ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume **1**, Issue 1, 1995

THE INTRINSIC INVARIANT OF AN APPROXIMATELY FINITE DIMENSIONAL FACTOR AND THE COCYCLE CONJUGACY OF DISCRETE AMENABLE GROUP ACTIONS

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Abstract. We announce in this article that i) to each approximately finite dimensional factor $\mathcal{R}$ of any type there corresponds canonically a group cohomological invariant, to be called the **intrinsic invariant** of $\mathcal{R}$ and denoted $\Theta(\mathcal{R})$, on which $\mathrm{Aut}(\mathcal{R})$ acts canonically; ii) when a group G acts on $\mathcal{R}$ via $\alpha : G \mapsto \mathrm{Aut}(\mathcal{R})$, the pull back of $\mathrm{Orb}(\Theta(\mathcal{R}))$, the orbit of $\Theta(\mathcal{R})$ under $\mathrm{Aut}(\mathcal{R})$,by α is a cocycle conjugacy invariant of α; iii) if G is a discrete countable amenable group, then the pair of the module, $\mathrm{mod}(\alpha)$, and the above pull back is a complete invariant for the cocycle conjugacy class of α. This result settles the open problem of the general cocycle conjugacy classification of discrete amenable group actions on an AFD factor of type III_1, and unifies known results for other types.

Introduction

The celebrated work of Connes, [1,3], surveyed in [2], Ocneanu's analysis, [12], and the previous work of Kawahigashi, and the second and third authors, [11], reveal the beatiful structure of the automorphism group of an approximately finite dimensional, or AFD, factor $\mathcal{R}$. In this note, we announce that it is possible to describe the structure of $\mathrm{Aut}(\mathcal{R})$ independently of the type of $\mathcal{R}$. This structure of $\mathrm{Aut}(\mathcal{R})$ enables us to define a cohomological invariant which we call the **intrinsic invariant** of $\mathcal{R}$. If a group G acts on $\mathcal{R}$ via α, then the pull back of the orbit of the intrinsic invariant of $\mathcal{R}$ under the natural action of the group $\mathrm{Aut}(\mathcal{R})$ gives a cocyle conjugacy invariant which is complete if G is a countable discrete amenable group. This completes the cocylce conjugacy classification of discrete amenable group actions on an AFD factor $\mathcal{R}$ including the type III_1 case.

Intrinsic Invariant and Main Theorem

Let $\mathcal{R}$ be an AFD factor, and let $\mathrm{Aut}(\mathcal{R})$ and $\mathrm{Int}(\mathcal{R})$ be the group of automorphisms and the group of inner automorphisms respectively. Let

$$\varepsilon : \alpha \in \mathrm{Aut}(\mathcal{R}) \longmapsto \dot{\alpha} \in \mathrm{Out}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R})/\mathrm{Int}(\mathcal{R})$$

Received by the editors May 17, 1995.
1991 *Mathematics Subject Classification.* 46L40.
This research is supported in part by NSF Grant DMS92-06984 and DMS95-00882, and also supported by the Australian Research Council Grant

be the canonical quotient map and set

$$\mathrm{Cnt}(\mathcal{R}) = \varepsilon^{-1}(\text{Center of } \mathrm{Out}(\mathcal{R})).$$

As $\mathrm{Int}(\mathcal{R})$ is not closed in $\mathrm{Aut}(\mathcal{R})$, we need to consider its closure $\overline{\mathrm{Int}}(\mathcal{R})$ and the quotient group $\mathrm{Mod}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R})/\overline{\mathrm{Int}}(\mathcal{R})$. We denote the canonical quotient map by $\mathrm{mod} : \alpha \in \mathrm{Aut}(\mathcal{R}) \longmapsto \mathrm{mod}(\alpha) \in \mathrm{Mod}(\mathcal{R})$. The map mod will be called the *module* and the image $\mathrm{mod}(\alpha)$ of $\alpha \in \mathrm{Aut}(\mathcal{R})$ in $\mathrm{Mod}(\mathcal{R})$ the *module* of α. Two more groups and a map are evidently associated with $\mathcal{R}$: the unitary group $\mathcal{U}(\mathcal{R})$, its center $\mathbf{T}$ which is the one dimensional torus group of complex numbers of modulus one, and the adjoint map:

$$\mathrm{Ad} : u \in \mathcal{U}(\mathcal{R}) \longmapsto \mathrm{Ad}(u) \in \mathrm{Int}(\mathcal{R}).$$

Of course, $\mathbf{T}$ is the kernel of the map Ad. Finally, we need to consider the flow of weights $F(\mathcal{R})$ and its cohomology groups: $B^1(F(\mathcal{R}))$, $Z^1(F(\mathcal{R}))$ and $H^1(F(\mathcal{R}))$. It is known,[5], that $\mathrm{Mod}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R})/\overline{\mathrm{Int}}(\mathcal{R})$ is canonically identified with $\mathrm{Aut}(F(\mathcal{R}))$. By the work of Wong, [15], the short exact sequence:

$$1 \longrightarrow \overline{\mathrm{Int}}(\mathcal{R}) \longrightarrow \mathrm{Aut}(\mathcal{R}) \longrightarrow \mathrm{Mod}(\mathcal{R}) \longrightarrow 1$$

splits, but not canonically. By [11; Theorem 1] and [5], there exists a canonical isomorphism from $H^1(F(\mathcal{R}))$ onto the Center of $\mathrm{Out}(\mathcal{R})$, which will be denoted by δ. These groups and maps are related as described in the following commutative diagrams of exact sequences:

$$\begin{array}{ccccccccc}
 & & & & \mathrm{Cnt}(\mathcal{R}) & \longrightarrow & \mathrm{H}^1(F(\mathcal{R})) & & \\
 & & & \nearrow & & & & \searrow & \\
1 & \longrightarrow & \mathrm{Int}(\mathcal{R}) & & \downarrow & & \downarrow \delta & & 1. \\
 & & & \searrow & & & & \nearrow & \\
 & & & & \mathrm{Aut}(\mathcal{R}) & \xrightarrow{\varepsilon} & \mathrm{Out}(\mathcal{R}) & &
\end{array}$$

The sequence involving $\mathrm{Cnt}(\mathcal{R})$ forms part of an exact square, as follows:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{T} & \longrightarrow & \mathcal{U}(F(\mathcal{R})) & \xrightarrow{\partial} & \mathrm{B}^1(F(\mathcal{R})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{U}(\mathcal{R}) & \longrightarrow & \widetilde{\mathcal{U}}(\mathcal{R}) & \longrightarrow & \mathrm{Z}^1(F(\mathcal{R}))) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Int}(\mathcal{R}) & \longrightarrow & \mathrm{Cnt}(\mathcal{R}) & \underset{\delta^{-1}\circ\varepsilon}{\longrightarrow} & \mathrm{H}^1(F(\mathcal{R})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Here, $\widetilde{\mathcal{U}}(\mathcal{R})$ is the semi direct product of $\mathcal{U}(\mathcal{R})$ by the extended modular action of $\mathrm{Z}^1(F(\mathcal{R}))$ as in [14]. Except for the lower right corner $\mathrm{H}^1(F(\mathcal{R}))$, all groups are Polish and all maps are continuous. As the above square of exact sequences is canonical, $\mathrm{Aut}(\mathcal{R})$ acts on the square, i.e. the above square is an equivariant square under the action of $\mathrm{Aut}(\mathcal{R})$. Let ν denote the map $\delta^{-1}\circ\varepsilon$, called the *modular invariant*. The middle vertical $\mathrm{Aut}(\mathcal{R})$ equivariant exact sequence of the above exact square:

$$1 \longrightarrow \mathcal{U}(F(\mathcal{R})) \longrightarrow \widetilde{\mathcal{U}}(\mathcal{R}) \longrightarrow \mathrm{Cnt}(\mathcal{R}) \longrightarrow 1$$

gives rise to a cohomological invariant, called the *characteristic invariant* $\chi \in \Lambda(\mathrm{Aut}(\mathcal{R}), \mathrm{Cnt}(\mathcal{R}), \mathcal{U}(F(\mathcal{R})))$. Thus we have the triplet:

$$(\mathrm{mod}, \chi, \nu) \in \mathrm{Hom}(\mathrm{Aut}(\mathcal{R}), \mathrm{Aut}(F(\mathcal{R})) \times \Lambda(\mathrm{Aut}(\mathcal{R}), \mathrm{Cnt}(\mathcal{R}), \mathcal{U}(F(\mathcal{R}))) \\ \times\ \mathrm{Hom}_{\mathrm{Aut}(\mathcal{R})}(\mathrm{Cnt}(\mathcal{R}), \mathrm{H}^1(F(\mathcal{R}))),$$

consiting of the action mod of $\mathrm{Aut}(\mathcal{R})$ on $F(\mathcal{R})$, the characteristic invariant and the $\mathrm{Aut}(\mathcal{R})$-equivariant homomorphism ν, which will be called the *intrinsic invariant* of the AFD factor $\mathcal{R}$ and denoted by $\Theta(\mathcal{R})$. Naturally, $\mathrm{Aut}(\mathcal{R})$ acts on $\mathrm{Hom}(\mathrm{Aut}(\mathcal{R}),$ $\mathrm{Aut}(F(\mathcal{R})))$, $\Lambda(\mathrm{Aut}(\mathcal{R}), \mathrm{Cnt}(\mathcal{R}), \mathcal{U}(F(\mathcal{R}))))$ and $\mathrm{Hom}_{\mathrm{Aut}(\mathcal{R})}(\mathrm{Cnt}(\mathcal{R})), \mathrm{H}^1(F(\mathcal{R})))$. Let $\mathrm{Orb}(\Theta(\mathcal{R}))$ denote the orbit of $\Theta(\mathcal{R})$ under the actionof $\mathrm{Aut}(\mathcal{R})$.

We are now at the position to state the main result:

Theorem 1. *Let $\mathcal{R}$ be an approximately finite dimensional separable factor and G be a countable discrete amenable group. If α is an action of G on $\mathcal{R}$, then the pull back $\alpha^*(\mathrm{Orb}(\Theta(\mathcal{R}))$ of the orbit of the intrinsic invariant is a complete invariant of the cocylce conjugacy class of α. More precisely, the inverse image, $N(\alpha) = \alpha^{-1}(\mathrm{Cnt}(\mathcal{R}))$, of $\mathrm{Cnt}(\mathcal{R})$ under α, the action* mod $\circ\,\alpha$ *of G on $F(\mathcal{R})$, the characteristic invariant $\chi(\alpha) \in \Lambda(G, N(\alpha), \mathcal{U}(F(\mathcal{R})))$ of α which is obtained as the pull back of $\Theta(\mathcal{R})$ and $\nu_\alpha \in \mathrm{Hom}_G(N(\alpha), \mathcal{U}(F(\mathcal{R})))$ determine the cocycle conjugacy class of α.*

