ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 1, Issue 2, 1995

ON COMPOSANTS OF SOLENOIDS

RONALD DE MAN

(Communicated by Krystyna Kuperberg)

Abstract. It is proved that any two composants of any two solenoids are homeomorphic.

1. Introduction

Solenoids were introduced by Van Dantzig [5]. His original description runs as follows. Let $P = (p_1, p_2, \dots)$ be a sequence of primes. The solenoid S_P is the intersection of a descending sequence of solid tori $T_1 \supset T_2 \supset T_3 \supset \dots$ such that T_{i+1} is wrapped around inside T_i longitudinally p_i times without folding back. Van Heemert proved that solenoids are indecomposable continua [8].

For solenoids a classification theorem exists, conjectured by Bing [3] and proved by McCord [10], giving necessary and sufficient conditions for two solenoids S_P and S_Q to be homeomorphic (see also [2]).

The composants of solenoids coincide with the arc components. Since solenoids are topological groups, any two composants of the same solenoid are homeomorphic. The main theorem of this paper is

Theorem 1. *Any two composants of any two solenoids are homeomorphic.*

The composants of solenoids are examples of orbits in dynamical systems that are not locally compact. A locally compact orbit is either a singleton, a simple closed curve, or a topological copy of the real line. For orbits which are not locally compact the situation is much more complicated. Fokkink has proved the existence of uncountably many of them [6, 7].

In [4] Bandt shows that any two nonzero composants of the bucket handle are homeomorphic. Following a suggestion of Fokkink, we shall adapt the ideas from that article to prove the result of this paper.

For the proof, we need a different description of solenoids. We define the cascade (C_P, σ) as follows. C_P is the Cantor set represented as the topological product $C_P = \prod_{i=1}^{\infty} \overline{p}_i$ of discrete spaces $\overline{p}_i = \{0, 1, \dots, p_i - 1\}$. The homeomorphism $\sigma : C_P \to C_P$ has the form

$$\sigma(x_1, x_2, \dots) = (x_1 + 1, x_2, x_3, \dots) \quad \text{if } x_1 < p_1 - 1,$$
$$\sigma(p_1 - 1, \dots, p_{k-1} - 1, x_k, \dots) = (0, 0, \dots, 0, x_k + 1, \dots) \quad \text{if } x_k < p_k - 1,$$
$$\sigma(p_1 - 1, p_2 - 1, \dots) = (0, 0, \dots).$$

Now we define the solenoid S_P to be the suspension $\Sigma(C_P, \sigma)$, obtained from the product $C_P \times [0, 1]$ by identifying each $(x, 1)$ with $(\sigma(x), 0)$.

Received by the editors June 26, 1995.
1991 *Mathematics Subject Classification.* Primary 54F15; Secondary 54F65, 54H20.

Since C_P is the topological group of P-adic integers and the map σ corresponds to the addition of $(1, 0, 0, \dots)$, it is easy to see that the composant of S_P containing the point zero can be represented as a suspension $\Sigma(\mathbf{Z}, \tau)$ where $\mathbf{Z}$ is the set of integers with the P-adic topology and $\tau : \mathbf{Z} \to \mathbf{Z}$ is defined by $\tau(x) = x + 1$.

In proving the theorem, we may clearly confine ourselves to the case of the composants of the zero point of the solenoids S_P and S_Q, where $P = (2, 2, \dots)$ and $Q = (q_1, q_2, \dots)$ is an arbitrary sequence of primes. We now describe these composants.

Let I denote the set of integers with the 2-adic topology. Then a local basis for $x \in I$ is given by $\{x + U_n\}_{n \geq 0}$ where $U_n = 2^n\mathbf{Z}$. Now $\Sigma(I, \tau)$ (with $\tau(x) = x + 1$) represents the composant of the zero point of the 2-solenoid $S_{(2,2,\dots)}$. Similarly, we let $\Sigma(J, \tau)$ represent the composant of the zero point of the solenoid S_Q by setting $J = \mathbf{Z}$ and taking $\{x + V_m\}_{m \geq 0}$ with $V_m = q_1 \cdots q_m\mathbf{Z}$ as a local basis for $x \in J$.

