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ON CHEREDNIK–MACDONALD–MEHTA IDENTITIES

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Abstract. In this note we give a proof of Cherednik's generalization of Macdonald–Mehta identities for the root system A_{n-1}, using representation theory of quantum groups. These identities give an explicit formula for the integral of a product of Macdonald polynomials with respect to a "difference analogue of the Gaussian measure". They were suggested by Cherednik, who also gave a proof based on representation theory of affine Hecke algberas; our proof gives a nice interpretation for these identities in terms of representations of quantum groups and seems to be simpler than that of Cherednik.

Introduction

In this note we give a proof of Cherednik's generalization of Macdonald–Mehta identities for the root system A_{n-1}, using representation theory of quantum groups. These identities, suggested and proved in [Ch2], give an explicit formula for the integral of a product of Macdonald polynomials with respect to a "difference analogue of the Gaussian measure". They can be written for any reduced root system, or, equivalently, for any semisimple complex Lie algebra $\mathfrak{g}$. Assuming for simplicity that $\mathfrak{g}$ is simple and simply laced, these identities are given by the following formula:

$$\frac{1}{|W|}\int \delta_k\overline{\delta_k}P_\lambda\overline{P_\mu}\gamma\,dx = q^{\lambda^2+(\mu,\mu+2k\rho)}P_\mu(q^{-2(\lambda+k\rho)}) \times q^{-2k(k-1)|R_+|}\prod_{\alpha\in R_+}\prod_{i=0}^{k-1}(1-q^{2(\alpha,\lambda+k\rho)+2i}) \tag{1}$$

where λ,μ are dominant integral weights, k is a positive integer, P_λ are Macdonald polynomials associated with the corresponding root system, with parameters $q^2, t = q^{2k}$(see [M1], [M2] or a review in [Kir2]), δ_k is the q-analogue of kth power of the Weyl denominator $\delta=\delta_1$:

$$\delta_k=\prod_{\alpha\in R^+}\prod_{i=0}^{k-1}(e^{\alpha/2}-q^{-2i}e^{-\alpha/2}), \tag{2}$$

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and γ is the Gaussian, which we define by

$$\gamma = \sum_{\lambda \in P} e^{\lambda} q^{\lambda^2}, \tag{3}$$

where P is the weight lattice. We consider γ as a formal series in q with coefficients from the group algebra of the weight lattice. In a more standard terminology γ is called the theta-function of the lattice P. All other notations, which are more or less standard, will be explained below.

These identities were formulated in the form we use in a paper of Cherednik [Ch2], who also proved them using double affine Hecke algebras (note: our notations are somewhat different from Cherednik's ones). We refer the reader to [Ch2] for the discussion of the history of these identities and their role in difference harmonic analysis.

The proof uses the following identity for the Gaussian, due to Kostant [Kos]:

$$\gamma = \left(\prod_{\alpha \in R_+} (1 - q^{2(\alpha,\rho)}) \right) \sum_{\nu \in P_+} q^{(\nu,\nu+2\rho)} (\dim_q L_\nu) \chi_\nu. \tag{4}$$

Here χ_ν is the character of the irreducible finite-dimensional module L_ν over $\mathfrak{g}$, and $\dim_q L_\nu := \chi_\nu(q^{2\rho})$ is the quantum dimension of L_ν.

Notations. We use the same notations as in [EK1], [EK2] with the following exceptions: we replace q by q^{-1} (note that this does not change Macdonald's polynomials) and we use the notation φ_λ for "generalized characters" (see below), reserving the notation χ_λ for usual (Weyl) characters. In particular, we define $\overline{e^\lambda} = e^{-\lambda}, \bar{q} = q$, and for $f \in \mathbb{C}_q[P]$, we define $f(q^\lambda), \lambda \in P$ by $e^\mu(q^\lambda) = q^{(\mu,\lambda)}$. For brevity, we also write λ^2 for (λ, λ). Finally, we denote by $\int \, dx : \mathbb{C}_q[P] \to \mathbb{C}_q$ the functional of taking the constant term: $\int e^\lambda \, dx = \delta_{\lambda,0}$.

The proof

In this section, we give a proof of the Cherednik–Macdonald–Mehta identities (1) for $\mathfrak{g} = \mathfrak{sl}_n$. The proof is based on the realization of Macdonald's polynomials as "vector-valued characters" for the quantum group $U_q\mathfrak{sl}_n$, which was given in [EK1]. For the sake of completeness, we briefly outline these results here, referring the reader to the original paper for a detailed exposition.

Let us fix $k \in \mathbb{Z}_+$ and denote by U the finite-dimensional representation of $U_q\mathfrak{sl}_n$ with highest weight $n(k-1)\omega_1$, where ω_1 is the first fundamental weight. We identify the zero weight subspace $U[0]$, which is one-dimensional, with $\mathbb{C}_q$.

