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GEODESIC LENGTH FUNCTIONS AND TEICHMÜLLER SPACES

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Abstract. Given a compact orientable surface Σ, let $\mathcal{S}(\Sigma)$ be the set of isotopy classes of essential simple closed curves in Σ. We determine a complete set of relations for a function from $\mathcal{S}(\Sigma)$ to $\mathbf{R}$ to be the geodesic length function of a hyperbolic metric with geodesic boundary on Σ. As a consequence, the Teichmüller space of hyperbolic metrics with geodesic boundary on Σ is reconstructed from an intrinsic combinatorial structure on $\mathcal{S}(\Sigma)$. This also gives a complete description of the image of Thurston's embedding of the Teichmüller space.

§1. Introduction

Let $\Sigma = \Sigma_{g,r}$ be a compact oriented surface of genus g with r boundary components. The Teichmüller space of isotopy classes of hyperbolic metrics with geodesic boundary on Σ is denoted by $T(\Sigma)$, and the set of isotopy classes of essential simple closed unoriented curves in Σ is denoted by $\mathcal{S} = \mathcal{S}(\Sigma)$. For each $m \in T(\Sigma)$ and $\alpha \in \mathcal{S}(\Sigma)$, let $l_m(\alpha)$ be the length of the geodesic representing α.

An interesting question is to characterize the geodesic length functions among all functions defined on $\mathcal{S}(\Sigma)$. We announce in this note a solution to this question.

The solution is expressed in terms of an intrinsic combinatorial structure on $\mathcal{S}(\Sigma)$.

Before stating the theorem, let us consider three basic examples which motivate the solution. We denote the isotopy class of a curve s by $[s]$, and $\{t \in \mathbf{R} | t > a\}$ by $\mathbf{R}_{>a}$.

Example 1. If the surface is the three-holed sphere $\Sigma_{0,3}$, then the set $\mathcal{S}(\Sigma_{0,3})$ consists of three elements which are the isotopy classes of the three boundary components of the surface. It is well known from the work of Fricke-Klein [FK] that any positive function from $\mathcal{S}(\Sigma)$ to $\mathbf{R}_{>0}$ is the geodesic length function of a unique element in the Teichmüller space.

For the rest of the note, we introduce the trace function $t_m(\alpha) = 2\cosh l_m(\alpha)/2$ from $\mathcal{S}(\Sigma)$ to $\mathbf{R}_{>2}$. We will deal with the trace function t_m instead of l_m.

Example 2. The surface is the one-holed torus $\Sigma_{1,1}$. Let $\mathcal{S}'$ be the set $\{[s] \in \mathcal{S} |$ s is not homotopic to the boundary $\partial\Sigma_{1,1}\}$. It is well known that $\mathcal{S}'$ is in one-one correspondence with the set of rational numbers $\mathbf{Q} \cup \{\infty\}$ where the map

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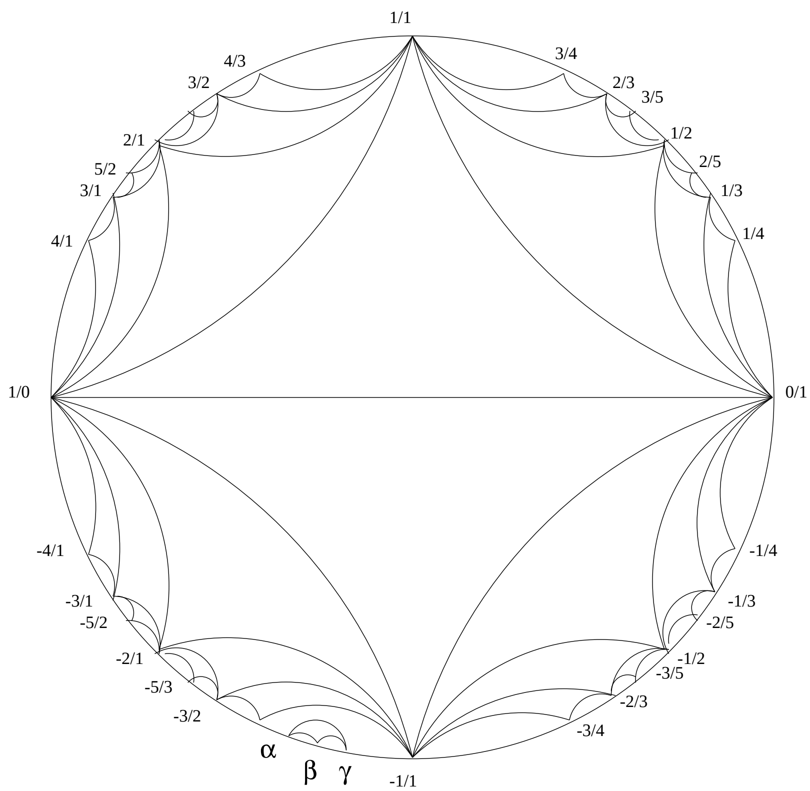


