TWISTED COCYCLES AND RIGIDITY PROBLEMS.

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We consider a class of cohomologies associated to a group action, outline a duality method for their calculation, and apply it to study different questions related to the group action. In particular, we prove a number of results on infinitesimal and cohomological rigidity of higher rank cocompact lattice actions on imaginary boundaries of some symmetric spaces (as well as results on cohomologies of some partially hyperbolic actions and lattice actions on a broader class of homogeneous spaces). We also obtain a very transparent proof of local C^3 rigidity of projective actions of cocompact lattices in $PSL(2,\mathbb{R})$.

1. Twisted cocycles. Definitions and notations.

Measurable twisted cocycles of a group action, as described below, are the first cohomologies of generalized induced representations (see [1]) associated with the action. (If the action is transitive then they are simply the representations induced by representations of a stationary subgroup.) In explicit form twisted cocycles has, probably, appeared for the first time in the work of Feldman and Moore [2] in the context of equivalence relations and their cohomologies. Our main motivation for giving rigorous definitions here is that we want to consider continuous and smooth cohomologies, and the essentially measurable methods and definitions of [2] are not immediately applicable.

Recall that if a group G acts on a space M then the cocycle of this action with coefficients in a group H is a map $\alpha : G \times X \to H$ such that

$$\alpha(g_1 g_2, m) = \alpha(g_1, g_2(m))\alpha(g_2, m).$$

Two cocycles α and β are called cohomologous if there exists a function $P : M \to H$ such that

$$\beta(g, m) = P(gm)^{-1}\alpha(g, m)P(m).$$

By $\mathrm{H}(G, H, M)$ we will denote a set of equivalence classes of cocycles. (Most of the times we will shorten the notation to $\mathrm{H}(G, M)$ since it will be clear from the text which coefficients we work with).

Imposing additional regularity condition on the functions $\alpha(g, \cdot)$ and P (C^∞, measurable, continuous, etc...), in the above definition, we define the spaces of cocycles of the corresponding regularity.

Before we define twisted cocycles, we would like to notice the close connection between cocycles and cohomologies. Let $\mathcal{R}$ be some abelian group. Then we can endow the space $F(M)$ — space of functions on M with values in $\mathcal{R}$, with a G-module structure defining

$$g \cdot f(m) = f(g^{-1}m).$$

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Consider first cohomologies of G with coefficients in this G-module. Then we notice that $z : G \to F(M)$ satisfies the cohomology equation if and only if

$$\alpha_z(g, m) = z(g^{-1})(m)$$

satisfies the equation for cocycles with coefficients in $\mathcal{R}$. And, it is easy to check that $z \sim z_1$ if and only if α_z is cohomological to α_{z_1} as cocycles. So, we have

$$\mathrm{H}^1(G, F(M)) \cong \mathrm{H}(G, \mathcal{R}, M).$$

Now, assume that $\mathcal{R}$ is endowed with a structure of H-module, for some group H. Then, aside from $g \cdot f(m) = f(g^{-1}m)$, we have other ways to turn $F(M)$ into G-module. Namely, in addition to the shift by g we can also multiply ("twist") $f \in F(M)$ by a function $x(g, m) : M \to H$. Then,

$$g \cdot f(m) = x(g, m) f(g^{-1}m)$$

defines the structure of G-module on $F(M)$ if and only if $x(g, m)$ satisfies the following equation

$$x(g_1 g_2, m) = x(g_1, m) x(g_2, g_1^{-1}m),$$

which is equivalent to $\alpha(g, m) = x(g^{-1}, m)^{-1}$ being a cocycle with coefficients in H. Denote the module constructed using cocycle α as $F_\alpha(M)$.