For $\lambda \in P_+$, we denote by Φ_λ the unique intertwiner

$$\Phi_\lambda : L_{\lambda+(k-1)\rho} \to L_{\lambda+(k-1)\rho} \otimes U$$

and define the generalized character $\varphi_\lambda \in \mathbb{C}_q[P] \otimes U[0] \simeq \mathbb{C}_q[P]$ by $\varphi_\lambda(q^x) = \operatorname{Tr}_{L_{\lambda+(k-1)\rho}}(\Phi_\lambda q^x)$.

We can now summarize the results of [EK1] as follows:

$$\begin{gathered} \varphi_0 = \prod_{\alpha \in R_+} \prod_{i=1}^{k-1} (e^{\alpha/2} - q^{-2i} e^{-\alpha/2}) = \delta_k/\delta, \\ \varphi_\lambda/\varphi_0 = P_\lambda, \end{gathered} \tag{5}$$

where P_λ is the Macdonald polynomial with parameters $q^2, t = q^{2k}$.

We can also rewrite Macdonald's inner product in terms of the generalized characters as follows. Recall that Macdonald's inner product on $\mathbb{C}_q[P]$ is defined by

$$\langle f, g\rangle_k = \frac{1}{|W|}\int \delta_k \overline{\delta_k} f \bar{g}\, dx$$

(this differs by a certain power of q from the original definition of Macdonald). Obviously, one has

$$\langle P_\lambda, P_\mu\rangle_k = \langle \varphi_\lambda, \varphi_\mu\rangle_1.$$

In order to rewrite this in terms of representation theory, let ω be the Cartan involution in $U_q\mathfrak{sl}_n$ (see [EK1]). For a $U_q\mathfrak{sl}_n$-module V, we denote by V^ω the same vector space but with the action of $U_q\mathfrak{sl}_n$ twisted by ω. Note that for finite-dimensional V, we have $V^\omega \simeq V^*$ (not canonically). Similarly, for an intertwiner $\Phi : L \to L\otimes U$ we denote by Φ^ω the corresponding intertwiner $L^\omega \to U^\omega \otimes L^\omega$. Finally, for $\Phi_1 : L_1 \to L_1 \otimes U, \Phi_2 : L_2 \to L_2 \otimes U$, define $\Phi_1 \odot \Phi_2^\omega \in \mathrm{End}(L_1 \otimes L_2^\omega)$ as the composition $L_1 \otimes L_2^\omega \to L_1 \otimes U \otimes U^\omega \otimes L_2^\omega \to L_1 \otimes L_2^\omega$, where the first arrow is given by $\Phi_\lambda \otimes \Phi_\mu^\omega$, and the second by the invariant pairing $U \otimes U^\omega \to \mathbb{C}_q$ (which is unique up to a constant). Then it was shown in [EK1] that

$$(\varphi_\lambda \overline{\varphi_\mu})(q^x) = \mathrm{Tr}_V((\Phi_\lambda \odot \Phi_\mu^\omega) q^{\Delta(x)}) = \sum_{\nu\in P_+} \chi_\nu(q^x) C^\nu_{\lambda\mu}$$

where $V = L_{\lambda+(k-1)\rho} \otimes L^\omega_{\mu+(k-1)\rho}$ and $C^\nu_{\lambda\mu}$ is the trace of $\Phi_\lambda \odot \Phi_\mu$ acting in the multiplicity space $\mathrm{Hom}(L_\nu, V)$. As a corollary, we get the following result:

$$\frac{1}{|W|}\int \delta\bar{\delta}\varphi_\lambda\overline{\varphi_\mu}\Big(\sum_{\nu\in P^+} a_\nu \chi_\nu\Big)\, dx = \sum_{\nu\in P^+} a_{\nu^*} C^\nu_{\lambda\mu}, \tag{6}$$

where $\nu^* = -w_0(\nu)$ is the highest weight of the module $(L_\nu)^*$ (here w_0 is the longest element of the Weyl group).

Of course, the coefficients $C^\nu_{\lambda\mu}$ are very difficult to calculate. However, the formula above is still useful. For example, it immediately shows that $\langle\varphi_\lambda, \varphi_\mu\rangle_1 = 0$ unless $\lambda = \mu$, which was the major part of the proof of the formula $\varphi_\lambda/\varphi_0 = P_\lambda$ in [EK1]. It turns out that this formula also allows us to prove the Cherednik–Macdonald–Mehta identities.

Theorem 1. *Let φ_λ be the renormalized Macdonald polynomials for the root system A_{n-1} given by* (5)*, and let γ be the Gaussian* (3)*. Then*

$$\begin{aligned}\frac{1}{|W|}\int \delta\bar{\delta}\varphi_\lambda\overline{\varphi_\mu}\gamma\, dx &= q^{(\lambda+k\rho)^2} q^{(\mu+k\rho)^2}\varphi_\mu(q^{-2(\lambda+k\rho)}) \\ &\quad\times \Big(\prod_{\alpha\in R_+}(1-q^{2(\alpha,\rho)})\Big) q^{-2\rho^2}\|P_\lambda\|^2 \dim_q L_{\lambda+(k-1)\rho},\end{aligned} \tag{7}$$

where $\|P_\lambda\|^2 = \langle P_\lambda, P_\lambda\rangle_k$.