FIGURE 1

sends the isotopy class $[s]$ to the "slope" of $[s]$. Two rational numbers p/q and p'/q' satisfying $pq' - p'q = \pm 1$ correspond to two isotopy classes which contain two simple closed curves a and b intersecting at one point transversely. Thus the modular configuration comes in as a combinatorial structure on $\mathcal{S}'$. A result of Fricke-Klein proved by L. Keen [Ke] gives a solution to the characterization problem. Namely, a function $f : \mathcal{S}(\Sigma_{1,1}) \to \mathbf{R}_{>2}$ is the trace t_m for some m in the Teichmüller space if and only if

$$f(\alpha)f(\beta)f(\gamma) + 2 = f^2(\alpha) + f^2(\beta) + f^2(\gamma) + f([\partial\Sigma_{1,1}]), \tag{1}$$

$$f(\gamma) + f(\gamma') = f(\alpha) + f(\beta), \tag{1'}$$

where (α, β, γ) and (α, β, γ') are two ideal triangles in the modular configuration.

Example 3. The surface is the four-holed sphere $\Sigma_{0,4}$. Let $\mathcal{S}'$ be the set $\{[s] \in \mathcal{S}|$ s is not homotopic to the boundary $\partial\Sigma_{0,4}\}$. It is well known [De], [Th] that $\mathcal{S}'$ is in one-one correspondence with the set of rational numbers $\mathbf{Q} \cup \{\infty\}$ so that two rational numbers p/q and p'/q' satisfy $pq' - p'q = \pm 1$ if and only if the isotopy classes p/q and p'/q' are distinct and contain two simple closed curves a and b which intersect at two points. Thus again the modular configuration comes in as a combinatorial structure on $\mathcal{S}'$. It is shown in [Lu1] that a function $f : \mathcal{S}(\Sigma_{0,4}) \to$

$\mathbf{R}_{>2}$ is the trace t_m for some m in the Teichmüller space if and only if

$$
\begin{aligned}
&f(\alpha)f(\beta)f(\gamma)+4\\
&\qquad = f^2(\alpha)+f^2(\beta)+f^2(\gamma)+f(\alpha)(f(b_i)f(b_j)+f(b_k)f(b_l))\\
&\qquad\quad + f(\beta)(f(b_i)f(b_k)+f(b_j)f(b_l))+f(\gamma)(f(b_i)f(b_l)+f(b_j)f(b_k))\\
&\qquad\quad + \sum_{s=1}^{4} f^2(b_s)+f(b_1)f(b_2)f(b_3)f(b_4), \quad i\neq j\neq k\neq i,
\end{aligned} \tag{2}
$$

$$
f(\gamma)+f(\gamma')=f(\alpha)f(\beta)-f(b_i)f(b_l)-f(b_j)f(b_k) \tag{2'}
$$

where (α,β,γ) and (α,β,γ') are two ideal triangles in the modular configuration, and b_s's ($s=1,2,3,4$) are the isotopy classes of the four boundary components of $\partial\Sigma_{0,4}$ so that each of the triples $\{\alpha,\ b_i,\ b_j\}$, $\{\beta,\ b_i,\ b_k\}$ and $\{\gamma, b_i, b_l\}$ bounds a 3-holed sphere in Σ.