Now, it is easy to check that $F_{\alpha_1}(M)$ and $F_{\alpha_2}(M)$ are isomorphic as G-modules if α_1 and α_2 are equivalent cocycles. Thus, we have constructed G-modules $F_\alpha(M)$ for each class of cocycles $\alpha \in \mathrm{H}(G, H, M)$. We will call this modules twisted modules. The trivial module $F(M)$ corresponds to the class of trivial cocycles: $\alpha(g, m) = e$, where e is the unit in H.

Consider cohomologies of G with coefficients in $F_\alpha(M)$. That naturally leads us to the following definition: map $\beta(g, m) : G \times M \to \mathcal{R}$ is called twisted cocycle iff

$$\{z : g \to \beta(g^{-1}, m) \in F(M))\} \in \mathrm{H}^1(G, F_\alpha(M)).$$

And two twisted cocycles are called cohomologous iff they generate the same element of $\mathrm{H}^1(G, F_\alpha(M))$.

Denote the space of equivalence classes of twisted cocycles constructed using $\alpha \in \mathrm{H}(G, H, M)$ by $\mathrm{H}(G, \mathcal{R}, M, \alpha)$ (again, most of the times we will omit reference to $\mathcal{R}$ in this notation, since it should be clear from the text which coefficients we work with). Then, it is well defined since by the definition

$$\mathrm{H}(G, M, \alpha) \cong \mathrm{H}^1(G, F_\alpha(M))$$

and, thus, only depends on the equivalence class of α, since the cohomologies with coefficients in isomorphic modules coincide.

We define twisted cocycles of different regularity by restricting $F(M)$ to be the space of functions of corresponding regularity.

From this definition of twisted cocycles we can easily derive the following equivalent definition, which is more convenient in some calculations.

Fix $\alpha \in \mathrm{H}(G, H, M)$. A map $\beta(g, m) : G \times M \to \mathcal{R}$ will be called twisted cocycle if it satisfies

$$\beta(g_1 g_2, m) = \beta(g_2, m) + \alpha(g_2, m)^{-1} \beta(g_1, g_2 m), \forall g_1, g_2 \in G, m \in M.$$

Two cocycles β and β_1 will be called cohomologous (or equivalent) if and only if there exists $f \in F(M)$ such that

$$\beta(g, m) - \beta_1(g, m) = f(m) - \alpha(g, m)^{-1} f(gm), \forall g \in G, m \in M.$$

Then, the set of equivalence classes can be identified with $\mathrm{H}(G, M, \alpha)$.

2. Duality method.

In this section we outline duality method for calculating cocycles and twisted cocycles of group actions.

The main idea is to try to lift the action to a well understood action on a bigger space. This lifting induces a map L on cocycle spaces. Then, naturally, the image and the kernel of this map contain most of the information about the cocycles of the original action. And since, presumably, we have a good understanding of the lifted action, we may hope to be able to easily obtain significant information about its cocycles, and, thus, the image of L. Then, we just need to get a hold on the kernel of L. It turns out, that the kernel subspaces enjoy some very useful duality properties, which make it possible, in some cases, to substitute a question about cocycles of an action by a question about cocycles of a different action, which dynamically may be completely different from the original one, and might turn out to be easier to study.

Suppose that G acts on X_1 and X_2 in such a way that the action on X_2 is a factor of the action on X_1. By $\mathrm{H}^{tr}(G, X_2)$ we will denote a set of those elements of $\mathrm{H}(G, X_2)$ that lift to a trivial element in $\mathrm{H}(G, X_1)$.

Also, assume that there is a certain structure on X_1 and X_2 preserved by the factor map. (It can be a structure of a topological space, measurable space, smooth manifold etc). And suppose that we consider some class of cocycles with respect to this structure (for example, continuous cocycles, measurable, smooth etc). If the reference to which class of cocycles we are working with is absent it means that we work with any of the above described classes, which are applicable to the factor maps involved. In that case cohomologous means cohomologous in the corresponding class.

Definitions for H^{tr}-spaces of twisted cocycles are absolutely analogous.