Proof. From (6) and (4), we get

$$\int \delta\bar{\delta}\varphi_\lambda\overline{\varphi_\mu}\gamma\, dx = \Big(\prod_{\alpha\in R_+}(1-q^{2(\alpha,\rho)})\Big)\sum_{\nu\in P^+} q^{(\nu,\nu+2\rho)}(\dim_q L_\nu) C^\nu_{\lambda\mu}. \tag{8}$$

On the other hand, let C be the Casimir element for $U_q\mathfrak{g}$ discussed above. Consider the intertwiner $(\Phi_\lambda \odot \Phi^\omega_\mu)\Delta(C) : V \to V$, where, as before, $V = L_{\lambda+(k-1)\rho} \otimes L^\omega_{\mu+(k-1)\rho}$. Then it follows from $C|_{L_\lambda} = q^{(\lambda,\lambda+2\rho)}$ that

$$\mathrm{Tr}_V((\Phi_\lambda \odot \Phi^\omega_\mu)\Delta(C)\Delta(q^{2\rho})) = \sum_{\nu\in P_+} C^\nu_{\lambda\mu} q^{(\nu,\nu+2\rho)} \dim_q L_\nu,$$

which is exactly the sum on the right-hand side of (8). On the other hand, using $\Delta(C) = (C \otimes C)(R^{21}R)$, we can write

$$\begin{aligned}&\mathrm{Tr}_V((\Phi_\lambda \odot \Phi^\omega_\mu)\Delta(C)\Delta(q^{2\rho}))\\ &\quad = q^{-2\rho^2} q^{(\lambda+k\rho)^2} q^{(\mu+k\rho)^2}\, \mathrm{Tr}_V((\Phi_\lambda \odot \Phi^\omega_\mu)(R^{21}R)\Delta(q^{2\rho})).\end{aligned}$$

This last trace can be calculated, which was done in [EK2, Corollary 4.2], and the answer is given by

$$\mathrm{Tr}_V((\Phi_\lambda \odot \Phi^\omega_\mu)(R^{21}R)\Delta(q^{2\rho})) = \varphi_\mu(q^{-2(\lambda+k\rho)})\|P_\lambda\|^2 \dim_q L_{\lambda+(k-1)\rho}.$$

Combining these results, we get the statement of the theorem. □

The norms $\|P_\lambda\|^2$ appearing on the right-hand side of (7) are given by Macdonald's inner product identities

$$\|P_\lambda\|^2 = \prod_{\alpha\in R_+} \prod_{i=1}^{k-1} \frac{1 - q^{-2(\alpha,\lambda+k\rho)-2i}}{1 - q^{-2(\alpha,\lambda+k\rho)+2i}},$$

which were conjectured in [M1], [M2] and proved for the root system A_{n-1} by Macdonald himself [M3]; see also [EK2] for the proof based on representation theory of $U_q\mathfrak{sl}_n$, and [Ch1] or a review in [Kir2] for a proof for arbitrary root systems. Using this formula and rewriting the statement of Theorem 1 in terms of Macdonald polynomials P_λ rather than φ_λ, we get the Cherednik–Macdonald–Mehta identities (1).

Remarks. 1. Note that the left-hand side of (7) is symmetric in λ, μ. Thus, the same is true for the right-hand side, which is exactly the statement of Macdonald's symmetry identity (compare with the proof in [EK2]).

2. The proof of Cherednik–Macdonald–Mehta identities given above easily generalizes to the case when q is a root of unity (see [Kir1] for the discussion of the appropriate representation-theoretic setup). In this case, we need to replace the set P_+ of all integral dominant weights by an appropriate (finite) Weyl alcove C (see [Kir1]), and the integral $\int \delta\bar{\delta} f\, dx$ should be replaced by $\mathrm{const} \sum_{\lambda\in C} f(q^{2(\lambda+\rho)}) \dim_q L_\lambda$. Using the following obvious property of the Gaussian:

$$\gamma(q^{2(\lambda+\rho)}) = q^{-(\lambda,\lambda+2\rho)}\gamma(q^{2\rho})$$

(which in this case coincides with formula (1.7) in [Kir1]), it is easy to see that in this case the Cherednik–Macdonald–Mehta identities are equivalent to

$$S^{-1}T^{-1}S = TST,$$

where the matrices S, T are defined in [Kir1, Theorem 5.4]. This identity is part of a more general result, namely, that these matrices S, T give a projective representation of the modular group $SL_2(\mathbb{Z})$ on the space of generalized characters (see [Kir1, Theorem 1.10] and references therein).

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