Our main result states that relations (1), (1′), (2) and (2′) are the set of all relations for a function from $\mathcal{S}(\Sigma)$ to $\mathbf{R}_{>2}$ to be the trace function t_m for an element m in the Teichmüller space.

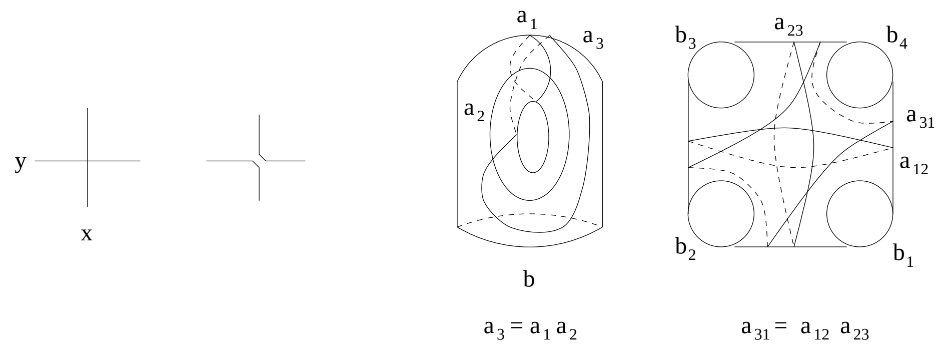
To be more precise, we introduce a combinatorial structure (corresponding to the modular configuration) on $\mathcal{S}$ as follows. Given two isotopy classes α and β, let $\mathrm{I}(\alpha,\beta)$ be the geometric intersection number between α and β in $\mathcal{S}(\Sigma)$, i.e., $\mathrm{I}(\alpha,\beta)$ $=\mathrm{Min}\{|a\cap b| \mid a\in\alpha \text{ and } b\in\beta\}$, where $|a\cap b|$ is the number of points in $a\cap b$. If two simple closed curves a and b intersect at one point transversely (resp. α, $\beta\in\mathcal{S}(\Sigma)$ with $\mathrm{I}(\alpha,\beta)=1$), we denote it by $a\perp b$ (resp. $\alpha\perp\beta$); if two simple closed curves a and b intersect at two points of different signs transversely and $\mathrm{I}([a],[b])=2$, we denote it by $a\perp_0 b$. In this case, we denote the relation between their isotopy classes by $[a]\perp_0[b]$. Suppose x and y are two arcs in Σ so that x intersects y transversely at one point. Then *the resolution of $x\cap y$ from x to y* is defined as follows. Take any orientation on x and use the orientation on Σ to determine an orientation on y. Now resolve the intersection point $x\cap y$ according to the orientations (see Figure 2(a)). If $a\perp b$ or $a\perp_0 b$, we define ab to be the curve obtained by resolving intersection points in $a\cap b$ from a to b. If $\alpha\perp\beta$ or $\alpha\perp_0\beta$, we define $\alpha\beta$ to be $[ab]$ where $a\in\alpha$ and $b\in\beta$ with $|a\cap b|=\mathrm{I}(\alpha,\beta)$. Note that in the examples 2 and 3, (γ,γ') is $(\alpha\beta,\beta\alpha)$.