In [3] we proved:

Theorem 2.1. *Let G_1 and G_2 be two groups acting on a space M. Consider cocycles with values in an arbitrary group H.*

If those actions commute, then:

$$\mathrm{H}^{tr}(G_1, M/G_2) \cong \mathrm{H}^{tr}(G_2, G_1\backslash M).$$

To be more precise, there is a canonically defined map

$$K(G_1, G_2) : \mathrm{H}^{tr}(G_1, M/G_2) \to \mathrm{H}^{tr}(G_2, G_1\backslash M)$$

such that $K(G_2, G_1) \circ K(G_1, G_2) = Id$ on $\mathrm{H}^{tr}(G_1, M/G_2)$.

Where M/G_2 and $G_1\backslash M$ are the factor spaces on which the other group acts in the obvious way, and H^{tr}-spaces defined with respect to the lifts of actions to the whole M.

And the duality for twisted cocycles is "built over" the duality for non-twisted ones:

Theorem 2.2. *Let G_1 and G_2 be two groups acting on a space M. Let H be any group, and $\mathcal{R}$ be any H-module. Let α be a cocycle with values in H. Assume that $\alpha \in \mathrm{H}^{tr}(G_1, H, M/G_2)$.*

If the G_1 and G_2 actions commute, then:

$$\mathrm{H}^{tr}(G_1, M/G_2, \alpha) \cong \mathrm{H}^{tr}(G_2, G_1\backslash M, K(G_1, G_2)(\alpha)).$$

To be more precise, there is a canonically defined group isomorphism (obviously, the spaces of twisted cocycles are naturally equipped with abelian group structure induced from $\mathcal{R}$):

$$K_1(G_1, G_2) : \mathrm{H}^{tr}(G_1, M/G_2, \alpha) \to \mathrm{H}^{tr}(G_2, G_1\backslash M, K(G_1, G_2)(\alpha))$$

such that $K_1(G_2, G_1) \circ K_1(G_1, G_2) = Id$ *on* $\mathrm{H}^{tr}(G_1, M/G_2, \alpha)$.

Remark: Whenever it is not likely to cause a misunderstanding we will denote the duality maps simply by K, instead of $K(G_1, G_2)$ and $K_1(G_1, G_2)$.

We will mostly use Theorems 2.1 and 2.2 in case of two closed subgroups P and Q acting on Lie group G, P from the left, Q from the right. Then for smooth, continuous and Hölder cocycles, for $\alpha \in \mathrm{H}^{tr}(P, G/Q)$ we have:

$$\mathrm{H}^{tr}(P, G/Q, \alpha) \cong \mathrm{H}^{tr}(Q, P\backslash G, K(\alpha)).$$

3. Cohomological rigidity

In this section we work with cocycles with coefficients in $\mathbb{R}^l$, $l \in \mathbb{N}$. Let G be a semi-simple connected Lie group with finite center and without compact factors. Let $\mathcal{G}$ be its Lie algebra, A — connected component of the maximal split Cartan subgroup, $\mathcal{A}$ the corresponding commutative subalgebra, N — closed subgroup containing A. Let Γ be an irreducible cocompact lattice in G.

In [4] we proved:

Theorem 3.1. *Let G be a Lie group, Γ a discrete subgroup, acting on G from the left. Then, for C^∞, continuous or Hölder cocycles we have:* $\mathrm{H}(\Gamma, G) = 0$.

Then, for $\alpha \in \mathrm{H}(\Gamma, N\backslash G)$, we have:

$$\mathrm{H}(\Gamma, N\backslash G, \alpha) = \mathrm{H}^{tr}(\Gamma, N\backslash G, \alpha) \cong \mathrm{H}^{tr}(N, G/\Gamma, K(\alpha)).$$

So, our duality method allows us to transform the question about cocycles of lattice action (which is “not nice in any way” — no obvious hyperbolicity properties, no invariant measures, we do not have a very good understanding of the structure of Γ) into a question about cocycles of N-action (which is “much better action” — partially hyperbolic, has finite smooth invariant measure, N is relatively “simple” group).