For a clopen subset A of I, we define the return map $\tau_A : A \to A$ by $\tau_A(x) = \min\{y \in A : y > x\}$. If there exist clopen subsets A and B of I and J, respectively, and a homeomorphism $f : A \to B$ such that $f \circ \tau_A = \tau_B \circ f$, we say that (I, τ) and (J, τ) are *first return equivalent*. ¿From Theorem 5.2 in [1] it follows that (I, τ) and (J, τ) are first return equivalent if and only if $\Sigma(I, \tau)$ and $\Sigma(J, \tau)$ are homeomorphic.

Theorem 1 is proved by constructing clopen subsets A and B. In the following section we sketch this construction. The details will appear elsewhere [9].

2. Sketch of the construction

First we choose two sequences of integers $0 = n_0 < n_1 < n_2 < \dots$ and $0 = m_0 < m_1 < m_2 < \dots$ We let $\{0\} = I^0 \subset I^1 \subset I^2 \subset \dots$ denote a chain of subsets of I such that I^k consists of $2^{n_k - n_{k-1}}$ consecutive copies of I^{k-1}, one of which is I^{k-1} itself. Analogously we define a chain $\{0\} = J_0 \subset J_1 \subset J_2 \subset \dots$ of subsets of J where J^k consists of $q_{m_{k-1}+1} \dots q_{m_k}$ copies of J^{k-1}.

We want to construct clopen subsets A and B and a homeomorphism $f : A \to B$ such that $f \circ \tau_A = \tau_B \circ f$. Observe that this last condition is satisfied iff the bijection f is strictly increasing.

We shall construct f in such a way that for each integer k, f induces a bijection between $I^k \cap A$ and $J^k \cap B$. Now fix an integer k. Suppose that for each $v \in \mathbf{Z}$ we can find $w \in \mathbf{Z}$ such that

(i) $(I^k + v2^{n_{k+1}}) \cap A = (I^k \cap A) + v2^{n_{k+1}}$ and $(J^k + wq_1 \cdots q_{m_k}) \cap B = (J^k \cap B) + wq_1 \cdots q_{m_k}$, and

(ii) the diagram

$$\begin{tikzcd} I^k \cap A \arrow[r, "f"] \arrow[d] & J^k \cap B \arrow[d] \\ (I^k + v2^{n_{k+1}}) \cap A \arrow[r, "f"] & (J^k + wq_1 \cdots q_{m_k}) \cap B \end{tikzcd}$$

commutes.

Then clearly, for $x \in I^k \cap A$ we have $x + U_{n_{k+1}} \subset A$, and for $x \in I^k \setminus A$ we have $x + U_{n_{k+1}} \subset I \setminus A$. Moreover, for $x \in I^k \cap A$ we find that $f(x + U_{n_{k+1}}) \subset f(x) + V_{m_k}$. So if these conditions are satisfied for, say, all even integers k, we have that A is clopen and f is continuous.

We can give similar conditions (say for odd integers k) that imply that B is clopen and f^{-1} is continous.

These remarks indicate a way to construct A, B and f. Let us call the sets $I^k + v2^{n_k}$ and $J^k + wq_1 \cdots q_{m_k}$ *k-blocks* and the sets $I^k + v2^{n_{k+1}}$ (for k even) and $J^k + wq_1 \cdots q_{m_{k+1}}$ (for k odd) *k-return blocks*. By an *interval* of k-blocks we mean a union of consecutive k-blocks. We define $f : I^l \cap A \to J^l \cap B$ by induction on l, and start by setting $f(0) = 0$. At each step, we use the definition of f on the blocks I^k and J^k to define f on the k-return blocks that are included in I^l and J^l. This will not define f on the whole of I^l and J^l, but we simply do not include the remaining parts in A and B.