Theorem. *For a surface $\Sigma_{g,r}$ of negative Euler number, a function l from $\mathcal{S}(\Sigma_{g,r})$ to $\mathbf{R}_{t>0}$ is the geodesic length function of a hyperbolic metric with geodesic boundary on $\Sigma_{g,r}$ if and only if*

(a) for any two simple closed curves a_1, a_2 with $a_1\perp a_2$, let $a_3=a_1a_2$ and b be the boundary of a regular neighborhood of $a_1\cup a_2$, then

$$
\begin{gathered}
t_1t_2t_3+2=t_1^2+t_2^2+t_3^2+t([b]),\\
t_3+t_3'=t_1t_2,
\end{gathered}
$$

where $t_i=2\cosh(l([a_i])/2)$ $(i=1,2,3)$, $t_3'=2\cosh(l([a_2a_1])/2)$, and $t([b])=2\cosh(l([b])/2)$, and



(a) (b)

FIGURE 2

(b) for any two simple closed curves a_{12} and a_{23} with $a_{12} \perp_0 a_{23}$, let $a_{31} = a_{12}a_{23}$ and b_1, b_2, b_3, b_4 be the four boundary components of a regular neighborhood of $a_{12} \cup a_{23}$ so that a_{ij}, b_i and b_j bound a subsurface $\Sigma_{0,3}$ $(i, j \leq 3)$, then

$$\begin{aligned} t_{12}t_{23}t_{31} + 4 = t_{12}^2 + t_{23}^2 + t_{31}^2 + t_{12}(t_1t_2 + t_3t_4) + t_{23}(t_2t_3 + t_1t_4) \\ + t_{31}(t_3t_1 + t_2t_4) + t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_1t_2t_3t_4, \end{aligned}$$

$$t_{31} + t'_{31} = t_{12}t_{23} - t_1t_3 - t_2t_4,$$

where $t_i = 2\cosh(l([b_i])/2)$, $t_{ij} = 2\cosh(l([a_{ij}])/2)$, *and* $t'_{31} = 2\cosh(l([a_{23}a_{12}])/2)$.

Figure 2 (surfaces have the right-hand orientations in the front face) shows the location of these curves. Note that the curves a_i and a_{ij} are symmetric in the sense that $[a_i][a_j] = [a_k]$ and $[a_{ij}][a_{jk}] = [a_{ki}]$ for $k \neq i, j$ and $(i, j) = (1, 2), (2, 3), (3, 1)$.

Relations (1), (1′), (2), and (2′) come from trace identities for SL(2,$\mathbf{R}$) matrices.

Thurston's compactification of the Teichmüller space $T(\Sigma)$ (see [Bo], [FLP], [Th]) uses the embedding $\pi : T(\Sigma) \to \mathbf{R}^{\mathcal{S}(\Sigma)}$ sending m to l_m. The theorem gives a complete description of the image of the embedding.

The proof of the theorem also shows the following result. Given a subset F of $\mathcal{S}(\Sigma)$, let $\pi_F : T(\Sigma) \to \mathbf{R}^F$ be the map $\pi_F(m) = l_m|_F$.

Corollary. *(a) For a surface $\Sigma_{g,r}$ of negative Euler number and $r > 0$, there exists a finite subset F in $\mathcal{S}(\Sigma_{g,r})$ consisting of $6g + 3r - 6$ elements so that the map $\pi_F : T(\Sigma_{g,r}) \to \mathbf{R}^F$ is a real analytic embedding onto an open subset which is defined by a finite set of (explicit) real analytic inequalities in the coordinates of π_F.*

(b) For a surface $\Sigma_{g,0}$ of negative Euler number, there exists a finite subset F of $\mathcal{S}(\Sigma_{g,0})$ consisting of $6g - 5$ elements so that $\pi_F : T(\Sigma_{g,0}) \to \mathbf{R}^F$ is an embedding whose image in $\mathbf{R}^F$ is defined by one real analytic equation and finitely many (explicit) real analytic inequalities in the coordinates of π_F.

It is shown by S. Wolpert [Wo] that the number $6g-5$ in part (b) of the corollary is minimal. The corollary without the statements about the image of the embedding was obtained previously by Okumura [Ok], Schmutz [Sc], and Sorvali [So]. Okumura [Ok1] has recently proven the corollary using a different method.