3.1. **Smooth cocycles.**

For partially hyperbolic actions we develop special analytic technique ([3]) similar to the technique used by Katok and Spatzier in their study of the cocycle rigidity of standard abelian actions ([5]).

The following result is proved in [4]:

Theorem 3.2. *Let G be of $\mathbb{R}$-rank $n \geq 2$.*

(*) Assume that $\mathcal{G}$ has no factors isomorphic to $\mathfrak{so}(m, 1)$ or $\mathfrak{su}(m, 1)$.

Then all C^∞ cocycles of the action of N on G/Γ are C^∞ cohomologous to constant cocycles.

Remark: The (*) condition is needed due to a non-uniformity in certain estimates on the decay of correlation coefficients for unitary representations of groups $\mathfrak{so}(m, 1)$ and $\mathfrak{su}(m, 1)$. It is quite likely that for the particular representations considered in the proofs of our results sufficient estimates still can be obtained and then the (*) condition will disappear from all our results (for details see [5] or [3]).

Due to Theorem 3.2, studying twisted cocycles of N-action we may restrict our attention to the case of constant twisting. Then the following result proved in [3] gives an "almost complete" description of all twisted cocycles of N-action:

Theorem 3.3. *Let G be of $\mathbb{R}$-rank $n \geq 3$.*

(*) Assume that $\mathcal{G}$ has no factors isomorphic to $\mathfrak{so}(m,1)$ or $\mathfrak{su}(m,1)$.

Let π be a non-zero linear form on $\mathcal{A}$, extendable to a representation of N and not proportional to a root of rank one factor of $\mathcal{G}$. Then every element of $\mathrm{H}(N, G/\Gamma, e^{-\pi})$ is C^∞ cohomologous to a constant twisted cocycle.

Moreover, if π is such that it is not proportional to any root of $\mathcal{G}$, then

$$\mathrm{H}(N, G/\Gamma, e^{-\pi}) = 0.$$

Since we now have a good understanding of cocycles of N-action, using duality, we easily get the following rigidity results for Γ-action ([4] and [3]):

Theorem 3.4. *Let G be of $\mathbb{R}$-rank $n \geq 2$.*

(*) Assume that $\mathcal{G}$ has no factors isomorphic to $\mathfrak{so}(m,1)$ or $\mathfrak{su}(m,1)$.

Then every C^∞ cocycle of the Γ action on $N\backslash G$ uniquely extends to a cocycle of the G action.

Theorem 3.5. *Let G be of $\mathbb{R}$-rank $n \geq 3$.*

(*) Assume that $\mathcal{G}$ has no factors isomorphic to $\mathfrak{so}(m,1)$ or $\mathfrak{su}(m,1)$.

Let α be any element of $\mathrm{H}(\Gamma, N\backslash G)$. Then for the C^∞ cocycles we have

1) if α's dual is not proportional to a root of $\mathcal{G}$, then

$$\mathrm{H}(\Gamma, N\backslash G, \alpha) = 0.$$

2) if α's dual is proportional to a root of $\mathcal{G}$, but not to a root of rank one factor of $\mathcal{G}$, then $\mathrm{H}(\Gamma, N\backslash G, \alpha)$ may be non-trivial, but its every element is extendable to a cocycle of the whole G-action twisted by the extension of α guaranteed by Theorem 3.4.

Notice that in the case when $\mathcal{G}$ does not have rank one factors Theorems 3.4 and 3.5 give a complete description of all twisted and non-twisted cocycles of the Γ action on $N\backslash G$.

3.2. **Continuous cocycles.** The usual non-twisted cocycles, are known to show some rigidity on measurable ([6]), C^∞ and Hölder ([7], [5]) levels and to be highly unstable on continuous level. In sharp contrast to it, the non-trivially twisted cocycles show strong regularity already on the continuous level. This phenomena proves to be extremely useful in our studies of differential rigidity ([8]).

