p_{1'}$. Then, using the same notations in the proof of Theorem 7.6, we have both $q_1 = L_1 \cap X, p_{1'} \in L$, and so does p_1 and $\sigma(1) = 1$.

If $f(L) = Q$, then L does not contain any indeterminant points. Also, any exceptional line E_i intersects Q at two indeterminant points. So there is a nonindeterminant intersection point $q_i = E_i \cap L$. It follows that $f(q_i) = p_i \in Q$ for all $i = 1, 2, 3$. Since all points of indeterminacy belong to Q , and f interchanges Q and L , we see that $f^{\text{even}} q_i \in L$ and $f^{\text{odd}} q_i \in Q$ and all n_i has to be odd. \square

8. UNREALIZABLE ESSENTIAL ELEMENTS

Let us consider $\omega = \kappa_{1,7,8} \circ s \in W_{13}$ such that the Cremona involution $\kappa_{1,7,8}$ is defined by

$$\begin{aligned} \kappa_{1,7,8} : e_0 &\mapsto 2e_0 - e_1 - e_7 - e_8 \\ e_1 &\mapsto e_0 - e_7 - e_8 \\ e_7 &\mapsto e_0 - e_1 - e_8 \\ e_8 &\mapsto e_0 - e_1 - e_7 \\ e_i &\mapsto e_i \quad \text{otherwise} \end{aligned}$$

and a cyclic permutation s is given by

$$s : e_0 \mapsto e_0, \quad e_i \mapsto e_{i+1}, \quad i = 1, \dots, 12, \quad \text{and} \quad e_{13} \mapsto e_1.$$

It follows that

$$\omega : \begin{cases} e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow e_4 \rightarrow e_5 \rightarrow e_6 \rightarrow e_0 - e_1 - e_8 \\ e_7 \rightarrow e_0 - e_1 - e_7 \\ e_8 \rightarrow e_9 \rightarrow e_{10} \rightarrow e_{11} \rightarrow e_{12} \rightarrow e_{13} \rightarrow e_0 - e_7 - e_8. \end{cases}$$

Thus if ω is realized by an automorphism F on a rational surface S , and F covers a birational map f , then we have

$$\text{Ind}(f^{-1}) = \{p_1, p_7, p_8\} \quad \text{and} \quad \text{Ind}(f) = \{p_{7'} = p_6, p_{8'} = p_7, p_{1'} = p_{13}\},$$

and hence the orbit data of f is given by 6, 1, 6 and a cyclic permutation. The direct computation shows that the spectral radius $\lambda(\omega)$ is given by the largest real zero of $t^6 - t^5 - t^3 - t + 1$. In summary, we have:

Lemma 8.1. *If ω is realized by an automorphism F on some rational surface S , then F covers a quadratic birational map f with orbit data 6, 1, 6 with a cyclic permutation. Furthermore the determinant of $f = \lambda(\omega) \sim 1.50614$, i.e., the restriction of f to the smooth locus of X is given by $f|_X(t) = \lambda(\omega)t + b$, $t \in \mathbb{C}$ for some $b \in \mathbb{C}$.*

Proposition 8.2. *ω cannot be realized by an automorphism F on an anticanonical rational surface (S, Y) such that $X = \pi(Y)$ is a cuspidal cubic.*

Proof. Suppose ω is realized by an automorphism F on an anticanonical rational surface (S, Y) . Then F covers a birational map f fixing X . By Diller [7, Theorem 1.3], f is uniquely determined up to linear conjugacy. We may assume f is the one constructed in Section 6. From direct computation, we see that an eigenvector $v = [v_0, v_1, \dots, v_{13}]$ of ω for the spectral radius is given by

$$\begin{aligned} v_2 = v_8, \quad v_3 = v_9, \quad v_4 = v_{10}, \quad v_5 = v_{11}, \quad v_6 = v_{12}, \quad \text{and} \\ \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_{13}\} \text{ is a set of 8 distinct numbers.} \end{aligned}$$

It follows that the base locus $\{p_1, \dots, p_{11}\}$ of the marked blowup satisfying p_{i+6} is infinitely near point to p_i of order 1 for $i = 2, \dots, 6$. Also by inspection, we see that there are 6 nodal roots for the marked blowup (S^v, Y^v, ϕ^v) :

$$\Phi(\phi^v)^+ = \{e_2 - e_8, e_3 - e_9, e_4 - e_{10}, e_5 - e_{11}, e_6 - e_{12}, e_0 - e_1 - e_7 - e_{13}\}$$

and

$$\omega(e_0 - e_1 - e_7 - e_{13}) = -(e_2 - e_8).$$

Thus, ω^{-1} does not preserve the positivity of the effective roots, and by Theorem 3.15 we see that $\omega \notin \text{Aut}(S^v, Y^v, \phi^v)$. \square

Proposition 8.3. *ω cannot be realized by an automorphism F on an anticanonical rational surface (S, Y) such that $X = \pi(Y)$ is either three lines joining at a single point or a conic with its tangent.*

Proof. Suppose f is a birational map fixing X , and F covers f . If X is three lines joining at a single point, by Theorem 7.6, there is no such f since σ is a cyclic permutation and $1 \not\equiv 6 \pmod{3}$. If X is a conic and its tangent, by Theorem 7.7, there is no such f since σ is a cyclic permutation, and 6 is not odd. \square

Theorem 8.4. *ω cannot be realized by an automorphism F on an anticanonical rational surface (S, Y) .*

Proof. By Theorem 7.1, the only possibility for X is either a cuspidal cubic, a conic and its tangent, or three lines joining at a single point. The result follows from Propositions 8.2 and 8.3. \square

Proof of Theorem C. The existence of an essential element that cannot be realized by an automorphism on an anticanonical rational surface is clear from Theorem 8.4. \square

Theorem 8.5. *Let s' be a reflection through a vector $e_2 - e_8$. Then $s' \circ \omega$ is realized by an automorphism on an anticanonical rational surface (S, Y) such that Y is the strict transform of a cuspidal cubic in \mathbb{P}^2 .*

Proof. Notice that $(e_2 - e_8) \cdot v = 0$. Thus ω and $s' \circ \omega$ have the same spectral radius, and an eigenvector v for ω associated with the spectral radius $\lambda(\omega)$ also satisfies

$$s' \circ \omega(v) = \lambda(\omega)v.$$

Furthermore, we have

$$s' \circ \omega : e_2 - e_8 \rightarrow e_3 - e_9 \rightarrow e_4 - e_{10} \rightarrow e_5 - e_{11} \rightarrow e_6 - e_{12} \rightarrow e_0 - e_1 - e_7 - e_{13} \rightarrow e_2 - e_8.$$

Thus $(s' \circ \omega)^{-1}$ preserves the positivity of nodal roots. By Theorem 3.15, we see that $s' \circ \omega \in (S^v, Y^v, \phi^v)$. \square

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