Some examples of the collection F and the images of the Teichmüller spaces are as follows. For $\Sigma_{2,0}$, take $F = \{[a_1], [a_2], [a_3], [a_4], [a_5], [a_6], [a_7]\}$ as in Figure 3. Then

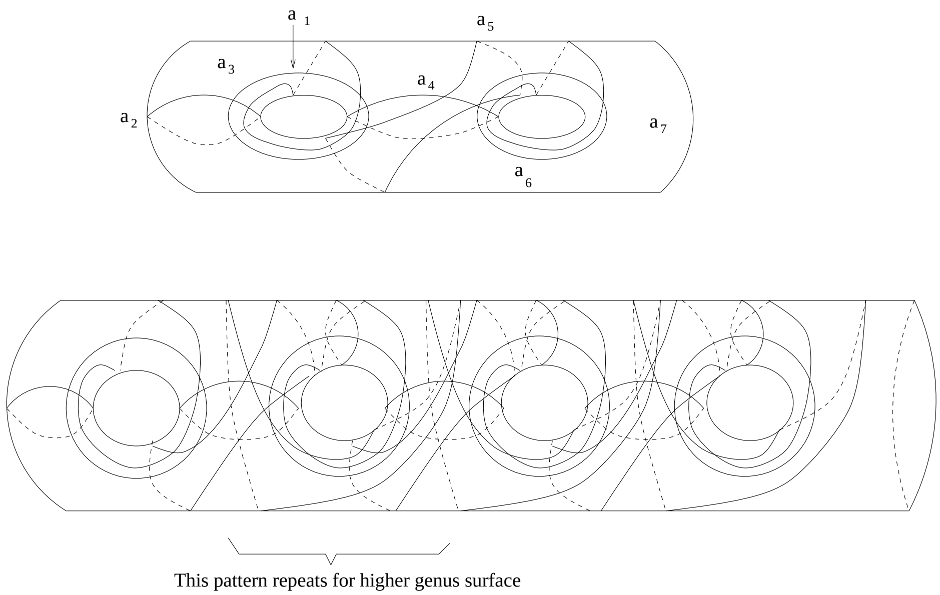


FIGURE 3

the map π_F is an embedding with image $\pi_F(T(\Sigma^0_{2,0})) = \{(t_1, t_2, t_3, t_4, t_5, t_6, t_7) \in \mathbf{R}^6_{>2} \mid t_8 > 2,\, t_9 > 2,\, t_8 = t_6t_7t_9 - t_6^2 - t_7^2 - t_9^2 + 2$, where $t_8 = t_1t_2t_3 - t_1^2 - t_2^2 - t_3^2 + 2$, and $(2 + t_2^2 + t_8)t_9^2 + 2t_2(t_4 + t_5)t_9 + 2t_2^2 + t_4^2 + t_5^2 + t_8^2 + t_2^2t_8 - t_4t_5t_8 - 4 = 0\}$.

Remark. The theorem and the corollary hold for hyperbolic metrics with cusp ends (see [Lu1]).

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§2. SKETCH OF THE PROOF OF THE THEOREM

There are several basic steps involved in the proof of the theorem. The proof is currently quite long. Below, we shall describe briefly the idea of the proof for a compact surface with non-empty boundary.

Step 1. We prove the result stated in Example 2 for $\Sigma_{0,4}$. This is essentially a careful application of the Maskit combination theorem [Mas] together with the trace relations for SL(2,$\mathbf{R}$) matrices.