It should be noted that this one theorem applies to all AFD factors of **any type**. Of course, the type of the carrier factor $\mathcal{R}$ affects on the nature of these invariants. We list the the invariants in each type as follows:
Type $\mathrm{I}_n, n \in \mathbf{N}$:

$$\mathrm{Int}(\mathcal{R}) = \mathrm{Cnt}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R}) \quad \textit{is compact}$$

$$\mathrm{Mod}(\mathcal{R}) = 1$$

$$G = N(\alpha),\ \chi_\alpha \in \mathrm{H}^2(G, \mathbf{T}), \nu_\alpha = 1.$$

Type I_∞:

$$\mathrm{Aut}(\mathcal{R}) = \mathrm{Cnt}(\mathcal{R}) = \mathrm{Int}(\mathcal{R}) \quad \textit{is not compact.}$$

$$\mathrm{Mod}(\mathcal{R}) = 1$$

$$G = N(\alpha),\ \chi_\alpha \in \mathrm{H}^2(G, \mathbf{T}), \nu_\alpha = 1.$$

Type II_1:

$$\mathrm{Int}(\mathcal{R}) = \mathrm{Cnt}(\mathcal{R}),\ \overline{\mathrm{Int}}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R}),\ \mathrm{Mod}(\mathcal{R}) = 1.$$

Therefore the characteristic invariant $\chi_\alpha \in \Lambda(G, N(\alpha), \mathbf{T})$ alone determines the cocylcle conjugacy of the action α of G.

Type II_∞:

$$\mathrm{Int}(\mathcal{R}) = \mathrm{Cnt}(\mathcal{R}), \quad F(\mathcal{R}) = \{L^\infty(\mathbf{R}), \textit{Translation}\}, \quad \mathrm{H}^1(F(\mathcal{R})) = 1,$$

$$\mathrm{Mod}(\mathcal{R}) = \mathbf{R}_+^*, \qquad \mathrm{Aut}(\mathcal{R}) = \overline{\mathrm{Int}}(\mathcal{R}) \rtimes \mathbf{R}_+^*$$

$$\widetilde{\mathcal{U}}(\mathcal{R}) = \mathcal{U}(\mathcal{R}) \times \mathcal{U}(L^\infty(\mathbf{R}))/\mathbf{T}.$$

Type III_0: All invariants are non-trivial in general.

$$\mathrm{Aut}(\mathcal{R}) = \overline{\mathrm{Int}}(\mathcal{R}) \rtimes \mathrm{Aut}(F(\mathcal{R})) \quad \text{by} \quad \text{Wong, [23];}$$

$$\widetilde{\mathcal{U}}(\mathcal{R}) = \mathcal{U}(\mathcal{R}) \rtimes \mathbf{Z}^1(F(\mathcal{R})) \quad \text{by} \quad [22].$$

Type $\mathrm{III}_\lambda, 0 < \lambda < 1$:

$$\mathcal{F}(\mathcal{R}) = \{L^\infty(\mathbf{R}/(-\log(\lambda)\mathbf{Z}), \textit{Translation}\},$$
$$\mathrm{H}^1(F(\mathcal{R})) = \mathbf{R}/T\mathbf{Z} \quad \textit{with} \quad T = -2\pi/\log\lambda;$$

$$\mathrm{Cnt}(\mathcal{R}) = \mathrm{Int}(\mathcal{R})\sigma(\mathbf{R}) \text{ where } \sigma(\mathbf{R}) = \text{ Modular Automorphism Group;}$$

$$\widetilde{\mathcal{U}}(\mathcal{R}) = (\mathcal{U}(\mathcal{R}) \rtimes Z^1(F(\mathcal{R})), \quad \mathrm{Aut}(\mathcal{R}) = \overline{\mathrm{Int}}(\mathcal{R}) \rtimes \mathbf{R}/(\log\lambda)\mathbf{Z};$$

and

$$Z^1(F(\mathcal{R})) = (\mathcal{U}(L^\infty(\mathbf{R}/(-\log(\lambda)\mathbf{Z})/\mathbf{T}) \rtimes \mathbf{T}.$$