The following results are proved in [3]:

Theorem 3.6. *Let G be of $\mathbb{R}$-rank $n \geq 2$. Then, for continuous cocycles, there is an open set U in $\mathrm{H}(N, G/\Gamma)$ such that for any $\alpha \in U$,*

$$\mathrm{H}(N, G/\Gamma, \alpha) = 0.$$

Moreover, U contains all constant cocycles with the exception, may be, of some constant cocycles, which are proportional to the roots of $\mathcal{G}$.

Also, if α is constant and proportional to a root of $\mathcal{G}$, $\mathrm{H}(N, G/\Gamma, \alpha)$ is not necessarily trivial, but may consist of only constant cocycles.

Theorem 3.7. *Let G be of $\mathbb{R}$-rank $n \geq 2$. Then there exists an open set $V \in \mathrm{H}(\Gamma, N\backslash G)$, such that for any $\alpha \in V$, $\mathrm{H}(\Gamma, N\backslash G, \alpha) = 0$.*

Moreover V contains all elements of $\mathrm{H}(\Gamma, N\backslash G)$,with constant duals, with the exception, may be, of those whose dual is proportional to a root of $\mathcal{G}$.

In particular, notice that V contains “almost all” C^∞ cocycles.

3.3. **Hölder cocycles.** For Hölder cocycles one can obtain results analogous to our C^∞ results, in the non-twisted case, and our continuous results in the twisted one. However, for the non-twisted Hölder results we have to impose additional condition that G is split.

4. Differential rigidity.

It turns out that in the study of differentiable perturbations of the action often one can single out the cohomology class of twisted cocycles which is responsible for the existence or non-existence of smooth conjugacy. That is, the perturbed action is smoothly conjugate to the original action (or one of its “canonical” perturbations) iff the corresponding cocycle is cohomologically trivial. This takes place for projective actions on S^1, actions by rotations, and some other classes of actions.

In [8] we show that for C^3 actions on S^1 the cohomology class of the Schwarzian twisted cocycle is the obstacle to the existence of C^3 conjugacy to a projective action. This, together with our duality technique yields a very transparent proof of C^3 local differential rigidity for projective actions of cocompact lattices in $PSL(2, \mathbb{R})$:

Theorem 4.1. *Let $\Gamma \subset PSL(2, \mathbb{R})$ be a cocompact lattice. Let $\tilde{\Gamma}$ be a representation of Γ into $Diff^3(S^1)$ which is sufficiently C^1 close to Γ among a set of finitely many generators. Then there exists another cocompact lattice $\Gamma_1 \subset PSL(2, \mathbb{R})$ and a C^3 diffeomorphism $h \in Diff^3(S^1)$ such that $\tilde{\Gamma} = h^{-1}\Gamma_1 h$.*

E.Ghys, in [9], proved the C^∞ version of this result. And we learned that recently Ghys ([10]) has extended his result to a global one and proved the C^3 version as well. The main motivation for giving our proof is to exhibit the intricate geometric structure associated to these actions by a simple cocycle.

Outline of a proof of Theorem 4.1. Recall that for a diffeomorphism $f \in Diff^3(S^1)$, its Schwarzian derivative $S(f)$ is defined by

$$S(f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 = \left(\frac{f''(x)}{f'(x)}\right)' - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Its most important properties are:

1. It is projectively invariant; in fact, $S(f) = S(g)$ if and only if
$$g(x) = \frac{af(x) + b}{cf(x) + d}$$
for some numbers a, b, c, d such that $ad - bc = 1$.
2. $S(f) = 0$ if and only if $f \in SL(2, \mathbb{R})$.
3. Cocycle property: $S(f \circ g)(x) = S(f)(g(x))[g'(x)]^2 + S(g)(x)$.

We observe the following fact.