Step 2. We prove a gluing lemma. The classical gluing lemma for surfaces is as follows. Take two hyperbolic surfaces X and Y with geodesic boundary so that they have the same lengths at two boundary components b_X and b_Y. Then one glues X to Y along b_X and b_Y by an isometry. However, the isometry is not unique due to the rotational symmetry of S^1. One obtains the so-called Fenchel-Nielsen twisting parameter for the gluing. We propose another procedure of gluing which will eliminate the parameter. Thus the result of the gluing produces a unique hyperbolic metric up to isotopy. The basic idea is as follows. Given a compact surface Σ, we decompose it as a union of two compact connected incompressible

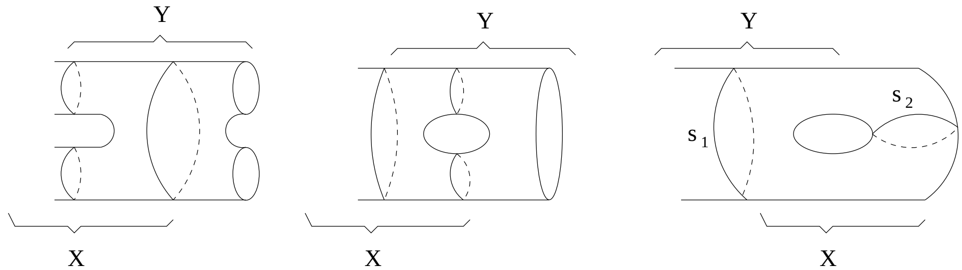

FIGURE 4

(meaning π_1-injective) subsurfaces X and Y so that $X \cap Y$ is homeomorphic to $\Sigma_{0,3}$ (see Figure 4). Let the three boundary components of $X \cap Y$ be a_1, a_2 and a_3. Then the gluing lemma states that for each hyperbolic metric m_X and m_Y on X and Y respectively so that a_i are geodesics in both metrics with $l_{m_X}(a_i) = l_{m_Y}(a_i)$, $(i = 1, 2, 3)$ there is a hyperbolic metric m in Σ unique up to isotopy so that the restriction of m to X is isotopic to m_X and the restriction of m to Y is isotopic to m_Y. The proof of the lemma is evident from the definition. For simplicity, we shall call this the gluing along 3-holed sphere lemma.

Step 3. We use the gluing along 3-holed sphere lemma to understand the hyperbolic metrics on $\Sigma_{1,2}$ by decomposing $\Sigma_{1,2}$ as a union $X \cup Y$, where $X \cong \Sigma_{1,1}$ and $Y \cong \Sigma_{0,4} - (b_1 \cup b_2)$, where b_1 and b_2 are the two boundary components. See Figure 4(c). A slight generalization of the version of the gluing lemma stated above is needed to take care of the non-compact Y.

Step 4. We set up the induction procedure as follows. Define the norm of a surface $\Sigma_{g,r}$ to be $3g+r$. Given a surface $\Sigma_{g,r}$, if $r \geq 2$, then $\Sigma_{g,r} = X \cup Y$, where $X = \Sigma_{g,r-1}$ and $Y = \Sigma_{0,4}$, and if $r = 1$, $\Sigma_{g,1} = X \cup Y$, where $X = \Sigma_{g-1,2}$ and $Y = \Sigma_{1,2}$ with $X \cap Y \cong \Sigma_{0,3}$ (see Figure 4). Thus given a function f from $\mathcal{S}(\Sigma_{g,r})$ to $\mathbf{R}_{>2}$ satisfying conditions (1), (1′), (2), and (2′), we consider the restrictions of f to $\mathcal{S}(X)$ and $\mathcal{S}(Y)$ (both are viewed as subsets of $\mathcal{S}(\Sigma)$ under the inclusion maps). By the induction hypothesis, there exist hyperbolic metrics m_X and m_Y on X and Y, respectively, so that $f|_{\mathcal{S}(X)} = t_{m_X}$ and $f|_{\mathcal{S}(Y)} = t_{m_Y}$. Now by the gluing lemma, there exits a metric m on Σ whose restriction to the two subsurfaces X and Y are isotopic to m_X and m_Y. Thus, we have constructed a hyperbolic metric m on Σ so that f is equal to t_m on the subset $\mathcal{S}(X) \cup \mathcal{S}(Y)$ of $\mathcal{S}(\Sigma)$.