Type III_1:

$$\mathcal{F}(\mathcal{R}) = \{\mathbf{C}, \textit{Trivial action of } \mathbf{R}\}, \overline{\mathrm{Int}}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R}), \ \mathrm{Mod}(\mathcal{R}) = 1;$$

$$\widetilde{\mathcal{U}}(\mathcal{R}) = \mathcal{U}(\mathcal{R}) \rtimes \mathbf{R}, \qquad \mathrm{H}^1(F(\mathcal{R})) = \mathbf{R}, \qquad \mathrm{Cnt}(\mathcal{R}) = \mathrm{Int}(\mathcal{R}) \rtimes \mathbf{R}.$$

It is interesting to note that the structure of the invariants in the type III_1case is simplest among type III cases yet the proof is the hardest. Special cases of the result in the III_1case have been established in [11]. The general case will appear in [10]. We state it here as an independent result:

Corollary 2. *If $\mathcal{R}$ is an AFD factor of type III_1, then with $N = \alpha^{-1}(\mathrm{Cnt}(\mathcal{R}))$ the pair*

$$(\chi_\alpha, \nu_\alpha) \in \Lambda(G, N, \mathbf{T}) \times \mathrm{Hom}_G(N, \mathbf{R})$$

is a complete invariant for the cocycle conjugacy class of the action α of a countable discrete amenable group G on $\mathcal{R}$. Every element $(\chi, \nu) \in \Lambda(G, N.\mathbf{T}) \times Hom_G(N, \mathbf{R})$ arises as the invariant of an action of G on $\mathcal{R}$.

References

[1] Connes, A., *Outerconjugacy classes of automorphisms of factors*, Ann. Sci. École Norm. Sup. **8** (1975), 383-419.

[2] Connes, A., *On the classification of von Neumann algebras and their automorphisms*, Symposia Math. **XX** (1976), 435-478.

[3] Conne, A., *Periodic automorphisms of the hyperfinite factor of type* II_1, Acta Sci. Math. **39** (1977), 39-66.

[4] Connes, A., *Factors of type* III_1*, property* L'_λ *and closure of inner automorphisms*, J. Operator Theory **14** (1985), 189-211.

[5] Connes, A. & Takesaki, M, *The flow of weights on factors of type III*, Tohoku Math. J. **29** (1977), 473-555.

[6] Haagerup, U., *Connes' bicentralizer problem and uniqueness of the injective factor of type* III_1, Acta Math. **158** (1987), 95-147.

[7] Haagerup, U. & Størmer, E., *Pointwise inner automorphisms of von Neumann algebras* with an appendix by C. Sutherland, J. Funct. Anal. **92** (1990), 177-201.

[8] Jones, V. F. R., *Actions of finite groups on the hyperfinite type* II_1 *factor*, Mem. Amer. Math. Soc. **237** (1980).

[9] Jones, V. F. R. & Takesaki, M., *Actions of compact abelian groups on semifinite injective factors*, Acta Math. **153** (1984), 213-258.

[10] Katayama, Y., Sutherland, C. E. & Takesaki, M., *The intrinsic invariant of an approximately finite dimensional factor and the cocycle conjugacy of discrete amenable group actions* , to appear.

[11] Kawahigashi, Y., Sutherland, C. E. & Takesaki, M., *The structure of the automorphism group of an injective factor and the cocycle conjugacy of discrete abelian group actions*, Acta Math. **169** (1992), 105-130.

[12] Ocneanu, A., *Actions of discrete amenable groups on factors*, vol. 1138, Lecture Notes in Math., Springer, Berlin, 1985.

[13] Sutherland, C. E. & Takesaki, M., *Actions of discrete amenable groups and groupoids on von Neumann algebras*, RIMS Kyoto Univ. **21** (1985), 1087-1120.

[14] Sutherland, C. E. & Takesaki, M., *Actions of discrete amenable groups on injective factors of type* III_λ*,* $\lambda \neq 1$, Pacific J. Math. **137** (1989), 405-444.

[15] Wong, S. Y. R., *On the dictionary between ergodic transformations, Krieger factors and ergodic flows*, Thesis, Univ. Newsouth Wales (1986), 72 + v.

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