Proposition 4.1. *A diffeomorphism $f \in Diff^3(S^1)$ is conjugate to a projective transformation $g \in SL(2, \mathbb{R})$ via a C^3 diffeomorphism h if and only if its Schwarzian*

$S(f)$ is cohomologous to 0, i.e. there exists a continuous function $\sigma(x)$ on S^1 such that

$$S(f)(x) = \sigma(f(x))[f'(x)]^2 - \sigma(x).$$

Thus, if group G acts on S^1 then the question whether the action is C^3 conjugate to some projective action is equivalent to the question whether its Schwarzian twisted cocycle $S \in \mathrm{H}(G, \mathbb{R}, S^1, D^{-2})$, where D is derivative cocycle, is cohomologically trivial as continuous twisted cocycle.

Now, let $\Gamma \subset PSL(2, \mathbb{R})$ be a cocompact lattice. And let $\tilde{\Gamma} \subset Diff^3(S^1)$. If $\tilde{\Gamma}$ is sufficiently C^1 close to Γ among a set of finitely many generators, then by [11] there exists a homeomorphism $h \in Homeo(S^1)$ such that $\tilde{\Gamma} = h^{-1}\Gamma h$. Via the conjugacy h the Schwarzian cocycle S of $\tilde{\Gamma}$ can be pulled back to a twisted cocycle T of the Γ action: $\Gamma \times S^1 \to \mathbb{R}$,

$$T(\gamma, x) = S(h^{-1}\gamma h, h^{-1}(x)),$$

which satisfies

$$T(\gamma_1\gamma_2, x) = T(\gamma_2, x) + T(\gamma_1, \gamma_2(x))[\tilde{\gamma_2'}(h^{-1}(x))]^2 =$$
$$= T(\gamma_2, x) + T(\gamma_1, \gamma_2(x))[\gamma_2'(x)]^2\delta(\gamma_2, x),$$

where $\delta(\gamma, x) = [\tilde{\gamma}'(h^{-1}(x))/\gamma'(x)]^2$. Moreover, if the perturbation $\tilde{\Gamma}$ is C^1 close to Γ among its finitely many generators, then δ is close to 1 among those generators. To finish the proof of Theorem 4.1 all we have to do now is to prove that T is a continuous coboundary.

Let $G = PSL(2, \mathbb{R})$, P be the subgroup consisting of upper triangular matrices, and Γ, as before, a cocompact Fuchsian subgroup. Then the action by projective transformations is isomorphic to the right action of Γ on $P\backslash G$. Let $\Delta_1 = (\gamma'(x))^{-2}$ be the inverse of the square of the derivative cocycle for the Γ action. Let, $\Delta = \delta(\gamma, x)^{-1}\Delta_1$ — be its pullback. Then T is an element of $\mathrm{H}(\Gamma, P\backslash G, \Delta)$. Then we have:

$$\mathrm{H}(\Gamma, P\backslash G, \Delta) = \mathrm{H}^{tr}(\Gamma, P\backslash G, \Delta) \cong \mathrm{H}^{tr}(P, G/\Gamma, K(\Delta)).$$

Direct calculation shows that $K(\Delta_1)$ is constant cocycle equal to the forth power of the bigger eigenvalue of $p \in P$.

So, now instead of dealing with cocycles of Γ-action twisted by cocycle close to inverse of square of derivative cocycle ("complicated" group and "complicated" twisting) we have a much simpler object to work with — cocycles of P action twisted by a cocycle close to a known constant cocycle ("simple" group and "simple" twisting). And, indeed, we fairly easily prove that if $K(\Delta)$ is a continuous cocycle close enough to $K(\Delta_1)$ then $\mathrm{H}(P, G/\Gamma, K(\Delta)) = 0$, and thus $\mathrm{H}(\Gamma, P\backslash G, \Delta) = 0$.