Step 5. This is the key step in the proof. The goal is to show that the above condition $f|_{\mathcal{S}(X)\cup\mathcal{S}(Y)} = t_m|_{\mathcal{S}(X)\cup\mathcal{S}(Y)}$ implies $f = t_m$.

Our observation is that the equations (1′) and (2′) give rise to an iteration process.

This shows that the value of f at $\beta\alpha$ is determined by the values of f at α, β, $\alpha\beta$ in case $\alpha \perp \beta$, and the values of f at α, β, $\alpha\beta$ and b_s's in case $\alpha \perp_0 \beta$. For instance, to determine f on $\mathcal{S}'(\Sigma_{1,1})$, it suffices to know the value of f at three vertices of an ideal triangulation in the modular configuration since the iteration equation (1′) will take care of the values of f on $\mathcal{S}'$.

We shall illustrate the proof of this major step by considering the special case of $\Sigma = \Sigma_{0,5}$. Take two disjoint, essential, non-boundary parallel, non-homotopic

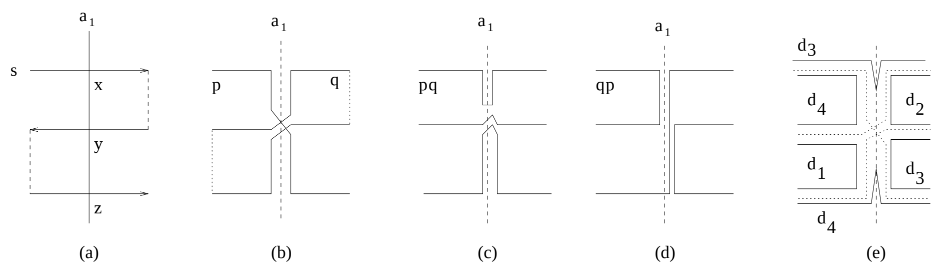

FIGURE 5

simple closed curves a_1 and a_2 in Σ. They give rise to a gluing along a 3-holed sphere decomposition of Σ, where we take X_i to be the 4-holed sphere bounded by a_i in Σ.

Now the given condition on f and t_m states that $f([a]) = t_m([a])$ for all simple closed curves a which are disjoint from either a_1 or a_2. A lemma of Lickorish below shows that if $f([a]) = t_m([a])$ holds for all simple closed curves a which intersect each a_i in at most two points, then $f = t_m$.

Lemma (Lickorish)**.** *Suppose s is an essential simple closed curve which intersects one of the two curves a_i in at least three points. Let the norm of a curve c be $|c \cap a_1| + |c \cap a_2|$. Then there exist two simple closed curves p and q with $p \perp_0 q$ so that $s = pq$ and the norms of the curves p, q, qp, and each of the four boundary components of a regular neighborhood $N(p \cup q)$ of $p \cup q$ are less than the norm of s.*

We sketch the proof of this lemma as follows. Suppose for definiteness that $|s \cap a_1| > 2$. Then since s is a separating simple closed curve, the intersection points of $s \cap a_1$ have alternating intersection signs in a_1. Pick up three adjacent intersection points x, y and z in a_1 and fix an orientation on s. Without loss of generality, we may assume that the arc from x to y (in curve s) in the orientation does not contain the point z. Then choose simple closed curves p and q as indicated in Figure 5. One verifies that the p, q and $\partial N(p \cup q)$ satisfy the condition in the lemma (see [Lu2] for more details on curves in surfaces).

Thus, to finish the proof of the theorem, by the lemma above and the iterate equation (2′), it suffices to show that $f([a]) = t_m([a])$ where a intersects each a_i at two points. There are only finitely many such a up to homeomorphisms of the surface leaving each a_i invariant. We verify the last condition $f([a]) = t_m([a])$ for $a \perp_0 a_i$ $(i = 1, 2)$ through iterated uses of the relations (2) and (2′).

This finishes the sketch of the proof.

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