So the construction boils down to selecting a k-block for each k-return block. As an example, we shall now describe the extension of $f : I^3 \cap A \to J^3 \cap B$ to $f : I^4 \cap A \to J^4 \cap B$.

For each $k \in \{0, 1, 2, 3\}$ we take a look at the conditions described above. First take $k = 3$. There is only one 3-return block in J^4, and that is J^3 that is already mapped by f^{-1} to the 3-block I^3.

Now take $k = 2$. The block I^4 consists of 3-blocks, and each 3-block contains a 2-return block. For these return blocks we select 2-blocks in J^4.

Next, we let $k = 1$. The complement of J^3 and the chosen 2-blocks in J^4 is a union of intervals of 2-blocks. To each interval corresponds an interval of 2-blocks in I^4. For each of the 1-return blocks in such an interval we choose a 1-block in the corresponding interval.

Finally, we consider $k = 0$. Now we have intervals of 1-blocks in I^4, and each 1-block contains a 0-return block.

So far, we have neglected the possibility that at a certain stage in the construction there might not be enough k-blocks where the return blocks can be mapped to. We deal with this problem by showing that the ratio of the number of blocks in two corresponding intervals can be kept within certain bounds. By choosing the k-blocks to which the return blocks will be mapped in a linear fashion, at each step in the construction these ratios will have approximately the same value. Let $P_k = 2^{n_{k+1} - n_k}$ and $Q_k = q_{m_k+1} \cdots q_{m_{k+1}}$. Then a $(k+1)$-block in I (in J respectively) contains P_k (Q_k respectively) k-blocks. Now when we extend f to I^{l+1} and J^{l+1}, and we try to select the k-blocks, the ratio of the number of k-blocks in two corresponding intervals will be approximately

$$\frac{P_k P_{k+1} \cdots P_l}{Q_k Q_{k+1} \cdots Q_l}.$$

This value can be bounded by choosing the numbers n_k and m_k in such a way that $1/2 < 2^{n_k}/q_1 \cdots q_{m_k} < 2$, since then $1/4 < (P_k \cdots P_l)/(Q_k \cdots Q_l) < 4$ for all $k \leq l$. By taking the numbers P_k and Q_k large enough, we can keep the error made in this approximation under control.

Acknowledgements. The author would like to thank Jan Aarts, Robbert Fokkink and K.P. Hart for their valuable comments and suggestions.

References

[1] J.M. Aarts, *The structure of orbits in dynamical systems*, Fund. Math. 129 (1988), 39-58.

[2] J.M. Aarts and R.J. Fokkink, *The classification of solenoids*, Proc. Amer. Math. Soc. 111 (1991), 1161-1163.

[3] R.H. Bing, *A simple closed curve is the only homogeneous bounded plane continuum that contains an arc*, Canad. J. Math. 12 (1960), 209-230.
[4] C. Bandt, *Composants of the horseshoe*, Fund. Math. 144 (1994), 231-241.
[5] D. van Dantzig, *Ueber topologisch homogene Kontinua*, Fund. Math. 15 (1930), 102-125.
[6] R.J. Fokkink, *The structure of trajectories*, PhD Thesis, Delft 1991.
[7] R.J. Fokkink, *There exist uncountably many orbits in flows*, Fund. Math. 136 (1990), 147-156.
[8] A. van Heemert, *Topologische Gruppen und unzerlegbare Kontinua*, Compositio Math. 5 (1938), 319-326.
[9] A.J. de Man, *On composants of solenoids*, Fund. Math. 147 (1995), 181-188.
[10] M.C. McCord, *Inverse limit sequences with covering maps*, Trans. Amer. Math. Soc. 114 (1965), 197-209.

Faculteit Wiskunde en Informatica, TU Eindhoven, Postbus 513, 5600 MB Eindhoven, The Netherlands
E-mail address: deman@win.tue.nl