5. Infinitesimal rigidity.

Recall that an action of a group G on a differentiable manifold X is called infinitesimally rigid if $\mathrm{H}^1(G, Vect(X)) = 0$, where $Vect(X)$ is the space of C^∞ vector fields on X (which is endowed with a structure of G-module in obvious way). The definition is, of course, motivated by Weil's local rigidity theorems ([12]) and the fact that $Vect(X)$ is the Lie algebra of the infinite dimensional Lie group of diffeomorphisms of X — $Diff(X)$, and the natural action of G on X is the composition of the action of G in $Diff(X)$ with the adjoint representation of $Diff(X)$ in its Lie algebra.

For different results on infinitesimal rigidity of lattice actions see [13] and [14].

In [3] we prove the following:

Theorem 5.1. *Let G be a semi-simple connected Lie group of $\mathbb{R}$-rank $n \geq 3$, with finite center, without compact factors and factors of rank one. Assume that G is split.*

Let M be the symmetric space corresponding to G. Let ∂M be its maximal boundary. Let Γ be an irreducible cocompact lattice in G.

Then, the natural action of Γ on ∂M is infinitesimally rigid.

Outline of the proof of Theorem 5.1. It is well known that ∂M can be identified with $P\backslash G$ ([15]), and the action of Γ on ∂M becomes an action by right multiplications on $P\backslash G$, where P is minimal parabolic subgroup in G.

From the Iwasawa decomposition we see that $P\backslash G$ is diffeomorphic to K — the maximal compact subgroup of G. In particular, $P\backslash G$ is paralelizable. Thus, the $Vect(\partial M)$ can be identified with $F(\partial M)$ — smooth functions on $P\backslash G$ with values in $\mathbb{R}^k$, where $k = dimK$.

Derivative of this action then can be written as a cocycle $D(\gamma, m)$ with values in $GL(m, \mathbb{R})$: $D(\gamma, m)$ is equal to the derivative of the transformation $m \to m\gamma^{-1}$, evaluated at point $m \in P\backslash G$ in the same coordinate system we used to trivialize the tangent bundle over $P\backslash G$.

The action of Γ on $Vect(\partial M)$ translates then into the following action on $F(\partial M)$:

$$\gamma \cdot f(m) = D(\gamma^{-1}, m\gamma^{-1}) f(m\gamma^{-1}) = D(\gamma, m)^{-1} f(m\gamma^{-1}), f(m) \in F(\partial M).$$

So, we see that this action gives $F(\partial M)$ the structure of twisted Γ-module $F_D(\partial M)$, and that the infinitesimal rigidity of Γ action on ∂M is equivalent to Γ having zero first cohomologies with coefficients in $F_D(\partial M)$. Or, passing, as usually, from cohomologies to twisted cocycles wee see that the infinitesimal rigidity is equivalent to $\mathrm{H}(\Gamma, \mathbb{R}^k, P\backslash G, D) = 0$.

First we prove that $D \in \mathrm{H}^{tr}(\Gamma, GL(k, \mathbb{R}), P\backslash G)$, and, thus, by the Theorem 3.1

$$\mathrm{H}(\Gamma, G, \overline{D}) = \mathrm{H}(\Gamma, G) = 0,$$

where $\overline{D}$ is lift of D to G. So, we have:

$$\mathrm{H}(\Gamma, P\backslash G, D) = \mathrm{H}^{tr}(\Gamma, P\backslash G, D) \cong \mathrm{H}^{tr}(P, G/\Gamma, K(D)).$$

It turns out that $K(D)$ is a constant cocycle, i.e. a representation of P. So, now, like in the proof of Theorem 4.1, we have to deal with reasonably uncomplicated group P and constant twisting. And, indeed, using smooth results on cohomological rigidity from Section 3 we manage to prove that every element of $\mathrm{H}(P, G/\Gamma, K(D))$ with the additional property of being smooth in group variable $p \in P$ (which is more then required by the definition of twisted cocycles) is a smooth coboundary.

Now, to finish the proof all we have to do is to notice that, obviously, every element of H^{tr}-spaces possesses this additional property.

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